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A NEW DEFINITION OF FRACTIONAL DERIVATIVES AND RELATED CONTENTS

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On the Occasion of Professor Ron DeVore's 80th Birthday

Abstract. Starting from the concept of the Mincowsky's space-time, a revised version for the definition of fractional derivatives is given. Adopting the properties of the classical derivatives, the newly defined fractional derivatives not only possess the properties of locality, duality, translation invariant, Leibnitz's rule etc. but also serve as the coefficients of the generalized fractional Taylor expansion which the classical fractional derivatives failed to. The new fractional derivatives as well as the model of the fractional differential equations can be got by the local differential analysis.

Based on the newly defined generalized fractional Taylor expansion, the time dependent functions of life processes are analyzed by utilizing the fractional derivatives. The life process functions can be reproduced both by tracking the histories forward and backward. Meanwhile, numerical formulas for calculating the new fractional derivatives and some examples of the mathematical models for the fractional differential equations are given.

1. Introduction

The classical literatures on Mincowsky space-time denoted the metric of time by ict. This endows the Mincowsky space-time a uniform metric both in space and time direction. Furthermore, on the time axis, there is a causal relationship. Einstein referred to an event as a function $F(x, y, z, ict)$ on the four-dimensional space-time. However, only the real part of the complex function $F(x, y, z, ict)$ can be observed in our real world. In this paper, we discuss the space coordinate free function $F : Im(C) \to C$, which only depends on time. We discuss $F(ict)$ or $F(it)$ for simplify.

Definition 1.1. A function $f(t) = Re(F(it))$ is the restriction of $F : Im(C) \to C$ which is from $Im(C)$ to $Re(C)$. Because in the real space, what we can observe is only the real part of the function $F(it)$.

Example 1.2. $F(it) = -i(it)(1+i(it)) = t(1-t)$ $(t \in (-\infty, \infty))$ is such a function and $f(t) = t(1-t)$ $(t \in (-\infty, \infty))$ is the function that we can observe in the real

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space for all time. $F(it) = \sqrt{-i(it)(1+i(it))} = \sqrt{t(1-t)}$ $(t \in (-\infty, \infty))$ is such a function too. However, in the real space, the function can be observed only in the time interval $[0,1]$. $f(t)$ will vanish in real space when $t < 0$ or $t > 1$. Therefore we have $f(t) = \sqrt{t_+(1-t)_+}$.

We discuss the life process function $f(t)$ to describe the occurrence and the developing progress of an event (e.g. the birth of life and its whole life process). The life process function $f(t)$ can not be analytic and must possess singularity at the occurrences and vanishing time points of the events. A life process exists in a finite interval, so a life process function should be compactly supported. For example, the function $\sqrt{t_+(1-t)_+}$ can perhaps be a candidate to describe a single life process such as the infect ability of one single virus. Moreover, a life process constructed by some sub-life processes can perhaps be described as a linear combination of Beta-functions

$$\sum c_j \frac{(t-t_j)_+^{\alpha_j}(t_j+d_j-t)_+^{\beta_j}}{\Gamma(1+\alpha_j)\Gamma(1+\beta_j)}, \quad d_j > 0, \tag{1.1}$$

where the j-th term of the life process function describes the j-th sub-life with the birth time t_j, the life length d_j and the death time $t_j + d_j$. The coefficient c_j characterizes the scale of the j-th sub-life, $c_j > 0$ means perhaps a new gene mutation of the virus with stronger pathogenicity occurs, while $c_j < 0$ means perhaps a vaccine is invented and used to the anti-spread of the virus. The power α_j $(\alpha_j > 0, \text{not an integer})$ and β_j $(\beta_j > 0, \text{not an integer})$ are the degrees of the birth and death of the j-th sub-life respectively. This is a simple example of the life process function under the Mincowsky's space-time framework, which is the summation of several sub-life processes.

If t_j acts as the occurrence time of the j-th event, the corresponding degree α_j can not be an integer, otherwise we shall have already observed the j-th event before time t_j. On the other hand, according to the conservation law, we can consider that the event (e.g. virus) stems from the imaginary space at time t_j.

For a life process function $f(t)$ with a given fractional power α, to discuss the behaviors of the fractional polynomial $f(t) = ct^\alpha$, we should recall the fractional derivatives [1] [2] [3].

The classical definition of fractional derivative was given independently by Riemann and Liouville. Hence, it goes by "the Riemann-Liouville definition" as follows

$$D^\alpha f(t) = \frac{d}{dt}\int_a^t f(s)(t-s)^{-\alpha} ds/\Gamma(1-\alpha), \qquad \alpha \in (0,1). \tag{1.2}$$

To see how the fractional integral operates in the formula (1.2), one begins with the integer order integral satisfying:

$$I_a^n f(t) = \int_a^t \int_a^{t_1} \cdots \int_a^{t_{n-1}} f(t_n) dt_n \cdots dt_1 = \int_a^t f(s)(t-s)^{n-1} ds/\Gamma(n).$$

Then one can generalize the above equation to get the fractional integral as follows

$$I_a^\alpha f(t) = \int_a^t f(s)(t-s)^{\alpha-1} ds/\Gamma(\alpha).$$

Thus, the Riemann-Liouville derivative of fractional order $\alpha\,(\alpha \in (0,1))$ can be defined as

$$D^{\alpha} := \frac{d}{dt} I^{1-\alpha}.$$

The operator D^{α} can be generalized for $\alpha > 1$,

$$D^{\alpha} := \frac{d^{\lfloor \alpha \rfloor + 1}}{dt^{\lfloor \alpha \rfloor + 1}} I^{\lfloor \alpha \rfloor + 1 - \alpha},$$

where $\lfloor \alpha \rfloor$ denotes the integer part of α.

The Riemann-Liouville definition of fractional derivatives possesses some uncommon properties compared to the classical integer derivatives. Some of them are even counter-intuitive. We make a few specific remarks below.

Remark 1.3. 1. Meaning of the lower bound: the Riemann-Liouville definition of fractional derivatives depends on the lower bound a of the integral in the formula (1.2), which could lead to ambiguity in the calculation of the value of the fractional derivative at t. Moreover, the physical meaning of the lower bound a is unclear.

2. Translation invariant: the definition of the fractional derivatives is not translation invariant. Generally speaking, we have

$$D^{\alpha} f(t+c) \neq D^{\alpha} f(t)|_{t=t+c}.$$

3. Locality: to find a fractional derivative, one needs the function values on the whole interval (a, t). This is in sharp contrast to the classical definition of the derivative, which is only a local concept. That is, one needs only to know the function values in an arbitrarily small neighborhood of t to calculate the classical derivative at t.

4. Null space: for any given $b > c > a$ (a is the lower bound of the integral of the Riemann-Liouville fractional derivatives), the functions $f_c(t) = (t-c)_+^{\alpha}/\Gamma(1+\alpha)$ satisfy the equation

$$D^{1+\alpha} f_c(t) = 0, \quad \text{for} \quad t \in (b, \infty),$$

which shows that the null space of the operator $D^{1+\alpha}$ for $t > b$ is infinitely dimensional. While the null space of classical derivative operator is always finitely dimensional.

5. Duality: the classical derivative possesses a dual functional that

$$D^{k}(t^{j}/j!)|_{t=0} = \delta_{jk}.$$

Therefore, this leads to the famous Taylor expansion for classical derivatives

$$f(t+\Delta t) = \sum f^{(k)}(t)(\Delta t)^{k}/k! + \text{Remainder},$$

which is an important technique to approximate the functions. Nevertheless, the Riemann-Liouville fractional derivatives do not enjoy the duality. This reinforces the notion that the Riemann-Liouville fractional derivative is intrinsically not local.

6. The numerical computation of Riemann-Liouville fractional derivatives runs into a singular integral, for which the classical numerical algorithms are unstable. This brings considerable difficulties in the implementation of the procedure.

In what follows, we will discuss and analyze these six items at issue.

2. Explanations of the life process functions

As a rule of thumb, we are not endowed with the right to choose arbitrarily the lower bound a of the integral in (1.2). This is the reason why there are restrictions on the behaviors of the underlying function $f(t)$ at the point a in the definition of Riemann-Liouville fractional derivatives. Another classical definition for fractional integrals is due to Riesz, in which he proposed the formula:

$$I^{\alpha} f(t) = \int_{-\infty}^{t} f(s)(t-s)^{\alpha-1} ds/\Gamma(\alpha), \quad \alpha > 0.$$

In an earlier paper by the authors [4], concerning the Fourier transform of radial basis functions, another definition for fractional integrals and derivatives was proposed. They took $-\infty$ as the lower bound of the integral, and proved that the set of fractional differential operators is a semigroup. If the underlying function is identically zero on the whole interval $(-\infty, a]$, i.e., nothing has happened during this time period, two definitions (Riesz and Riemann-Liouville) give the same result. Nevertheless, setting the lower bound of the integral to be $-\infty$, we are able to clearly explain the presence of the factor $1/\Gamma(\alpha)$ in the definition of fractional derivatives [5]. Let $s := t - s$. Then

$$\begin{aligned} I^{\alpha} f(t) &= \int_0^{\infty} f(t-s)(s)^{\alpha-1} ds/\Gamma(\alpha) \\ &= e^t \int_0^{\infty} [e^{-(t-s)} f(t-s)] \cdot [\tfrac{e^{-s} s^{\alpha-1}}{\Gamma(\alpha)}] ds. \end{aligned}$$

Notice that $\int_0^{\infty} e^{-s} s^{\alpha-1} ds/\Gamma(\alpha) = 1$ is the integral of the density function of the Gamma distribution.

Theorem 2.1. *Taking the α-th order fractional integral is equivalent to first taking the average of Gamma distribution of order α to the function $f(t)e^{-t}$ and then multiplying e^t.*

In the following, we take $-\infty$ as the lower bound of the integral in the fractional differential and integral operators (1.2).

Suppose that we have a function $F : Im(C) \to C$ defined in the Mincowsky space-time and its restriction from $Im(C)$ to $Re(C)$ takes the form:

$$f(t) = \sum_{j=0}^{N} c_j \frac{(t-t_j)_+^{\alpha_j} (t_j + d_j - t)_+^{\beta_j}}{\Gamma(1+\alpha_j)\Gamma(1+\beta_j)}, d_j > 0, \tag{2.1}$$

where t_j $(t_j \leq t_{j+1})$ is the birth time, d_j is the life length and $t_j + d_j$ is the death time of the j-th event.

As aforementioned, a life process function $f(t)$ manifests itself near t_j in two different parts, analytic and non analytic. The analytic part can be treated by classical method such as the Taylor expansion. The non analytic part should be treated separately by considering the jump of left and the right derivative at this point. The current paper is mainly concerned with describing the occurrence of new events by utilizing right fractional derivative, as opposed to the dying-out of events described by left fractional derivative.

In the following, we discuss such life process functions in which series of events had occurred in the past and continue to have effect on the present time and the dying-out of events is ignored. Thus those functions can be written as

$$f(t) = \sum_{j=0}^{N} c_j \frac{(t - t_j)_+^{\alpha_j}}{\Gamma(1 + \alpha_j)}, \quad t_j \leq t_{j+1}. \tag{2.2}$$

The term $\sum_{k=0}^{j-1} c_k \frac{(t-t_k)_+^{\alpha_k}}{\Gamma(1+\alpha_k)}$ at t_j in (2.2) can be considered to be the analytic continuation of the effect of past events and have both their "causing" and "resulting", whereas the term $c_j \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$ indicates the occurrence of new event at time t_j.

The left and right derivatives of the term $\sum_{k=0}^{j-1} c_k \frac{(t-t_k)_+^{\alpha_k}}{\Gamma(1+\alpha_k)}$ at t_j are equal. Their "causing" and "resulting" can be traced by using the laws of inertia or conservation. However the left and right derivatives possess a jump for the term $c_j \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$. Therefore what are "trickier" is in the sense that some information or energy comes from the imaginary part of the complex space. What we can observe in the real space is: a new event has occurred or a new species has been born. Therefore, its left and right derivatives are not matched.

More precisely, for each $0 \leq j \leq N$, an event occurred at time t_j, which continues to have an effect on time $t > t_j$. Furthermore, this effect can be quantitatively written as $c_j \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$. The function written above in (2.2) somewhat resembles a Taylor expansion, which we will elaborate in the next section. Utilizing results from the approximation theory (e.g. spline, however generalized to fractional order), any function can be approximated by certain combinations of this kind of functions (fractional cut-off polynomial).

3. Generalized fractional Taylor expansion

If $f(t)$ is an analytic function, we can use its analytic expansion at any point to backtrack and predict the whole process of $f(t)$.

Let us recall the following basic facts. Suppose $f(t) = c\frac{(t-t_0)_+^{\alpha}}{\Gamma(1+\alpha)}$, where $\alpha > 0$. According to the Riemann-Liouville definition of fractional derivative, if $t > t_0$, we have $D^{\alpha} f(t) = c(t-t_0)_+^0 \triangleq D^{\alpha} f(t)|_{t \to t_{0+}} (t-t_0)_+^0 \equiv c$. Alternatively, $D^{\alpha} f(t_{0+}) = c$. It follows that

$$f(t) = D^{\alpha} f(t_{0+}) \frac{(t - t_0)_+^{\alpha}}{\Gamma(1 + \alpha)} = I^{\alpha} D^{\alpha} f(t). \tag{3.1}$$

Note that $D^{\alpha} f(t)$ is a jump function whose derivative is equal to zero almost everywhere, then

$$I^{1+\alpha} D^{1+\alpha} f(t) = 0. \tag{3.2}$$

Put it succinctly, the function $f(t) = c\frac{(t-t_0)_+^\alpha}{\Gamma(1+\alpha)}$ precisely describes the following: there occurred an event that can be characterized as having scale c and degree α at time t_0. This justifies that the function has the derivative of order α at t_0, whose exact value is c. This has completely characterized the following statement: the event $D^\alpha f(t_{0+}) = c$ occurred at time t_0.

If one cuts off the history at time t_0, that is $f(t) \equiv 0$ when $t < t_0$. This means that nothing had occurred before time t_0. The above discussion of (3.1) and (3.2) also applies to the cut-off polynomial of the integer order for $t > t_0$. However, as has been pointed out in the previous section, an integer degree α describes a drastically different situation compared to what is described by a non-integer degree α. At time t_0, α being an integer indicates that the event is the analytic continuation of a polynomial, or the continuous effect of a past event; whereas α being a non-integer shows that a new event occurred at time t_0, or the causing stems from the imaginary space.

Let us first consider a special interesting case in which all the events occurred at the same time t_0, and $0 < \alpha_j < \alpha_{j+1}$, $j = 0, \cdots, \gamma$. The life process function in this case can be written as:

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}.$$

Comparing to the classical Taylor expansion, we hope to see (formally) that c_j is the coefficient in the sense of the Taylor expansion and therefore we derive a new fractional derivative and the generalized fractional Taylor expansion.

We define the generalized fractional Taylor expansion of order k $(1 \le k \le \gamma+1)$ for the given $\{\alpha_j\}_{j=0}^{\gamma}$ as follows:

$$T_k f(t) = \sum_{j=0}^{k-1} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}. \tag{3.3}$$

Then we have

$$D^{\alpha_k}(f(t) - T_k f(t)) = c_k + \mathcal{O}(t-t_0)_+^{\alpha_{k+1}-\alpha_k}. \tag{3.4}$$

Since the (right) fractional derivative of order $(1+\alpha)$ of the function $c\frac{(t-t_0)_+^\alpha}{\Gamma(1+\alpha)}$ is identically zero, that is $D^{1+\alpha}(c\frac{(t-t_0)_+^\alpha}{\Gamma(1+\alpha)}) = 0$, we can derive a killing operator. Suppose we have known the values of $\{\alpha_j\}$ $(0 \le j \le k-1)$.

Let operator $k_j = I^{1+\alpha_{j-1}} D^{1+\alpha_{j-1}}$, then the operator k_j applies on

$$c_{j-1}\frac{(t-t_0)_+^{\alpha_{j-1}}}{\Gamma(1+\alpha_{j-1})}$$

is equal to zero. Furthermore we define a killing operator

$$K_k = \prod_{j=1}^{k} k_j = \prod_{j=1}^{k} I^{1+\alpha_{j-1}} D^{1+\alpha_{j-1}}. \tag{3.5}$$

We have

$$K_k f(t) = f(t) - T_k f(t) \tag{3.6}$$

to get the remainder of the generalized fractional Taylor expansion. K_k has killed the generalized fractional Taylor expansion of order k. The $(k+1)$-th term (indexed by k) can be calculated as follows

$$\alpha_k = \lim_{t \to t_{0+}} \frac{(t-t_0)(K_k f)'(t)}{K_k f(t)}, \tag{3.7}$$

and

$$c_k = \lim_{t \to t_{0+}} D^{\alpha_k} K_k f(t). \tag{3.8}$$

This allows us to recover the original function by finding its classical right derivatives term by term. Thus the original function can be written down by a generalized fractional Taylor expansion of the first $k+1$ terms at t_0:

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)} = \sum_{j=0}^{k} (D^{\alpha_j} K_j f)(t_{0+}) \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)} + o(t-t_0)_+^{\alpha_k}, \tag{3.9}$$

which contains only the terms with the fractional degrees. We can obtain the degree α_j by (3.7) and the scale c_j by (3.8) respectively.

In the next section, we will define the new fractional derivative of order α_j of the function $f(t)$ whose value is the scale c_j that

$$D_{New}^{\alpha_j} f(t_{0+}) := D^{\alpha_j} K_j f(t_{0+}).$$

4. History reproducing and back-tracking

Suppose $f(t)$ describes the value of a life process (such as the number of the infected people by virus). An equivalent form of the equation (2.2) is as follows. At time $\{t_j\}_{j=0}^N$ $(t_j < t_{j+1})$, some events happened with scales $\{c_{jk}\}$ and degrees $\{\alpha_{jk}\}$ $(0 < \alpha_{jk} < \alpha_{j(k+1)})$, then the life process function performs as:

$$f(t) = \sum_{j=0}^{N} \sum_{k=0}^{\gamma_j} c_{jk} \frac{(t-t_j)_+^{\alpha_{jk}}}{\Gamma(1+\alpha_{jk})}, \qquad \sum_{j=0}^{N} \sum_{k=0}^{\gamma_j} 1 < \infty. \tag{4.1}$$

This models the hypothesis of punctuated equilibrium in the evolutionary process of species. Here $\{t_j\}_{j=0}^N$ are the moments in the evolutionary process when gene mutations occurred. There is a corresponding **event table**

$$\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{N}$$

which describes completely what have happened in the history.

Here the interesting mathematical question is: how to find the event table based upon appropriately sampled data of the function (especially the recently sampled data, which we want to use for back tracking the history). More precisely, how can we sample the data from $f(t)$ to get its representation by the generalized fractional Taylor expansion (4.1) or the event table (chronicle of events).

4.1. **History Reproducing.** We read the history page by page and find the first occurrence point of the event t_0 from which $f(t) \neq 0$. By using the method described above, we can obtain $\{(t_0, \alpha_{0k}, c_{0k})\}_{k=0}^{\gamma_0}$. Suppose we have studied all the events happened at time $\{t_j\}_{j=0}^{l-1}$ and the first m events happened at time t_l, that is the event table $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{l-1}$ and $\{(t_l, \alpha_{lk}, c_{lk})\}_{k=0}^{m-1}$ have already been found.

Define the generalized fractional Taylor expansion of order lm as to be

$$T_{lm}f(t) = \sum_{j=0}^{l-1}\sum_{k=0}^{\gamma_j} c_{jk}\frac{(t-t_j)_+^{\alpha_{jk}}}{\Gamma(1+\alpha_{jk})} + \sum_{k=0}^{m-1} c_{lk}\frac{(t-t_l)_+^{\alpha_{lk}}}{\Gamma(1+\alpha_{lk})}. \tag{4.2}$$

Furthermore define the killing operator as

$$K_{lm}f(t) = f(t) - T_{lm}f(t). \tag{4.3}$$

Then a "new starting point" t_{l+1} or the next event happened at time t_l is the starting point for the remained function $f(t)-T_{lm}f(t)$. In this way, we can go event by event to get the event table $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{N}$ and eventually reproduce the function $f(t)$. If $t < t_l$, $f(t) - T_{lm}f(t) = 0$. Thus it is easy to find t_l.

The above process entails computing or finding the values in the event table one-by-one. In particular, one needs to find the exact moments of $\{t_j\}_{j=0}^{N}$ when these events occurred. To find the values of the fractional derivatives at t_{j+}, one only needs to do computing work near t_{j+}. Figuratively speaking, one needs to do on-site inspections so as to characterize $(t_j, \alpha_{jk}, c_{jk})$ one by one. Furthermore, one has to follow the order of time, starting from the very beginning.

The above analysis tells us that the values of fractional derivatives become more scientifically significant when the time variable t approaches to the occurrence time of the events, and the event table truly characterizes the quantitative nature of the life process function (the occurrence time, scales and the degrees of the events).

There are two components at the occurrence time t_l of the event. The first is the continuous effect of the previous events $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{l-1}$. The second is how new events $\{(t_l, \alpha_{lk}, c_{lk})\}_{k=0}^{\gamma_l}$ occur. For example, we consider the life process function $f(t) = c\frac{(t-t_0)_+^{\alpha}}{\Gamma(1+\alpha)}$. If $t > t_0$, $D^{\alpha}f(t) = c$. This is a jump function, whose continuous parts are determined by the effect of inertia, while the jump-discontinuity point t_0 indicates the time when a new event occurs. In characterizing the event, it is important to find and record when a new event occurs and how it occurs. As for the effect of inertia, it can be obtained via the computation of the analytic continuation of the corresponding terms of the life process function. In essence, the event table $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{N}$ is the characterization of the life process function, where $c_{jk} = D_{New}^{\alpha_{jk}}f(t_{j+})$ will be defined in the section 5. Thus the corresponding historical process is represented in such an elegant and efficient way.

4.2. **History Back-tracking.** There is a more interesting and important question: is it possible to know the entire history by observing a segment of the recent information of $f(t)$? Of course, this pertains to a causal relationship. There is a cause, there is a consequence. But on the contrary, the same consequence may be the result of different causes.

If $t_1 > t_0$ and $f(t) = c(t-t_0)_+^0 - c(t-t_1)_+^0$, we find that the function value is identically zero if $t > t_1$ regardless of what value c takes. The event $\{(t_0, 0, c), (t_1, 0, -c)\}$ lived in (t_0, t_1), however had totally vanished after t_1 that we can not track back what had occurred in the history based on present information. We do not have the faintest idea of knowing what events had occurred before, as well as when and how they occurred.

Another such example is $f(t) = c(t - t_0)_+^0$. The behavior of the function near the current time $t > t_0$ does not give any hint about t_0 when the event occurred. To describe the event accurately and clearly, one needs to go to the occurrence time t_0 to witness the history on-site. Alternatively speaking, deriving cause from consequence is an inverse problem that may has multiple solutions.

We ask ourselves the question: what kinds of historical events can we recover and reenact by only observing a segment of recent history?

To answer this question, we first need to analyze the occurrence of a single event. Let $f(t) = c\frac{(t-t_0)_+^\alpha}{\Gamma(1+\alpha)}$ ($\alpha > 0$, not an integer). This expression has three parameters: $\{(t_0, \alpha, c)\}$. As such, we need three data to recover the original event. Obviously, when $t > t_0$, we have

$$\frac{f(t)f''(t)}{(f'(t))^2} = \frac{\alpha - 1}{\alpha}.$$

It follows that

$$\alpha = \left[1 - \frac{f(t)f''(t)}{(f'(t))^2}\right]^{-1}.$$

We also have

$$\frac{f(t)}{f'(t)} = \frac{t - t_0}{\alpha}.$$

Hence

$$t_0 = t - \alpha\frac{f(t)}{f'(t)}.$$

Finally, from

$$f(t) = c\frac{(t - t_0)_+^\alpha}{\Gamma(1 + \alpha)},$$

we obtain

$$c = \frac{\Gamma(1 + \alpha)f(t)}{(t - t_0)_+^\alpha}.$$

Here the calculation requires that $f'(t) \neq 0$. Because $f(t)$ is a polynomial of fractional power, $f(t)$ and its derivatives of all orders have isolated zeros. Therefore, it is easy to locate such an observation point $\bar{t}$ where $f^{'}(\bar{t}) \neq 0$. Then we can compute the degree α, occurrence time t_0 and scale c of the event via the above formulas and the current state of $f(t)$ near $\bar{t}$.

We now consider a more complicated case. Let

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t - t_0)_+^{\alpha_j}}{\Gamma(1 + \alpha_j)},$$

where $\{\alpha_j\}_{j=0}^{\gamma}$ are all positive non-integers. This function describes the case that several events occurred at the single time t_0. The function is analytic for $t > t_0$, so its analytic expansion at $\bar{t}$ ($\bar{t} > t_0$) is

$$f(t) = \sum f^{(n)}(\bar{t})\frac{(t-\bar{t})^n}{n!} \tag{4.4}$$

and the Cauchy-Hadamard convergence radius is $R = (\overline{\lim}_{n\to\infty}|\frac{f^{(n)}(\bar{t})}{n!}|^{1/n})^{-1}$. Since t_0 is the only singular point of $f(t)$, then $R = \bar{t} - t_0$. Calculating $f^{(n)}(\bar{t})$ and furthermore the convergence radius R of the corresponding analytic expansion (4.4), we can obtain the value of t_0 which is the occurrence time of the events. Let $\bar{t}$ move near the current time and denote $t := \bar{t}$, then

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)} = \sum_{j=0}^{\gamma}\frac{c_j}{\Gamma(1+\alpha_j)}R^{\alpha_j}.$$

Define

$$f_0(R) = f(t_0 + R)$$

and

$$f_l(R) = R\cdot\frac{d}{dR}(f_{l-1}(R)), l = 1,2,3,\cdots .$$

These data can be obtained at the observation point t. Thus we have

$$\begin{aligned} f_l(R) &= \sum_{j=0}^{\gamma}\frac{c_j}{\Gamma(1+\alpha_j)}\alpha_j^l R^{\alpha_j} \\ &= (\alpha_0^l,\cdots,\alpha_\gamma^l)\cdot diag(R^{\alpha_j})\cdot(\frac{c_0}{\Gamma(1+\alpha_0)},\cdots,\frac{c_\gamma}{\Gamma(1+\alpha_\gamma)})^T. \end{aligned} \tag{4.5}$$

For $j,k = 0,1,\cdots\gamma$, we define the vector

$$F_l = (f_l,\cdots,f_{l+\gamma})^T = (\alpha_k^j)\cdot diag(\alpha_j^l)\cdot diag(R^{\alpha_j})\cdot(\frac{c_0}{\Gamma(1+\alpha_0)},\cdots,\frac{c_\gamma}{\Gamma(1+\alpha_\gamma)})^T$$

where α_k^j is the entry of the matrix (α_k^j) at the row j and column k, and the matrix

$$(f_{j+k+l})_{(\gamma+1)\times(\gamma+1)} = (F_l, F_{l+1},\cdots,F_{l+\gamma})$$

whose entry at row j and column k is f_{j+k+l}.

We have

$$F_{l+\gamma} = (\alpha_k^j)\cdot \text{diag}(\alpha_j^\gamma)\cdot((\alpha_k^j))^{-1}\cdot F_l, \quad j,k = 0,\cdots\gamma,$$

and

$$(f_{j+k+\gamma})_{(\gamma+1)\times(\gamma+1)} = (\alpha_k^j)\cdot\text{diag}(\alpha_j^\gamma)\cdot(\alpha_k^j)^{-1}(f_{j+k})_{(\gamma+1)\times(\gamma+1)}.$$

Thus the matrix $\text{diag}(\alpha_j)$ raised to power γ is the eigenvalue matrix of the matrix

$$(f_{j+k+\gamma})\cdot(f_{j+k})^{-1}.$$

Once we find all the eigenvalues of the matrix $(f_{j+k+\gamma})\cdot(f_{j+k})^{-1}$, a simple calculation allows us to capture $\{\alpha_j\}_{j=0}^{\gamma}$. Finally, the scales $\{c_k\}$ are the solutions of the entries of the equation $(\alpha_k^j)\cdot diag(R^{\alpha_j})\cdot(\frac{c_j}{\Gamma(1+\alpha_j)}) = F_0$ and

$$(c_0,\cdots,c_\gamma)^T = diag(\frac{\Gamma(1+\alpha_j)}{R^{\alpha_j}})\cdot(\alpha_k^j)^{-1}\cdot F_0.$$

Therefore, we have obtain the following theorem.

Theorem 4.1. *If some events with various degrees* $\{\alpha_j\}_{j=0}^{\gamma}(0 < \alpha_j < \alpha_{j+1})$ *and scales* $\{c_j\}_{j=0}^{\gamma}$ *occurred at the single time* t_0*, the life process function* $f(t)$ *can be represented as* $\sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$*. Furthermore, assume that the matrix* $(f_{j+k})_{(\gamma+1)\times(\gamma+1)}$ *is non-singular at the observation time* $\bar{t}$ *near the current time. Then one can use the information near* $\bar{t}$ *and the above algorithm to recover the entire history process which* $f(t)$ $(t < \bar{t})$ *describes, that is we can obtain the event table* $\{(t_0, \alpha_j, c_j)\}_{j=0}^{\gamma}$*. Since the determinant of the matrix* $(f_{j+k})_{(\gamma+1)\times(\gamma+1)}$ *is meromorphic, its zeros are isolated, then we can easily find such a observation time* $\bar{t}$ *near the current time.*

Moreover, if $f(t) = \sum_{j=0}^{N}\sum_{k=0}^{\gamma_j} c_{jk}(t-t_j)_+^{\alpha_{jk}}/\Gamma(1+\alpha_{jk})$ and α_{jk} are all positive non-integers, we have

$$f(t) = \sum_{k=0}^{\gamma_N} c_{Nk}(t-t_N)_+^{\alpha_{Nk}}/\Gamma(1+\alpha_{Nk}) + A(t)$$

for $t > t_N$, where $A(t)$ is an analytic function even at t_N. Therefore

$$A(t) = \sum_{j=0}^{\infty} a_j(t-t_N)^j/j!,$$

and

$$f(t) = \sum_{k=0}^{\gamma_N} c_{Nk}(t-t_N)_+^{\alpha_{Nk}}/\Gamma(1+\alpha_{Nk}) + \sum_{j=0}^{\infty} a_j(t-t_N)^j/j!.$$

By the same argument, the convergence radius of the analytic expansion at $\bar{t}$ of the function $f(t)$ is $\bar{t} - t_N$, so we can get the time t_N. If there are only finitely many events occurred, we can use the approach and algorithm of the theorem (4.1) and consider the integer-power and non-integer-power terms simultaneously. The problem is equivalent to find the eigenvalues of the matrix $(f_{j+k+\gamma})(f_{j+k})^{-1}$ as $\gamma \to \infty$, which is theoretically solvable. In practice, one can take a relatively large γ and find the eigenvalues α_j^{γ} of the matrix$(f_{j+k+\gamma})(f_{j+k})^{-1}$ (This applies both to non-integer-power terms and the first few terms of the analytic expansion $A(t)$). Then select out the eigenvalues whose $1/\gamma$-power are fractions to construct the fractional terms of the generalized fractional Taylor expansion at t_{N+}. Since the analytic expansion absolutely converges in its convergence interval, eigenvalues calculated by the algorithm is convergent.

In this fashion, we start from N-th event going backward, separate and analyze the events on a one-by-one basis. Paying a close attention to the occurrence times, the degrees, and the scales of these events allows us to obtain the generalized fractional Taylor expansion (4.1) and therefore the entire history process is known by back-tracking.

Theorem 4.2. *If* $f(t) = \sum_{j=0}^{N}\sum_{k=0}^{\gamma_j} c_{jk}\frac{(t-t_j)_+^{\alpha_{jk}}}{\Gamma(1+\alpha_{jk})}$ *in which both* N *and* γ_j *are finite (there were finitely many events occurred), we are able to use the information (or behavior)* $\{f^{(j)}(\bar{t})\}_{j=0}^{\infty}$ *of the life process function to track back the history and obtain the event table* $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{N}$*, where* $\bar{t}$ *is near the current time.*

Corollary 4.3. *If we use the time $-t$ instead of t, the right derivative becomes the left derivative. This allows us to use the current state to predict when and how the departed dying-out occur in future time.*

Remark 4.4. Nevertheless, we are unable to predict when and what kind of new events will occur in the future. In lay terms, we may be able to predict the time when a certain species becomes extinct, however we are unable to predict when the species will have a gene mutation to become a new kind of species, nor we are able to predict the time of an abnormal dying-out of the species.

In the above, we have used mathematical analysis in proving this interesting piece of common knowledge.

5. New definition of fractional derivatives

Summarizing the discussion above, we observe that the value of fractional derivative at t_j has clear and striking physical meanings. Furthermore, they are closely related to the generalized fractional Taylor expansion. However, the classical Riemann-Liouville definition of fractional derivatives does not address this relations, nor does it treat the two concepts (fractional derivatives and the generalized fractional Taylor expansion) with individual attention.

Based on these observations, we first give a new definition of the fractional derivative of order α_k at t_0 for a life process function

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$$

which is denoted by $D_{New}^{\alpha_k} f(t_{0+})$.

Definition 5.1.

$$D_{New}^{\alpha_k} f(t_{0+}) = \lim_{t \to t_{0+}} D^{\alpha_k}(f(t) - T_k f(t)) = \lim_{t \to t_{0+}} D^{\alpha_k} K_k f(t) = c_k.$$

Furthermore, if a life process function is

$$f(t) = \sum_{j=0}^{N} \sum_{k=0}^{\gamma_j} c_{jk} \frac{(t-t_j)_+^{\alpha_{jk}}}{\Gamma(1+\alpha_{jk})},$$

a new definition for the fractional derivative of order α_{lm} at t_l is denoted by $D_{New}^{\alpha_{lm}} f(t_{l+})$.

Definition 5.2.

$$D_{New}^{\alpha_{lm}} f(t_{l+}) = \lim_{t \to t_{l+}} D^{\alpha_{lm}}(f(t) - T_{lm} f(t)) = \lim_{t \to t_{l+}} D^{\alpha_{lm}} K_{lm} f(t) = c_{lm}.$$

Remark 5.3. The introduction of the killing operator (3.5), (3.6) and (4.3) is very important. A differential operator of higher integer-power automatically kill the polynomials of the lower degree. But fractional derivatives do not possess such property. In finding the derivative of order α_k, information carried by $T_k f(t)$ may be masked and diluted in the developing process of the later-occurring events. The classical Riemann-Liouville definition fails to address this issue. Killing operators are capable of separating the terms of the generalized fractional Taylor expansion and extract the coefficients therein.

The new definition of fractional derivatives includes a special case of the relationship between the classical derivatives and Taylor expansion. That is, if $\{\alpha_k\}$ are integers, the result is the same as if one expands Taylor series with classical derivatives. An analytic function can be always expanded by the classical Taylor expansion. With the generalized fractional Taylor expansion, however, when new events take part in the process at their occurrence time, we see that the life process function some what like a tree branch, you can track a branch from one bifurcation point to the next bifurcation point. This motivates us to state the following theorems.

Theorem 5.4. *If the event table is* $\{(t_0, \alpha_j, c_j)\}_{j=0}^{\gamma}$ *(*$0 < \alpha_j < \alpha_{j+1}$*), the events occurred at the single time* t_0*. We have*

$$f(t) = \sum_{j=0}^{\gamma} c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)},$$

then $D_{New}^{\alpha_j} f(t_{0+}) := c_j$*. That is, the corresponding life process function can be represented by its generalized fractional Taylor expansion:*

$$f(t) = \sum_{j=0}^{\gamma} D_{New}^{\alpha_j} f(t_{0+}) \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}.$$

$D_{New}^{\alpha_j} f(t_{0+}) = c_j$ *is a necessary and sufficient condition for the generalized fractional Taylor expansion of* $f(t)$ *containing the term* $c_j \frac{(t-t_0)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$*, i.e.*

$$D_{New}^{\alpha} \frac{(t-t_0)_+^{\beta}}{\Gamma(\beta+1)}|_{t=t_{0+}} = \delta_{\alpha\beta} \ (kronecker).$$

Theorem 5.5. *If the event table is* $\{(t_j, \alpha_j, c_j)\}_{j=0}^{N}$*, then the occurrence time* $\{t_j\}_{j=0}^{N}$ $(t_j < t_{j+1})$ *of the events are isolated. We have*

$$f(t) = \sum_{j=0}^{N} c_j \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)},$$

then $D_{New}^{\alpha_j} f(t_{j+}) := c_j$*. That is, the corresponding life process function can be represented by its generalized fractional Taylor expansion:*

$$f(t) = \sum_{j=0}^{N} D_{New}^{\alpha_j} f(t_{j+}) \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}.$$

$D_{New}^{\alpha_j} f(t_{j+}) = c_j$ *is a necessary and sufficient condition for the generalized fractional Taylor expansion of* $f(t)$ *containing the term* $c_j \frac{(t-t_j)_+^{\alpha_j}}{\Gamma(1+\alpha_j)}$*, i.e.*

$$D_{New}^{\alpha} \frac{(t-t_j)_+^{\alpha}}{\Gamma(\alpha+1)}|_{t=t_{k+}} = \delta_{jk} \ (kronecker).$$

Theorem 5.6. *If the event table is* $\{(t_j, \alpha_{jk}, c_{jk})\}_{k=0}^{\gamma_j}{}_{j=0}^{N}$ $(t_j < t_{j+1},\ 0 < \alpha_{jk} < \alpha_{j(k+1)})$*, We have*

$$f(t) = \sum_{j=0}^{N} \sum_{k=0}^{\gamma_j} c_{jk} \frac{(t - t_j)_+^{\alpha_{jk}}}{\Gamma(1 + \alpha_{jk})},$$

then $D_{New}^{\alpha_{jk}} f(t_{j+}) := c_{jk}$*. That is, the corresponding life process function can be represented by its generalized fractional Taylor expansion:*

$$f(t) = \sum_{j=0}^{N} \sum_{k=0}^{\gamma_j} D_{New}^{\alpha_{jk}} f(t_{j+}) \frac{(t - t_j)_+^{\alpha_{jk}}}{\Gamma(1 + \alpha_{jk})}.$$

$D_{New}^{\alpha_{jk}} f(t_{j+}) = c_{jk}$ *is a necessary and sufficient condition for the generalized fractional Taylor expansion of* $f(t)$ *containing the term* $c_{jk}\frac{(t-t_j)_+^{\alpha_{jk}}}{\Gamma(1+\alpha_{jk})}$*, i.e.*

$$D_{New}^{\alpha} \frac{(t - t_j)_+^{\beta}}{\Gamma(\beta + 1)}|_{t=t_{k+}} = \delta_{\alpha\beta}\delta_{jk}\ (kronecker).$$

(This formula and the Taylor expansion still hold true when $\{\alpha, \beta\}$ *take integer values).*

Remark 5.7. We classify the time in the domain of a life process function in two categories. The first is the analytic points. They pertain to the analytic continuation of the effect of the events occurred in the past. The second is the new-event-occurring points. They are reflected in the newly added fraction-power terms. As such, they are non-analytic points. Our new definition of fractional derivatives is mainly utilized to reveal the nature of the second-category points and describes the occurrence of new events. Furthermore the definition is friendly with the classical definition of the derivative.

If the right factional derivative is non-zero at time t_j, it means that a new event occurs. Accordingly, if the left factional derivative is non-zero at time t_j, it means that some earlier events must die out at that time. In evolutionary terms, some species become extinct at that point. On the contrary, at any analytic point, fractional derivatives are always zero.

Our new definition of fractional derivatives has the following properties:

1. Locality: as the classical derivatives, the value of $D_{New}^{\alpha} f$ at t is completely determined by the function values in a small neighborhood $(t, t + \epsilon)$ of t. If the $D_{New}^{\alpha} f(t) \neq 0$ with a non-integer α, it indicates that a new event of the scale and degree $\{D_{New}^{\alpha} f(t), \alpha\}$ occurs at time t.

2. Independent of the lower bound: our definition of the fractional derivatives does not rely on any stipulation on the choice of a time as origin.

3. Translation invariant: the operator D_{New}^{α} is translation invariant in the sense that the differential operator and translation operator are commutable.

4. Duality: the new defined fractional derivatives possess a duality that

$$D_{New}^{\alpha}(t-t_j)_+^{\beta}/\Gamma(\beta+1)|_{t=t_{j+}} = \delta_{\alpha\beta}\ (kronecker).$$

This formula holds true automatically when $\{\alpha, \beta\}$ are integers. Furthermore,

$$D_{New}^{\alpha}\frac{(t-t_j)_+^{\beta}}{\Gamma(\beta+1)}|_{t=t_{k+}} = \delta_{\alpha\beta}\delta_{jk}\ (kronecker).$$

5. The Leibnitz rule of differentiation: if $\gamma = \alpha_j + \beta_k$, it obeys

$$D_{New}^{\gamma}[u(t)v(t)] = \sum_{\alpha_j+\beta_k=\gamma}\frac{\Gamma(1+\alpha_j)\Gamma(1+\beta_k)}{\Gamma(1+\gamma)}D_{New}^{\alpha_j}u(t)\cdot D_{New}^{\beta_k}v(t).$$

This formula holds true automatically when γ and $\{\alpha, \beta\}$ are integers (the classical Leibnitz rule).

6. It can be numerically approximated by using a finite-difference formula. The formula is highly effective and requires only function values from a small neighborhood of the point of interest.

7. It is capable of effectively modeling problems in sciences by incorporating fractional differential equations, just as modeling with the classical ordinary differential equations.

8. The essential purpose of solving a fractional differential equation is to find the occurrence time, the degree and scale of a certain new event.

Properties 1 – 5 above are obvious. We hereby elaborate on property 6. Under the framework of the classical fractional derivative, a singular integral has to be computed, for which the numerical procedure is usually unstable. This has brought difficulties in implementations and applications. With the new definition, its locality admits the following method of calculation. Utilizing the killing operator K_α, where K_α kills the terms whose degree are smaller than α of the expansion of $f(t)$. One first finds the following function

$$\bar{f}(t) = K_\alpha f(t) = c(t-\bar{t})_+^{\alpha}/\Gamma(1+\alpha) + o(t-\bar{t})_+^{\alpha}$$

in a small neighborhood of the point $\bar{t}$. Then by the new definition, we have

$$\alpha = \lim_{s\to 0_+}\lim_{\bar{s}\to 0_+}\frac{s\cdot(\bar{f}(\bar{t}+s+\bar{s})-\bar{f}(\bar{t}+s))}{\bar{s}\cdot\bar{f}(\bar{t}+s)}.$$

Modeling s^2 with $\bar{s}$, we obtain that the degree and scale of the event are

$$\alpha = \lim_{s\to 0_+}\frac{(\bar{f}(\bar{t}+s+s^2)-\bar{f}(\bar{t}+s))}{s\cdot\bar{f}(\bar{t}+s)},$$

and

$$c = D_{New}^{\alpha}\bar{f}(\bar{t}_+) = \lim_{s\to 0_+}\Gamma(1+\alpha)\bar{f}(\bar{t}+s)/(s)^{\alpha}.$$

We may use $s = \Delta t$ to model the fractional forward divided differences, which leads to an approximate value of $D_{New}^{\alpha}f(\bar{t})$ and $f(\bar{t}+\Delta t) \sim T_\alpha f(\bar{t}+\Delta t)$

$+D^{\alpha}_{New}f(\bar{t}_+)(\Delta t)^{\alpha}$, where $T_{\alpha}f$ contains the terms whose degree is smaller than α of the generalized expansion of $f(t)$.

Formula for the forward divided difference is associated with the new definition of the factional derivative (also applicable with classical divided difference). Let $\bar{f}(t) = K_{\alpha}f(t)$. Then at time $\bar{t}$, we have

$$D^{\alpha}_{New}f(\bar{t}) \sim \Gamma(1+\alpha)\bar{f}(\bar{t}+\Delta t)/(\Delta t)^{\alpha},$$

where

$$\alpha \sim \frac{(\bar{f}(\bar{t}+\Delta t+\Delta t^2) - \bar{f}(\bar{t}+\Delta t))}{\Delta t\bar{f}(\bar{t}+\Delta t)}.$$

We introduce our new definition for fractional derivatives on the occurrence time of an event $D^{\alpha_{jk}}_{New}(t_j)$ to characterize a life process function. Because the function of the life process is determined by its event table, the fractional differential equation which models the relations of the function and its derivatives will turn to be an algebraic equation of the entries of the event table. Since we introduce fractional differential equations to describe the nature of the newly occurring events, special attentions are devoted to non-analytic points $\{t_j\}_{j=0}^N$ where the fractional differential equation $\delta_{t_j}P(D^{\alpha}_{New})f(t) = 0$ holds true. δ_{t_j} in above is a Dirac operator. More precisely, a fractional differential operator is only applied on the non-analytic points, since the fractional derivatives are zero at the analytic points.

In what follows, we discuss a few examples stemming from typical problems of the mathematical models in applications.

Homogeneous linear equations: The homogeneous linear fractional differential equations are de facto constraints or conditions to be satisfied at the occurrence time of events. For example, there holds true that the (fractional) differential equation $D^{\alpha}_{New}f = D^{\beta}_{New}f$ at $t = t_j$ $(j = 0, \cdots N)$. Then the solution should have the form:

$$f(t) = \sum_{j=0}^{N} c_j[\frac{(t-t_j)^{\alpha}_+}{\Gamma(1+\alpha)} + \frac{(t-t_j)^{\beta}_+}{\Gamma(1+\beta)}].$$

More generally, a homogeneous linear fractional differential equation takes the form

$$\sum_{k=0}^{K} \bar{c}_k D^{\alpha_k}_{New} f = 0, \quad \forall t = t_j,\ j = 0, \cdots N, \tag{5.1}$$

where $\{t_j\}$ are the given event occurrence time. Then the solution of the equation (5.1) should be written as

$$f(t) = \sum_{j=0}^{N}\sum_{k=0}^{K} c_{jk}\frac{(t-t_j)^{\alpha_k}_+}{\Gamma(1+\alpha_k)},$$

where $\sum_{k=0}^{K}\bar{c}_k c_{jk} = 0$ is the algebraic form of the fractional differential equation (5.1). Equivalently, all the rows of the matrix (c_{jk}) are in the dual space of the vector $(\bar{c}_k)$. These characters of the new occurring events are all in the subspace at every t_j. We remind readers that the structure of the solution space is a $K(N+1)$-dimensional linear function space which is determined by $\{t_j\}_{j=0}^N$ (the occurrence time points) and $\{\bar{c}_k\}_{k=0}^K$ (the constraint of the event). For every j, the new occurring events are restricted by the same condition which is $\sum_{k=0}^K \bar{c}_k c_{jk} = 0$. All the solutions are in a fiber bundle which is constructed by equipping the dual space of $(\bar{c}_k)$ on every t_j.

Characteristic equation: A characteristic fractional differential equation takes the form

$$D_{New}^\alpha f = \lambda f, \quad \text{at} \quad \{t_j\}_{j=0}^N. \tag{5.2}$$

This is an archetypical phenomenon in which the accumulative effect of the past events leads to the occurrence of a new event. In other words, the occurrence of the new event is caused by the past events. If a new event occurs at t_j, we denote $f(t) = f_j(t),\ t_{j-1} \le t < t_j, j = 1, \cdots, N$, where $f_j(t)$ is an analytic function at t_j which describes the effects of the events occurred before the time t_j. For $t_j \le t < t_{j+1}$, the function develops to be

$$f(t)|_{t_j<t<t_j+1} = f_{j+1}(t) = f_j(t) + \lambda f_j(t_j)(t-t_j)_+^\alpha/\Gamma(1+\alpha).$$

More precisely, we have

$$f(t) = f_0(t) + \sum c_j(t-t_j)_+^\alpha/\Gamma(1+\alpha),$$

in which $c_0 = \lambda f_0(t_0)$ and $c_j = \lambda[f_0(t_j) + \sum_{k=0}^{j-1} c_k(t_j - t_k)_+^\alpha/\Gamma(1+\alpha)],\quad j \ge 1$. We expect that the above formula serves as an excellent model for the evolution of many species. For this very purpose, λ is usually negative. An occurrence time t_j of an event can be determined as follows. When the population of a certain species reaches a critical level, i.e., there is a ceiling C such that $f(t) > C$, then disasters may strike. New events accompanying a drastic reduction of the population will occur. The larger the $f(t) - C$, the more likely the new events occur.

Remark 5.8. In the massive mathematical literature devoted to fractional derivatives, a focus is on expanding target functions in terms of the appropriately chosen bases. The basis functions are usually taken to be infinitely differentiable. Spline functions are frequently used before as they have a certain integer-order smoothness. The discussion fails to the functions that have a certain fractional degree of continuity. The results obtained lack physical meanings that dropped the information $(t_j, \alpha_{jk}, c_{jk})$ coming from the imaginary space.

Simply put, Riemann- Liouville fractional derivatives primarily deal with functions that have a certain fractional degree of continuity at the lower bound of the Riemann- Liouville integral, which has a tendency of being misplaced. Our definition can handle such functions at any event-occurring time points.

The methodology here suits the viewpoint that all the occurrence time of the events are isolated (the hypothesis of punctuated equilibrium in the evolutionary process of species), which is akin to that a typical meromorphic function has isolated singular point. We have also taken the same approach to locate an occurrence time of the event as to find a singular point of a meromorphic function.

The kernel $(t-s)_+^\alpha$ generalizes the basis $\{(t-s_j)_+^\alpha\}$ which consists of cut-off polynomials of the degree α. Based on the theory of the spline, the space spanned by $\{(\cdot - s)_+^\alpha\}$ is infinitely dimensional, which is dense in the space of continuous functions. With an appropriately-stipulated decay, span$\{(\cdot - s)_+^\alpha\}$ is dense in larger function spaces.

Then the continuous function $f(t)$ can be expressed as a limit of Riemann sum

$$f(t) = \lim_{\Delta_j \to 0} \sum g(s_j)(t - s_j)_+^\alpha \Delta s_j / \Gamma(1+\alpha) = \int g(s)(t-s)_+^\alpha / \Gamma(1+\alpha) ds,$$

where $g(s)$ is usually a distribution in the Schwarz class (Borel's measure), but in rare cases $g(s)$ becomes a continuous function in the application. In other words, for any life process function $f(t)$, one can find a suitable generalized function $g(s)$ so that the above equation holds true.

Remark 5.9. In the application, people always want to have major principle phases (e.g. the event table $\{t_j, \alpha_{jk}, D_{New}^{\alpha_{jk}} f\}$) to describe a life process function and drop the small events ($\{\alpha_{jk} \sim 0, D_{New}^{\alpha_{jk}} f \sim 0\}$) as the errors in the sense of the hypothesis of punctual equilibrium. Thus we do like to represent

$$f(t) = \sum g(s_j)(t - s_j)_+^\alpha \Delta s_j / \Gamma(1+\alpha) + Err.$$

Based on the learning theory, one can use the Lasso to minimize the error to find the major events occurred at s_j in the history.

At the end of the paper, we reiterate that the classical definition of fractional derivatives is pointedly given to deal with functions that have a certain fractional degree of continuity at the lower bound a of the integral presented in the definition. Using the new definition given in the current paper, we are able to deal with functions that have fractional degrees of continuity at multiple occurrence time $\{t_j\}$ of the events. In solving classical fractional differential equations, one often utilizes the spectrum method in which "super smooth" functions (e.g. analytic functions) are employed as basis functions. These "super smooth" functions designed to characterize the life process functions always have profound deviations from the life process functions. In our opinion, this has defeated the original purpose. We believe in the principle that it is more approachable to characterize functions with fractional degree continuity by using functions of the same types. We can fully and deeply characterize the occurrence time, the scales and degrees of a life process function only in this way.

As the idiom goes: "Those fit yourself are the best." The generalized fractional Taylor expansion embodies the new approach. Our new definition of fractional derivatives gives an elegant and complete characterization of the entire process in which all the pieces including event tables and generalized fractional Taylor expansions fit in the right places.

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References

[1] R. Hermann, *Fractional Calcules: An Introduction to Physicists*, World Scientific Publishing, 2014.
[2] K. S. Mille, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Son, 1993.
[3] S. G. Saniko and A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, 1993.
[4] R. Schaback and Z. M. Wu, *Operators on radial functions*, Journal of Computational and Applied Mathematics **73** (1996), 257–270.
[5] Z. M. Wu, *Fractional Derivatives from the perspective of the Minkowski's space-time (in Chinese)*, Sci. Sin. Math. **47** (2017), 1335–1344.

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