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ZEROS OF CHEBYSHEV POLYNOMIALS ON JORDAN CURVES

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Dedicated to Ronald DeVore

ABSTRACT. In connection with a problem of J. Christiansen, B. Simon and M. Zinchenko it is shown that there is a Jordan curve σ such that no subarc of σ is analytic, but along some subsequence of the natural numbers the zero distribution of the n-th Chebyshev polynomial of σ does not converge to the equilibrium distribution of σ.

1. Introduction

Let K be a compact subset on the complex plane consisting of infinitely many points, and let $T_n(z) = z^n + \cdots$ be the unique monic polynomial of degree $n = 1, 2, \ldots$ which minimizes the supremum norm $\|T_n\|_K$ on K among all monic polynomials of the same degree. This T_n is called the n-th Chebyshev polynomial on K. Chebyshev polynomials have connection to a number of areas in mathematics, for their importance and various uses and appearances we refer to [11] by M. Sodin and P. Yuditskii.

Many of the properties of Chebyshev polynomials can be found in the series of papers [2]– [5] by J. Christiansen, B. Simon, M. Zinchenko and P. Yuditskii. One of the most important properties is Szegő's theorem:

$$\lim_{n\to\infty} \|T_n\|_K^{1/n} = \mathrm{cap}(K),$$

where $\mathrm{cap}(K)$ denotes logarithmic capacity.[1]

In this paper we are interested in the zeros of T_n, more precisely in their limit distribution, i.e., in the behavior of the normalized zero counting measures ν_{T_n} of T_n. By an old theorem of H. Widom [12] if Ω denotes the unbounded component of $\overline{\mathbf{C}} \setminus K$, then T_n has $o(n)$ zeros in any compact subset of Ω. Thus, any weak* limit ν of ν_{T_n}, $n = 1, 2, \ldots$, is supported on $\mathrm{Pc}(K) := \overline{\mathbf{C}} \setminus \Omega$, which is called the polynomial convex hull of K. It is also a general fact (see e.g. [10, Theorem III.3.9]) that if $\mathrm{cap}(K) > 0$, then any such ν satisfies $U^\nu(z) = U^{\omega_K}(z)$ for all large z, where ω_K is the equilibrium measure of K and

$$U^\nu(z) = \int \log \frac{1}{|z-t|} d\nu(t) \tag{1.1}$$

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[1]In what follows we shall use potential theoretic concepts such as logarithmic capacity and equilibrium measure, see [6], [7] or [10] for these and their properties.

denotes the logarithmic potential of ν. In other words, any limit measure must generate in Ω the same logarithmic potential as the equilibrium measure. Now in the case when K has empty interior and connected complement, i.e., when $\mathrm{Pc}(K)$ has empty interior, then (1.1) implies (see e.g. Carleson's unicity theorem [10, Theorem II.4.13]) that $\nu = \omega_K$, i.e., in this case the limit distribution of the zeros is the equilibrium distribution, which is a result of H.-P. Blatt, E. B. Saff and M. Simkani [1] (see also [10, Theorem III.3.6]).

What happens with the zeros when $\mathrm{Pc}(K)$ has non-empty interior is a complete mystery. The simplest such situation is when $K = \sigma$ is a Jordan curve (homeomorphic image of a circle), which we shall assume from now on. If σ is an analytic Jordan curve, then the zeros stay away from σ (see [9]), and, conversely, if the zeros do not accumulate at any point of σ, then σ is analytic. More generally, by [5, Theorem 1.1] if U is a neighborhood of a subarc J of σ and $\nu_{T_n}(U) \to 0$ as $n \to \infty$, then J is analytic. The converse is not true, if σ has a corner point with outer angle $< \pi$, then by a result of E. B. Saff and N. Stylionopoulos in [8] $\nu_n \to \omega_\sigma$ in the weak* topology, irrespective if the rest of σ is analytic or not (recall also that ω_σ is supported on σ). In connection with these results Christiansen, Simon and Zinchenko conjectured ([5, Conjecture 2.4]) that if σ is a Jordan curve no subarc of which is analytic, then the asymptotic zero distribution of the zeros of P_n is the equilibrium distribution ω_σ. In this note we show that this is not the case.

Theorem 1.1. *There is a Jordan curve σ such that no subarc of σ is analytic (actually C^1), but along some subsequence of the natural numbers the zero distribution of the n-th Chebyshev polynomial of σ does not converge to the equilibrium distribution of σ.*

It should be mentioned that the construction below gives a σ and a subsequence of $\{\nu_n\}$ along which the convergence to ω_σ does not take place. It may very well happen that the Christiansen–Simon–Zinchenko conjecture is true in the sense that along some subsequence the zero distribution is, indeed, the equilibrium measure.

The construction in Proposition 1.1 is somewhat technical, so first we give a sketch.

1.1. **Outline of the proof.** σ will be the limit of some lemniscates $\sigma_{P_n} = \{z \mid |P_n(z)| = 1\}$ for some monic polynomials P_n of some degree N_n. These σ_{P_n} will be Jordan curves with a parametrization $\gamma_{P_n} : C_1 \to \mathbf{C}$ that uniformly converge to a continuous function $\gamma : C_1 \to \mathbf{C}$ that gives the parametrization of σ. The curve σ will be of distance $< 1/4$ from the unit circle C_1, but P_n will have more than half of its zeros in the disk $\Delta_{1/2}$ of radius $1/2$ about the origin, and σ_{P_n} will be so close to σ that the same is true for the N_n-th degree Chebyshev polynomial of σ (note that P_n is the N_n-th Chebyshev polynomial of σ_{P_n}, and if σ and σ_{P_n} are sufficiently close, then so are their Chebyshev polynomials of the given degree N_n). This rules out that the zero distribution of the Chebyshev polynomials converge to the equilibrium measure of σ.

There are two other issues to be taken care of. The first is that the limit of Jordan curves is not necessarily a Jordan curve, and this problem is resolved by ensuring that the image under γ_{P_n} of any two points $e^{iu}, e^{iv} \in C_1$ that are of distance $\geq 1/j$

is bigger than some $\delta_j > 0$ for all $j \le n$. Then the same is true for the images under γ for all j, so γ defines a Jordan curve.

The other issue is to make sure that no part of σ is analytic. This will be achieved by ensuring that there is a dense set of points w_n on σ such that close to any w_n (closer than, say, $1/n$) there are two points $\tilde{w}_{n,1}$ and $\tilde{w}_{n,2}$ of σ such that they cut σ into two arcs both of which are of diameter $> n|\tilde{w}_{n,1} - \tilde{w}_{n,2}|$ — a property (call it the crosscut property) that clearly cannot hold if any subarc of σ is analytic (or even C^1). Both the zero accumulation property in $\Delta_{1/2}$ and the just mentioned crosscut property will be guaranteed for $\sigma_{P_{n+1}}$ by selecting $P_{n+1}(z) = P_n(z)^{m_n}(z - \alpha_n)$ with some very large m_n and some α_n close to w_{n+1}. The main effort will be to ensure that $\sigma_{P_{n+1}}$ is a Jordan curve with the described crosscut property around a given point w_{n+1}. This can be done because $\sigma_{P_{n+1}}$ is very close to σ_{P_n} for very large m_n, except for a "bubble" close to w_{n+1} containing α_n, and the "neck" of the "bubble" can be as narrow as we wish by suitably adjusting α_n (see the figures below).

2. Proof of Theorem 1.1

We shall use the following notations. Let $\Delta_r(z)$ denotes the open disk of radius r about z, and set $\Delta_r \equiv \Delta_r(0)$ and $C_1 = \partial\Delta_1(0)$, which is the unit circle. We use $U_r(E)$ for the open r-neighborhood of a set E. For a polynomial P we define the level set

$$\sigma_P = \{z \mid |P(z)| = 1\}$$

which we call a lemniscate.

2.1. Jordan lemniscates and their natural parametrization.

Proposition 2.1. *Let P be a polynomial of degree N. If σ_P is a Jordan curve, then $P^{1/N}$ is defined and univalent on the domain*

$$\{z \mid |P(z)| \ge c\}$$

for some $c < 1$. In particular, all zeros of P' lie inside σ_P.[2]

Proof. All zeros of P lie inside σ_P, so if we use the main branch of logarithm, then $P^{1/N}$ is defined and single-valued outside σ_P. As z runs through σ_P in the counterclockwise direction, the value $P(z)$ runs through the unit circle N-times, so there are consecutive arcs $\sigma_1, \dots, \sigma_N$ on σ_P such that they are disjoint except for their endpoints and $P(z)$ runs through the unit circle once in the counterclockwise

[2]This last property, which is apparently a folklore result, can also be verified as follows. P' cannot have a zero on σ_P by the assumed Jordan-curve property, and assume, to the contrary, that for some z_0 lying outside σ_P we have $P'(z_0) = 0$. Then the lemniscate $\{z \mid |P(z)| = |P(z_0)|\}$ has a multiple point, so the set $\{z \mid |P(z)| < |P(z_0)|\}$ has at least two components each containing at least one zero of P. But σ_P lies inside this set and contains all zeros of P in its interior, which is impossible.

A third proof follows from the fact that by conformality and $|P(z)| = 1$ on $z \in \sigma_P$, the vector $P(z)/P'(z)$ is perpendicular to the curve σ_P at any $z \in \sigma_P$. Hence, its change of argument as we circle σ_P once counterclockwise is 1, and so the claim follows from the argument principle. This proof also shows that if a Jordan curve σ is the level set of a function g that is analytic on and inside σ, then g has precisely 1 more zeros inside σ than its derivative.

The proofs also give that if σ_P is the union of k Jordan curves, then inside these curves P' has altogether precisely $N - k$ zeros.

direction as z runs trough σ_j, $j = 1, \ldots, N$. At this point all we can say is that the total change of the argument of $P(z)$ as z runs through σ_j is 2π, but actually the argument of $P(z)$ increases monotonically. In fact, if this was not the case, then there would be a σ_j and two points z_j, z_j^* in σ_j that are different from its endpoints such that $P(z_j) = P(z_j^*) =: w$. But on each other σ_i there is a point z_i with $P(z_i) = w$, and we would get a contradiction, since then P would take the value w at least $(N+1)$-times (at each z_i and also at z_j^*).

Thus, the argument of P monotonically increases[3], and hence so is the argument of $P^{1/N}$. Therefore, $P^{1/N}$ is 1–to–1 on σ_P, and since the exterior domain of σ_P (including the point infinity) is simply connected, it follows from the argument principle that $P^{1/N}$ is 1–to–1 also in that exterior domain.

The same can be told instead of σ_P with any lemniscate

$$\sigma_P^* = \{z \mid |P(z)| = c\}$$

that lies sufficiently close to σ_P (see the next subsection), and that proves the claim.

In particular, the derivative of $P^{1/N}$ cannot vanish on and outside σ_P, hence all zeros of P' lie inside σ_P. □

We can take $\gamma_P(\zeta) = (P^{1/N})^{-1}(\zeta)$, $\zeta \in C_1$, as a parametrization of σ_P, which we call its natural parametrization.

2.2. **How to recognize when σ_P is a Jordan curve?** If T is a connected set that contains all zeros of P and if $|P| < 1$ on T, then σ_P is a Jordan curve. Indeed, every component of $\{|P| < 1\}$ must contain a zero of P, so, under the assumption, this set is connected. In particular, $P' \neq 0$ on σ_P (a zero of P' on σ_P would create a multiple point and then the set $\{|P| < 1\}$ could not be connected), so σ_P is locally an analytic Jordan arc, and since it has only one component, it is a Jordan curve.

The converse is also true: if σ_P is a Jordan curve, then $\{z \mid |P(z)| < 1\}$ is a connected set containing all the zeros of P, so these zeros can be connected by a system of broken lines T on which $|P| < 1$.

In a similar fashion, if T does not connect all zeros of P, then we can still conclude that the zeros of P that lie in T lie in one connected component of $\{|P| < 1\}$.

2.3. **Local inverses and their properties.** Let f be analytic on $\overline{\Delta_r}$ and assume that

$$0 < d \leq |f'| \leq D \tag{2.1}$$

there. Assuming $f(0) = 0$ we can write

$$f(z) = a_1 z + \cdots, \qquad |a_1| \geq d,$$

and without loss of generality we may assume that a_1 is real and $a_1 \geq d$. We have

$$f'(z) = a_1 + \sum_{j \geq 2} j a_j z^{j-1}$$

[3]By an observation of E. Rahmanov, this is also the consequence of the Cauchy-Riemann equations for $\log P$ — with some local branch of the logarithm — at the points of σ_P

and here

$$j|a_j| = \left|\frac{1}{2\pi i}\int_{|\xi|=r}\frac{f'(\xi)}{\xi^j}d\xi\right| \le \frac{D}{r^{j-1}},$$

so

$$\left|\sum_{j\ge 2} j a_j z^{j-1}\right| \le D\sum_{j\ge 2}(|z|/r)^{j-1} = D\frac{|z|/r}{1-|z|/r} \le d/2$$

if $|z| \le dr/4D$, and hence

$$\Re f'(z) \ge \frac{a_1}{2} \ge \frac{d}{2}, \qquad |z| \le \frac{dr}{4D}.$$

Now if $u, v \in \Delta_{dr/4D}$ are two distinct points, then

$$|f(u) - f(v)| = \left|\int_u^v f'(t)dt\right| = \left|\int_0^1 f'(u + (v-u)s)(v-u)ds\right| \ge \frac{d}{2}|u - v|$$

because with $\xi = (u-v)/|u-v|$

$$\Re\left\{f'(u + (v-u)s)(v-u)/\xi\right\} \ge \frac{d}{2}|u - v|$$

and $|\xi| = 1$. This implies, in particular, that f is univalent in $\Delta_{dr/4D}$, and by Koebe's 1/4-theorem[4] the image of $\Delta_{dr/4D}$ under f contains the disk about the origin and of radius

$$(dr/4D)|f'(0)|/4 \ge d^2r/16D,$$

i.e. $\Delta_{d^2r/16D} \subseteq f(\Delta_{dr/4D})$. In general, for $\delta \le dr/4D$ the image of Δ_δ under f contains the disk $\Delta_{d\delta/4}$. It also follows from the formula for the derivative of inverse functions that

$$|(f^{-1})'(z)| \le \frac{1}{d}, \qquad z \in \Delta_{d^2r/16D}.$$

Suppose now that, in addition to the f we have been considering, there is another analytic function g on Δ_r such that

$$|f - g| \le \theta$$

with some constant θ. By Cauchy's formula (applied on disks Δ_ρ, $\rho < r/2$ and then letting $\rho \to r/2$)

$$|f' - g'| \le \frac{2\pi r}{2\pi}\frac{\theta}{(r - r/2)^2} = \frac{4\theta}{r}, \qquad |z| \le r/2,$$

and so

$$\frac{d}{2} \le |g'| \le 2D \qquad \text{on } D_{r/2} \text{ if } \qquad \theta \le dr/8.$$

Therefore, according to what we have proven above, g is univalent on $\Delta_{dr/32D}$ (note that now instead of r, d, D we have to use $r/2, d/2, 2D$), and $g(0) + \Delta_{d^2r/256D}$ is in the range of g when restricted to $\Delta_{dr/32D}$. Since $|g(0)| \le \theta$, it follows that if

[4]Koebe's theorem claims that if $h(z) = z + \cdots$ is univalent in the unit disk, then the image of the unit disk under h contains the disk $\Delta_{1/4}$.

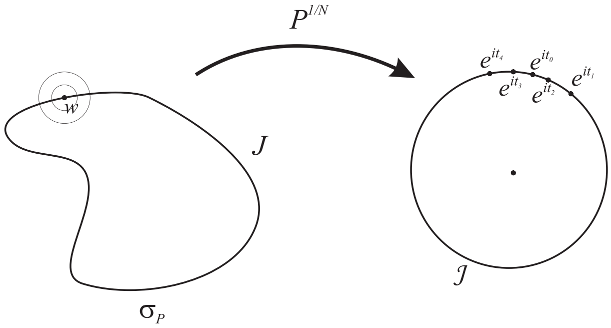

FIGURE 1. Schematic figure of σ_P, its arcs $J^* \subset J$ that are the portions of σ that lie outside the two disks around w, as well as their images $\mathcal{J}^* \subset \mathcal{J}$ under $P^{1/N}$ on the unit circle with endpoints e^{it_3}, e^{it_2} resp. e^{it_4} and e^{it_1}

$\theta \le d^2r/512D$ is also satisfied, then $\Delta_{d^2r/512D}$ is in the common range of f and g when they are restricted to $\Delta_{dr/32D}$.

Let $w \in \Delta_{d^2r/512D}$, and set $u = g^{-1}(w)$ and $v = f^{-1}(w)$. As we have seen, $|f(u) - f(v)| \ge \frac{d}{2}|u - v|$, and if we combine this with $|f(u) - g(u)| \le \theta$ then we obtain

$$0 = |f(v) - g(u)| \ge \frac{d}{2}|u - v| - \theta,$$

which implies $|u - v| \le 2\theta/d$, i.e.

$$|g^{-1} - f^{-1}| \le \frac{2\theta}{d} \qquad \text{on} \qquad \Delta_{d^2r/512D}. \tag{2.2}$$

So far we have assumed that $f(0) = 0$. If $f(0) \ne 0$, then $f(0) + \Delta_{d^2r/512D}$ is in the range of both f and g when restricted to $\Delta_{dr/32D}$, and (2.2) is true when $\Delta_{d^2r/512D}$ is replaced by $f(0) + \Delta_{d^2r/512D}$.

2.4. **Properties of the natural parametrization.** Let P, σ_P as before, and choose ρ_0 so small that for each $w \in \sigma_P$ the intersection $\sigma_P \cap \Delta_\rho(w)$ is a Jordan arc for all $\rho < 2\rho_0$, all zeros of P' lie of distance $> 2\rho_0$ from σ_P, and $P^{1/N}$ is univalent on the $2\rho_0$-neighborhood $U_{2\rho_0}(\sigma_P)$ of σ_P (see Section 2.1). Choose $d > 0$, $D > 1$ so that $d \le |(P^{1/N})'| \le D$ on $U_{2\rho_0}(\sigma_P)$, and finally choose a number $0 < \rho < \rho_0 d/4D$.

Let $w \in \sigma_P$ be given, and consider the arcs $J = \sigma_P \setminus \Delta_\rho(w)$ and $J^* = \sigma_P \setminus \Delta_{2\rho}(w)$. The function $P^{1/N}$ maps J into an arc $\mathcal{J}$ of the unit circle, and it maps J^* into a subarc $\mathcal{J}^*$ of $\mathcal{J}$, see Figure 1. If $P(w)^{1/N} = e^{it_0}$ and the endpoints of $\mathcal{J}$ resp. $\mathcal{J}^*$ in their counterclockwise orientation are e^{it_3}, e^{it_2} resp. e^{it_4} and e^{it_1}, then what we have shown in Section 2.3 implies that with some $c_1, c_2 > 0$ we have for all $\rho < \rho_0 d/4D$

$$c_1\rho \le t_2 - t_1,\ t_0 - t_2,\ t_3 - t_0,\ t_4 - t_3 \le c_2\rho. \tag{2.3}$$

We have also seen that if $\theta \le d^2\rho/1024D$ is given and for an analytic function g we have $|g - P^{1/N}| \le \theta$ on $U_{\rho/2}(J)$, then

$$|g^{-1} - \gamma_P| \le \frac{2\theta}{d} \qquad \text{on} \quad \mathcal{J}. \tag{2.4}$$

For a large number V (to be selected below) we set $R(z) = P(z)^m(z-\alpha)$, where m is a large number and $|\alpha - w| < \rho/4V$ will be selected later. The degree of R is $mN + 1$ and

$$R(z)^{1/(mN+1)} = P(z)^{m/(mN+1)}(z-\alpha)^{1/(mN+1)}. \tag{2.5}$$

We also request that if $\theta \leq d^2\rho/1024D$ is given, then the m in the definition of R be so large that irrespectively of the actual choice of α (with $|\alpha - w| < \rho/4V$)

: 1) $|R| > 1$ in the unbounded component of $\mathbf{C} \setminus \overline{U_{\rho/2V}(\sigma_P)}$,

: 2) $|R| < 1$ in the bounded component of $\mathbf{C} \setminus \overline{U_{\rho/2V}(\sigma_P)}$.

: 3) with the main branch of the logarithm in defining the powers $P^{1/N}$ and $R^{1/mN+1}$ we have

$$|P^{1/N} - R^{1/(mN+1)}| < \theta \qquad \text{on} \quad U_{\rho/2}(J) \tag{2.6}$$

(note that any $z \in U_{\rho/2}(J)$ and $\alpha \in \Delta_{\rho/4V}(w)$ are at least of distance $\rho/4$ apart). Therefore, according to what we have just said in (2.4), it follows that if σ_R is also a Jordan curve, then

$$|\gamma_P - \gamma_R| \leq \frac{2\theta}{d} \qquad \text{on} \quad \mathcal{J}. \tag{2.7}$$

This is an estimate on the two parametrizations γ_P and γ_R only on the arc $\mathcal{J}$ of the unit circle, and note that it is independent of the actual choice of $|\alpha - w| < \rho/4V$ and $\rho < \rho_0 d/4D$. On the whole of C_1 we prove

Claim 2.2. *If* $\theta < \min\{d^2\rho/1024D, c_1\rho/8\}$, *then*

$$|\gamma_P - \gamma_R| \leq 4\rho, \qquad \textit{on} \quad C_1. \tag{2.8}$$

Here c_1 is from (2.3).

Proof. The inequality

$$|\gamma_P(e^{it}) - \gamma_R(e^{it})| \leq 4\rho$$

follows from (2.7) if $e^{it} \in \mathcal{J}$ because $\theta < d\rho$, so in what follows we may assume that $e^{it} \notin \mathcal{J}$, i.e. $t_2 < t < t_3$. In that case $|\gamma_P(e^{it}) - w| \leq \rho$, therefore if $|\gamma_R(e^{it}) - w| \leq 3\rho$ is also true, then (2.8) follows. We shall show that for sufficiently large V this is indeed the case.

In fact, suppose to the contrary that $\gamma_R(e^{it})$ lies outside $\Delta_{3\rho}(w)$. It definitely lies in $U_{\rho/V}(\sigma_P)$ by properties 1)–2) above (note that $|R(\gamma_R(e^{it}))| = 1$), so it lies in a ball of radius ρ/V about a point $z_0 \in J^*$: $|z_0 - \gamma_R(e^{it})| \leq \rho/V$. Then

$$|P^{1/N}(z_0) - P^{1/N}(\gamma_R(e^{it}))| \leq D\frac{\rho}{V},$$

$$|P^{1/N}(\gamma_R(e^{it})) - R^{1/(mN+1)}(\gamma_R(e^{it}))| \leq \theta,$$

which imply, in view of $R^{1/(mN+1)}(\gamma_R(e^{it})) = e^{it}$, the inequality

$$|P^{1/N}(z_0) - e^{it}| \leq c_1(\rho/4),$$

provided $V > 8D/c_1$ and $\theta < c_1\rho/8$, where c_1 is the constant from (2.3). However, this is impossible, since $P^{1/N}(z_0) \in \mathcal{J}^*$, so its distance from any point of the arc $\{e^{it} \mid t_2 \leq t \leq t_3\}$ is at least $c_1\rho/2$ by the choice of c_1 in (2.3). This contradiction proves the claim. □

We summarize this section: if $\varepsilon > 0$ is given, then choosing $\rho < \varepsilon/4$ so that it satisfies all the requirements above as well as choosing $\theta < \min\{d^2\rho/1024D, c_1\rho/8\}$ we get that if P and R are related as in (2.5)–(2.6), then, assuming that σ_R is a Jordan curve, we have

$$|\gamma_P - \gamma_R| < \varepsilon. \tag{2.9}$$

The order of the choice of the parameters for a given ε is: $\rho < \varepsilon/4$, $\theta < \min(d^2\rho/1024D, c_1\rho/8)$, and m so large that 1)–3) are true where $V > 8D/c_1$ is a fixed number.

2.5. **The choice of R.** As before, let $w \in \sigma_P$ be given. In this section we shall choose α in $R(z) = P(z)^m(z-\alpha)$ close to w: $|w-\alpha| \le \rho/4V$ (see the previous section).

We shall consider the polynomials $R_\beta(z) = P(z)^m(z-\beta)$ with $|w-\beta| \le \rho/4V$. First of all, for a given $\varepsilon > 0$ choose a $\rho < \varepsilon/4$, θ and m_1 so that when α is replaced by β, everything we have discussed is valid for $m \ge m_1$, irrespectively of the actual choice of β with $|w-\beta| < \rho/4V$, except for the property that σ_{R_β} is a Jordan curve. For $R = R_\alpha$ that Jordan curve property has been assumed in the discussion above, and in what follows we have to verify it during our process of selecting α. In general, for R_β we do not need this property.

We may also assume $\rho > 0$ so small that $P'(z) \ne 0$ in the 2ρ-neighborhood of σ_P (this is possible for P' is not zero on σ_P).

Let $\overline{wa}$ be the outer normal segment to σ_P at w of some length $|w-a| < \rho/8V$, and parametrize $\overline{wa}$ as $w + t(a-w)$, $0 \le t \le 1$. Let, furthermore, a^* be the reflection of a onto w. For small $|w-a|$ the point a lies in the outer, and the point a^* lies in the inner domain to σ_P. Select a this way, and we consider the polynomials R_β with $\beta \in \overline{wa}$. Connect by a system T of broken lines the point a^* with all zeros of P and P' inside the inner domain to σ_P (recall Section 2.1 that this is possible). There is an $m_2 \ge m_1$ such that for $m \ge m_2$ we have

: a) $|R_\beta| < 1$ on T irrespectively of the choice of $\beta \in \overline{wa}$,

: b) $|R_a(z)| > 1$ for all $z \in \Sigma$ where Σ is the set of points that lie outside σ_P and are of distance $|a-w|/2$ from σ_P, as well as $|R_a(z)| > 1$ for all z lying in the outer domain to σ_P that are of distance $\ge |a-w|/2$ from both σ_P and a.

: c) for all $\beta \in \overline{wa}$ we have $|R_\beta(z)| > 1$ when z lies outside σ_P and is of distance $\ge \rho/2$ from σ_P and $|R_\beta(z)| < 1$ when z lies inside σ_P and is of distance $\ge \rho/2$ from σ_P,

: d) for all $\beta \in \overline{wa}$ we have $R'_\beta(z) \ne 0$ when z lies in the $\rho/2$-neighborhood of σ_P and $|z-a| \ge \rho/2$.

That for sufficiently large m properties a)–c) can be achieved is simple to see, and for property d) consider that

$$R'_\beta(z) = P(z)^{m-1}\Big(mP'(z)(z-\beta) + P(z)\Big)$$

and recall that $|P'(z)|$ has a positive lower bound in the ρ-neighborhood of σ_P by the choice of ρ (note also that $|z-a| \ge \rho/2$ implies $|z-\beta| \ge \rho/4$ for all $\beta \in \overline{wa}$ because $|w-a| < \rho/8V \le \rho/8$).

For $\beta = a$ the lemniscate σ_{R_β} has two connected components: one containing all zeros of P^m (by property a) and Section 2.2 these zeros lie in the same component of all $\{|R_\beta| < 1\}$), and one containing the point a. In fact, the set Σ in part b) separates these two components. On the other hand, for $\beta = w$ the lemniscate σ_{R_β} has only one component because of property a) and because of the fact that $|R_\beta| < 1$ on the segment $\overline{a^*w}$ (note that $T \cup \overline{a^*w}$ connects all zeros of R_β for $\beta = w$). Therefore, there must be a smallest $t_0 \in [0, 1]$ with the property that for all $t_0 < t \le 1$ and for $\beta = w + t(a - w)$ the lemniscate σ_{R_β} has two components (it cannot have more by property a) and the fact that T connects all zeros of P^m).

Let $\alpha_0 = w + t_0(a - w)$. Then the lemniscate $\sigma_{R_{\alpha_0}}$ is connected (otherwise for all $t < t_0$ sufficiently close to t_0 and for $\beta = w + t(a - w)$ the lemniscate σ_{R_β} would also be disconnected, but this contradicts the definition of t_0). Moreover, $\sigma_{R_{\alpha_0}}$ is not a Jordan curve. Indeed, if it was, then by Section 2.2 there would be a connected set T_1 of broken lines inside $\sigma_{R_{\alpha_0}}$ that connects the zeros of R_{α_0} such that $|R_{\alpha_0}| \le 1 - \kappa$ on T_1 with some $\kappa > 0$. But then for $t > t_0$ sufficiently close to t_0 and for $\beta = w + t(a - w)$ we would have $|R_\beta| < 1$ on $T_1 \cup \overline{\alpha_0\beta}$, and this set connects the zeros of R_β. This would mean that σ_{R_β} is connected, which is impossible again by the definition of t_0.

Thus, $\sigma_{R_{\alpha_0}}$ is connected but it is not a Jordan curve, so it has a multiple point W, where necessarily $R'_{\alpha_0}(W) = 0$. We have

$$R'_{\alpha_0}(z) = P(z)^{m-1}\left[mP'(z)(z - \alpha_0) + P(z)\right],$$

and this has $(m - 1)N$ zeros at the zeros of P. Furthermore, by Rouche's theorem, the expression in the square bracket has a zero (counting multiplicity) close to any zero of P' provided m is sufficiently large, and these $N - 1$ zeros lie in the component of $\{|R_{\alpha_0}| < 1\}$ that contain the zeros of P (recall that $|R_{\alpha_0}| < 1$ on T which connects the zeros of P and P') for all $m \ge m_3$ with some $m_3 \ge m_2$. So, assuming $m \ge m_3$, we conclude that R'_{α_0} has $mN - 1$ zeros in the just mentioned component, and as a consequence we conclude that W can only be a simple zero of R'_{α_0}. Then in a neighborhood U of W the intersection $U \cap \sigma_{R_{\alpha_0}}$ consists of two analytic arcs intersecting at W at right angle.

Note that α_0 and the other zeros of R_{α_0} lie in different components of $\{|R_{\alpha_0}| < 1\}$. Consider the "cross" $ABCD$ with center at W, where $\overline{AC}$ and $\overline{BD}$ bisects the angles between the two tangent lines to $\sigma_{R_{\alpha_0}}$ at W, and where C lies in the same component of $\{|R_{\alpha_0}| < 1\}$ as α_0 (and then A lies in the other component of this set), see Figure 2. Connect now α_0 and C by a broken line T_2, and A and a zero of P by a broken line T_3 inside $\{|R_{\alpha_0}| < 1\}$ (T_2 and T_3 lie in different components of this set) so that $|R_{\alpha_0}| < 1$ on $T_2 \cup T_3$. Let α be on the segment $\overline{\alpha_0 W}$ lying close to α_0. If α is sufficiently close to α_0 and the cross $ABCD$ is sufficiently small, then $|z - \alpha| < |z - \alpha_0|$ is true for all $z \in \overline{AC}$. Therefore, if α is sufficiently close to α_0, then $|R_\alpha| < 1$ on $T \cup T_2 \cup T_3 \cup \overline{AC} \cup \overline{\alpha_0\alpha}$. But this latter set is connected and contains all zeros of R_α, hence σ_{R_α} is a Jordan curve by Section 2.2.

On the other hand, $|R_{\alpha_0}(B)| > 1$ and $|R_{\alpha_0}(D)| > 1$, so if α is sufficiently close to α_0, then we have $|R_\alpha(B)| > 1$ and $|R_\alpha(D)| > 1$. But on the segment $\overline{BD}$ we have $|R_\alpha| < 1$ at the point $W \in \overline{AC}$, so σ_{R_α} must intersect the segments $\overline{BW}$ and $\overline{WD}$

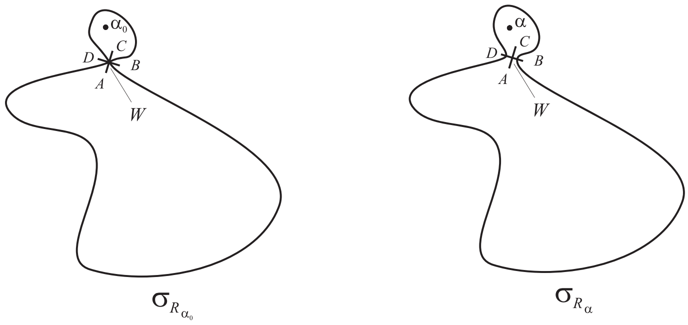

FIGURE 2. Schematic figure of $\sigma_{R_{\alpha_0}}$ and the cross $ABCD$ with center at the double point W on the left figure. The final choice α is close to α_0 on the segment $\overline{\alpha_0 W}$, and the resulting Jordan curve σ_{R_α} is depicted on the right figure.

in some points, say w_1 and w_2. So w_1 and w_2 are two different points on σ_{R_α} such that $|w_1 - w_2| < |D - B|$.

Let τ be the smaller of the diameters of the two components of $\sigma_{R_{\alpha_0}} \setminus \{W\}$. If α lies sufficiently close to α_0, then the two components of $\sigma_{R_\alpha} \setminus \{w_1, w_2\}$ have diameters $> \tau/2$. Now if M is arbitrarily given, and the cross is so small that $|D - B| < \tau/2M$, then it follows that both components of $\sigma_{R_\alpha} \setminus \{w_1, w_2\}$ have diameters bigger than $M|w_1 - w_2|$, i.e. the small crosscut $\overline{w_1 w_2}$ of σ_{R_α} is producing two large components relative to the length $|w_1 - w_2|$ of the crosscut.

Finally, since we had $|\alpha_0 - w| < \rho/4V$, we can choose α so close to α_0 on the segment $\overline{\alpha_0 W}$ that $|\alpha - w| < \rho/4V$ is also satisfied.

This completes the choice of α, for which σ_{R_α} is a Jordan curve.

Let us summarize our findings.

Claim 2.3. *Given $w \in \sigma_P$, $\varepsilon > 0$, $\delta > 0$ and M, there is a polynomial $R(z) = P(z)^m(z - \alpha)$ such that σ_R is a Jordan curve, $|\gamma_R - \gamma_P| < \varepsilon$, and σ_R has two points w_1, w_2 in $\Delta_\delta(w)$ such that both components of $\sigma_{R_\alpha} \setminus \{w_1, w_2\}$ have diameters bigger than $M|w_1 - w_2|$.*

Indeed, all we need to do is to set $\rho < \min(\varepsilon/4, \delta/4)$, $|a - w| < \delta/2$ in the construction above, choose m large (independently of $|\alpha - w| < \rho/4V$) then choose the cross $ABCD$ small, and finally choose α sufficiently close to α_0. All statements in Claim 2.3 were obtained during the construction except for w_1, w_2 being in $\Delta_\delta(w)$. This last property follows from the requirements c) and d) in Section 2.5. Indeed, by c) the lemniscate $\sigma_{R_{\alpha_0}}$ lies in the $\rho/2$-neighborhood of σ_P, and then by property d) the point W (where R'_{α_0} is zero) on that lemniscate must satisfy $|a - W| \le \rho/2$. Therefore, if α lies sufficiently close to α_0, then w_1 and w_1 lie in $\Delta_{3\rho/4}(a)$, which is part $\Delta_\delta(w)$.

2.6. **Proof of Theorem 1.1.** σ will be the limit of a sequence of Jordan curves σ_{P_n} with P_n a polynomial of degree N_n having the following properties:

I. If γ_{P_n} is the natural parametrization of σ_{P_n}, then

$$|\gamma_{P_n} - \gamma_{P_{n+1}}| \le \varepsilon_n, \qquad n = 1, 2, \dots, \tag{2.10}$$

where $\varepsilon_n < 1/4n$ will be selected later so that they also satisfy

$$\sum_{n=1}^{\infty} \varepsilon_n < 1/4, \qquad \sum_{m>n} \varepsilon_m < \varepsilon_n \quad \text{for all } n = 1, 2, \dots. \tag{2.11}$$

II. More than half of the zeros of P_n lie in $\Delta_{1/2}$.
III. $|P_n| \ge \delta_n^*$ on $|z| = 1/2$ with some $\delta_n^* > 0$.

We start from $P_1(z) = z$, $\sigma_{P_1} = C_1$, $\varepsilon_1 = 1/8$, $\delta_1^* = 1/3$, and once P_n is defined, we choose $P_{n+1}(z) = P_n(z)^{m_n}(z - \alpha_n)$ as in the preceding sections so that they satisfy the properties to be discussed below.

In view of (2.10) the functions γ_{P_n} uniformly converge to some $\gamma : C_1 \to \mathbf{C}$,

$$\gamma = \lim_{n\to\infty} \gamma_{P_n} = \gamma_{P_1} + \sum_{n=1}^{\infty} (\gamma_{P_{n+1}} - \gamma_{P_n}), \tag{2.12}$$

for which

$$|\gamma - \gamma_{P_n}| \le \sum_{m=n}^{\infty} |\gamma_{P_{m+1}} - \gamma_{P_m}| \le \sum_{m=n}^{\infty} \varepsilon_m < 2\varepsilon_n.$$

In particular, if σ is the continuous curve that γ describers, then σ lies in the 1/4-neighborhood of C_1. Below we shall detail the construction so that σ becomes a Jordan curve (that does not have to be so just from σ being the limit of the Jordan curves σ_{P_n}), and then $\overline{\Delta_{1/2}}$ lies in the interior of σ.

We mention first of all that P_n is the N_n-th degree Chebyshev polynomial for σ_{P_n}. Indeed, if $S(z) = z^{N_n} + \cdots$ is an arbitrary monic polynomial of degree N_n, then, since

$$\lim_{z\to\infty} \frac{S(z)}{P_n(z)} = 1,$$

it follows from the maximum principle that in the exterior of σ_{P_n} that $\|S\|_{\sigma_{P_n}} \ge 1$, meaning that P_n has the smallest norm on σ_{P_n} among such polynomials.

From the unicity of Chebyshev polynomials for a compact set (consisting of infinitely many points) it follows that there is an $\varepsilon_{n,1} > 0$ such that if σ is a Jordan curve such that the Hausdorff distance $\mathrm{dist}(\sigma, \sigma_{P_n})$ is smaller than $\varepsilon_{n,1}$, then for the Chebyshev polynomial $T_{N_n}^{\sigma}$ of degree N_n of σ we have

$$|T_{N_n}^{\sigma} - P_n| < \frac{\delta_n^*}{2} \qquad \text{on} \quad \sigma_{P_n}. \tag{2.13}$$

Then, by the maximum principle, the same inequality holds on $|z| = 1/2$ (this circle lies inside every σ_{P_n}), and by property III. and Rouche's theorem we can conclude that for $\varepsilon_n < \varepsilon_{n,1}/2$ the polynomials P_n and $T_{N_n}^{\sigma}$ have the same number of zeros in $\Delta_{1/2}$, so $T_{N_n}^{\sigma}$ has more than half of its zeros in $\Delta_{1/2}$. This shows that the zero counting measures of $T_{N_n}^{\sigma}$, $n = 1, 2, \dots$, do not converge to the equilibrium distribution of σ (which lies on σ).

Next, suppose that for some numbers δ_j, $j = 1, 2, \dots, n$ we have

$$\min_{|e^{iu} - e^{iv}| \ge 1/j} |\gamma_{P_n}(e^{iu}) - \gamma_{P_n}(e^{iv})| > \delta_j, \qquad j = 1, 2, \dots, n. \tag{2.14}$$

Then there is an $\varepsilon_{n,2} > 0$ such that if for some function $\gamma : C_1 \to \mathbf{C}$ we have $|\gamma_{P_n} - \gamma| < \varepsilon_{n,2}$, then

$$\min_{|e^{iu}-e^{iv}|\geq 1/j} |\gamma(e^{iu}) - \gamma(e^{iv})| > \delta_j, \qquad j = 1, 2, \ldots, n, \tag{2.15}$$

is also true. If $\varepsilon_n < \varepsilon_{n,2}/2$ is also satisfied, then this will hold for the γ in (2.12). Since the inequality in (2.15) is true for all j, it follows that γ defines a Jordan curve σ.

Assume now that $w_n \in \sigma_{P_n}$ is a point on σ_{P_n} such that there are $w_{n,1}, w_{n,2} \in \Delta_{1/n}(w_n)$ distinct points on σ_{P_n} with the property that if $H^1(w_{n,1}, w_{n,2}; \sigma_{P_n})$ and $H^2(w_{n,1}, w_{n,2}; \sigma_{P_n})$ are the two components of $\sigma_{P_n} \setminus \{w_{n,1}, w_{n,2}\}$, then

$$\min\big(\mathrm{diam}(H^1(w_{n,1}, w_{n,2}; \sigma_{P_n})), \mathrm{diam}(H^2(w_{n,1}, w_{n,2}; \sigma_{P_n}))\big) > n|w_{n,1} - w_{n,2}|.$$

Assume that $w_{n,1} = \gamma_{P_n}(e^{it_{n,1}})$ and $w_{n,2} = \gamma_{P_n}(e^{it_{n,2}})$. Then there exists an $\varepsilon_{n,3} > 0$ such that if $\gamma : C_1 \to \mathbf{C}$ is a parametrization of a Jordan curve σ such that $|\gamma_{P_n} - \gamma| < \varepsilon_{n,3}$, then for $\tilde{w}_{n,j} = \gamma(e^{it_{n,j}})$ we have

$$\min\big(\mathrm{diam}(H^1(\tilde{w}_{n,1}, \tilde{w}_{n,2}; \sigma)), \mathrm{diam}(H^2(\tilde{w}_{n,1}, \tilde{w}_{n,2}; \sigma)\big) > n|\tilde{w}_{n,1} - \tilde{w}_{n,2}|, \tag{2.16}$$

and of course we shall require of our ε_n that $\varepsilon_n < \varepsilon_{n,3}/2$ hold, so the just discussed property is true for the γ from (2.12). Let $\{t_k\}$ be an enumeration of the rational numbers on $\mathbf{R}$, and assume that $w_n = \gamma_{P_n}(e^{it_n})$ for all n. Then

$$\begin{aligned} |\tilde{w}_{n,j} - \gamma(e^{it_n})| &\leq |\tilde{w}_{n,j} - w_{n,j}| + |w_{n,j} - \gamma_{P_n}(e^{it_n})| + |\gamma_{P_n}(e^{it_n}) - \gamma(e^{it_n})| \\ &\leq 2\varepsilon_n + \frac{1}{n} + 2\varepsilon_n < \frac{2}{n}. \end{aligned}$$

But the points $\gamma(e^{it_n})$, $n = 1, 2 \ldots$, form a dense set on σ, therefore (2.16) shows that no subarc of σ is analytic (or even C^1).

To complete the induction, set

$$\varepsilon_n = \frac{1}{4}\min(\varepsilon_{n,1}, \varepsilon_{n,2}, \varepsilon_{n,3}, \varepsilon_{n-1}),$$

and $P_{n+1}(z) = P_n(z)^{m_n}(z - \alpha_n)$ of degree $N_{n+1} = m_n N_n + 1$, where this polynomial is constructed in Section 2.5 for the numbers $\varepsilon = \varepsilon_n$ and for the point $w = \sigma_{P_n}(e^{it_n}) \in \sigma_{P_n}$ (with an α_n lying close to w_n as in Section 2.5). Then (2.12) defines a Jordan curve σ with all the discussed properties. If P_n has more than $N_n/2$ of its zeros in $\Delta_{1/2}$, say $\geq \frac{N_n}{2} + q_n$ zeros with $q_n > 0$, then $P_{n+1}(z) = P_n(z)^{m_n}(z - \alpha_n)$ has at least

$$m_n\left(\frac{N_n}{2} + q_n\right) > \frac{m_n N_n + 1}{2}$$

zeros in $\Delta_{1/2}$ provided m_n is so large that $m_n q_n > 1/2$, so the assumption II. on the zeros is preserved when going from P_n to P_{n+1}.

Finally, if we set

$$\delta^*_{n+1} = \min_{|z|=1/2} |P_{n+1}(z)| \geq (\delta^*_n)^{m_n}\frac{1}{4}$$

and

$$\delta_{n+1} = \min_{|e^{iu}-e^{iv}|\geq 1/(n+1)} |\gamma_{P_{n+1}}(e^{iu}) - \gamma_{P_{n+1}}(e^{iv})|,$$

then property III., as well as (2.14) are preserved while going from P_n to P_{n+1}, and this completes the proof.

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