Pure and Applied Functional Analysis
Volume 8, Number 2, 2023, 603–626

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

LONG TIME AND LARGE CROWD DYNAMICS OF FULLY DISCRETE CUCKER-SMALE ALIGNMENT MODELS

EITAN TADMOR

To Ron DeVore for many years of great friendship

Abstract. We provide a bird's eye view on developments in analyzing the long time, large crowd behavior of Cucker-Smale alignment dynamics. We consider a class of (fully-)discrete models, paying particular attention to general alignment protocols in which agents, with possibly time-dependent masses, are driven by a large class of heavy-tailed communication kernels. The presence of time-dependent masses allows, in particular, non-symmetric communication. While revisiting known results in the literature, we also shed new light of various aspects on the long time flocking/swarming behavior, driven by the decay of energy fluctuations and heavy-tailed connectivity. We also discuss the large crowd dynamics in terms of the hydrodynamic description of the corresponding Euler alignment models.

1. The Cucker-Smale model

In 1998, Craig Reynolds won a Scientific and Engineering Award of the Academy of Motion Picture Arts and Sciences for "*pioneering contributions to the development of three dimensional computer animation for motion picture production*", [121]. Reynolds was recognized for his work on realistic simulations of *flocking*, [120], proposing a collective dynamics of 'bird-like objects' (or 'boids') which are driven by pairwise interactions acting in three zones of *repulsion*, *alignment* and *attraction*. A similar 3Zone protocol is found in a broad spectrum of models for collective dynamics in different contexts: in modeling swarming dynamics in *ecology* — from fish, birds and sheep to bacteria, locust and insects, [1–3, 7, 8, 10, 12, 13, 19, 20, 23, 24, 32, 40–44, 51, 53, 54, 60, 61, 64, 65, 77, 78, 84, 87, 88, 94, 97, 111, 112, 114, 115, 120, 128, 139–143, 145, 148]; modeling social dynamics of *human interactions* — from pedestrians, exchange of opinions and ratings to markets and marketing, [4–6,11,16,17,21,25,47,57,73–76,89,98,117,123,146,147]; and in modeling the dynamics of *sensor-based networks* — ranging from macro-molecules and metallic rods to control and mobile robot networks, [18,39,82,83,91,107–110,122,149,150]. The common theme of the different models is crowd dynamics dictated by pairwise interactions between members of the crowd which are viewed as *agents*. A main

2020 *Mathematics Subject Classification.* 92D25, 70F40, 35Q35.
Key words and phrases. Flocking, alignment, spectral analysis, strong solutions, critical thresholds.
Research was supported in part by ONR grant N00014-2112773.

question of interest is to understand how the small scale pairwise interactions within the crowd, are self-organized into a large scale patterns of the whole crowd, so that "the whole is greater than the sum of its parts". One then refers to the *emergent behavior* of the crowd, where the larger patterns are realized by a crowd forming a flock, reaching a consensus, admitting a synchronized state, aggregate into one or more clusters, etc.

The class Cucker-Smale alignment models. Pairwise attraction and repulsion are familiar from particle physics, for example, particle dynamics driven Coulomb and other singular potentials, [92, 93, 124–126]. Here, we focus our attention on *alignment* dynamics, driven by pairwise interactions in which agents steer towards *average heading*. We consider the agent-based system in which N agents, identified with (position, velocity) pairs $(\mathbf{x}_i(t), \mathbf{v}_i(t)) : \mathbb{R}_+ \mapsto (\Omega, \mathbb{R}^d)$ and subject to prescribed initial conditions, $(\mathbf{x}_i(0), \mathbf{v}_i(0)) = (\mathbf{x}_{i0}, \mathbf{v}_{i0}) \in (\Omega, \mathbb{R}^d)$, are driven by

$$\left\{ \begin{aligned} &\mathbf{x}_i(t+\tau) = \mathbf{x}_i(t) + \tau\mathbf{v}_i(t) \\ &\mathbf{v}_i(t+\tau) = \mathbf{v}_i(t) + \tau \sum_{j\in\mathcal{N}_i} m_j\phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)). \end{aligned} \right. \tag{1.1}$$

The dynamics is dictated by a symmetric *communication kernel*,

$$\phi(\mathbf{x}, \mathbf{x}') = \phi(\mathbf{x}', \mathbf{x}) \geqslant 0.$$

Its dynamic values, $\phi_{ij}(t) = \phi(\mathbf{x}_i(t), \mathbf{x}_j(t))$, encode the 'rule of engagement' between agents, and in particular the neighborhood $\mathcal{N}_i = \{j : \phi_{ij}(t) > 0\}$, which contributes to the steering of a 'boid' positioned at $\mathbf{x}_i$. The spatial domain Ω is either $\mathbb{T}^d$ or $\mathbb{R}^d$, so that boundaries are avoided, and $\tau > 0$ is a small, possibly variable time-step, $\tau = \tau(t)$. Different agents, $(\mathbf{x}_i, \mathbf{v}_i)$, are assumed to have different masses, m_i, or other constant traits attributed to an agent positioned at $\mathbf{x}_i$.

We refer to (1.1) as the class of Cucker-Smale (C-S) models for alignment dynamics. Different models are attached to different ϕ's and different m_i's. The original model of Cucker & Smale (C-S) [43, 44] is *the* canonical model for the class of alignment dynamics (1.1) with $\phi(\mathbf{x}, \mathbf{x}') \sim (1 + |\mathbf{x} - \mathbf{x}'|)^{-\beta}$, $\beta > 0$, which assumes a uniform mass distribution $m_i \equiv 1/N$,

$$\mathbf{v}_i(t+\tau) = \mathbf{v}_i(t) + \frac{\tau}{N} \sum_{j\in\mathcal{N}_i} \phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)). \tag{1.2}$$

The work of Cucker & Smale attracted a considerable attention in the literature and motivated the study of many variants of the C-S alignment models; we refer to [9, 33,34,113,116,127,131,137] and the references therein. In particular, a more general alignment model based on the formation of 'blobs' or multi-flocks of agents with different masses was derived in [136]. In other models, different m_i's can be identified with different intrinsic 'traits' of different agents, such as degree, temperature, [33, 66, 71, 85, 105]. We further elaborate on one example.

The Motsch-Tadmor model. If each of the terms contributing to the C-S alignment on the right of (1.2), $\sum_j \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)$, is of the same $\mathscr{O}(1)$-order, then its total action of order $\mathscr{O}(N)$ will peak at time $t = \mathscr{O}(1/N)$. Thus, as noted in [136, §2], the pre-factor $1/N$ is C-S model (1.2) is in fact a *scaling* factor, so that the dynamics peaks at the desired time $t \sim \mathscr{O}(1)$.

In [105] we advocated a more realistic scaling which is adapted to spatial variability in the intensity of different alignment terms,

$$\mathbf{v}_i(t+\tau) = \mathbf{v}_i(t) + \frac{\tau}{\sum\limits_{k\in\mathcal{N}_i} \phi_{ik}(t)} \sum_{j\in\mathcal{N}_i} \phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)). \tag{1.3}$$

Here the scaling depends on the *degree* of different agents,

$$deg_i := \sum_{k\in\mathcal{N}_i} \phi_{ik}(t).$$

It should be emphasized that the communication array in M-T model, $\left\{\frac{1}{deg_i}\phi_{ij}\right\}$ is not symmetric. Nevertheless, it does fit the general symmetric framework of C-S class (1.1) with a proper choice of 'masses' $m_i = \frac{1}{L}deg_i$, and *symmetric* interactions $\widetilde{\phi}_{ij} = L\phi_{ij}\frac{1}{deg_i}\frac{1}{deg_j}$, recovering (1.3),

$$\begin{aligned} \mathbf{v}_i(t+\tau) &= \mathbf{v}_i(t) + \tau \sum_{j\in\mathcal{N}_i} m_j\widetilde{\phi}_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)), \\ m_i = \frac{1}{L}deg_i, &\quad \widetilde{\phi}_{ij} = \frac{1}{L}\phi_{ij}\frac{1}{m_i}\frac{1}{m_j}. \end{aligned} \tag{1.4}$$

The scaling parameter L has no effect on the alignment and was introduced here in order to re-scale the total mass[1] so that $M := \sum_i m_i = \mathcal{O}(1)$. In this case, however, the degrees vary in time, $m_i = \frac{1}{L}deg_i(t)$ and the discussion below needs to be modified to include time-dependent masses. This will be further explored in section 3.3 below.

As another example, we mention a similar situation that arises in the context of *thermodynamic* C-S model [33, 66], where m_i's can be identified with the different temperatures $m_i = \theta_i(t)$ of agents with re-scaled velocities $\frac{1}{\theta_i}\mathbf{v}_i$. Again, one needs to address the time-dependence of the temperatures which are dictated by a separate dynamics.

2. Communication kernels

The dynamics of (1.1) is dictated by a symmetric *communication kernel*, $\phi(\cdot,\cdot) \geqslant 0$. Where do these communication kernels come from? they arise from a combination of empirical and phenomenological considerations. A sample of the large literature can be found in [41, 42, 64, 65, 81, 82, 89, 91, 128, 139, 144–147, 149, 150] and the references therein. We mention several primary examples.

A large part of current literature is devoted to the generic class of *metric-based* kernels,

$$\phi(\mathbf{x}, \mathbf{x}') = \varphi(|\mathbf{x} - \mathbf{x}'|).$$

The choice of metric kernels $\varphi(r) = \mathbb{1}_{[0,R_0]}$ and $\varphi(r) = (1+r)^{-\beta},\ 0 < \beta < 1$ are found in the seminal works of Vicsek et. al. [143] and respectively Cucker & Smale [43]. They are motivated by a *phenomenological* reasoning that the strength of pairwise interactions is short-range or at least decreasing with the relative distance, "*birds of feather flock together*" [101]; this should be contrasted with an opposite

[1] For example,, in the case of long range all-to-all communication where $\phi_{ij} = \mathcal{O}(1)$, then $deg_j = \mathcal{O}(N)$ and we set $L = N^2$ so that $M = \frac{1}{L}\sum_j deg_j = \mathcal{O}(1)$.

heterophilous protocol, [106], based on tendency to attract diverse groups so that $\varphi(r)$ is *increasing* over its compact support. A particular sub-class of such metric-based protocols are the *singular* kernels, $\varphi(r)=r^{-\beta}, 0<\beta<d+2$, which emphasize near-by neighbors, $r\ll 1$, over those farther away, [27,49,103,116,119,132,134]. The case of non-summable kerenls, $\beta=d+2s, s\in(0,1)$ correspond to Riesz kernels and could be properly interpreted as principle values of summation in the commutator form [132]

$$\sum_j \phi_{ij}m_j(\mathbf{v}_j-\mathbf{v}_i)=\sum_j\frac{m_j\mathbf{v}_j-m_i\mathbf{v}_i}{|\mathbf{x}_j-\mathbf{x}_i|^{d+2s}}-\sum_j\frac{m_j-m_i}{|\mathbf{x}_j-\mathbf{x}_i|^{d+2s}}\mathbf{v}_i.$$

An important source for communication kernels are detailed observations. As a prime example we mention the class of *topologically-based* kernels, dictated by the *size* of the crowd in between agents positioned at $\mathbf{x}$ and $\mathbf{x}'$

$$\phi(\mathbf{x},\mathbf{x}')=\varphi(\mu(\mathbf{x},\mathbf{x}')),\quad \mu(\mathbf{x},\mathbf{x}'):=\frac{1}{N}\#\{k:\,\mathbf{x}_k\in\mathcal{C}(\mathbf{x},\mathbf{x}')\}. \tag{2.1}$$

Here, $\mathcal{C}(\mathbf{x},\mathbf{x}')$ is a pre-determined communication region enclosed between $\mathbf{x}$ and $\mathbf{x}'$. In particular, if $\mathcal{C}$ is shifted to R-ball centered at $\mathbf{x}$, one ends up with the *non-symmetric* topological kernel [105] $\phi(\mathbf{x},\mathbf{x}')=\frac{\varphi(|\mathbf{x}-\mathbf{x}'|)}{\mu(B_R(\mathbf{x}))}$. Topologically-based communication was observed in starflag project reported in [7, 24, 31, 32], where birds react to the number of closest neighbors rather than their metric distance, and in pedestrian dynamics [123], where communication is decreasing in more crowded regions, and was analyzed in [7, 14, 135].
More on topologically-based kernels can be found in [15, 24, 70, 95, 110]
As a third example, we mention *random-based* communication protocols found in chemo- and photo-tactic dynamics, [67], the Elo rating system, [52, 80], voter and related opinion-based models, [11], or a random-batch method and consensus-based optimization. [25, 45, 61, 86, 118]. Another class of communication kernels are those learned from the data, [22, 99, 100]. Finally, we mention communication kernels which are derived from 'higher order' principles; for example, a minimum entropy principle [12, 13], and the paradigm of anticipation [63].

3. Long time dynamics

A key aspect in the long time behavior of (1.1) is the decay in time of the fluctuations of velocities $\{\mathbf{v}_i-\mathbf{v}_j\}$. Velocity fluctuations can be measured in a weighted-ℓ^2 average sense quantifying *energy fluctuations*, or in a uniform sense quantifying the ℓ^∞-*diameter* of the discrete crowd of velocities.

3.1. **Energy fluctuations.**

We let $\delta\mathscr{E}(t)$ denote the *energy fluctuations*, scaled by the total mass[2]

$$\delta\mathscr{E}(t):=\frac{1}{2M^2}\sum_{i,j}|\mathbf{v}_i(t)-\mathbf{v}_j(t)|^2m_im_j,\qquad M=\sum_i m_j.$$

[2] Here and below, $|\cdot|$ denotes an arbitrary vector norm on $\mathbb{R}^d$.

Thus, $\delta\mathscr{E}(t)$ is the weighted ℓ^2-diameter of the set of velocities $\{\mathbf{v}_i\}_{i=1}^N$ at time t. Equivalently, we can express it as fluctuations around the mean velocity $\overline{\mathbf{v}}$

$$\delta\mathscr{E}(t) = \frac{1}{M}\sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}(t)|^2 m_i, \qquad \overline{\mathbf{v}} := \frac{1}{\sum_i m_i}\sum_i m_i\mathbf{v}_i(t) \tag{3.1}$$

The energy balance encoded in $(1.1)_2$ implies (for simplicity we suppress the time dependence on t on the right-hand side)

$$\left\{\begin{aligned} &\frac{1}{M}\sum_i m_i|\mathbf{v}_i(t+\tau)|^2 - \frac{1}{M}\sum_i m_i|\mathbf{v}_i(t)|^2 \\ &\quad = \frac{2\tau}{M}\sum_i \big\langle m_i\mathbf{v}_i, \sum_j m_j\phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\big\rangle + \frac{\tau^2}{M}\sum_i m_i\big|\sum_j m_j\phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\big|^2. \end{aligned}\right. \tag{3.2}$$

Since the communication kernel is symmetric, $\phi_{ij} = \phi_{ji}$, the total momentum is conserved

$$\mathscr{M}(t+\tau) - \mathscr{M}(t) = \frac{\tau}{M}\sum_{i,j} m_i m_j\phi_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0, \quad \mathscr{M}(t) := \frac{1}{M}\sum_i m_i\mathbf{v}_i(t). \tag{3.3}$$

This implies that the incremental change in energy of the left of (3.2) is the same as the incremental change of energy fluctuations. Indeed,

$$\begin{aligned} \frac{1}{M}\sum_i m_i|\mathbf{v}_i(t)|^2 &\equiv \frac{1}{2M^2}\sum_{i,j} |\mathbf{v}_i(t) - \mathbf{v}_j(t)|^2 m_i m_j + \frac{1}{M^2}\big|\sum_i m_i\mathbf{v}_i(t)\big|^2 \\ &= \delta\mathscr{E}(t) + \big|\mathscr{M}(t)\big|^2, \end{aligned}$$

and the same applies at $t+\tau$,

$$\frac{1}{M}\sum_i m_i|\mathbf{v}_i(t+\tau)|^2 \equiv \delta\mathscr{E}(t+\tau) + \big|\mathscr{M}(t+\tau)\big|^2,$$

and since the squared terms on the right of the last two equalities are the same, we find

$$\frac{1}{M}\sum_i m_i|\mathbf{v}_i(t+\tau)|^2 - \frac{1}{M}\sum_i m_i|\mathbf{v}_i(t)|^2 = \delta\mathscr{E}(t+\tau) - \delta\mathscr{E}(t). \tag{3.4a}$$

We now come to the main point, namely, that the alignment operator of CS dynamics is coercive in the sense that

$$\frac{2}{M}\sum_i \big\langle m_i\mathbf{v}_i, \sum_j m_j\phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\big\rangle = -\frac{1}{M}\sum_{i,j} \phi_{ij}|\mathbf{v}_i - \mathbf{v}_j|^2 m_i m_j. \tag{3.4b}$$

The *weighted* fluctuations on the right is identified as the *enstrophy*. We can bound the last squared term on the right of (3.2) in terms of the enstrophy and the maximal *weighted degree*, $deg_+(t) := \max_i \sum_j \phi_{ij}(t)m_j$

$$\frac{1}{M}\sum_i m_i\big|\sum_j m_j\phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)\big|^2 \leqslant deg_+(t)\frac{1}{M}\sum_{i,j} \phi_{ij}\big|\mathbf{v}_j - \mathbf{v}_i\big|^2 m_i m_j; \tag{3.4c}$$

Inserting (3.4) back into the energy balance (3.2) we find

$$\delta\mathscr{E}(t+\tau)-\delta\mathscr{E}(t) \leqslant -\tau(t)\big(1-deg_+(t)\cdot\tau(t)\big)\frac{1}{M}\sum_{i,j}\phi_{ij}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)\big|^2 m_i m_j. \tag{3.5}$$

Observe that we now pay attention to the time dependence on the right; in particular, the possibly variable time step, $\tau = \tau(t)$, and the time-dependent communication weights, $\phi_{ij}(t) = \phi(\mathbf{x}_i(t), \mathbf{x}_j(t))$.

We let $\mathbb{A}(t)$ denote the $N \times N$ *adjacency matrix* $\mathbb{A}(t) = \{\phi_{ij}(t)\}$ encoding the edges of communication at time t, and define $\Delta_{\mathbf{m}}\mathbb{A}(t)$ is the *weighted graph Laplacian*

$$(\Delta_{\mathbf{m}}\mathbb{A})_{\alpha\beta} = \begin{cases} -\phi_{\alpha\beta}\sqrt{m_\alpha m_\beta}, & \alpha \neq \beta \\[2ex] \displaystyle\sum_{\gamma\neq\alpha}\phi_{\alpha\gamma}m_\gamma & \alpha = \beta. \end{cases} \tag{3.6}$$

The weighted graph Laplacian, weighted by the masses $\mathbf{m} = (m_1, \ldots, m_N)$, has real eigenvalues, $\lambda_1 = 0 \leqslant \lambda_2 \leqslant \ldots \lambda_N$. This generalizes the usual notion of graph Laplacian, e.g., [35, 102], corresponding to the case of uniform weight, $m_i = \mathcal{O}(1/N)$. We now summarize the computations above, quantifying the decay of energy fluctuations in terms of the spectral gap, $\lambda_2\big(\Delta_{\mathbf{m}}\mathbb{A}(t)\big)$.

Theorem 3.1 (**Decay of energy fluctuations**). *Consider the C-S dynamics* (1.1) *with time-steps small enough such that*

$$\tau(t)\cdot\max_i\sum_j\phi_{ij}(t)m_j \leqslant \frac{1}{2}. \tag{3.7}$$

Then the following bound of energy fluctuations holds

$$\begin{aligned} &\delta\mathscr{E}(t_n) \leqslant exp\Big\{-\sum_{k=0}^{n-1}\lambda_2(t_k)\tau(t_k)\Big\}\delta\mathscr{E}_0, \\ &\lambda_2(t) = \lambda_2\big(\Delta_{\mathbf{m}}\mathbb{A}(t)\big), \quad t_{k+1} = t_k + \tau(t_k). \end{aligned} \tag{3.8}$$

Proof. We return to the energy fluctuations bound (3.5). It remains to relate the enstrophy on the right of (3.5) to the energy fluctuations on the left. To this end, we use the following sharp lower bound on the enstrophy [72, §3], expressed in terms of its spectral gap $\lambda_2(t) = \lambda_2\big(\Delta_{\mathbf{m}}\mathbb{A}(t)\big)$,

$$\begin{aligned} &\sum_{i,j}\phi_{ij}(t)|\mathbf{v}_j(t)-\mathbf{v}_i(t)\big|^2 m_i m_j \geqslant \frac{\lambda_2(t)}{M}\sum_{i,j}|\mathbf{v}_j(t)-\mathbf{v}_i(t)\big|^2 m_i m_j, \\ &\lambda_2 = \lambda_2(\Delta_{\mathbf{m}}\mathbb{A}). \end{aligned} \tag{3.9}$$

Inserted into (3.5), the time-step restriction (3.7) and (3.9) yield

$$\begin{aligned} \delta\mathscr{E}(t+\tau) &\leqslant \delta\mathscr{E}(t) - \frac{\tau(t)}{2}\lambda_2(t)\frac{1}{M^2}\sum_{i,j}|\mathbf{v}_j(t)-\mathbf{v}_i(t)\big|^2 m_i m_j \\ &= \Big(1-\tau(t)\lambda_2(t)\Big)\delta\mathscr{E}(t) \leqslant e^{-\tau(t)\lambda_2(t)}\delta\mathscr{E}(t), \end{aligned}$$

and (3.8) follows. □

Remark 3.2 (**Graph connectivity**)**.** The weighted graph Laplacian $\Delta_{\mathbf{m}}\mathbb{A}$ is symmetrizable, with real eigenvalues $\lambda_1 = 0 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_N$. The weighted Poincaré inequality (3.9) provides a sharp lower bound on the enstrophy in terms of the spectral gap $\lambda_2(\Delta_{\mathbf{m}}\mathbb{A}) > 0$, which reflects the connectivity of the weighted graph (V, E), where vertices of V tag the positions $\{\mathbf{x}_i\}$ and the edges E quantify the connections $\{\phi_{ij}\}$. The intricate aspect here is the interplay between the graph which is time dependent, $(\mathsf{V}(t), \mathsf{E}(t))$, hence its various properties are dictated by the alignment dynamics on the graph, and at the same time, as we observe in theorem 3.1, the fluctuations of alignment dynamics are dictated by the connectivity of the underlying graph.
The spectral gap, $\lambda_2(\Delta_{\mathbf{m}}\mathbb{A})$, generalizes the usual notions of graph connectivity in terms of the *Fiedler number* in case of uniform weights $m_i \equiv 1/N$. In particular, we point out that the weighted Poincaré bound (3.9) depends only on the total mass M but otherwise is independent of the condition number, $\frac{\max_i m_i}{\min_i m_i}$.

Theorem 3.1 describes the long time behavior of a fully-discrete C-S dynamics (1.1) under a general setup based on symmetric communication kernel, $\phi_{ij} = \phi_{ji}$, which involves variable spatial weights m_i and variable time stepping, $\tau = \tau(t_k)$ satisfying a CFL-like time-step restriction (3.7). In particular, letting $\max_k \tau(t_k) \to 0$ we recover the semi-discrete CS model

$$\left\{\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t)\\ &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \sum_{j\in\mathcal{N}_i} m_j(t)\phi_{ij}(t)(\mathbf{v}_j(t) - \mathbf{v}_i(t)). \end{aligned}\right. \tag{3.10}$$

and theorem 3.1 tells that

$$\delta\mathscr{E}(t) \leqslant exp\Big\{-\int_0^t \lambda_2(s)\mathrm{d}s\Big\}\delta\mathscr{E}_0. \tag{3.11}$$

3.2. **Fluctuations revisited— ℓ^∞-diameter of fluctuations.** We measure the fluctuations of velocities in terms of the the ℓ^∞-diameter of the collection of velocities $\{\mathbf{v}_i\}$

$$\delta\mathscr{V}(t) := \max_{i,j} |\mathbf{v}_i(t) - \mathbf{v}_j(t)|.$$

It will be convenient to trace the scalar components which form this diameter. To this end we fix an arbitrary unit vector[3], $|\boldsymbol{\omega}|_* = 1$. Since $|\mathbf{v}| = \max_{|\boldsymbol{\omega}|_*=1}\langle \mathbf{v}, \boldsymbol{\omega}\rangle$ then

$$\delta\mathscr{V}(t) := \max_{|\boldsymbol{\omega}|_*=1}\max_{p,q}\big(v_p(t) - v_q(t)\big), \qquad v_p(t) := \langle \mathbf{v}_p(t), \boldsymbol{\omega}\rangle.$$

We now trace the decay of these scalar components of velocity fluctuations, considering an arbitrary (p, q) pair, $v_p(t) - v_q(t)$, where as before we suppress the

[3] The vector norm $|\cdot|$ is assumed to have it dual $|\boldsymbol{\omega}|_* = \sup_{|\mathbf{v}|=1}\langle \mathbf{v}, \boldsymbol{\omega}\rangle$.

dependence on time t on the right,

$$\begin{aligned}
v_p(t+\tau) - v_q(t+\tau) \\
&= v_p - v_q + \tau\sum_j m_j\phi_{pj}(v_j - v_p) - \tau\sum_j m_j\phi_{qj}(v_j - v_q)\\
&= \big(1-\tau\sum_j m_j\phi_{pj}\big)v_p - \big(1-\tau\sum_j m_j\phi_{qj}\big)v_q\\
&\quad + \tau\sum_j m_j\phi_{pj}v_j - \tau\sum_j m_j\phi_{qj}v_j\\
&= \big(1-\tau\sum_j m_j\phi_{pj}\big)v_p - \big(1-\tau\sum_j m_j\phi_{qj}\big)v_q\\
&\quad + \tau\sum_j m_j(\phi_{pj}-c_j)v_j - \tau\sum_j m_j(\phi_{qj}-c_j)v_j.
\end{aligned}$$

In the last step we introduced arbitrary scalars c_j's — their contribution to the last two terms on the right cancel out. By the CFL condition (3.7), the first two parenthesis on the right are positive. We now set $c_j := \min_{p,q}\{\phi_{pj},\phi_{qj}\}$ — with this choice the last two parenthesis on the right are also non-negative. Hence, if we let v_+ and v_- denote the extreme values $v_+ := \max_p v_p$ and $v_- := \min_q v_q$ we conclude

$$\begin{aligned}
v_p(t+\tau) - v_q(t+\tau) \\
&\leqslant \big(1-\tau\sum_j m_j\phi_{pj}\big)v_+ - \big(1-\tau\sum_j m_j\phi_{qj}\big)v_-\\
&\qquad + \tau\sum_j m_j(\phi_{pj}-c_j)v_+ - \tau\sum_j m_j(\phi_{qj}-c_j)v_-\\
&= v_+ - v_- - \tau\sum_j m_jc_j(v_+ - v_-) = \Big(1-\tau\sum_j m_jc_j\Big)\Big(\max_p v_p - \min_q v_q\Big)\\
&= \big(1-\tau\kappa(\mathbb{A})\big)\max_{p,q}\big(v_p - v_q\big), \qquad \kappa(\mathbb{A}) := \sum_j m_j\min_{p,q}\{\phi_{pj},\phi_{qj}\}.
\end{aligned}$$

Since (p,q) is an arbitrary pair we conclude

$$\delta\mathscr{V}(t+\tau) = \max_{|\boldsymbol{\omega}|_*=1}\max_{p,q}\big(v_p(t+\tau) - v_q(t+\tau)\big) \leqslant \big(1-\tau\kappa(\mathbb{A}(t)\big)\delta\mathscr{V}(t). \tag{3.12}$$

Theorem 3.3 (**Decay of uniform fluctuations**). *Consider the C-S dynamics* (1.1) *with time-steps small enough such that* (3.7) *holds*

$$\tau(t)\cdot\max_i\sum_j \phi_{ij}(t)m_j(t) \leqslant \frac{1}{2}.$$

Then the following bound of the diameter of fluctuations holds

$$\begin{aligned}
&\delta\mathscr{V}(t_n) \leqslant exp\Big\{-\sum_{k=0}^{n-1}\kappa\big(\mathbb{A}(t_k)\big)\tau(t_k)\Big\}\delta\mathscr{V}_0,\\
&\kappa\big(\mathbb{A}(t)\big) = \sum_j m_j(t)\min_{p,q}\{\phi_{pj}(t),\phi_{qj}(t)\}.
\end{aligned} \tag{3.13}$$

We emphasize that the bound (3.13) applies to C-S dynamics with general communication, $\{m_j(t)\phi_{ij}\}$, which need *not* be symmetric, as it allows for time-dependent masses. In particular, it applies to both the Cucker-Smale alignment model with symmetric interactions, (1.2), $m_j = 1/N$ and the Motsch-Tadmor alignment model with non-symmetric, time-dependent interactions (1.3), $m_i(t) = \frac{1}{L}deg_i(t)$.
In case of a uniform-in-time lower bound $\kappa\big(\mathbb{A}(t_k)\big) \geqslant \eta > 0$, e.g., see the particular case of all-to-all connectivity in (3.16), we end up with the exponential decay

$$\delta\mathscr{V}(t) \leqslant e^{-\eta t}\delta\mathscr{V}_0, \qquad \eta = \min_{t_k}\sum_j m_j(t_k)\min_{p,q}\{\phi_{pj}(t_k), \phi_{qj}(t_k)\}. \tag{3.14}$$

Remark 3.4 (**Spectral gap vs. coefficient of ergodicity).** The role of spectral gap in the present context of connectivity of graph goes back to Fiedler [58,59]. In the case of equal weights, the so-called Fiedler number $\lambda_2(\Delta\mathbb{A}) > 0$ quantifies the algebraic connectivity of the graph (V,E) supported at vertices $\mathsf{V} = \{i : \mathbf{x}_i\}$ with weighted edges $\mathsf{E} = \{(i,j) : \phi_{ij} > 0\}$.
The inequality (3.9) is sharp in the sense that

$$\frac{1}{M}\lambda_2(\Delta_{\mathbf{m}}\mathbb{A}) = \min\frac{\sum_{i,j}\phi_{ij}|\mathbf{v}_i - \mathbf{v}_j|^2 m_i m_j}{\sum_{i,j}|\mathbf{v}_i - \mathbf{v}_j|^2 m_i m_j}.$$

The obvious bound that follows,

$$\lambda_2(\Delta_{\mathbf{m}}\mathbb{A}) \geqslant M\min_{i,j}\phi_{ij}, \tag{3.15}$$

shows that $\phi_{ij}(t) > 0 \ \rightsquigarrow\ \lambda_2(\Delta_{\mathbf{m}}\mathbb{A})(t) > 0$. This is the scenario of a global, all-to-all connectivity between every pair of agents. The bound (3.15) is not sharp: we may have certain edges vanish while still maintaining a connected graph, that is, the strict inequality $\lambda_2 > M\min_{ij}\phi_{ij} = 0$ holds. A positive coefficient of ergodicity allows more general scenarios, in which pairs of agents, positioned at say $\mathbf{x}_p$ and $\mathbf{x}_q$, may lack direct communication, $\phi_{pq} = 0$, but they still communicate through an intermediate agent positioned at $\mathbf{x}_k$. That is, for each (p,q) there exists (at least) one agent positioned at $\mathbf{x}_k$, $k = k(p,q)$, which is the ‘ go between ’ agent so that[4] $\min\{\phi_{pk}, \phi_{qk}\} > 0$. This one-layer of communication is captured by the refined lower-bound

$$\lambda_2(\Delta_{\mathbf{m}}\mathbb{A}) \geqslant \kappa(\mathbb{A}) = \sum_j m_j\min_{p,q}\{\phi_{pj}, \phi_{qj}\} \geqslant M\min_{i,j}\phi_{ij}. \tag{3.16}$$

The estimate (3.12) in its ℓ^1-dual form for goes back to Dobrushin [50], quantifying the contractivity of column-stochastic matrices in terms of the so-called coefficient of ergodicity, denoted here $\kappa(\mathbb{A})$, [79]. It was revisited in many follow-up works, e.g., its used to quantify the relative entropy in discrete Markov processes [37,38] scrambling in models of opinion dynamics [89,90] and flocking dynamics [106, §2.1].

3.3. **Energy fluctuations revisited — time dependent masses.** The study of long time behavior based on ℓ^∞-diameter of velocity fluctuations enjoyed the advantage of addressing time-dependent masses. In contrast, our study of energy fluctuations in section 3.1 was restricted to constant masses. Here we observe that

[4] Of course the special case $k = p$ recovers the direct pairwise communication.

the proof of theorem 3.1 can be adapted to include the case of time-dependent masses, $m_i = m_i(t)$. Indeed, the time variability of the masses enters at precisely in two places: the time invariant total momentum in (3.3)

$$\mathscr{M}(t+\tau) = \mathscr{M}(t), \qquad \mathscr{M}(t) := \sum_i m_i(t)\mathbf{v}_i(t), \tag{3.17a}$$

and the evaluation of the incremental energy fluctuations (3.4a)

$$\delta\mathscr{E}(t+\tau) - \delta\mathscr{E}(t) = \sum_i m_i(t+\tau)|\mathbf{v}_i(t+\tau)|^2 - \sum_i m_i(t)|\mathbf{v}_i(t)|^2. \tag{3.17b}$$

To pursue our line of proof when $m_i = m_i(t)$, the momentum $\mathscr{M}(t+\tau)$ and energy fluctuations $\delta\mathscr{E}(t+\tau)$ need to be weighted by $m_i(t+\tau)$ rather than $m_i(t)$. Thus, the two qualities above admit the additional terms

$$\sum_i |m_i(t+\tau) - m_i(t)| \cdot |\mathbf{v}_i(t+\tau)|,$$

and, respectively,

$$\sum_i |m_i(t+\tau) - m_i(t)| \cdot |\mathbf{v}_i(t+\tau)|^2,$$

so one needs to control the incremental changes $\sum_i |m_i(t+\tau) - m_i(t)|$. Consider the example of the M-T model (1.3) with metric kernel $\phi(\mathbf{x},\mathbf{x}') = \varphi(|\mathbf{x}-\mathbf{x}'|)$ where the time-dependent masses are then given by the degrees

$$m_i(t) = \frac{1}{L} deg_i(t) = \sum_j \phi_{ij}(t), \qquad \phi_{ij}(t) = \varphi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|).$$

Assume that φ is a *smooth* metric communication kernel satisfying a localized Lip bound in the sense that

$$|\varphi(r) - \varphi(s)| \leqslant C_1 \max\{|\varphi(r)|, |\varphi(s)|\}|r-s|.$$

For this large class of localized Lip bounded φ's (which includes for example, $\varphi(r) = (1+r)^{-\beta}$ with $C_1 = \beta$), we have

$$\begin{aligned}
&|m_i(t+\tau) - m_i(t)| \\
&\quad \leqslant \sum_j \big(\varphi(|\mathbf{x}_i(t+\tau) - \mathbf{x}_j(t+\tau)|) - \varphi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)\big) \\
&\quad \leqslant C_1 \sum_j \max\{\phi_{ij}(t+\tau), \phi_{ij}(t)\} \cdot \big|\big(\mathbf{x}_i(t+\tau) - \mathbf{x}_j(t+\tau)\big) - \big(\mathbf{x}_i(t) - \mathbf{x}_j(t)\big)\big|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\sum_i |m_i(t+\tau) - m_i(t)| \cdot |\mathbf{v}_i(t+\tau)| \\
&\qquad\qquad \leqslant C_1 \sum_{i,j} \max\{\phi_{ij}(t+\tau), \phi_{ij}(t)\} \cdot \tau|\mathbf{v}_i(t) - \mathbf{v}_j(t)| \cdot |\mathbf{v}_i(t+\tau)|.
\end{aligned}$$

Now, using a uniform bound on the velocities, $\max_i |\mathbf{v}_i(t+\tau)| \leqslant C_2$, the exponential decay of $\delta\mathscr{V}(t)$, (3.14), and recalling $\phi_{ij} = L\widetilde{\phi}_{i,j}m_i m_j$ in (1.4), we find

$$
\begin{aligned}
&\sum_i |m_i(t+\tau) - m_i(t)| \cdot |\mathbf{v}_i(t+\tau)| \\
&\qquad \leqslant C_1 C_2 \tau \cdot \delta\mathscr{V}(t) \sum_{i,j} \max\left\{\widetilde{\phi}_{ij}(t) m_i(t) m_j(t), \widetilde{\phi}_{ij}(t+\tau) m_i(t+\tau) m_j(t+\tau)\right\} \\
&\qquad \leqslant C' C_2 \tau e^{-\eta t}, \qquad\qquad C' := C_1 M^2 \cdot \max|\phi| \cdot \delta\mathscr{V}_0.
\end{aligned}
$$

Hence, the equalities (3.17) in the case of constant masses are now replaced by the corresponding

$$
|\mathscr{M}(t+\tau) - \mathscr{M}(t)| \leqslant C'\tau e^{-\eta t}, \tag{3.18a}
$$

and, respectively,

$$
\begin{aligned}
\Big|\Big(\delta\mathscr{E}(t+\tau) - \sum_i m_i(t+\tau)|\mathbf{v}_i(t+\tau)|^2\Big) - \Big(\delta\mathscr{E}(t) - \sum_i m_i(t)|\mathbf{v}_i(t)|^2\Big)\Big| \\
\leqslant C' C_2^2 \tau e^{-\eta t}.
\end{aligned} \tag{3.18b}
$$

Thus, presence of smoothly varying time-dependent masses, accounts for additional terms which have a bounded accumulated effect. One can then study the long time behavior based on energy fluctuations in the presence of time-dependent masses, similar to our discussion in the next section, of flocking/swarming phenomena with constant masses.

4. Flocking and Swarming

The phenomena of *flocking* or *swarming* require the emergence of coordinated long time behavior of velocities, while the crowd of agents remains contained within finite diameter

$$
D(t) := \max_{i,j} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| \leqslant D_+ < \infty. \tag{4.1}
$$

The emerging behavior of velocities in intimately linked to the decay bounds of energy fluctuations. Indeed, (3.8) and its corresponding semi-discrete (3.11) imply that *if* the weighted graph of communication remains sufficiently strongly connected in the sense that $\lambda_2(t)$ has diverging tail, then by (3.1)

$$
\int^{\infty} \lambda_2(s)\mathrm{d}s = 0 \rightsquigarrow \sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}(t)|^2 m_i \leqslant exp\Big\{-\int_0^t \lambda_2(s)\mathrm{d}s\Big\}\delta\mathscr{E}_0 \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.2}
$$

In particular, since the mean velocity is an invariant of the flow,

$$
\overline{\mathbf{v}}(t) := \frac{1}{M}\sum_j m_j \mathbf{v}_j(t) = \overline{\mathbf{v}}_0,
$$

(4.2) tells us that a heavy-tailed $\lambda_2(t)$ implies the long time behavior of the velocities that align along the initial mean, $\mathbf{v}_i(t) \stackrel{t\to\infty}{\longrightarrow} \overline{\mathbf{v}}_0$

Remark 4.1 (**Emerging velocity in presence of time-dependent masses).** In case of constant masses, the mean velocity $\overline{\mathbf{v}}_(t)$ remains invariant in times, and the decay of velocity fluctuations implies the emergence of $\overline{\mathbf{v}}_0$ as the limiting velocity. The presence of time-dependent masses, however, leaves open the question of what is the emerging velocity. Thus, for example, in case of the M-T dynamics (1.3), we expect that velocities will align along the corresponding mean $\overline{\mathbf{v}}$

$$|\mathbf{v}_i(t) - \overline{\mathbf{v}}(t)| \stackrel{t\rightarrow\infty}{\longrightarrow} 0, \qquad \overline{\mathbf{v}}(t) := \frac{1}{\sum_j deg_j(t)} \sum_j deg_j(t)\mathbf{v}_j(t).$$

The question is if and when the emerging *limiting* velocity,
$\lim_{t\rightarrow\infty} \frac{1}{\sum_j deg_j(t)} \sum_j deg_j(t)\mathbf{v}_j(t)$, exists.

4.1. **Long-range interactions.** But when does $\lambda_2(t)$ satisfy the 'heavy-tail' condition sought in (4.2)? this is a central question for tracing the phenomenon of flocking. It was addressed in many references, starting with the original [43, 44] followed by [68, 69]; see [30, 131] and the references therein. A definitive answer is provided in case of *long-range kernels*,

$$\phi(\mathbf{x}, \mathbf{x}') \gtrsim \frac{1}{(1+|\mathbf{x}-\mathbf{x}'|)^\beta}, \quad \beta > 0. \tag{4.3}$$

In this case we bound the tail of the spectral gap, $\lambda_2(t) = \lambda_2\big(\Delta_{\mathbf{m}}\mathbb{A}(t)\big)$,

$$\lambda_2\big(\Delta_{\mathbf{m}}\mathbb{A}(t)\big) \geqslant M \min \phi_{ij}(t) \gtrsim \frac{M}{\big(1+D(t)\big)^\beta} \geqslant \frac{M}{(1+D_0+\delta\mathscr{V}_0\cdot t)^\beta}. \tag{4.4}$$

The first inequality on the right follows from (3.15), the second follows from (4.3) and the third follows from a uniform bound on the diameter of velocities[5],

$$D(t) \leqslant D_0 + \int^t \delta\mathscr{V}(s)\mathrm{d}s \leqslant D_0 + \delta\mathscr{V}_0 \cdot t,$$

and hence the heavy-tailed bound, $\lambda_2(t) \gtrsim (1+t)^{-\beta}$ for $\beta \leqslant 1$. We conclude that the C-S dynamics (1.1) with long-range communication (4.3), $\beta \leqslant 1$, admits *unconditional flocking*

$$\sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}_0|^2 m_i \lesssim \left\{ \begin{array}{ll} exp\big\{-\frac{M}{(1-\beta)\delta\mathscr{V}_0}\big(1+D_0+\delta\mathscr{V}_0\cdot t\big)^{1-\beta}\big\} & \beta<1 \\ \\ \big(1+D_0+\delta\mathscr{V}_0\cdot t\big)^{-M/\delta\mathscr{V}_0} & \beta = 1 \end{array}\right\} \stackrel{t\rightarrow\infty}{\longrightarrow} 0.$$

We can now use a bootstrap argument — the fractional exponential decay of the fluctuations of order $1-\beta > 0$ implies that the diameter remains uniformly bounded

[5]The result follows without appealing to the bound on diameter of velocities (3.12). Instead, a simpler maximum principle argument follows from the CFL condition (3.7),

$$|\mathbf{v}_i(t+\tau)| \leqslant \Big(1-\tau\sum_j \phi_{ij}m_j\Big)|\mathbf{v}_i(t)| + \tau\sum_j \phi_{ij}m_j|\mathbf{v}_j(t)| \leqslant \max_j |\mathbf{v}_j(t)| \leqslant \ldots \leqslant v_+(0),$$
$$v_+(0) = \max_i |\mathbf{v}_i(0)|,$$

and integration of $(1.1)_1$, and likewise, $(3.10)_1$ in the semi-discrete case, imply $D(t) \leqslant D_0 + 2v_+(0)\cdot t$.

and hence uniform bounded connectivity

$$D(t) \leqslant D_0 + \int^t \delta\mathscr{V}(s)\mathrm{d}s \leqslant D_+ \quad \rightsquigarrow \quad \lambda_2(t) \geqslant \eta := \frac{1}{(1+D_+)^\beta}.$$

Revising (4.4) with a finite diameter $\leqslant D_+$, yields the improved exponential bound $\lesssim e^{-\eta t}$. A similar argument applies in the borderline case of $\beta = 1$: clearly, if $\delta\mathscr{V}_0 < M$ then the finite tail of $\lesssim (1+t)^{-M/\delta\mathscr{V}_0}$ will lead to a finite diameter; and indeed, since $\delta\mathscr{V}(t)$ is decaying, we will eventually reach the threshold $\delta\mathscr{V}(t_c) < M$ and exponential decay follows thereafter. We summarize.

Theorem 4.2 (**Flocking/swarming with long range kernels**). *Consider the C-S dynamics* (1.1) *driven by long-range kernel* (4.3), $\beta \leqslant 1$. *Then the crowd of agents has finite support* D_+ *and there is exponential decay of fluctuations around the mean velocity,*

$$\sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}_0|^2 m_i \lesssim e^{-\eta t}\delta\mathscr{E}_0, \qquad \eta = \frac{1}{(1+D_+)^\beta}. \tag{4.5}$$

The precise exponential bound, $\eta = \eta(\beta, D_0, v_+)$, was captured in [68] using an elegant argument based on a proper Liapunov functional for C-S with metric kernel and uniform masses.

Remark 4.3 (**No uniform bound**). We distinguish between two types of bounds on velocity fluctuations — the ℓ^2-based energy fluctuations, theorem 3.1, and the ℓ^∞ bounds, theorem 3.3 or at least the uniform bound on velocities, see footnote 5. Suppose we try to pursue a purely ℓ^2-based argument for flocking behavior. The energy bound (3.8) implies the uniform-in-time bound $\max_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}_0| \leqslant C\sqrt{N}$ which in turn yields a bound on the diameter $D(t) \leqslant D_0 + 2C\sqrt{N}t$. We now use the same bootstrap argument as before to find a uniform-in-time bound on the diameter $D(t) \lesssim D_+(N) := N^{\frac{\beta}{2(1-\beta)}}$. We conclude an exponential flocking of rate

$$\sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}_0|^2 m_i \lesssim D_+(N) e^{-D_+(N)t}, \qquad D_+(N) = N^{\frac{\beta}{2(1-\beta)}}.$$

As expected, the fluctuations bound grows with N. However, the point to note here is that the exponential decay in time enforces exponential alignment bound, uniform in t *and* N when $t \gg N^{\frac{\beta}{2(1-\beta)}}$. For example, $\beta = 1/4$ requires a moderate time of $t \gg N^{1/6}$ before exponential decay takes place.

Theorem 4.2 was derived based on considerations of energy fluctuations. Similarly, we can proceed using the ℓ^∞-diameter fluctuations of theorem 3.3. Its semi-discrete limit $\max_k \tau(t_k) \to 0$ reads

$$\max_{i,j} |\mathbf{v}_i(t) - \mathbf{v}_j(t)| \leqslant exp\Big\{ - \int_0^t \sum_j m_j(s) \min_{p,q}\{\phi_{pj}(s), \phi_{qj}(s)\}\mathrm{d}s\Big\}\delta\mathscr{V}_0.$$

Here, we generalize theorem 4.2 to the case of time-dependent masses. Using a bootstrap argument as before we end up with

Theorem 4.4 (Flocking/swarming with long range kernels — time-dependent masses). *Consider the C-S dynamics* (1.1) *with possibly time-dependent masses, $m_i = m_i(t)$, driven by long-range kernel* (4.3)*, $\beta \leqslant 1$. Then the crowd of agents has finite support D_+ and there is exponential decay of fluctuations of velocities,*

$$\max_{i,j} |\mathbf{v}_i(t) - \mathbf{v}_j(t)| \lesssim exp\Big\{ -\eta \int_0^t \sum_j m_j(s)\mathrm{d}s \Big\} \delta\mathscr{V}_0, \qquad \eta = \frac{1}{(1+D_+)^\beta}. \tag{4.6}$$

Observe that in the example of M-T model (1.3) the scaling alluded in footnote 1, $M = \mathcal{O}(1)$, implies $\displaystyle\int_0^t \sum_j m_j(s)\mathrm{d}s \geqslant Ct$ and hence we end up with an exponential decay $e^{-\eta Ct}$.

The arguments that led to theorem 4.2 and the new theorem 4.4 demonstrate a rather general methodology for studying flocking, swarming and more general emerging phenomena in alignment based dynamics. It consists of two main ingredients:

- Decay of energy fluctuations. This is tied to spectral analysis of the dynamic graph $\big(\mathsf{V}(t), \mathsf{E}(t)\big)$. In typical cases, the dynamics is equipped with an intrinsic 'energy' and energy fluctuations.
- Bound on the velocities — either a uniform bound on velocities or on ℓ^∞-diameter of velocities fluctuations. In either case, the purpose is to trace the size of the spatial diameter and show that the crowd does not disperse, $D(t) \leqslant D_+$. In general, this is the more intricate bound to prove.

As an example we mention alignment dynamics with external forcing [129]. Other examples include C-S dynamics with *matrix* communication kernels, and C-S dynamics in which both, alignment and attraction, take place. We continue with this example in the context of *anticipation dynamics*.

4.2. **From anticipation to Cucker-Smale dynamics.** Particles are driven by the external forces induced by the environment and/or by other particles. The dynamics of social particles, on the other hand, is driven by *probing* the environment — living organisms, human interactions and sensor-based agents have *senses and sensors*, with which they actively probe the environment (and hence they are commonly viewed as 'active particles' [9]). A distinctive feature of active particles in probing the environment is *anticipation* — the dynamics is not driven instantaneously, but reacts to positions $\mathbf{x}^\tau(t) := \mathbf{x}(t) + \tau\mathbf{v}(t)$, *anticipated* at $t+\tau$, where $\tau > 0$ is an anticipation time increment. A general framework for anticipation dynamics, driven by pairwise interactions induced by radial potential $U = U(r)$, reads

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = -\dfrac{1}{N}\displaystyle\sum_{j=1}^N \nabla U(|\mathbf{x}_i^\tau(t) - \mathbf{x}_j^\tau(t)|), \qquad \mathbf{x}_k^\tau(t) := \mathbf{x}_k(t) + \tau\mathbf{v}_k(t). \end{cases}$$

The alignment is encoded here in the anticipated time — indeed, expanding the RHS in (the assumed small) τ we obtain, [130],

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \overbrace{-\frac{1}{N}\sum_j \nabla U(|\mathbf{x}_j - \mathbf{x}_i|)}^{\text{repulsion+attraction}} + \overbrace{\frac{\tau}{N}\sum_{j\in\mathcal{N}_i} \Phi_{ij}(\mathbf{v}_j - \mathbf{v}_i)}^{\text{alignment}}, \tag{4.7}$$
$$\Phi_{ij} = D^2U(|\mathbf{x}_i - \mathbf{x}_j|).$$

Thus, we *derive* a general class of 3Zone models (4.7), where the first terms on the right account for repulsion/attraction, depending whether their scalar amplitudes $U'_{ij} := U'(|\mathbf{x}_i - \mathbf{x}_j|) < 0$ or, respectively, $U'_{ij} > 0$, while the second term on the right accounts for an alignment with *matrix* coefficients, $\Phi_{ij} = D^2U(|\mathbf{x}_i - \mathbf{x}_j|)$, see figure 4.1.

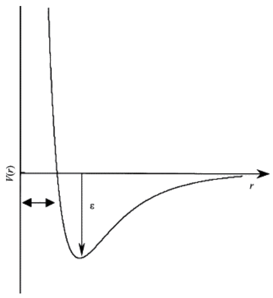

FIGURE 4.1. Potential $U(r)$

This leads us to consider an even larger class of 3Zone models with repulsion/attraction induced by potential U and alignment term induced by a separate scalar symmetric kernel, $\phi_{ij} = \phi(\mathbf{x}_i, \mathbf{x}_j)$ (independent of U),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = -\frac{1}{N}\sum_j \nabla U(|\mathbf{x}_j - \mathbf{x}_i|) + \frac{\tau}{N}\sum_{j:\phi_{ij}>0} \phi(\mathbf{x}_i, \mathbf{x}_j)(\mathbf{v}_j - \mathbf{v}_i). \tag{4.8}$$

The special case of metric-based kernel $\phi_{ij} = \varphi(|\mathbf{x}_i - \mathbf{x}_j|)$ recovers the C-S dynamics (1.2), $m_i \equiv 1/N$. For the special case of anticipation (4.7) we have $\Phi_{ij} \geqslant U''(|\mathbf{x}_i - \mathbf{x}_j|)\mathbb{I}$.

The energy fluctuations associated with (4.8)

$$\delta\mathscr{E}(t) := \frac{1}{2N}\sum_i |\mathbf{v}_i(t) - \overline{\mathbf{v}}|^2 + \frac{1}{2N^2}\sum_{i,j} U(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|),$$

are dissipated due to alignment at a precise rate dictated by local velocity fluctuations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\mathscr{E}(t) = -\frac{\tau}{2N^2}\sum_{i,j}\phi_{ij}(t)|\mathbf{v}_i - \mathbf{v}_j|^2.$$

We assume a smooth radial potential so that $U'(0)=U(0)=0$. It follows that the class of convex potentials and 'fat-tailed' kernels such that

$$U''(|\mathbf{x}_i-\mathbf{x}_j|)+\phi_{ij}\gtrsim\langle|\mathbf{x}_i-\mathbf{x}_j|\rangle^{-\gamma},\quad \gamma<4/5, \tag{4.9}$$

guarantee decay of energy fluctuations, $\delta\mathscr{E}(t)\leqslant C_0 exp\{-t^{\frac{4-5\gamma}{4-3\gamma}}\}$, which in turn implies asymptotic flocking towards the average velocity, $\overline{\mathbf{v}}=\frac{1}{N}\sum_j\mathbf{v}_j$, [130]. Moreover, agents asymptotically congregate in space, forming a traveling wave dictated by the presence of an attractive potential U, e.g., a quadratic U leads to a limiting harmonic oscillator. [129].

Open questions. The arguments above exclude two important features in collective dynamics: since (4.9) implies U is increasing, it does not address the role of *repulsion* in shaping the emergent behavior. The large time behavior of 2Zone repulsion-attraction models were discussed in, e.g., [28, 29, 36, 46, 55, 56]. The corresponding question for the full 3Zone model, in which attraction, alignment and repulsion co-exist, is mostly open.

Another key aspect is the long-range alignment sought by the 'heavy-tailed' kernels in (4.9) which does not address the local character of self-organized dynamics. The long time collective behavior based on *short-range* protocols hinges on the *graph connectivity* of the crowd, realized by the *adjacency matrix* $\mathbb{A}(t):=\{\phi_{ij}(t)\}$. Short-range interactions may lead to instability. This can be traced by the *graph Laplacian* $\Delta\mathbb{A}(t)$: while the initial configuration of the crowd is assumed to form one connected cluster expressed by the positivity of its spectral gap $\lambda_2(\Delta\mathbb{A}(0)>0$, it may break down into two or more disconnected clusters at a finite time when $\lambda_2(\Delta\mathbb{A}(t_c)=0$. Flocking analysis with short-range kerenls can be found in [26,48,62,83,104,135,138].

5. Large crowd dynamics

The question of instability for a fixed number of N agents governed by short-range alignment is better addressed in the context of *large crowd* dynamics of $N\gg 1$ agents. The latter is realized by the empirical distribution

$$f_N(t,\mathbf{x},\mathbf{v}):=\frac{1}{M}\sum_i m_i\delta(\mathbf{x}-\mathbf{x}_i(t))\otimes\delta(\mathbf{v}-\mathbf{v}_i(t)).$$

The large crowd dynamics is captured by its first two $\mathbf{v}$ moments which are assumed to exist, [131]:

$$\rho(t,\mathbf{x})=\lim_{N\to\infty}\int f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v},\quad \rho\mathbf{u}(t,\mathbf{x})=\lim_{N\to\infty}\int \mathbf{v}f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$$

These are the density and momentum which encode the macroscopic description of the agents based (4.8) (we abbreviate $\square=\square(t,\mathbf{x}),\square'=\square(t,\mathbf{x}')$)

$$\begin{cases}\rho_t+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u})=0\\ (\rho\mathbf{u})_t+\nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u}+\mathbb{P})=\tau\displaystyle\int_{\mathbf{x}'\in\Omega}\phi(\mathbf{x},\mathbf{x}')(\mathbf{u}'-\mathbf{u})\rho\rho'\mathrm{d}\mathbf{x}'-\rho\nabla U*\rho(t,\mathbf{x})\end{cases} \tag{5.1}$$

There are several ingredients in the macroscopic description: the pressure (Reynolds stress) tensor, $\mathbb{P}(t,\mathbf{x}) := \lim_{N\to\infty}\int(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})^\top f_N(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$, encodes the second-order $\mathbf{v}$ moments of f_N. The closure of (5.1) is imposed by assuming a limiting distribution at thermal equilibrium – a Maxwellian. But there is no generic closure in the present context of collective dynamics, since agents maintain their own detailed energy balance which is beyond the realm of collective motion. The two terms on the right capture scalar alignment and respectively attraction/repulsion induced by the potential U.

5.1. **Short-range interactions.** For simplicity, we ignore the role of attraction/ repulsion and conclude with three examples which trace the flocking behavior of the purely alignment hydrodynamics (5.1) with $U \equiv 0$.

Non-vacuous dynamics. In the first example, we consider the dynamics in the 2π-torus driven by bounded short-range kernels, $\phi(\mathbf{x},\mathbf{x}')$, localized along the diagonal

$$\frac{1}{\Lambda}\mathbb{1}_{R_0}(|\mathbf{x}-\mathbf{x}'|) \leqslant \phi(\mathbf{x},\mathbf{x}') \leqslant \Lambda\mathbb{1}_{2R_0}(|\mathbf{x}-\mathbf{x}'|), \quad R_0 \ll \pi. \tag{5.2}$$

It follows that strong solutions with non-vacuous density $\rho(t,\cdot) \gtrsim (1+t)^{-1/2}$ flock around the limiting velocity $\overline{\mathbf{v}}$ due to the decay of energy fluctuations $\delta\mathscr{E}(t) \to 0$, [135, Theorem 1.1]. As we noted in [138, theorem 3.3], the decay of energy fluctuations is independent of the specific closure of the pressure — what really matters is the non-vanishing density, the connectivity of the supp$\rho(t,\cdot)$ which enables to propagate information of alignment.

Topological interactions. The non-vacuous lower bound $(1+t)^{-1/2}$ is not sharp. As a second example we mention a topologically-based *singular* communication kernel, corresponding to (2.1)

$$\phi(\mathbf{x},\mathbf{x}') \sim \mathbb{1}_{R_0}(|\mathbf{x}-\mathbf{x}'|) \times \frac{1}{\mathrm{dist}^d_\rho(\mathbf{x},\mathbf{x}')}, \tag{5.3}$$

which involves the density weighted distance $\mathrm{dist}_\rho(\mathbf{x},\mathbf{x}') = \big(\int_{\mathcal{C}(\mathbf{x},\mathbf{x}')}\mathrm{d}\rho(t,\mathbf{z})\big)^{1/d}$. it follows that smooth solutions satisfying the relaxed non-vacuous condition, $\rho(t,\cdot) \gtrsim (1+t)^{-1}$, must flock, [135]. Again, no vacuum is a key aspect which enables the propagation of information: as long as no vacuous islands are formed, alignment dictates flocking behavior.

Multi-species. Our third example involves multi-species dynamics

$$\begin{cases} (\rho_\alpha)_t + \nabla_{\mathbf{x}}\cdot(\rho_\alpha\mathbf{u}_\alpha) = 0 \\ (\rho_\alpha\mathbf{u}_\alpha)_t + \nabla_{\mathbf{x}}\cdot(\rho_\alpha\mathbf{u}_\alpha\otimes\mathbf{u}_\alpha + \mathbb{P}_\alpha) = \tau\displaystyle\int_{\mathbf{x}'\in\Omega}\varphi_{\alpha\beta}(|\mathbf{x}-\mathbf{x}'|)(\mathbf{u}'_\beta-\mathbf{u}_\alpha)\rho_\alpha\rho'_\beta\mathrm{d}\mathbf{x}'. \end{cases}$$

In this case, different species tagged by the identifiers $\alpha,\beta \in \mathcal{I}$, are distinguished by their different protocol of communication with the environment of other species, $\phi_{\alpha\beta}$. In [72] it was shown that if the different species maintain non-vacuous densities $\rho_\alpha(t,\cdot) \gtrsim (1+t)^{-1}$ and if the communication array $\mathbb{A}(r) := \{\phi_{\alpha\beta}(r)\}$ forms a *connected graph*, $\lambda_2\big(\Delta\mathbb{A}(r)\big) \gtrsim (1+r)^{-\beta}$ with heavy-tail, $\beta < 1$, then flocking follows, $\mathbf{u}_\alpha \stackrel{t\to\infty}{\longrightarrow} \overline{\mathbf{u}} := \frac{1}{|\mathcal{I}|}\sum_{\alpha\in\mathcal{I}}\mathbf{u}_\alpha$.

References

[1] I. Aoki, *A simulation study on the schooling mechanism in fish*, Bulletin of the Japan Society of Scientific Fisheries. **48** (1982), 1081–1088.

[2] G. Ariel and A. Ayali, *Locust collective motion and its modeling*, PLoS Comput Biol **11** (2015): e1004522.

[3] G. Ariel, M. Sidortsov, S. Ryan, S. Heidenreich, M. Bär and A. Be'er, *Collective dynamics of two-dimensional swimming bacteria: Experiments and models*, Phys. Rev. E **98** (2018): 032415.

[4] R. Axelrod, The Evolution of Cooperation, New York: Basic Books, 1984.

[5] R. Axelrod, *The Complexity of Cooperation: Agent-Based Models of Competition and Collaboration*, Princeton University Press, Princeton, NJ, 1997.

[6] R. Bailo, José A. Carrillo and P. Degond, *Pedestrian models based on rational behaviour*, in: Crowd Dynamics, vol. 1, Modeling and Simulation in Science, Engineering and Technology, Birkhauser, 2019, pp. 259-292.

[7] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, *Interaction ruling animal collective behavior depends on topological rather than metric distance*, PNAS **105** (2008), 1232–1237.

[8] A. Barbaro, B. Einarsson, B. Birnir, S. Sigurosson, H. Valdimarsson, Ó. K. Pálsson, S. Sveinbjörnsson and S. Sigurasson, *Modeling and simulations of migration of pelagic fish*, ICES J.of Marine Science **66** (2009), 826–838.

[9] N. Bellomo, P. Degond and E. Tadmor (eds), *Active Particles*, Advances in Theory, Models, and Applications, Vol **1** and **2**, Modeling and Simulation in Science, Engineering and Technology Birkhäuser, 2017 & 2019

[10] E. Ben-Jacob, *Bacterial self-organization: co-enhancement of complexification and adaptability in a dynamic environment*, Phil. Trans. R. Soc. Lond. A. **361** (2003), 1283–1312.

[11] E. Ben-Naim, *Opinion dynamics: rise and fall of political parties*, Europhys. Lett. **6** (2005), 671–677.

[12] W. Bialek, A. Cavagna, I. Giardina, T. Morad, E. Silvestri, M. Viale and A. M. Walczake, *Statistical mechanics for natural flocks of birds* PNAS **109** (2012), 4786-4791.

[13] W. Bialek, A. Cavagna, I. Giardina, T. Morad, O. Pohl, E. Silvestri, M. Viale, and A. M. Walczak, *Social interactions dominate speed control in poising natural flocks near criticality*, PNAS **111** (2014), 7212–7217.

[14] A. Blanchet and P. Degond, *Topological interactions in a Boltzmann-type framework*, J. Stat. Phys. **163** (2016), 41–60.

[15] A. Blanchet and P. Degond *Kinetic models for topological nearest-neighbor interactions*, J. Stat. Phys. **169** (2017), 929–950.

[16] V. D. Blondel, J. M. Hendrickx, A. Olshevsky and J. N. Tsitsiklis, *Convergence in multiagent coordination, consensus, and flocking*, in: Proc. 44th IEEE Conference on Decision and Control, Sevilla (Spain), December 2005.

[17] V. D. Blondel, J. M. Hendricks and J. N. Tsitsiklis, *On Krause' s multi-agent consensus model with state-dependent connectivity*, IEEE Trans Automat. Control **54** (2009), 2586–2597.

[18] M. Bodnar and J. J. L. Velázquez, *Friction dominated dynamics of interacting particles locally close to a crystallographic lattice*, Math Methods APpl Sci. **36** (2013), 1206–1228.

[19] E Bonabeau, M Dorigo and G Theraulaz, Swarm Intelligence: from Natural to Artificial Systems, Oxford University Press, 1999.

[20] A. M. Bruckstein, N. Cohen and A. Efrat, *Ants, crickets and frogs in cyclic Pursuit* CIS report #9105, Center for Intelligent Systems, Technion Israel Inst. of Tech., 1991.

[21] C. Brugna and G. Toscani, *Kinetic models of opinion formation in the presence of personal conviction*, Phys Rec E **92** (2015): 052818.

[22] S. Brunton, J. Proctor and N. Kutz, *Discovering governing equations from data by sparse identification of nonlinear dynamical systems*, PNAS **113** (2016), 3932–3937.

[23] D. S. Calovi, A. Litchinko, V. Lecheval, U. Lopez, A. Pérez Escudero, H. Chaté, C. Sire, G. Theraulaz, *Disentangling and modeling interactions in fish with burst-and-coast swimming*

reveal distinct alignment and attraction behaviors PLoS Comput Biol **14** (2018): e1005933. https://doi.org/10.1371/journal.pcbi.1005933.

[24] M. Camperi, A. Cavagna, I. Giardina, G. Parisi and E. Silvestri, *Spatially balanced topological interaction grants optimal cohesion in flocking models*, Interface Focus **2** (2012), 715–725.

[25] C. Castellano, S. Fortunato, V. Loreto, *Statistical physics of social dynamics*, Rev. Modern Phys. **81** (2009), 591–646.

[26] J. Carrillo, *Swarming models with local alignment effects: phase transitions and hydrodynamics*, in: Collective Dynamics, Control and Imaging, ETH-ITS Ki-Net meeting, 2017, https: // www.math.umd.edu/ ∼tadmor/ ki_net/ activities/ presentations/31_757_Carrillo on Swarming Zurich2017.pdf

[27] J. A. Carrillo, Y.-P.Choi, P. B. Mucha and J. Peszek *Sharp conditions to avoid collisions in singular Cucker-Smale interactions*, Nonlinear Anal. Real World Appl. **37** (2017), 317–328.

[28] J. Carrillo, Y.-P. Choi and S. Perez, *A review on attractive-repulsive hydrodynamics for consensus in collective behavior*, in: Active Particles. Advances in Theory, Models, and Applications, N. Bellomo, P. Degond, and E. Tadmor (eds), vol. 1, Birkhäuser, 2017.

[29] J. A. Carrillo, A. Figalli and F.S. Patacchini, *Geometry of minimizers for the interaction energy with mildly repulsive potentials*, Ann. I. H. PoincaréAN **34** (2017), 1299–1308.

[30] J. A. Carrillo, M. Fornasier, G. Toscani and F. Vecil, *Particle, kinetic, and hydrodynamic models of swarming*, in: Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences, Series: Modelling and Simulation in Science and Technology, Naldi, G., Pareschi, L., Toscani, G. (eds.), Birkhauser, 2010, pp. 297–336.

[31] A. Cavagna, I. Giardina, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, *The starflag handbook on collective animal behaviour. 1: Empirical methods*, Animal Behaviour **76** (2008), 217–236; and
A. Cavagna, I. Giardina, A. Orlandi, G. Parisi and A. Procaccini *The starflag handbook on collective animal behaviour. 2: Three-dimensional analysis*, Animal Behaviour **76** (2008), 237–248.

[32] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini and M. Viale, *Scale-free correlations in starling flocks*, Proc. Nat. Academy Sci., U.S.A. **107** (2010), 11865–11870.

[33] Y.-P. Choi, S.-Y. Ha, J. Jung and J. Kim, *Global dynamics of the thermodynamic Cucker-Smale ensemble immersed in incompressible viscous fluids*, Nonlinearity **32** (2019), 1597-1640.

[34] Y.-P. Choi, S.-Y. Ha and Z. Li, *Emergent dynamics of the Cucker-Smale flocking model and its variants*, in: Active Particles. Theory, Models, Applications, N. Bellomo, P. Degond and E. Tadmor (eds.) vol. I., Modeling and Simulation in Science and Technology, Birkhauser-Springer, 2017.

[35] F. R K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, No. 92, AMS, 1997.

[36] Y.-L. Chuang, M. R. D'Orsogna, D. Marthaler, A. Bertozzi and L. Chayes, *State transitions and the continuum limit for a 2D interacting, self-propelled particle system*, Physica D **232** (2007), 33–47.

[37] J. E. Cohen, Y. Derriennic and Gh. Zbaganu, *Majorization, monotonicity of relative entropy, and stochastic matrices*, Contemp. Math. **149** (1993), 251–259.

[38] J. E. Cohen, Y. Iwasa, Gh. Rautu, M. B. Ruskai, E. Seneta and Gh. Zbaganu, *Relative entropy under mappings by stochastic matrices*, Linear Algebra Appl. **179** (1993), 211–235.

[39] J. Cortés, S. Martinez, and F. Bullo, *Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions*, IEEE Trans. Automat. Control **51** (2006), 1289–1298.

[40] I. Couzin, J. Krause, R. James, G. D. Ruxton and N. R. Franks, *Collective memory and spatial sorting in animal groups*, J. Theor. Biol. **218** (2002), 1–11.

[41] I. Couzin and N. Franks, *Self-organized lane formation and optimized traffic flow in army ants* Proc. R. Soc. Lond. B **270** (2003), 139–146.

[42] I. D. Couzin, J. Krause, N. R. Franks and S. A. Levin, *Effective leadership and decision making in animal groups on the move*, Nature **433** (2005), 513–516.

[43] F. Cucker and S. Smale, *On the mathematics of emergence*, Japan. J. Math. **2** (2007), 197–227.

[44] F. Cucker and S. Smale, *Emergent behavior in flocks*, IEEE Trans. Automat. Control **52** (2007), 852–862.

[45] G. Deffuant, F. Amblard, G. Weisbuch and T. Faure, *How can extremism prevail? A study based on the relative agreement interaction model*, J. Artificial Societies and Social Simulation **5** (2002), http://jasss.soc.surrey.ac.uk/5/4/1.html.

[46] P. Degond, G. Dimarco, T. B. N. Mac and N. Wang, *Macroscopic models of collective motion with repulsion*, Commun. Math. Sci. **13** (2015), 1615–1638.

[47] M. H. DeGroot, *Reaching a consensus*, J. Amer. Stat. Asso. **69** (1974), 118–121.

[48] H. Dietert and R. Shvydkoy, *On Cucker–Smale dynamical systems with degenerate communication*, Analysis and Applications **19** (2021), 551–573.

[49] T. Do, A. Kiselev, L. Ryzhik and C. Tan, *Global regularity for the fractional Euler alignment system*, Arch. Ration. Mech. Anal. **228** (2018), 1–37

[50] R. L. Dobrushin, *Central limit theorem for nonstationary Markov chains. I*, Theory Probab. Appl. **1** (1956), 65–80.

[51] M. R. D'Orsogna, Y.-L. Chuang, A. Bertozzi, and L. Chayes, *Self-propelled particles with soft-core interactions: patterns, stability, and collapse*, Phys. Rev. Lett. **96** (2006): 104302.

[52] B. Düring, M. Torregross and M.-T. Wolfram, *Boltzmann and Fokker–Planck equations modelling the Elo rating system with learning effects*, Journal of Nonlinear Science **29** (2019), 1095–1128.

[53] L. Edelstein-Keshet, *Mathematical models of swarming and social aggregation*, International Symposium on Nonlinear Theory and its Applications, (NOLTA 2001) Miyagi, Japan, 2001.

[54] L. Edelstein-Keshet, J. Watmough and D. Grunbaum, *Do travelling band solutions describe cohesive swarms? An investigation for migratory locusts* Journal of mathematical biology **36** (1998), 515–549.

[55] R. C. Fetecau, Y. Huang, and T. Kolokolnikov, *Swarm dynamics and equilibria for a nonlocal*, aggregation model, Nonlinearity **24** (2011), 2681–2716.

[56] R. C. Fetecau and Y. Huang, *Equilibria of biological aggregations with nonlocal repulsive–attractive interactions*, Physica D **260** (2013), 49–64.

[57] G. Fibich and R. Gibori, *Aggregate diffusion dynamics in agent-based models with a spatial structure*, Operations Research **58** (2010), 1450–1468.

[58] M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Math. J. **23** (1973), 298–305.

[59] M. Fiedler, *Laplacian of graphs and algebraic connectivity*, Combin. Graph Theory **25** (1989), 57–70.

[60] E. Fonio, Y. Heyman, L. Boczkowski, A. Gelblum, A. Kosowski, A. Korman, O. Feinerman, *A locally-blazed ant trail achieves efficient collective navigation despite limited information*, eLife, Ecology, Computational and System Biology, (2016) https://doi.org/10.7554/eLife.20185.001.

[61] A. Galante, S. Wisen, D. Bhaya and D. Levy, *Modeling local interactions during the motion of cyanobacteria*, J. Theoretical Biology **309** (2012), 147–158.

[62] J. Garnier, G. Papanicolaou and T.-W. Yang, *Consensus convergence with stochastic effects*, Vietnam Journal of Mathematics **45** (2016), 51–75.

[63] P. Gerlee, K. Tunstrøm, T. Lundh, and B. Wennberg, *Impact of anticipation in dynamical systems*, Phys. Rev. E **96** (2017): 062413.

[64] F. Ginelli, F. Peruani, M.-H. Pillot, H. Chaté, G. Theraulaz, and R. Bon, *Intermittent collective dynamics emerge from conflicting imperatives in sheep herds*, PNAS **112** (2015), 12729–12734.

[65] V. Grimm and S. F Railsback, Individual-Based Modeling and Ecology, Princeton Univ Press, 2005.

[66] S-Y. Ha, J. Kim and T. Ruggeri, *Emergent behaviors of thermodynamic Cucker–Smale particles*, SIMA **50** (2018), 3092–3121.

[67] S. Y Ha and D. Levy, *Particle, kinetic and fluid models for phototaxis*, Discrete Cont. Dynamical Systems Ser. B. **12** (2009), 77–108.

[68] S. Y Ha and J. G Liu, *A simple proof of the Cucker-Smale flocking dynamics and mean-field limit*, Communications in Mathematical Sciences **7** (2009), 297–325.

[69] S.-Y. Ha and E. Tadmor, *From particle to kinetic and hydrodynamic descriptions of flocking*, Kinetic and Related Models **1** (2008), 415–435.

[70] J. Haskovec, *Flocking dynamics and mean field limit of the Cucker-Smale-type model with topological interactions*, Phys. D **261** (2013), 42–51.

[71] J. Haskovec, *A simple proof of asymptotic consensus in the Hegselmann–Krause and Cucker-Smale Models with normalization and delay* SIADS **20** (2021), 130–148.

[72] S. He and E. Tadmor, *A game of alignment: collective behavior of multi-species*, AIHP (c) Nonlinear Analysis **38** (2021), 1031–1053.

[73] R. Hegselmann and U. Krause, *Opinion dynamics and bounded confidence: models, analysis and simulation*, J. Artificial Societies and Social Simulation **5** (2002).

[74] D. Helbing, *Traffic and related self-driven many particle systems*, Reviews of Modern Physics **73** (2001), 1067–1141.

[75] D. Helbing, *Pattern formation, social forces, and diffusion instability in games with success-driven motion*, *Eur. Phys. J. B* **67** (2009), 345–356.

[76] D. Helbing, *Quantitative Sociodynamics: Stochastic Methods and Models of Social Interaction Processes* Springer-Verlag, 2010.

[77] C. K Hemelrijk and H. Hildenbrandt, *Self-organized shape and frontal density of fish schools* Ethology **114** (2008), 245–254.

[78] E. Hensor, I. D. Couzin and J. Krause, *Modelling density-dependent fish shoal distributions in the laboratory and filed*, OIKOS **110** (2005), 344–352.

[79] I. Ipsen and T. Selee, *Ergodicity coefficients defined by vector norms*, SIAP **32** (2011), 153–200.

[80] P.-E. Jabin and S. Junca, *A continuous model for ratings*, SIMA **75** (2015), 420–442.

[81] M. O. Jackson, Social and Economic Networks, Princeton University Press, 2010.

[82] A. Jadbabaie, J. Lin, and A.S. Morse, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, *IEEE Trans. Automat. Control* **48** (2003), 988–1001.

[83] M. Ji and M. Egerstedt, *Distributed coordination control of multi-agent systems while preserving connectedness*, *IEEE Trans. Robot.* **23** (2007), 693–703.

[84] L. Jiang, L. Giuggioli, A. Perna, R. Escobedo, V. Lecheval, C. Sire, Z. Han and Gu. Theraulaz, *(Identifying influential neighbors in animal flocking*, PLoS Comput Biol **13** (2017): e1005822.

[85] C. Jin, *Flocking of the Motsch-Tadmor model with a cut-off interaction function*, J Stat Phys **171** (2018), 345–360.

[86] S. Jin, L. Li and J.-G. Liu, *Random batch methods (RBM) for interacting particle systems*, JCP **400** (2020): 108877

[87] E. W. Justh and P. S. Krishnaprasad, Extremal Collective Behavior, in: Proc. 49th IEEE Conf. Decision and Control, 2010, pp. 5432–5437.

[88] D. A. Kessler and H. Levine, *Pattern formation in dictyostelium via the dynamics of cooperative biological entities*, Phys. Rev. E **48** (1993), 4801–4804.

[89] U. Krause, *A discrete nonlinear and non-autonomous model of consensus formation*, Communications in Difference Equations (2000), 227–236.

[90] U. Krause, *Positive Dynamical Systems in Discrete Time*, Studies in Mathematics, De Gruyter, 2015.

[91] Y. Kuramoto, *Self-entrainment of a population of coupled non-linear oscillators*, in: *International Symposium on Mathematical Problems in Mathematical Physics*, Lecture Notes in Physics, vol. 39, Springer, 1975, pp. 420–422.

[92] T. Leblé and S. Serfaty, *Large deviation principle for empirical fields of Log and Riesz gases*, Invent. Math. **210** (2017), 645–757.

[93] T. Leblé and S. Serfaty, *Fluctuations of two-dimensional Coulomb gases*, Geom. Funct. Anal. (GAFA) **28** (2018), 443–508.

[94] X. Li and J. Xiao, *Swarming in homogeneous environments: A social interaction based framework*, Journal Theor. Biology **264** (2010), 747–759.

[95] W. Li, *Stability analysis of swarms with general topology*, IEEE Trans. Systems, Man, Cyber., Part B **38** (2008), 1084–1097.
[96] Z. Li, *Effectual leadership in flocks with hierarchy and individual preference*, DCDS A **34** (2014), 3683–3702.
[97] L. Liu, R. Escobedo, C. Sire and G. Theraulaz, *Computational and robotic modeling reveal parsimonious combinations of interactions between individuals in schooling fish*, Plos Computational Biology **16** (2020): e1007194.
[98] J. Lorenz, *Continuous opinion dynamics of multidimensional allocation problems under bounded confidence. A survey*, Internat. J. Modern Phys. C **18** (2007), 1819–1838.
[99] F. Lu, M. Zhong, S. Tang and M. Maggioni, *Nonparametric inference of interaction laws in systems of agents from trajectory data*, PNAS **116** (2019), 14424–14433.
[100] Z. Mao, Z. Li, G. Em Karniadakis *Nonlocal flocking dynamics: Learning the fractional order of PDEs from particle simulations*, Comm. on Applied Math. and Computation **1** (2019), 597–619.
[101] M. McPherson, L. Smith-Lovin and J. M Cook, *Birds of a feather: Homophily in social networks*, Annual Review of Sociology **27** (2001), 415–444.
[102] R. Merris, *Laplacian matrices of graphs: A survey*, Linear Algebra Applications **197,198** (1994), 143–176.
[103] P. Minakowski, P. B. Mucha, J. Peszek and E. Zatorska, Singular Cucker-Smale dynamics, in: Active Particles, Advances in Theory, Models, and Applications, vol. 2, N. Bellomo, P. Degond and E. Tadmor (eds.), Birkhäuser, 2019.
[104] J. Morlaes, J. Peszek and E. Tadmor, *Flocking with short-range interactions*, Journal of Statistical Physics **176** (2019), 382–397.
[105] S. Motsch and E. Tadmor, *A new model for self-organized dynamics and its flocking behavior*, J. Stat. Physics **144** (2011), 923–947.
[106] S. Motsch and E. Tadmor, *Heterophilious dynamics enhances consensus*, SiRev **56** (2014), 577–621.
[107] A. Muntean and F. Toschi (eds), Collective Dynamics from Bacteria to Crowds: An Excursion Through Modeling Analysis and Simulation, CISM Intl. Ctr Mech. Sci., vol. 553, Springer, 2014.
[108] K. Okuda and Y. Kuramoto, *Mutual entertainment between populations of coupled oscillators*, Progr. Theoret. Phys. **86** (1991), 1159–1176.
[109] R. Olfati-Saber, *Flocking for multi-agent dynamic systems: algorithms and theory*, IEEE Trans. Auto. Control **51** (2006), 401–420.
[110] R. Olfati-Saber and R. M. Murray, *Consensus problems in network of agents with switching topology and time delays*, IEEE Trans. on Auto. Control **49** (2004), 1520–1533.
[111] E. Ott, T. Antonsen, P. Guzdar and N. Mecholsky, *Dynamics of continuous model for flocking*, IPAM workshop "Swarming by Nature and by Design", 2006.
[112] B. L. Partridge, *The structure and function of fish schools*, Sci. Am. **246** (1982): 114.
[113] J. Park, H. Kim and S.-Y Ha, *Cucker-Smale flocking with inter-particle bonding forces*, IEEE Trans. Automatic Control **55** (2010), 2617–2623.
[114] J. K. Parrish, S. V. Viscido and D. Grünbaum, *Self-orgainzed fish schools: an examination of emergent properties*, The Biological Bulletin **202** (2002), 296-305.
[115] A Perna and G Theraulaz, *When social behaviour is moulded in clay: on growth and form of social insect nests* Journal of Experimental Biology **220** (2017), 83–91.
[116] J. Peszek, *Discrete Cucker-Smale flocking model with a weakly singular weight*, SIAM J. Math. Anal., **47** (2015), 3671–3686.
[117] B. Piccoli and A. Tosin, *Time-evolving measures and macroscopic modeling of pedestrian flow* Arch. Rat. Mech. Anal. **199** (2011), 707–738.
[118] R. Pinnau, C. Totzeck, O. Tse and S. Martin, *A consensus-based model for global optimization and its mean-field limit*, Math. Models and Methods in Applied Sciences **27** (2017),183–204.
[119] D. Poyato and J. Soler, *Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models, Math. Models Methods Appl. Sci.* **27** (2017), 1089–1152.

[120] C. Reynolds, *Flocks, herds and schools: A distributed behavioral model*, ACM SIGGRAPH **21** (1987), 25–34.
[121] C. Ryenolds, https://www.imdb.com/event/ev0000003/1998/1.
[122] C. Ringhofer *Traffic flow models and service rules for complex production systems*, in: Decision Policies for Production Networks, K. Kempf, D. Armbruster (eds), Springer, 2012, pp. 209–233.
[123] K. W. Rio, G. C. Dachner and W. H. Warren *Local interactions underlying collective motion in human crowds* Proc. R. Soc. B **285** (2018): 20180611.
[124] E. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Comprehensive Studies in Math. vol. 316, Springer-Verlag, Heidelberg, 1997.
[125] S. Serfaty, *Microscopic description of Log and Coulomb gases* Lecture notes of the PCMI, ArXiv:1709.04089.
[126] S. Serfaty, *Mean field limit for Coulomb flows*, EMS Newsletter Dec (2018) 16-21 (ArXiv:1803.08345).
[127] J. Shen, *Cucker-Smale flocking under hierarchical leadership*, SIAM J. Appl. Math. **68** (2007/08), 694–719.
[128] A. Shklarsh, G. Ariel,, E. Schneidman and E. Ben-Jacob, *Smart swarms of bacteria-inspired agents with performance adaptable interactions*, PLoS Comput. Biology **7** (2011), 1–11.
[129] R. Shu and E. Tadmor, *Flocking hydrodynamics with external potentials* ARMA **238** (2020), 347–381.
[130] R. Shu and E. Tadmor, *Anticipation breeds alignment*, ARMA **240** (2021), 203–241.
[131] R. Shvydkoy, *Dynamics and Analysis of Alignment Models of Collective Behavior*, Springer, 2021.
[132] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing*, Transactions of Mathematics and its Applications **1** (2017), 1–26.
[133] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing II: flocking*, Discrete and Continuous Dynamical Systems-A **37** (2017), 5503–5520.
[134] R. Shvydkoy and E. Tadmor, *Eulerian dynamics with a commutator forcing III: Fractional diffusion of order* $0 < \alpha < 1$, Physica D **376-377** (2018), 131–137.
[135] R. Shvydkoy and E. Tadmor, *Topologically-based fractional diffusion and emergent dynamics with short-range interactions*, SIMA **52** (2020): 5792-5839.
[136] R. Shvydkoy and E. Tadmor, *Multi-flocks: emergent dynamics in systems with multi-scale collective behavior*, MMS **19** (2021), 1115–1141.
[137] E. Tadmor, *Self-organized Dynamics. From emerging consensus to hydrodynamic flocking*, Nachdiplom lecture Notes Spring 2017, ETH-Zurich
[138] E. Tadmor, *On the mathematics of swarming: emergent behavior in alignment dynamics* Notices of the AMS **68** (2021), 493–503.
[139] J. Toner and Y. Tu, *Flocks, herds, and schools: A quantitative theory of flocking*, Physical Review E. **58** (1998), 4828–4858.
[140] C. M. Topaz and A. Bertozzi, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math. **65** (2004), 152–174.
[141] C. M. Topaz, M. R. D'Orsogna, L. Edelstein-Keshet and A. J. Bernoff, *Locust dynamics: behavioral phase change and swarming* PLoS Comput. Biol. **8** (2012): e1002642. pmid:22916003.
[142] H. Ueda, K. Stephens, K. Trivisa and W. Bentley, *Bacteria floc, but do they flock? Insights from population interaction models of quorum sensing*, mBio **10** (2019): e00972-19.
[143] T. Vicsek, Czirók, E. Ben-Jacob, I. Cohen and O. Schochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett. **75** (1995), 1226–1229.
[144] T. Vicsek and A. Zefeiris, *Collective motion*, Physics Reprints, **517** (2012), 71–140.
[145] I. Wagner and A. Bruckstein, *From Ants to A(ge)nts: A Special Issue on Ant-Robotics*, Annals of Math. and Artificial Intelligence**31** (2001), 1–5..
[146] G. Weisbuch, G. Deffuant and F. Amblard, *Persuasion dynamics*, Phys. A **353** (2005), 555–575.

[147] G. Weisbuch, *Social opinion dynamics*, in: Econophysics and Sociophysics: Trends and Perspectives, B. K. Chakrabarti, A. Chakrabarti and A. Chatterjee (eds), Wiley, 2006, pp. 67–94.
[148] L. Youseff, A. Barbaro, P. Trethewey, B. Birnir, and J. Gilbert, *Parallel modeling of fish interaction*, in: Computational Science and Engineering, 11th IEEE International Conference, 2008, pp. 234–241.
[149] M. Zavlanos G. Pappas, Potential fields for maintaining connectivity of mobile networks IEEE Transactions on Robotics **2** (2007), 812–816.
[150] M. Zavlanos, M. Egerstedt, and G. J. Pappas, *Graph theoretic connectivity control of mobile robot networks*, Proceedings of the IEEE. **99** (2011), 1525–1540.

Manuscript received January 5 2022
revised April 1 2022

Eitan Tadmor
Department of Mathematics and Institute for Physical Sciences & Technology (IPST), University of Maryland, College Park
E-mail address: tadmor@umd.edu