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APPROXIMATION AND QUADRATURE BY WEIGHTED LEAST SQUARES POLYNOMIALS ON THE SPHERE

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Dedicated to Ronald DeVore on the occasion of his 80th birthday

Abstract. Given a sequence of Marcinkiewicz-Zygmund inequalities in L_2 on a usual compact space $\mathcal{M}$, Gröchenig in [11] introduced the weighted least squares polynomials and the least squares quadrature from pointwise samples of a function, and obtained approximation theorems and quadrature errors. In this paper we investigate the problems confined on the sphere and obtain approximation theorems and quadrature errors which are order optimal. We also give upper bounds of the operator norms of the weighted least squares operators on the sphere.

1. Introduction

Let

$$\mathbb{S}^d := \{x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \mid |x|^2 = \sum_{k=1}^{d+1} |x_k|^2 = 1\} \quad (d \geq 2)$$

be the unit sphere in $\mathbb{R}^{d+1}$ endowed with the rotationally invariant measure μ normalized by $\int_{\mathbb{S}^d} d\mu = 1$. We denote by $L_2(\mathbb{S}^d)$ the usual Hilbert space of square-integrable functions on $\mathbb{S}^d$ with the inner product

$$\langle f, g\rangle := \int_{\mathbb{S}^d} f(x)g(x)d\mu(x) \tag{1.1}$$

and the norm $\|f\|_2 = \langle f, f\rangle^{1/2}$, and by $C(\mathbb{S}^d)$ the space of continuous functions on $\mathbb{S}^d$ with supremum norm

$$\|f\| := \|f\|_\infty := \sup_{x\in\mathbb{S}^d} |f(x)|.$$

The space Π_n^d of spherical polynomials on $\mathbb{S}^d$ of degree at most n consists of the restrictions to $\mathbb{S}^d$ of all polynomials on $\mathbb{R}^{d+1}$ of total degree at most n. The dimension of Π_n^d is given by

$$d_n := \dim(\Pi_n^d) = \frac{(2n+d)\Gamma(n+d)}{\Gamma(d+1)\Gamma(n+1)}.$$

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Let S_n be the orthogonal projection from $L_2(\mathbb{S}^d)$ onto Π_n^d, i. e., for $f \in L_2(\mathbb{S}^d)$,

$$S_n f(x) = \int_{\mathbb{S}^d} f(y) E_n(x, y) d\mu(y), \tag{1.2}$$

where $E_n(x, y)$ is the reproducing kernel for Π_n^d with respect to the inner product (1.1). That is, $E_n(x, y) = \sum_{k=1}^{d_n} p_k(x) p_k(y)$, where $\{p_k \,|\, 1 \le k \le d_n\}$ is an orthonormal basis in Π_n^d.

This paper is concerned with constructive polynomial approximation on $\mathbb{S}^d$ that uses function values (the samples) at selected well-distributed points (sometimes called standard information). Here the "well-distributed" points indicate that those points constitute a Marcinkiewicz-Zygmund (MZ) family on $\mathbb{S}^d$ defined as follows.

Definition 1.1. Let $\mathcal{X} = \{\mathcal{X}_n\} = \{x_{n,k} : n = 1, 2, \dots, k = 1, \dots, l_n\}$ be a doubly-indexed set of points in $\mathbb{S}^d$ and $\tau = \{\tau_{n,k} : n = 1, 2, \dots, k = 1, \dots, l_n\} \subset (0, +\infty)$ be a family of positive weights. Then $\mathcal{X}$ is called a Marcinkiewicz-Zygmund (MZ) family with associated weight τ, if there exist constants $A, B > 0$ independent of n such that

$$A\|p\|_2^2 \le \sum_{k=1}^{l_n} |p(x_{n,k})|^2 \tau_{n,k} \le B\|p\|_2^2 \qquad \text{for all } p \in \Pi_n^d. \tag{1.3}$$

The ratio $\kappa = B/A$ is the global condition number of the Marcinkiewicz-Zygmund family $\mathcal{X}$, and $\mathcal{X}_n = \{x_{n,k} : k = 1, \dots, l_n\}$ is the nth layer of $\mathcal{X}$.

The MZ inequality (1.3) means that the L_2-norm of a spherical polynomial of degree at most n is comparable to the discrete version given by the weighted ℓ_2-norm of its restriction to $\mathcal{X}_n$. It follows from [2, 7, 19] that such MZ families exist if the families are dense enough. We remark that Fekete points of degree $\lfloor n(1+\varepsilon) \rfloor$ ($\varepsilon > 0$) on the sphere are MZ families with the equal weights $\tau_{n,k} = \frac{1}{d_n}$, see [17], where $\lfloor a \rfloor$ is the largest integer not exceeding $a \in \mathbb{R}$. Also, sufficient conditions and necessary density conditions for MZ families with the equal weights $\tau_{n,k} = \frac{1}{d_n}$ on the sphere are obtained in [18] and [16], respectively.

Given a MZ family on a usual compact space $\mathcal{M}$, Gröchenig in [11] introduced the weighted least squares polynomials and the least squares quadrature from pointwise samples of a function, and obtained approximation theorems and quadrature errors. However, the obtained error estimates in [11] are not optimal due to the generality of $\mathcal{M}$. In this paper we confine to the sphere $\mathbb{S}^d$ and obtain the optimal error estimates.

Let $\mathcal{X}$ be a MZ family with associated weight τ. Given the samples $\{f(x_{n,k})\}$ of a continuous function f on $\mathbb{S}^d$ on the nth layer $\mathcal{X}_n$, we want to approximate f using only these samples. For this we solve a sequence of weighted least squares problems with samples taken from the samples $\mathcal{X}_n$:

$$L_n f = \operatorname*{arg\,min}_{p \in \Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}. \tag{1.4}$$

This procedure yields a sequence $\{L_n f\}$ of the best weighted ℓ_2-approximation of the data $\{f(x_{n,k})\}$ by a spherical polynomial in Π_n^d for every $f \in C(\mathbb{S}^d)$. We call $L_n f$

the weighted least squares polynomial, and L_n the weighted least squares operator. Clearly, L_n is the projection onto Π_n^d, i.e., L_n is a bounded linear operator on $C(\mathbb{S}^d)$ satisfying that $L_n^2 = L_n$ and the range of L_n is Π_n^d. It follows from (1.3) that for $p \in \Pi_n^d$, $p = 0$ whenever $p(x_{n,k}) = 0$. This means that usually $\mathcal{X}_n$ contains more than $d_n = \dim \Pi_n^d$ points, so that it is not an interpolating set for Π_n^d. Therefore, $L_n f$ is usually a quasi-interpolant.

Let $E_n(x, y)$ be the reproducing kernel of Π_n^d with respect to the inner product (1.1). We note that $E_n(x, y)$ satisfies the following properties:

(1) For any $x, y \in \mathbb{S}^d$, $E_n(x, y) = E_n(y, x)$;

(2) For any fixed $y \in \mathbb{S}^d$, $E_n(\cdot, y) \in \Pi_n^d$;

(3) For any $p \in \Pi_n^d$ and $x \in \mathbb{S}^d$, $p(x) = S_n p(x) = \langle p, E_n(\cdot, x)\rangle$.

Then the Marcinkiewicz-Zygmund inequalities (1.3) say that every set $\{\tilde{e}_{n,k} : k = 1, \dots, l_n\}$, $\tilde{e}_{n,k} = \tau_{n,k}^{1/2} E_n(\cdot, x_{n,k})$ is a frame for Π_n^d with uniform frame bounds $A, B > 0$, i.e., for all $p \in \Pi_n^d$, we have

$$A\|p\|_2^2 \le \sum_{k=1}^{l_n} |\langle p, \tilde{e}_{n,k}\rangle|^2 \le B\|p\|_2^2.$$

It follows that the associated frame operator

$$T_n p(\cdot) = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k}\rangle \tilde{e}_{n,k}(\cdot) = \sum_{k=1}^{l_n} \tau_{n,k} p(x_{n,k}) E_n(\cdot, x_{n,k})$$

is invertible on Π_n^d for every $n \in \mathbb{N}$. We obtain the dual frame $\{e_{n,k} = T_n^{-1}(\tilde{e}_{n,k}),\ k = 1, \dots, l_n\}$ for Π_n^d with uniform frame bounds B^{-1} and A^{-1}, and every polynomial $p \in \Pi_n^d$ can be reconstructed from the samples on $\mathcal{X}_n$ by

$$p = T_n^{-1} T_n p = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k}\rangle T_n^{-1} \tilde{e}_{n,k} = \sum_{k=1}^{l_n} \tau_{n,k}^{1/2} p(x_{n,k}) e_{n,k}.$$

The weights for the quadrature rules are defined by

$$w_{n,k} = \langle \tau_{n,k}^{1/2} e_{n,k}, 1\rangle = \tau_{n,k}^{1/2} \int_{\mathbb{S}^d} e_{n,k}(x) d\mu(x)$$

and the corresponding quadrature rule is defined by

$$I_n(f) = \sum_{k=1}^{l_n} w_{n,k} f(x_{n,k}). \tag{1.5}$$

Such quadrature rules are usually named the least squares quadrature.

Let $\mathcal{X}$ be a MZ family with associated weight τ. For a function f in the Sobolev space $H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$ (see Subsection 2.1 for definition of $H^\sigma(\mathbb{S}^d)$), Gröchenig obtained in [11] that

$$\|f - L_n f\|_2 \le c(1 + \kappa^2)^{1/2} n^{-\sigma + d/2} \|f\|_{H^\sigma},$$

and

$$\Big| \int_{\mathbb{S}^d} f(x) d\mu(x) - I_n(f) \Big| \le c(1 + \kappa)^{1/2} n^{-\sigma + d/2} \|f\|_{H^\sigma},$$

where c depends on d, σ, but not on f or κ or the MZ family $\mathcal{X}$.

We remark that the condition $\sigma > d/2$ is the well known necessary and sufficient condition for $H^\sigma(\mathbb{S}^d)$ to be continuously embedded in $C(\mathbb{S}^d)$, and is unavoidable in any approximation scheme that employs function values.

In this paper we improve the above results and obtain the optimal estimates. One of our main results can be formulated as follows.

Theorem 1.2. Let $\mathcal{X}$ be a MZ family on $\mathbb{S}^d$ with associated weight τ and global condition number $\kappa = B/A$, L_n be the weighted least squares operators defined by (1.4), and let I_n be the least squares quadratures defined by (1.5). If $f \in H^\sigma(\mathbb{S}^d)$, $\sigma > d/2$, then we have

$$\|f - L_n f\|_2 \le c(1+\kappa^2)^{1/2} n^{-\sigma} \|f\|_{H^\sigma}, \tag{1.6}$$

and

$$\Big| \int_{\mathbb{S}^d} f(x) d\mu(x) - I_n(f) \Big| \le c(1+\kappa^2)^{1/2} n^{-\sigma} \|f\|_{H^\sigma}, \tag{1.7}$$

where c depends on d, σ, but not on f or κ or the MZ family $\mathcal{X}$.

Remarkably, the convergence rates for the weighted least squares approximation and for the least squares quadrature errors are optimal (as explained in Remark 1.3 below) in a variety of Sobolev space settings. When the global condition number $\kappa = 1$, the weighted least squares operators are just the hyperinterpolation operators on the sphere, and the least squares quadratures are just the positive quadrature on the sphere (see Subsection 2.5). In this case, the inequality (1.6) and (1.7) were obtained in [28] and [1], respectively.

Remark 1.3. For $N \in \mathbb{N}$, the sampling numbers (or the optimal recovery) of the Sobolev classes $BH^\sigma(\mathbb{S}^d)$ (the unit ball of the space $H^\sigma(\mathbb{S}^d)$) in $L_2(\mathbb{S}^d)$ are defined by

$$g_N(BH^\sigma(\mathbb{S}^d), L_2(\mathbb{S}^d)) := \inf_{\substack{\xi_1,\dots,\xi_N \in \mathbb{S}^d \\ \varphi:\, \mathbb{R}^N \to L_2(\mathbb{S}^d)}} \sup_{f \in BH^\sigma(\mathbb{S}^d)} \|f - \varphi(f(\xi_1), \dots, f(\xi_N))\|_2,$$

where the infimum is taken over all N points $\xi_1, \dots, \xi_N$ in $\mathbb{S}^d$ and all mappings φ from $\mathbb{R}^N$ to $L_2(\mathbb{S}^d)$. The optimal quadrature errors of the Sobolev classes $BH^\sigma(\mathbb{S}^d)$ are defined by

$$e_N(\mathrm{Int}; BH^\sigma(\mathbb{S}^d)) := \inf_{\substack{\lambda_1,\dots,\lambda_N \in \mathbb{R} \\ \xi_1,\dots,\xi_N \in \mathbb{S}^d}} \sup_{f \in BH^\sigma(\mathbb{S}^d)} \Big| \int_{\mathbb{S}^d} f(x)\, d\mu(x) - \sum_{j=1}^N \lambda_j f(\xi_j) \Big|.$$

It follows from [31], [1] and [13] that

$$g_N(BH^\sigma(\mathbb{S}^d), L_2(\mathbb{S}^d)) \asymp N^{-\sigma/d} \quad \text{and} \quad e_N(\mathrm{Int}; BH^\sigma(\mathbb{S}^d)) \asymp N^{-\sigma/d}, \tag{1.8}$$

where the notation $A(N) \asymp B(N)$ means that $A(N) \ll B(N)$ and $A(N) \gg B(N)$, $A(N) \ll B(N)$ means that there exists a positive constant c independent of N such that $A(N) \le cB(N)$, and $A(N) \gg B(N)$ means $B(N) \ll A(N)$.

According to [16–18] there exist MZ families with $l_n \asymp d_n \asymp N \asymp n^d$. For such MZ families it follows from (1.8) that

$$\sup_{f \in BH^\sigma(\mathbb{S}^d)} \|f - L_n(f)\|_2 \asymp N^{-\sigma/d} \asymp g_N(BH^\sigma(\mathbb{S}^d), L_2(\mathbb{S}^d)),$$

which implies that the weighted least squares operators L_n are order optimal algorithms in the sense of optimal recovery. Also, we have

$$\sup_{f\in BH^\sigma(\mathbb{S}^d)}\Big|\int_{\mathbb{S}^d} f(x)\,d\mu(x) - I_n(f)\Big| \asymp N^{-\sigma/d} \asymp e_N(\mathrm{Int}; BH^\sigma(\mathbb{S}^d)),$$

which means that the least squares quadrature rules are the order optimal quadrature formulas.

For a linear operator L on $C(\mathbb{S}^d)$, the operator norm $\|L\|$ of L is given by

$$\|L\| := \sup\{\|Lf\| \mid f \in C(\mathbb{S}^d),\ \|f\| \le 1\}.$$

The quantity $\|L\|$ of a projection L is also called the Lebesgue constant of L and the estimation of $\|L\|$ is extremely important in numerical computation. We use the Christoffel functions to get the following estimation.

Theorem 1.4. Let $\mathcal{X}$ be a MZ family on $\mathbb{S}^d$ with associated weight τ and global condition number $\kappa = B/A$, and let L_n be the weighted least squares operators defined by (1.4). Then we have

$$n^{(d-1)/2} \ll \|L_n\| \ll \kappa^{1/2} n^{d/2}.$$

Remark 1.5. When $\kappa = 1$, the weighted least squares operators are just the hyperinterpolation operators on the sphere (see Subsection 2.5). In this case, we have (see [10, 20, 23])

$$\|L_n\| \asymp n^{(d-1)/2}.$$

Hence, we conjecture that the upper bound of the operator norm of L_n is $n^{(d-1)/2}$ multiplied by a positive constant independent of n. However, we have not been able to prove it.

Remark 1.6. Since the weighted least squares operator L_n is the projection onto Π_n^d, by the Lebesgue theorem the hyperinterpolation error in the uniform norm can be estimated as

$$\|f - L_n(f)\| \le (1 + \|L_n\|)E_n(f) \ll \kappa^{1/2} n^{d/2} E_n(f),$$

where

$$E_n(f) := \inf_{p\in\Pi_n^d} \|f - p\|$$

is the best approximation of f by Π_n^d.

This paper is organized as follows. Section 2 contains 5 subsections. In Subsections 2.1 we introduce some basic facts about spherical harmonics and the definition of the Sobolev spaces $H^\sigma(\mathbb{S}^d)$. Subsection 2.2 is devoted to giving two examples of MZ families on the sphere. In Subsections 2.3 and 2.4 we give the formal expression of the weighted least squares polynomial $L_n f$ and show that the least squares quadrature $I_n(f)$ is just the integration of the weighted least squares polynomial $L_n f$. In Subsection 2.5 we show that the hyperinterpolation is just the weighted least squares polynomial for a MZ family with the global condition number $\kappa = 1$. Finally, in Section 3 we give the proofs of Theorems 1.2 and 1.4.

We remark that the results of Subsections 2.3-2.5 can be extended to a general compact space $\mathcal{M}$.

2. Preliminary

This section is devoted to giving some basic facts about spherical harmonics (see for example, [7]) and some preliminary knowledge.

2.1. **Harmonic analysis on the sphere.**

Let $\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} \mid |x| = 1\}$ denote the unit sphere in $\mathbb{R}^{d+1}$, where $(x, y) = x \cdot y$ is the usual inner product and $|x| = (x, x)^{1/2}$ is the Euclidean norm. We denote by $\mathcal{H}_\ell^d$ the space of all spherical harmonics of degree l on $\mathbb{S}^d$, i.e., the space of the restrictions to $\mathbb{S}^d$ of all harmonic homogeneous polynomials of exact degree ℓ on $\mathbb{R}^{d+1}$. It is well known that the dimension of $\mathcal{H}_\ell^d$ is

$$N(d, \ell) = \dim \mathcal{H}_\ell^d = \begin{cases} 1, & \text{if } \ell = 0, \\ \frac{(2\ell+d-1)\,(\ell+d-2)!}{(d-1)!\ l!}, & \text{if } \ell = 1, 2, \ldots. \end{cases}$$

The spaces $\mathcal{H}_\ell^d$, $\ell = 0, 1, 2, \ldots$ are just the eigenspaces corresponding to the eigenvalues $-\ell(\ell + d - 1)$ of the Laplace-Beltrami operator $\triangle$ on the sphere $\mathbb{S}^d$ and are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{\mathbb{S}^d} f(x)g(x)d\mu(x).$$

Let

$$\{Y_{\ell,k} \equiv Y_{\ell,k}^d \mid k = 1, \ldots, N(d, \ell)\}$$

be a fixed orthonormal basis for $\mathcal{H}_\ell^d$. We have the addition theorem for the spherical harmonics of degree ℓ:

$$\sum_{k=1}^{N(d,\ell)} Y_{\ell,k}(x)Y_{\ell,k}(y) = \frac{\ell + \lambda}{\lambda} C_\ell^\lambda(x \cdot y), \quad x, y \in \mathbb{S}^d, \ \lambda = \frac{d-1}{2}, \quad d \geq 2,$$

where $C_\ell^\lambda(t)$ is the usual ultraspherical (Gegenbeuer) polynomial of order λ normalized by $C_\ell^\lambda(1) = \frac{\Gamma(\ell+2\lambda)}{\Gamma(2\lambda)\Gamma(\ell+1)}$ and generated by

$$\frac{1}{(1 - 2zt + z^2)^\lambda} = \sum_{k=0}^{\infty} C_k^\lambda(t) z^k \quad (0 \leq z < 1).$$

(See ([26, p. 81]).

Since the space Π_n^d can be expressed as a direct sum of

$$\Pi_n^d = \mathcal{H}_0^d \bigoplus \mathcal{H}_1^d \bigoplus \cdots \bigoplus \mathcal{H}_n^d,$$

we obtain that

$$\{Y_{\ell,k} \mid k = 1, \ldots, N(d, \ell), \ \ell = 0, 1, \ldots, n\}$$

forms an L_2-orthonormal basis of Π_n^d. Hence, the L_2-reproducing kernel $E_n(x, y)$ of Π_n^d has the explicit expression:

$$E_n(x, y) = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} Y_{\ell,k}(x)Y_{\ell,k}(y) = \sum_{\ell=0}^{n} \frac{\ell + \lambda}{\lambda} C_\ell^\lambda(x \cdot y), \ \lambda = \frac{d-1}{2}, \ x, y \in \mathbb{S}^d.$$

Specifically, we have

$$E_n(x,x) = \sum_{\ell=0}^{n} \frac{\ell+\lambda}{\lambda}\frac{\Gamma(\ell+2\lambda)}{\Gamma(2\lambda)\Gamma(\ell+1)} \asymp n^d, \quad \lambda = \frac{d-1}{2}, \quad x \in \mathbb{S}^d. \tag{2.1}$$

We remark that

$$\{Y_{\ell,k} \mid k = 1,\dots,N(d,\ell),\ \ell = 0,1,2,\dots\}$$

is an orthonormal basis for the Hilbert space $L_2(\mathbb{S}^d)$. Thus any $f \in L_2(\mathbb{S}^d)$ can be expressed by its Fourier (or Laplace) series:

$$f = \sum_{\ell=0}^{\infty}\sum_{k=1}^{N(d,\ell)} \langle f, Y_{\ell,k}\rangle Y_{\ell,k},$$

where

$$\langle f, Y_{\ell,k}\rangle = \int_{\mathbb{S}^d} f(x) Y_{\ell,k}(x)\, d\mu(x)$$

are the Fourier coefficients of f. The Sobolev space $H^\sigma(\mathbb{S}^d)$, where $\sigma > 0$, is defined as the space of all functions in $L_2(\mathbb{S}^d)$ for which the norm

$$\|f\|_{H^\sigma} := \left(\sum_{\ell=0}^{\infty}\Big(1+\ell(\ell+d-1)\Big)^{\sigma}\sum_{k=1}^{N(d,\ell)} \big|\langle f, Y_{\ell,k}\rangle\big|^2\right)^{1/2}$$

is finite. The space $H^\sigma(\mathbb{S}^d)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{H^\sigma} := \sum_{\ell=0}^{\infty}\Big(1+\ell(\ell+d-1)\Big)^{\sigma}\sum_{k=1}^{N(d,\ell)} \langle f, Y_{\ell,k}\rangle\, \langle g, Y_{\ell,k}\rangle,$$

which induces the norm $\|\cdot\|_{H^\sigma}$. It is easily seen that for $f \in H^\sigma(\mathbb{S}^d)$,

$$\|f - S_n f\|_2 \le c n^{-\sigma}\|f\|_{H^\sigma}, \tag{2.2}$$

where $S_n f$ is given by (1.2).

If $\sigma > d/2$, then the space $H^\sigma(\mathbb{S}^d)$ is continuously embedded in $C(\mathbb{S}^d)$ and is a reproducing kernel Hilbert space with the reproducing kernel

$$K_\sigma(x,y) = \sum_{\ell=0}^{\infty}\Big(1+\ell(\ell+d-1)\Big)^{-\sigma}\sum_{k=1}^{N(d,\ell)} Y_{\ell,k}(x)Y_{\ell,k}(y).$$

2.2. **Two examples of MZ families.**

We shall give two examples of MZ families with equal weights. Let X be a finite subset of $\mathbb{S}^d$ with cardinality $d_n := \dim \Pi_n^d$. The points $\xi_1,\dots,\xi_{d_n}$ in X maximizing a Vandermonde-type determinant

$$\big|\Delta(\xi_1,\dots,\xi_{d_n})\big| = \big|\det\big(Q_i(\xi_j)\big)_{i,j=1}^{d_n}\big|$$

are called Fekete points of degree n for $\mathbb{S}^d$ (these points are sometimes called extremal fundamental systems of points, as in [24]), where $Q_i,\ i = 1,\dots,d_n$ are a basis of Π_n^d. It is known that Fekete points are independent of the choice of the polynomial basis. For any fixed $\varepsilon > 0$, if $\mathcal{X} = \{\mathcal{X}_n\}$ and $\mathcal{X}_n$ are Fekete points of degree $\lfloor n(1+\varepsilon)\rfloor$ for $\mathbb{S}^d$, then $\mathcal{X}$ is a MZ family with equal weights $\tau_{n,k} = \frac{1}{d_n}$ (see [17]).

Let X be a finite subset of $\mathbb{S}^d$. The mesh norm of X is defined by

$$\rho(X) = \sup_{u\in\mathbb{S}^d} \inf_{z\in X} d(u,z),$$

where $d(x,y)$ denotes the standard geodesic distance $\arccos x\cdot y$ between two points x and y on $\mathbb{S}^d$. Let $\mathcal{X} = \{\mathcal{X}_n\} = \{x_{n,k} : n = 1,2,\dots,\ k = 1,\dots,l_n\}$ be a doubly-indexed set of points in $\mathbb{S}^d$. We say that $\mathcal{X}$ is uniformly separated if there is a positive number $\varepsilon > 0$ such that

$$d(x_{n,k}, x_{n,l}) \geq \frac{\varepsilon}{n+1} \quad \text{if} \quad k \neq l$$

for all $n \in \mathbb{N}$. Let $\mathcal{X} = \{\mathcal{X}_n\}$ be a uniformly separated array in $\mathbb{S}^d$ such that for all $n \geq 1$,

$$\rho(\mathcal{X}_n) \leq \frac{\eta}{n},$$

where $\eta < \pi/2$. Then $\mathcal{X}$ is a MZ family with the equal weights $\tau_{n,k} = \frac{1}{d_n}$ (see [18]).

However, in [17] and [18], the authors did not give estimates of the global condition numbers of two MZ families on the sphere.

2.3. **The weighted least squares polynomials $L_n f$.**

Now let $\mathcal{X}$ be a MZ family with associated weight τ. We use the weighted discretized inner product

$$\langle f, g\rangle_{(n)} := \sum_{k=1}^{l_n} f(x_{n,k}) g(x_{n,k}) \tau_{n,k}, \tag{2.3}$$

and the discretized norm

$$\|f\|_{(n)}^2 = \langle f, f\rangle_{(n)}.$$

We consider the corresponding orthogonal polynomial projection L_n onto Π_n^d with respect to the weighted discretized inner product $\langle\cdot,\cdot\rangle_{(n)}$, namely the weighted least squares polynomial $L_n f$ defined by

$$L_n f = \arg\min_{p\in\Pi_n^d} \|f-p\|_{(n)}^2 = \arg\min_{p\in\Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}.$$

We shall give a formal expression for $L_n f$. Let $\varphi_i,\ i = 1,\dots,d_n$ be an orthonormal basis of Π_n^d with respect to the weighted discrete scalar product (2.3), i.e.,

$$\langle \varphi_i, \varphi_j\rangle_{(n)} = \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad 1 \leq i,j \leq d_n.$$

Such $\varphi_i,\ i = 1,\dots,d_n$ can be obtained by applying the Gram-Schmidt orthogonalization process to the basis $\{Y_{\ell,k} \mid k = 1,\dots,N(d,\ell),\ \ell = 0,1,\dots,n\}$ of Π_n^d. We set

$$D_n(x,y) = \sum_{k=1}^{d_n} \varphi_k(x)\varphi_k(y).$$

Clearly, the weighted least squares polynomial $L_n f$ is just the orthogonal projection of f onto Π_n^d with respect to the weighted discrete scalar product (2.3), i. e.,

$$L_n f(x) = \langle f, D_n(\cdot, x)\rangle_{(n)} = \sum_{k=1}^{l_n} f(x_{n,k}) D_n(x_{n,k}, x)\tau_{n,k}. \tag{2.4}$$

Here, $D_n(x, y)$ is the reproducing kernel for Π_n^d with respect to the weighted discrete scalar product (2.3) satisfying the following properties:

(1) For any $x, y \in \mathbb{S}^d$, $D_n(x, y) = D_n(y, x)$;
(2) For any fixed $y \in \mathbb{S}^d$, $D_n(\cdot, y) \in \Pi_n^d$;
(3) For any $p \in \Pi_n^d$ and $x \in \mathbb{S}^d$,

$$p(x) = L_n p(x) = \langle p, D_n(\cdot, x)\rangle_{(n)} = \sum_{k=1}^{l_n} p(x_{n,k}) D_n(x_{n,k}, x)\tau_{n,k}. \tag{2.5}$$

According to the definition of the operator norm and (2.4), by the standard argument we have

$$\|L_n\| = \max_{x\in\mathbb{S}^d} \sum_{k=1}^{l_n} |D_n(x_{n,k}, x)|\tau_{n,k}. \tag{2.6}$$

2.4. **The least squares quadratures.**

Let $\mathcal{X}$ be a MZ family with a weight τ and $E_n(x, y)$ be the L_2-reproducing kernel of Π_n^d. Then the frame operator

$$T_n p(\cdot) = \sum_{k=1}^{l_n} \langle p, \tilde{e}_{n,k}\rangle \tilde{e}_{n,k}(\cdot) = \sum_{k=1}^{l_n} \tau_{n,k} p(x_{n,k}) E_n(\cdot, x_{n,k})$$

is invertible on Π_n^d, where $\tilde{e}_{n,k} = \tau_{n,k}^{1/2} E_n(\cdot, x_{n,k})$. We set

$$e_{n,k} = T_n^{-1}(\tilde{e}_{n,k}) \quad \text{and} \quad w_{n,k} = \langle \tau_{n,k}^{1/2} e_{n,k}, 1\rangle = \tau_{n,k}^{1/2} \int_{\mathbb{S}^d} e_{n,k}(x) d\mu(x).$$

Then the least squares quadrature is defined by

$$I_n(f) = \sum_{k=1}^{l_n} w_{n,k} f(x_{n,k}).$$

In this subsection we shall show that

$$I_n(f) = \int_{\mathbb{S}^d} L_n f(x) d\mu(x) = \sum_{k=1}^{l_n} f(x_{n,k})\tau_{n,k} \int_{\mathbb{S}^d} D_n(x_{n,k}, x) d\mu(x). \tag{2.7}$$

In order to prove (2.7) it suffices to show that

$$\tau_{n,k}^{1/2} e_{n,k} = \tau_{n,k} D_n(x_{n,k}, \cdot).$$

Indeed, by (2.4) and (2.5) we have for $x, y \in \mathbb{S}^d$

$$T_n(D_n(\cdot, y))(x) = \sum_{k=1}^{l_n} \tau_{n,k} D_n(x_{n,k}, y) E_n(x, x_{n,k})$$
$$= L_n(E_n(\cdot, x))(y) = E_n(x, y).$$

It follows that

$$\tau_{n,k}^{1/2} e_{n,k} = \tau_{n,k} T_n^{-1}(E_n(\cdot, x_{n,k})) = \tau_{n,k} D_n(x_{n,k}, \cdot).$$

Hence, (2.7) holds. By (2.7) and the Hölder inequality we obtain that

$$\Big| \int_{\mathbb{S}^d} f(x) d\mu(x) - I_n(f) \Big| \leq \int_{\mathbb{S}^d} |f(x) - L_n f(x)| d\mu(x) \leq \|f - L_n f\|_2. \tag{2.8}$$

2.5. **Hyperinterpolation and the weighted least squares polynomials.**

Hyperinterpolation was originally introduced by I. H. Sloan (see [22]). It uses the Fourier orthogonal projection of a function which can be expressed in the form of integrals, but approximates the integrals used in the expansion by means of a positive quadrature formula. Hence, hyperinterpolation is a discretized orthogonal projection on polynomial subspaces and provides a polynomial approximation which relies only on a discrete set of data. In recent years, hyperinterpolation has attracted much interest, and a great number of interesting results have been obtained (see [3–6, 10, 12, 14, 15, 20–23, 25, 27–30, 32]).

Assume that $Q_n(f) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k})$, $n = 1, 2, \ldots$ is a sequence of positive quadrature formulas on $\mathbb{S}^d$ which are exact for Π_{2n}^d, i.e., $\tau_{n,k} > 0$, and for all $f \in \Pi_{2n}^d$,

$$\int_{\mathbb{S}^d} f(x) d\mu(x) = Q_n(f) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}).$$

For any $p \in \Pi_n^d$, using $f = p^2$ in the above equality we obtain that the family $\mathcal{X}$ is a MZ family with the constants $A = B = 1$ and the global condition number κ is equal to 1.

The hyperinterpolation operator H_n on $\mathbb{S}^d$ is defined by

$$H_n f(x) = \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k}) E_n(x, x_{n,k}), \quad f \in C(\mathbb{S}^d).$$

We use the discretized inner product

$$\langle f, g \rangle_{(n)} := \sum_{k=1}^{l_n} f(x_{n,k}) g(x_{n,k}) \tau_{n,k}$$

and the discretized norm $\|f\|_{(n)}^2 = \langle f, f \rangle_{(n)}$. Since the quadrature formula Q_n is exact for Π_{2n}^d, we get for all $p, q \in \Pi_n^d$,

$$\langle p, q \rangle = \int_{\mathbb{S}^d} p(x) q(x) d\mu(x) = Q_n(p\, q) = \langle p, q \rangle_{(n)}.$$

Hence,

$$\{Y_{\ell,k} \mid k = 1, \dots, N(d, \ell),\ \ell = 0, 1, \dots, n\}$$

forms an orthonormal basis of Π_n^d with respect to $\langle \cdot, \cdot \rangle_{(n)}$. It follows that

$$D_n(x, y) = E_n(x, y),$$

and

$$H_n f(x) = \langle f, E_n(\cdot, x) \rangle_{(n)} = \sum_{\ell=0}^{n} \sum_{k=1}^{N(d,\ell)} \langle f, Y_{\ell,k} \rangle_{(n)} Y_{\ell,k}(x),$$

which means that the hyperinterpolation operator H_n is just the discretized orthogonal projection on the polynomial subspace Π_n^d with respect to $\langle \cdot, \cdot \rangle_{(n)}$, and satisfies

$$\|f - H_n f\|_{(n)}^2 = \min_{p \in \Pi_n^d} \|f - p\|_{(n)}^2 = \min_{p \in \Pi_n^d} \sum_{k=1}^{l_n} |f(x_{n,k}) - p(x_{n,k})|^2 \tau_{n,k}.$$

Hence, the hyperinterpolation $H_n f$ is just the weighted least squares polynomial for a sequence of positive quadrature formulas on $\mathbb{S}^d$ which are exact for Π_{2n}^d. It follows from (2.7) that the least squares quadrature is

$$I_n(f) = \sum_{k=1}^{l_n} f(x_{n,k}) \tau_{n,k} \int_{\mathbb{S}^d} D_n(x_{n,k}, x) d\mu(x) = \sum_{k=1}^{l_n} f(x_{n,k}) \tau_{n,k},$$

which is just the positive quadrature formula $Q_n(f)$.

On the other hand, if the global condition number κ of a MZ family $\mathcal{X}$ with the weight τ is equal to 1, then for all $p \in \Pi_n^d$, we have

$$A\|p\|_2^2 = \sum_{k=1}^{l_n} |p(x_{n,k})|^2 \tau_{n,k}.$$

It follows that for $f, g \in \Pi_n^d$,

$$A\langle f, g \rangle = \sum_{k=1}^{l_n} f(x_{n,k}) g(x_{n,k}) \tau_{n,k}.$$

Since any function in Π_{2n}^d can be expressed as a linear combination of the product of the functions x^α and x^β, where $x \in \mathbb{S}^d,\ \alpha, \beta \in \mathbb{N}_0^{d+1}, |\alpha| \le n,\ |\beta| \le n,\ x^\alpha := x_1^{\alpha_1} \dots x_{d+1}^{\alpha_{d+1}},\ |\alpha| = \alpha_1 + \cdots + \alpha_{d+1}$, we obtain that $\mathcal{X}$ determines a sequence of positive quadrature formulas $Q_n(f) = \frac{1}{A} \sum_{k=1}^{l_n} \tau_{n,k} f(x_{n,k})$ on $\mathbb{S}^d$ which are exact for Π_{2n}^d. This means that the weighted least squares operators L_n are just the hyperinterpolation operators H_n.

Hence, the hyperinterpolation is just the weighted least squares polynomial for a MZ family with the global condition number $\kappa = 1$, and the weighted least squares polynomial may be viewed as a generalization of the hyperinterpolation.

3. Proofs of Theorems 1.2 and 1.4

In order to prove Theorem 1.2, we shall use the following lemma.

Lemma 3.1. [6, Theorem 2.1] Let Γ be a finite subset of $\mathbb{S}^d$, and let $\{\mu_\omega : \ \omega \in \Gamma\}$ be a set of positive numbers satisfying

$$\sum_{\omega\in\Gamma} \mu_\omega |f(\omega)|^{p_0} \le C_1 \int_{\mathbb{S}^d} |f(x)|^{p_0}\, d\mu(x), \quad \forall f \in \Pi_N^d,$$

for some $0 < p_0 < \infty$ and some positive integer N. If $0 < q < \infty$, $M \ge N$ and $f \in \Pi_M^d$, then

$$\sum_{\omega\in\Gamma} \mu_\omega |f(\omega)|^{q} \le CC_1 \Big(\frac{M}{N}\Big)^d \int_{\mathbb{S}^d} |f(y)|^{q}\, d\mu(y),$$

where $C > 0$ depends only on d and q.

Proof of Theorem 1.2.

We rewrite the orthonormal basis $\{Y_{\ell,k} \mid k = 1, \dots, N(d,\ell),\ \ell = 0, 1, \dots, n\}$ of Π_n^d as $\{\phi_k \mid k = 1, \dots, d_n\}$. Let

$$\mathbf{y}_n = (\tau_{n,1}^{1/2} f(x_{n,1}), \dots, \tau_{n,l_n}^{1/2} f(x_{n,l_n}))^*$$

be the given sampling data column vector, U_n be the $l_n \times d_n$ matrix with entries

$$(U_n)_{kl} = \tau_{n,k}^{1/2} \phi_l(x_{n,k}), \quad k = 1, \dots, l_n,\ l = 1, \dots, d_n,$$

and let $R_n = U_n^* U_n$, where U_n^* is the conjugate transpose of the matrix U_n. Suppose that the solution to the weighted least squares problem (1.4) is the polynomial

$$L_n f = \sum_{k=1}^{d_n} a_{n,k} \phi_k \in \Pi_n^d$$

with coefficient column vector $\mathbf{a}_n$. According to the standard formula for the solution of a least squares problem by means of the Moore-Penrose inverse $U_n^+ = (U_n^* U_n)^{-1} U_n^* = R_n^{-1} U_n^*$, we obtain that

$$\mathbf{a}_n = U_n^+ \mathbf{y}_n = R_n^{-1} U_n^* \mathbf{y}_n.$$

For any $c \in \mathbb{R}^{d_n}$, $p = \sum_{k=1}^{d_n} c_k \phi_k$, we have

$$\|p\|_2^2 = |c|^2 \ \text{ and } \ (R_n c, c) = (U_n c, U_n c) = \sum_{k=1}^{n} |p(x_{n,k})|^2 \tau_{n,k}.$$

It follows from (1.3) that

$$A|c|^2 \le (R_n c, c) \le B|c|^2,$$

which means that the spectrum of every R_n is contained in the interval $[A, B]$. We obtain that the operator norm of R_n^{-1} is bounded by A^{-1} and the operator norm of U_n^* is bounded by

$$\|U_n^*\| = \|U_n\| = \|U_n^* U_n\|^{1/2} = \|R_n\|^{1/2} \le B^{1/2},$$

where the operator norm of a matrix A is defined by $\|A\| = \sup_{|x|=1} |Ax|$.

We use the orthogonal decomposition

$$\|f - L_n f\|_2^2 = \|f - S_n f\|_2^2 + \|S_n f - L_n f\|_2^2.$$

We recall that

$$S_n f = \sum_{k=1}^{d_n} f_{n,k}\phi_k$$

with coefficient column vector $\mathbf{f}_n$, where $f_{n,k} = \langle f, \phi_k\rangle$. By the Parseval equality we obtain that

$$\begin{aligned}\|S_n f - L_n f\|_2^2 &= |\mathbf{f}_n - \mathbf{a}_n|^2 = |\mathbf{f}_n - R_n^{-1}U_n^*\mathbf{y}_n|^2 \\ &= |R_n^{-1}U_n^*(U_n\mathbf{f}_n - \mathbf{y}_n)|^2 \\ &\le A^{-2}B|U_n\mathbf{f}_n - \mathbf{y}_n|^2.\end{aligned}$$

Finally, we have

$$(U_n\mathbf{f}_n)_k = \tau_{n,k}^{1/2} S_n f(x_{n,k}).$$

Hence,

$$|U_n\mathbf{f}_n - \mathbf{y}_n|^2 = \sum_{k=1}^{l_n} |f(x_{n,k}) - S_n f(x_{n,k})|^2 \tau_{n,k} = \|f - S_n f\|_{(n)}^2.$$

It follows that

$$\|S_n f - L_n f\|_2^2 \le A^{-2}B\|f - S_n f\|_{(n)}^2. \tag{3.1}$$

For $f \in H^\sigma(\mathbb{S}^d),\ \sigma > d/2$, we define

$$A_0 f = S_n f, \quad A_k f = S_{2^k n} f - S_{2^{k-1}n} f \quad \text{for} \quad k \ge 1.$$

Then for $f \in H^\sigma(\mathbb{S}^d),\ \sigma > d/2$, the series $\sum\limits_{k=0}^{\infty} A_k f(x)$ converges to $f(x)$ uniformly in $\mathbb{S}^d$. By (2.2) we have

$$\|A_k f\|_2 \le \|f - S_{2^k n} f\|_2 + \|f - S_{2^{k-1}n} f\|_2 \ll 2^{-k\sigma} n^{-\sigma} \|f\|_{H^\sigma}. \tag{3.2}$$

Note that

$$f(x) - S_n f(x) = \sum_{k=1}^{\infty} A_k f(x),$$

and the right series converges uniformly on $\mathbb{S}^d$. Hence, by (3.1) and the trigonometric inequality we have

$$\|S_n f - L_n f\|_2 \le A^{-1}B^{1/2}\|f - S_n f\|_{(n)} \le A^{-1}B^{1/2} \sum_{k=1}^{\infty} \|A_k f\|_{(n)}.$$

Note that $A_k f \in \Pi^d_{2^k n}$. Using Lemma 3.1 with $p_0 = q = 2$ and (3.2), we obtain that

$$\begin{aligned}\|A_k f\|^2_{(n)} &= \sum_{k=1}^{l_n} |A_k f(x_{n,k})|^2 \tau_{n,k} \\ &\le cB\Big(\frac{2^k n}{n}\Big)^d \int_{\mathbb{S}^d} |A_k f(x)|^2 \, d\mu(x) \\ &\ll B 2^{kd} 2^{-2k\sigma} n^{-2\sigma} \|f\|^2_{H^\sigma}.\end{aligned}$$

We have

$$\begin{aligned}\|S_n f - L_n f\|_2 &\le A^{-1} B^{1/2} \sum_{k=1}^{\infty} \|A_k f\|_{(n)} \\ &\ll A^{-1} B \sum_{k=1}^{\infty} 2^{kd/2} 2^{-k\sigma} n^{-\sigma} \|f\|_{H^\sigma} \\ &\ll A^{-1} B n^{-\sigma} \|f\|_{H^\sigma}.\end{aligned}$$

It follows that

$$\begin{aligned}\|f - L_n f\|_2 &= \big(\|f - S_n f\|_2^2 + \|S_n f - L_n f\|_2^2\big)^{1/2} \\ &\ll (1 + \kappa^2)^{1/2} n^{-\sigma} \|f\|_{H^\sigma}.\end{aligned}$$

This completes the proof of (1.6). Inequality (1.7) follows from (1.6) and (2.8) directly.

The proof of Theorem 1.2 is now finished. □

Remark 3.2. The inequality (3.1) was proved in [11]. For the convenience of the reader, we provide details of the proof.

Proof of Theorem 1.4.

In order to give the lower estimate of the operator norm $\|L_n\|$, we use the Daugavet theorem to obtain that

$$\|L_n\| \ge \|S_n\| \asymp n^{(d-1)/2}.$$

So it suffices to get the upper estimate of $\|L_n\|$.

Let

$$\Phi_n(x) = \inf_{p \in \Pi^d_n,\ |p(x)|=1} \|p\|_2^2, \quad x \in \mathbb{S}^d,$$

and

$$\Psi_n(x) = \inf_{p \in \Pi^d_n,\ |p(x)|=1} \|p\|_{(n)}^2, \quad x \in \mathbb{S}^d$$

be the Christoffel functions with respect to the inner $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{(n)}$, respectively. We remark that estimates for Christoffel functions are useful in comparing different norms of functions in Π^d_n, and they are also basic tools in the theory of orthogonal polynomials.

It follows from (1.3) that

$$A\Phi_n(x) \le \Psi_n(x) \le B\Phi_n(x), \quad x \in \mathbb{S}^d. \tag{3.3}$$

Now let $E_n(x,y)$ and $D_n(x,y)$ be the reproducing kernels of Π_n^d with respect to the inner $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{(n)}$, respectively. According to [9, Theorem 3.5.6], we have

$$E_n(x,x)^{-1} = \Phi_n(x) \quad \text{and} \quad D_n(x,x)^{-1} = \Psi_n(x).$$

It follows from (3.3) that

$$B^{-1}E_n(x,x) \le D_n(x,x) \le A^{-1}E_n(x,x), \quad x \in \mathbb{S}^d. \tag{3.4}$$

Using $p = 1$ in (1.3) we get

$$A \le \sum_{k=1}^{l_n} \tau_{n,k} \le B. \tag{3.5}$$

Using (2.6), the Cauchy-Schwartz inequality, (2.5), (3.5), (3.4), and (2.1), we have

$$\begin{aligned}
\|L_n\| &= \max_{x\in\mathbb{S}^d} \sum_{k=1}^{l_n} \tau_{n,k}|D_n(x,x_{n,k})| \\
&\le \max_{x\in\mathbb{S}^d} \Big(\sum_{k=1}^{l_n} \tau_{n,k}(D_n(x,x_{n,k}))^2\Big)^{1/2}\Big(\sum_{k=1}^{l_n}\tau_{n,k}\Big)^{1/2} \\
&\le B^{1/2}\max_{x\in\mathbb{S}^d} D_n(x,x)^{1/2} \\
&\ll \kappa^{1/2}\max_{x\in\mathbb{S}^d} E_n(x,x)^{1/2} \\
&\ll \kappa^{1/2} n^{d/2}.
\end{aligned}$$

This completes the proof of Theorem 1.4. □

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