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ON A REFINEMENT OF MARCINKIEWICZ–ZYGMUND TYPE INEQUALITIES IN L^q, $q > 0$

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Dedicated to Ronald DeVore on his 80th birthday

Abstract. The main goal of this paper is to verify for every $1 \le q < \infty$ a refined Marcinkiewicz–Zygmund type inequality with a precise error term

$$\frac{1}{2}\sum_{j=0}^{m-1}(x_{j+1}-x_{j-1})w(x_j)|t_n(x_j)|^q = (1+O(\varepsilon))\int_{-\pi}^{\pi} w(x)|t_n(x)|^q\,dx,$$

where t_n is any trigonometric polynomial of degree at most n, $-\pi = x_0 < x_1 < \cdots < x_m = \pi$, $\max_{0\le j\le m-1}(x_{j+1}-x_j) = O\big(\frac{\sqrt{\varepsilon}}{n}\big)$, $m, n \in \mathbb{N}$, and w is a Jacobi type weight. Moreover, some new Marcinkiewicz–Zygmund type inequalities for $0 < q < 1$ are given, as well.

1. Introduction

In the past 15-20 years the problem of discretisation of uniform and L^q norms in various finite dimensional spaces has been widely investigated. In case of $L^q, 1 \le q < \infty$ norms for trigonometric polynomials this problem is usually referred to as the *Marcinkiewicz–Zygmund type problem.* On the other hand when uniform norm and algebraic polynomials are considered then the terms *norming sets* or *optimal meshes* are usually used in the literature. Historically the first discretisation result was given by S. N. Bernstein [1] in 1932 who showed that for any trigonometric polynomial t_n of degree $\le n$ and any $0 = x_0 < x_1 < \cdots < x_m < 2\pi = x_{m+1}$ with $\max_{0\le j\le m}(x_{j+1}-x_j) \le \frac{2\sqrt{\varepsilon}}{n}$, $\varepsilon > 0$ we have

$$\max_{x\in[0,2\pi]}|t_n(x)| \le (1+\varepsilon)\max_{0\le j\le m}|t_n(x_j)|. \tag{1.1}$$

The above estimate essentially shows that the uniform norm of trigonometric polynomials of degree $\le n$ can be discretized with accuracy ε using $m \sim \frac{n}{\sqrt{\varepsilon}}$ properly chosen nodes. A standard substitution $x = \cos t$ leads to an extension of (1.1) for algebraic polynomials when $\max_{0\le j\le m}(\arccos x_{j+1} - \arccos x_j) \le \frac{2\sqrt{\varepsilon}}{n}$. (See also [3, pp. 91-92] for details.)

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The first result on the discretisation of the L^q, $1 < q < \infty$ norm is due to Marcinkiewicz and Zygmund [12] who verified in 1937 that for any univariate trigonometric polynomial t_n of degree at most n and every $1 < q < \infty$ we have

$$\int |t_n|^q \sim \frac{1}{n} \sum_{s=0}^{2n} \left| t_n \left(\frac{2\pi s}{2n+1} \right) \right|^q \tag{1.2}$$

where the constants involved in the above equivalence relation depend only on q. The above relation provides discretization of the L^q, $1 < q < \infty$ norm of trigonometric polynomials of degree $\leq n$ with $2n+1$ nodes.

Various generalizations of the Marcinkiewicz–Zygmund type results to the multivariate setting were given in the literature, see for instance the survey paper [11]. Feng Dai [4] gave some analogues of Marcinkiewicz–Zygmund type inequalities for multivariate algebraic polynomials on the sphere and ball in $\mathbb{R}^d$. Extensions for multivariate exponential sums can be found in [5], [14], [8], [9]. In [6] using Bernstein–Markov, Schur and Videnskii type polynomial inequalities, various extensions of the Marcinkiewicz–Zygmund type bounds for multivariate polynomials on more general multivariate domains were verified, which in particular include polytopes, cones, spherical sectors, toruses.

Recently, the followng refinement of the classical Marcinkiewicz–Zygmund result in the spirit of Bernstein's estimate (1.1) was given in [7] in case when $q \geq 2$.

For any $-\pi = x_0 < x_1 < \dots < x_m = \pi$, $x_{-1} = x_{m-1} - 2\pi$ *with*

$$\max_{0 \leq j \leq m-1} (x_{j+1} - x_j) < \frac{\sqrt{\varepsilon}}{qn}, \quad 0 < \varepsilon < 1,$$

and for every $t_n \in T_n$ *we have whenever* $q \geq 2$

$$(1-\varepsilon) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} |t_n(x_j)|^q \leq \int_{-\pi}^{\pi} |t_n(x)|^q \, dx \leq (1+\varepsilon) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} |t_n(x_j)|^q. \tag{1.3}$$

This is a Marcinkiewicz–Zygmund type estimate of precision ε similar to Bernstein's uniform bound (1.1). In particular, choosing equidistant nodes $x_j := \frac{\pi j}{m}\, |j| \leq m$ with $m > \left[\frac{\pi q n}{\sqrt{\varepsilon}}\right]$, $|j| \leq m$, the last estimate gives

$$\frac{1-\varepsilon}{2m} \sum_{j=-m}^{m-1} |t_n(x_j)|^q \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x)|^q \, dx \leq \frac{1+\varepsilon}{2m} \sum_{j=-m}^{m-1} |t_n(x_j)|^q.$$

It should be noted that the spacing needed above can be achieved with discrete meshes of cardinality $m \sim \frac{n}{\sqrt{\varepsilon}}$. This upper bound for cardinality turned out to be sharp with respect to ε, as well.

The main goal of the present note is to extend (1.3) for any $1 \le q < \infty$. Similarly to [7] we will accomplish this for weighted L^q norms with Jacobi type weights. Moreover we will also prove new L^q Marcinkiewicz–Zygmund type results for trigonometric polynomials in case when $0 < q < 1$. Finally, possible extensions to multivariate setting and sharpness of our results will be also discussed.

2. The case $1 \le q < \infty$

In order to extend the precise Marcinkiewicz–Zygmund type result (1.3) for $1 \le q < 2$ we will need a technical lemma which is a certain variation of integration by parts.

Lemma 2.1. *Let $W(x)$ be an absolutely continuous 2π-periodic function, and let $t(x)$ be a trigonometric polynomial. Then for every $\alpha > 0$ we have*

$$\alpha \left| \int_{-\pi}^{\pi} W(x)|t(x)|^{\alpha-1} t'(x)\, dx \right| \le \int_{-\pi}^{\pi} |W'(x)||t(x)|^{\alpha}\, dx\,.$$

Proof. In case $t(x)$ has no sign-change, an integration by parts shows the validity of the lemma. Otherwise, set $F(x) := W(x)|t(x)|^{\alpha}$. Note that the assumptions of the lemma ensure that F is an absolutely continuous 2π-periodic function and thus $F'(x) \in L^1[-\pi,\pi]$. Clearly

$$F'(x) = W'(x)|t(x)|^{\alpha} + \alpha W(x)|t(x)|^{\alpha-1} t'(x) \operatorname{sgn} t(x), \quad x \in [-\pi,\pi].$$

Hence

$$\int_{-\pi}^{\pi} F'(x) \operatorname{sgn} t(x)\, dx = \int_{-\pi}^{\pi} W'(x)|t(x)|^{\alpha} \operatorname{sgn} t(x)\, dx + \alpha \int_{-\pi}^{\pi} W(x)|t(x)|^{\alpha-1} t'(x)\, dx\,.$$

Denote now by $(-\pi) \le \xi_1 < \xi_2 < \cdots < \xi_k (< \pi)$ the sign changes of $t(x)$ on $[-\pi,\pi)$. Since $\alpha > 0$ it follows that $F(\xi_j) = 0$, $1 \le j \le k+1$, where $\xi_{k+1} = \xi_1 + 2\pi$. Using in addition that F is absolutely continuous we obtain

$$\begin{aligned}
\left| \int_{-\pi}^{\pi} F'(x) \operatorname{sgn} t(x)\, dx \right| &= \left| \int_{\xi_1}^{\xi_{k+1}} F'(x) \operatorname{sgn} t(x)\, dx \right| \\
&= \left| \sum_{j=1}^{k} (-1)^j \int_{\xi_j}^{\xi_{j+1}} F'(x)\, dx \right| \\
&= \left| \sum_{j=1}^{k} (-1)^j (F(\xi_{j+1}) - F(\xi_j)) \right| \\
&= 0\,.
\end{aligned}$$

Therefore

$$\alpha \int_{-\pi}^{\pi} W(x)|t(x)|^{\alpha-1} t'(x)\, dx = -\int_{-\pi}^{\pi} W'(x)|t(x)|^{\alpha} \operatorname{sgn} t(x)\, dx$$

which clearly yields the statement of the lemma. ☐

Consider a trigonometric Jacobi type weight

$$w(x) := \prod_{k=1}^{s} \left| \sin \frac{x - y_k}{2} \right|^{a_k}, \tag{2.1}$$

$x \in [-\pi, \pi]$, $-\pi \le y_1 < \cdots < y_s < \pi$, $a_k > 1$, $1 \le k \le s$, $s \in \mathbb{N}$. The defect of this weight denoted by d_w is defined as $d_w := \max_k a_k$. The Bernstein and Schur type inequalities given in [13] for trigonometric polynomials with doubling weights hold, in particular for trigonometric Jacobi type weights, as well.

Theorem 2.2. *Let w be a trigonometric Jacobi type weight* (2.1)*. Then for any $0 < \varepsilon < 1$, $n, m \in \mathbb{N}$, $1 \le q < \infty$ and discrete nodes $-\pi = x_0 < x_1 < \cdots < x_m = \pi$ with $\max_{0 \le j \le m-1}(x_{j+1} - x_j) \le \frac{\sqrt{\varepsilon}}{\sqrt{c}qn}$ and every $t_n \in T_n$ we have*

$$\begin{aligned} (1-\varepsilon) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} w(x_j) |t_n(x_j)|^q \\ \le \int_{-\pi}^{\pi} w(x) |t_n(x)|^q \, dx \le (1+\varepsilon) \sum_{j=0}^{m-1} \frac{x_{j+1} - x_{j-1}}{2} w(x_j) |t_n(x_j)|^q, \end{aligned} \tag{2.2}$$

where $c > 0$ is a constant depending only on the weight.

In particular, if $x_j = \frac{\pi j}{m}$, $|j| \le m$, $m \ge \sqrt{\frac{c}{\varepsilon}} \pi q n$ then we get

$$\begin{aligned} \frac{1-\varepsilon}{2m} \sum_{j=-m}^{m-1} w(x_j) |t_n(x_j)|^q \le \frac{1}{2\pi} \int_{-\pi}^{\pi} w(x) |t_n(x)|^q \, dx \\ \le \frac{1+\varepsilon}{2m} \sum_{j=-m}^{m-1} w(x_j) |t_n(x_j)|^q. \end{aligned}$$

Proof. Set $h_m := \max_{0 \le j \le m-1}(x_{j+1} - x_j)$. We will apply now the estimate

$$\begin{aligned} R &:= \left| \int_{-\pi}^{\pi} w(x) |t_n(x)|^q \, dx - \frac{1}{2} \sum_{j=0}^{m-1} (x_{j+1} - x_{j-1}) w(x_j) |t_n(x_j)|^q \right| \\ &\le \frac{h_m^2}{2} \int_{-\pi}^{\pi} \big[q(q-1) w (t_n')^2 |t_n|^{q-2} + q w |t_n|^{q-1} |t_n''| \\ &\quad + 2q |t_n|^{q-1} |t_n' w'| + |t_n|^q |w''| \big] \, dx \\ &=: \frac{h_m^2}{2} (R_1 + R_2 + R_3 + R_4) \end{aligned} \tag{2.3}$$

verified in [7]. In addition it is also shown in [7] that for every $1 < q < \infty$

$$R_2 + R_3 + R_4 \le c q^2 n^2 \|t_n\|_{L_q(w)}^q$$

with a constant $c > 0$ depending only on w. Here and in what follows $\| \cdot \|_{L_q(w)}$ stands for the usual L_q norm with weight w. However a similar upper bound for the first term R_1 is verified in [7] only for $q \ge 2$. Now the lemma proved above will yield a complete upper estimate for R_1 in case of any $1 \le q < \infty$. Assume first that

$1 < q < \infty$. Setting in this lemma $\alpha = q - 1 > 0$, $W(x) := w(x)t_n'(x)$, $g(x) := t_n(x)$ and using the 2π periodicity of all functions involved we obtain

$$\begin{aligned} R_1 &= q(q-1)\int_{-\pi}^{\pi} w(x)|t_n(x)|^{q-2}(t_n'(x))^2\,dx \\ &\le q\int_{-\pi}^{\pi} (|w'(x)t_n'(x)| + w(x)|t_n''(x)|)|t_n(x)|^{q-1}\,dx \\ &= R_2 + \frac{R_3}{2} \\ &\le cq^2n^2\|t_n\|^q_{L_q(w)} \end{aligned}$$

with a constant $c > 0$ depending only on w. Thus whenever $1 < q < \infty$ we have using (2.3)

$$R \le cq^2n^2h_m^2\int_{-\pi}^{\pi} w(x)|t_n(x)|^q\,dx$$

where $c > 0$ depends only on w. Obviously, letting $q \to 1$ the last estimate extends for every $1 \le q < \infty$. □

Remark 2.3. When $w = 1$ in Theorem 2.2 a more precise upper bound can be given. Indeed, in this case $R_3 = R_4 = 0$ and $R_2 \le q^2n^2\|t_n\|^q_{L_q(w)}$ (see [7]). This in turn implies

$$R \le \frac{h_m^2}{2}(R_1 + R_2) \le h_m^2R_2 \le q^2n^2h_m^2\|t_n\|^q_{L_q(w)}$$

in the proof of Theorem 2.2. Thus when $w \equiv 1$ Theorem 2.2 holds with the explicit constant $c = 1$.

Theorem 2.2 implies a Marcinkiewicz–Zygmund type result for univariate algebraic polynomials with a remainder term of quadratic accuracy for a Jacobi type weight (2.1) on $[-1, 1]$. Indeed, the next corollary easily follows from Theorem 2.2 by a standard trigonometric substitution $x = \cos y$.

Corollary 2.4. *Let $m, n \in \mathbb{N}$, $1 \le q < \infty$, $0 < \varepsilon < 1$ and set $x_j := \cos\frac{\pi j}{m}$, $1 \le j \le m-1$ with $m \ge \sqrt{\frac{c}{\varepsilon}}qn$. Then for every $p_n \in P_n^1$ and any Jacobi type weight w on $[-1, 1]$ we have*

$$\begin{aligned} \frac{1-\varepsilon}{m}\sum_{j=0}^{m-1}\sqrt{1-x_j^2}w(x_j)|p_n(x_j)|^q &\le \frac{1}{\pi}\int_{-1}^{1} w(x)|p_n(x)|^q\,dx \\ &\le \frac{1+\varepsilon}{m}\sum_{j=0}^{m-1}\sqrt{1-x_j^2}w(x_j)|t_n(x_j)|^q. \end{aligned}$$

Using the technique developed in [6] and [7] one can obtain similar results in the multivariate case for d dimensional ball and simplex. In multivariate case one can apply the Fubini theorem in order to reduce the consideration to lower dimensions with weighted integrals of trigonometric polynomials t_n of the form $\int w|t_n|^q$ appearing with Jacobi type weights w related to the Jacobians of the transformations of the corresponding domains. The main tools needed above: Bernstein and Schur type inequalities hold for such weighted integrals of trigonometric polynomials, too.

This yields Marcinkiewicz–Zygmund type result for d variate algebraic polynomials $p_n \in P_n^d$ of degree $\le n$ with a remainder term of quadratic accuracy. In particular if D is the d dimensional ball or simplex then there exists meshes $x_1, \dots, x_m \in D$ of cardinality $m \sim (\frac{n}{\sqrt{\varepsilon}})^d$, $0 < \varepsilon < 1$, so that with some positive weights w_j, $1 \le j \le m$ we have for every $1 \le q < \infty$ and $p_n \in P_n^d$

$$\int_D |p_n|^q \sim (1+\varepsilon) \sum_{1 \le j \le m} w_j |p_n(x_j)|^q.$$

Theorem 2.2 and Corollary 2.4 essentially show that the L^q, $1 \le q \le \infty$, norm of trigonometric and algebraic polynomials of degree $\le n$ can be discretized with accuracy ε using $m \sim \frac{n}{\sqrt{\varepsilon}}$ properly chosen nodes. This raises the natural question whether the quantity $\sqrt{\varepsilon}$ in the cardinality of discrete mesh is sharp. The affirmative answer to this question in case when $q = \infty$ or 2, can be found in [7]. Our next result verifies that in general this is true for any $1 \le q < \infty$.

Theorem 2.5. *Let $m, n \in \mathbb{N}$, $1 \le q < \infty$. Then there exists a discrete point set $-\pi = x_0 < x_1 < \cdots < x_m = \pi$, $x_{-1} = x_m - 2\pi$ satisfying $\max_{0 \le j \le m-1}(x_{j+1} - x_j) \le \frac{c}{m}$, so that*

$$\frac{1}{2} \sum_{1 \le j \le m-1} (x_{j+1} - x_{j-1}) |\sin n x_j|^q \le \left[1 - c_q \left(\frac{n}{m}\right)^2\right] \int_{-\pi}^{\pi} |\sin nx|^q \, dx \tag{2.4}$$

The above estimate shows that in order to achieve the precision ε in Theorem 2.2 for the trigonometric polynomial $\sin nx$ we need in general at least $m \ge c\frac{n}{\sqrt{\varepsilon}}$ properly chosen nodes.

Proof. Let $f_n(x) := |\sin nx|^q$, and consider

$$\begin{aligned} R(q) &:= \left| \int_{-\pi}^{\pi} f_n(x)\, dx - \frac{1}{2} \sum_{j=0}^{m-1} (x_{j+1} - x_{j-1}) f_n(x_j) \right| \\ &= \frac{1}{2} \left| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} f_n''(x)(x - x_j)(x_{j+1} - x)\, dx \right| \end{aligned}$$

(see [7, (11)]). Let

$$f_n''(x) = q(q-1)n^2 \left|\sin nx\right|^{q-2} - q^2 n^2 \left|\sin nx\right|^q . \tag{2.5}$$

We will choose the x_j's symmetric around the origin, translate them from the interval $[0, \pi/(2n)]$ to $[-\pi, \pi)$. Because of this and the π/n periodicity of $f_n''(x)$, we can write

$$R(q) = 2n \left| \sum_{x_j \in [0, \frac{\pi}{2n})} \int_{x_j}^{x_{j+1}} f_n''(x)(x - x_j)(x_{j+1} - x)\, dx \right| . \tag{2.6}$$

Thus in what follows, we have to define $[\frac{m}{4n}]$ nodes in the interval $\left[0, \frac{\pi}{2n}\right)$. In this way we may get $m \pm c$ nodes instead of m, but changing the role of m in the proof we can get the statement of the theorem.

Let first $q = 1$. Then $f_n''(x) = -n^2|\sin nx|$, and let $x_j = \frac{2\pi j}{m}$, $0 \le j \le m-1$. We get from (2.6)

$$R(1) \ge c\frac{n^3}{m^3} \sum_{0\le j\le \frac{m}{4n}} \sin nx_j$$
$$\ge c\frac{n^4}{m^3} \sum_{0\le j\le \frac{m}{4n}} x_j$$
$$\ge c\left(\frac{n}{m}\right)^2 .$$

Next, let $q > 1$. With some $\lambda > 2\pi$ to be defined below, set

$$\mu_{n,m} := \left[\frac{m}{\lambda n}\sqrt{\frac{q-1}{2q}}\right] .$$

Consider the discrete mesh

$$x_j = \begin{cases} \frac{\lambda j}{m}, & 0 \le j \le \mu_{n,m}, \\ \frac{\lambda\mu_{n,m}+2\pi(j-\mu_{n,m})}{m}, & \mu_{n,m} < j \le j_0 := \mu_{n,m} + \left[\frac{m}{4n} - \frac{\lambda\mu_{n,m}}{2\pi}\right] \le \frac{Cm}{n} \end{cases}$$

in the half-open interval $[0, \pi/(2n))$, where $C > 0$ is an absolute constant. The distance of the adjacent nodes is less than λ/m. We extend this system of nodes from $[0, \pi/(2n))$ just like in the case of $q = 1$, and obtain a system of points $\{x_j\}$. (Their number is $m \pm cn$, but one can easily transform this to the situation of m points.)

Since

$$\int_{x_j}^{x_{j+1}} (x - x_j)(y_{j+1} - x)\,dx = \frac{1}{6}(x_{j+1} - x_j)^3, \quad 0 \le j \le j_0 ,$$

using the mean value theorem for the integrals of $(x - x_j)(x_{j+1} - x)f_n''(x)$ when $0 \le j < \mu_{n,m}$ we get from (2.6)

$$R(q) \ge \frac{n}{3}\left(\frac{\lambda}{m}\right)^3 \left|\sum_{0\le j\le \mu_{n,m}} f_n''(y_j)\right| - \frac{n}{3}\left(\frac{\pi}{m}\right)^3 \sum_{\mu_{n,m}\le j<\frac{Cm}{n}} |f_n''(y_j)| \tag{2.7}$$

$$=: A - B, \qquad x_j \le y_j \le x_{j+1} . \tag{2.8}$$

First we give an upper estimate for B, using the representation (2.5) and the elementary inequalities

$$(\sin ny_j)^{q-2} \le \begin{cases} (\sin nx_j)^{q-2} & \text{if } 1 < q \le 2, \\ (\sin nx_{j+1})^{q-2} & \text{if } 2 \le q < \infty \end{cases} \quad \text{and} \quad (\sin ny_j)^q \le (\sin nx_{j+1})^q,$$

$1 \le j < \frac{m}{4n}$. We get

$$\begin{aligned} B &\le c\left(\frac{n}{m}\right)^3 \sum_{0\le j\le \frac{Cm}{n}} [(nx_j + nx_{j+1})^{q-2} + (nx_{j+1})^q] \\ &\le c\left(\frac{n}{m}\right)^3 \sum_{0\le j\le \frac{Cm}{n}} \left[\left(\frac{nj}{m}\right)^{q-2} + \left(\frac{nj}{m}\right)^q\right] \\ &\le c\left(\frac{n}{m}\right)^{q+1} \sum_{0\le j\le \frac{Cm}{n}} j^{q-2} + c\left(\frac{n}{m}\right)^{q+3} \sum_{0\le j\le \frac{Cm}{n}} j^q \\ &\le c_q\left(\frac{n}{m}\right)^2, \end{aligned}$$

where $c_q > 0$ depends on q.

Next we give a lower estimate for A. We obtain from (2.7),

$$\begin{aligned} A &\ge \frac{n}{3}\left(\frac{\lambda}{m}\right)^3 \left|\sum_{j=0}^{\mu_{n,m}} f_n''(y_j)\right| \\ &= \frac{q}{3}\left(\frac{\lambda n}{m}\right)^3 \left|\sum_{j=0}^{\mu_{n,m}} (\sin ny_j)^{q-2}(q-1-q\sin^2 ny_j)\right| \end{aligned}$$

Note that when $1 \le j \le \mu_{n,m} - 1$ we have

$$q - 1 - q\sin^2 ny_j \ge q - 1 - qn^2 x_{\mu_{n,m}}^2 \ge q - 1 - qn^2\left(\frac{1}{n}\sqrt{\frac{q-1}{2q}}\right)^2 = \frac{q-1}{2}.$$

Therefore we obtain from the previous lower bound

$$\begin{aligned} A &\ge \frac{q(q-1)}{6}\left(\frac{\lambda n}{m}\right)^3 \sum_{j=0}^{\mu_{n,m}} (\sin ny_j)^{q-2} \\ &\ge c_q'\left(\frac{\lambda n}{m}\right)^3\left(\frac{\lambda n}{m}\right)^{q-2} \sum_{j=0}^{\mu_{n,m}} j^{q-2} \\ &\ge c_q'\left(\frac{\lambda n}{m}\right)^{q+1}\left(\frac{m}{\lambda n}\sqrt{\frac{q-1}{2q}}\right)^{q-1} \\ &\ge c_q'\left(\frac{\lambda n}{m}\right)^2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} R(q) \ge A - B &\ge c_q'\left(\frac{\lambda n}{m}\right)^2 - c_q\left(\frac{n}{m}\right)^2 \\ &> c_q\left(\frac{n}{m}\right)^2, \end{aligned}$$

if we set $\lambda > \max\left(\sqrt{\frac{2c_q}{c'_q}}, 4\pi\right)$. □

Remark 2.6. The case $q = 2$ plays a special role here. Namely, then $f''_n(x) = 2n^2 \cos 2nx$ which is antisymmetric with respect to the point $x = \frac{\pi}{4n}$, thus $R(2) = 0$ (see (2.6)). This means that the error in this case, for equidistant nodes is $o\left(\left(\frac{n}{m}\right)^2\right)$. It would be interesting to determine the exact error for all $2 \le q < \infty$, for equidistant nodes.

3. The case $0 < q < 1$

Now we turn our attention to the case $0 < q < 1$. There seem to be no available Marcinkiewicz–Zygmund type upper bounds for integrals of trigonometric polynomials via discrete sums in this case. So Theorem 3.3 proved below appears to be the first results in this direction. In what follows we will denote by c possibly distinct positive constants depending only on q and weight w. We need two lemmas to prove Theorem 3.3 below.

Lemma 3.1. *Let w be a trigonometric Jacobi type weight* (2.1) *and* $0 < q < 1$. *For any $t_n \in T_n$ set $f_n(x) := w^{\frac{1}{q}} t_n(x)$. Then*

$$\left(\int_{-\pi}^{\pi} |f'_n(x)|\, dx\right)^q \le cn \int_{-\pi}^{\pi} |f_n(x)|^q\, dx$$

where $c > 0$ depends only on q and w.

Proof. Clearly,

$$|f'_n(x)| \le \frac{1}{q} w^{\frac{1}{q}-1} |w'(x) t_n(x)| + w^{\frac{1}{q}} |t'_n(x)|.$$

Furthermore it is easy to see that with $r(x) := \prod_{1\le k\le s} \sin\frac{|x-y_k|}{2}$ we have for the trigonometric Jacobi type weight (2.1) $r(x)|w'(x)| \le cw(x)$. Note that $r(x)$ is a trigonometric Jacobi type weight of defect 1. Then using the Schur type inequality from [13, Theorem 4.4], for the trigonometric polynomial t_n with the doubling weight $w^{\frac{1}{q}-1}|w'(x)|$ it follows that

$$\begin{aligned}\int_{-\pi}^{\pi} w^{\frac{1}{q}-1} |w'(x) t_n(x)|\, dx &\le cn \int_{-\pi}^{\pi} w^{\frac{1}{q}-1} |w'(x)| r(x) |t_n(x)|\, dx \\ &\le cn \int_{-\pi}^{\pi} w^{\frac{1}{q}} |t_n(x)|\, dx\,.\end{aligned}$$

In addition, by the L^1 Bernstein type inequality [13], Theorem 4.1 with the doubling weight $w^{\frac{1}{q}}$ we have

$$\int_{-\pi}^{\pi} w^{\frac{1}{q}} |t'_n(x)|\, dx \le cn \int_{-\pi}^{\pi} w^{\frac{1}{q}} |t_n(x)|\, dx.$$

Thus

$$\int_{-\pi}^{\pi} |f'_n(x)|\, dx \le cn \int_{-\pi}^{\pi} w^{\frac{1}{q}} |t_n(x)|\, dx\,. \tag{3.1}$$

Now we need to recall that $w^{\frac{1}{q}}|t_n(x)|$ is a generalized trigonometric polynomial of degree $n + \frac{1}{2q}\sum_{1\le k\le s} a_k$ where the a_k-s are the exponents of the Jacobi type weight (2.1). Then using the Nikolskii type inequality [2], Theorem A.4.3, p. 394 for $w^{\frac{1}{q}}|t_n(x)|$ yields

$$\int_{-\pi}^{\pi} w^{\frac{1}{q}}|t_n(x)|\,dx \le cn^{\frac{1}{q}-1}\left(\int_{-\pi}^{\pi} w|t_n(x)|^q\right)^{\frac{1}{q}} dx.$$

Hence recalling estimate (3.1) we finally arrive at

$$\int_{-\pi}^{\pi} |f_n'(x)|\,dx \le cn\int_{-\pi}^{\pi} w^{\frac{1}{q}}|t_n(x)|\,dx \le cn^{\frac{1}{q}}\left(\int_{-\pi}^{\pi} w|t_n(x)|^q\right)^{\frac{1}{q}} dx. \quad \square$$

Lemma 3.2. *Let $0<q<1$, $\xi_k \in [-\pi,\pi)$, $a_k \in \mathbb{R}^+$, $k=1,\dots,r$, be arbitrary and consider a positive trigonometric polynomial u_n of degree n. Then the derivative of the function*

$$W(x) := u_n(x)^q \prod_{k=1}^{r}\left|\sin\frac{x-\xi_k}{2}\right|^{a_k} \tag{3.2}$$

has at most $2n+2r+1$ sign changes in $[-\pi,\pi)$.

Proof. Clearly, we have

$$W'(x) = W(x)\left(\frac{qu_n'}{u_n} + \frac{1}{2}\sum_{k=1}^{r} a_k \cot\frac{x-\xi_k}{2}\right).$$

Here, when r is even the quantity in the brackets is a rational trigonometric function of the form $\frac{p_n}{u_n\prod_{k=1}^r \sin\frac{x-\xi_k}{2}}$ where p_n is a trigonometric polynomial of degree $n+\frac{r}{2}$ which has at most $2n+r$ roots in $[-\pi,\pi)$. Thus $W'(x)$ has at most $2n+2r$ sign changes if r is even. When r is odd, we can multiply the numerator and denominator of the above ratio by, say, $\sin\frac{x}{2}$. $\square$

Theorem 3.3. *Let $n,m\in\mathbb{N}$, $0<q<1$, $-\pi=x_0<x_1<\cdots<x_m=\pi$ and $t_n\in T_n$. With the weight (2.1) we have*

$$\sum_{j=0}^{m-1}(x_{j+1}-x_j)w(x_j)|t_n(x_j)|^q = (1+\varepsilon)\int_{-\pi}^{\pi} w(x)|t_n(x)|^q\,dx, \tag{3.3}$$

where

$$\varepsilon = c\min(nh_m^q, n^2h_m), \quad h_m = \max_{0\le j\le m-1}(x_{j+1}-x_j).$$

Remark 3.4. Both estimates given in the theorem have some advantages. The first estimate is sharper if $nh_m^{1-q}\ge c$. The second estimate is sharper when $nh_m^{1-q}\le c$. Whether the optimal order cnh_m holds, is an open problem.

Proof. Using the notation from Lemma 3.1 we have

$$\begin{aligned}
R &:= \left| \int_{-\pi}^{\pi} |f_n(x)|^q \, dx - \sum_{j=0}^{m-1} (x_{j+1} - x_j) |f_n(x_j)|^q \right| \\
&= \left| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} [|f_n(x)|^q - |f_n(x_j)|^q] \, dx \right| \\
&\leq \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} |f_n(x) - f_n(x_j)|^q \, dx \\
&= \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} \left| \int_{x_j}^{x} f_n'(u) \, du \right|^q dx \\
&\leq \sum_{j=0}^{m-1} (x_{j+1} - x_j) \left(\int_{x_j}^{x_{j+1}} |f_n'(x)| \, dx \right)^q \\
&= 2\pi \sum_{j=0}^{m-1} t_j c_j^q
\end{aligned}$$

where we used the notations

$$c_j := \int_{x_j}^{x_{j+1}} |f_n'(x)| \, dx, \quad t_j := \frac{x_{j+1} - x_j}{2\pi}, \ 0 \leq j \leq m-1, \ \sum_{j=0}^{m-1} t_j = 1.$$

Since the function x^q is concave for $0 < q < 1$ and $\sum_{j=0}^{m-1} t_j = 1$ it follows that

$$\sum_{j=0}^{m-1} t_j c_j^q \leq \left(\sum_{j=0}^{m-1} t_j c_j \right)^q .$$

Combining this with the previous upper bound we obtain

$$\begin{aligned}
R &\leq 2\pi \sum_{j=0}^{m-1} t_j c_j^q \\
&= (2\pi)^{1-q} \left(\sum_{j=0}^{m-1} (x_{j+1} - x_j) \int_{x_j}^{x_{j+1}} |f_n'(x)| \, dx \right)^q \\
&\leq (2\pi)^{1-q} h_m^q \left(\int_{-\pi}^{\pi} |f_n'(x)| \, dx \right)^q .
\end{aligned}$$

Hence applying Lemma 3.1 to the integral on the right hand side it follows that

$$R \leq c h_m^q n \int_{-\pi}^{\pi} |f_n(x)|^q \, dx.$$

Obviously, this verifies the statement of Theorem 3.3 with $\varepsilon = cnh_m^q$.

To prove the statement with $\varepsilon = cn^2 h_m$ we use the identity

$$f(b)(b-a) - \int_a^b f(x)\,dx = \int_a^b f'(x)(x-a)\,dx$$

with $f(x) = w(x)|t_n(x)|^q$ to obtain

$$\begin{aligned}
(3.4)\qquad R^* &:= \left| \sum_{j=0}^{m-1} w(x_{j+1})|t_n(x_{j+1})|^q (x_{j+1}-x_j) - \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} w(x)|t_n(x)|^q\,dx \right| \\
&= \left| \sum_{j=0}^{m-1} \int_{x_j}^{x_{j+1}} [w(x)|t_n(x)|^q]'(x-x_j)\,dx \right| \\
&\le h_m \int_{-\pi}^{\pi} |[w(x)|t_n(x)|^q]'|\,dx\,.
\end{aligned}$$

Let

$$t_n(x) = u_{n-\ell}(x) \prod_{k=1}^{\ell} \sin\frac{x-z_k}{2},$$

where $z_1, \dots, z_\ell$ are the real roots of $t_n(x)$ (possibly with multiplicity), and $0 < u_{n-\ell}(x) \in T_{n-\ell}$. Thus with $w(x)$ as defined in (2.1) we have

$$f(x) = w(x)|t_n(x)|^q = c \prod_{k=1}^{s} \left|\sin\frac{x-y_k}{2}\right|^{a_k} \prod_{k=1}^{\ell} \left|\sin\frac{x-z_k}{2}\right|^q u_{n-\ell}(x)^q\,.$$

Then we can apply Lemma 3.2 to conclude that $f'(x)$ has at most $2(n-\ell)+2s+2\ell+1 = 2n+2s+1 =: d \le cn$ sign changes. Denote these sign changes by $\eta_1, \dots, \eta_d$. Thus (3.4) yields

$$\begin{aligned}
R^* &\le \int_{-\pi}^{\pi} |[w(x)|t_n(x)|^q]'|\,dx \\
&= \sum_{i=1}^{d-1} \left| \int_{\eta_i}^{\eta_{i+1}} [w(x)|t_n(x)|^q]'\,dx \right| + \left| \int_{\eta_d}^{\eta_1+2\pi} [w(x)|t_n(x)|^q]'\,dx \right| \\
&= \sum_{i=1}^{d-1} |w(\eta_{i+1})|t_n(\eta_{i+1})|^q - w(\eta_i)|t_n(\eta_i)|^q| + |w(\eta_1+2\pi)|t_n(\eta_1+2\pi)|^q \\
&\quad - w(\eta_d)|t_n(\eta_d)|^q| \\
&\le 2d\|w|t_n|^q\|_{L_\infty} \\
&\le cn\|w|t_n|^q\|_{L_\infty} \\
&\le cn^2\|w|t_n|^q\|_{L_q}^q
\end{aligned}$$

where in the last step we applied again the Nikolskii type inequality [2, Theorem A.4.3, p. 394], between the L_∞ and L_q norms for the function $w^{1/q}|t_n|$. $\square$

By Theorem 3.3 when $0 < q < 1$ the Marcinkiewicz–Zygmund type asymptotic relations

$$\sum_{j=0}^{m-1}(x_{j+1}-x_j)w(x_j)|t_n(x_j)|^q = (1+\varepsilon)\int_{-\pi}^{\pi} w(x)|t_n(x)|^q\,dx$$

for trigonometric polynomials t_n can hold for discrete meshes with

$$\max_{0\le j\le m-1}(x_{j+1}-x_j)\sim \frac{\varepsilon}{n^2}.$$

In particular, uniformly distributed nodes of cardinality $m\sim\frac{n^2}{\varepsilon}$ or $m\sim(n/\varepsilon)^{1/q}$ (whichever is smaller) will accomplish this goal. The question of optimality of this bound for cardinality is open when $0 < q < 1$.

However, we can easily give a reasonable lower estimate in case of equidistant nodes and unweighted polynomials.

Theorem 3.5. *Let $0 < q < 1$ and $x_j = \pi j/m$, $0 \le |j| \le m$. Then we have*

$$\sum_{|j|\le m}(x_{j+1}-x_j)|\sin nx_j|^q \le \Big(1-\frac{c_q n}{m}\Big)\int_{-\pi}^{\pi}|\sin nx|^q\,dx\,.$$

Proof. For the error of approximation we have (compare the proof of Theorem 2.5)

$$\begin{aligned}
S &:= c\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}}|(|\sin nx|^q)'|(x-x_j)\,dx\\
&\ge cn\sum_{\frac{m}{4n}<j<\frac{m}{3n}}\int_{x_j}^{x_{j+1}} n(\sin nx)^{q-1}\cos nx\cdot(x-x_j)\,dx\\
&\ge cn^2\sum_{\frac{m}{4n}<j<\frac{m}{3n}}\int_{x_j}^{x_{j+1}}(x-x_j)\,dx\\
&= \frac{cn}{m}\,,
\end{aligned}$$

which implies the statement of the theorem, since

$$\int_{-\pi}^{\pi}|\sin nx|^q\,dx = \int_{-\pi}^{\pi}|\sin x|^q\,dx. \quad \square$$

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