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WEIGHTED STATISTICAL APPROXIMATION PROPERTIES OF JAIN-MARKOV OPERATORS

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Dedicated to Professor Ronald A. DeVore on the occasion of his 80 th birthday, with high esteem.

Abstract. The present paper deals with the weighted statistical approximation processes of the Jain-Markov operators and their bivariate extension.

1. Introduction

There are many sequences of linear positive operators in literature that their Korovkin type approximation properties are investigated (see [6] for details).

The main aim of this study is to analyze the statistical approximation behavior of the Jain-Markov operators and their bivariate modification on some weighted spaces.

In [34], G. C. Jain introduced the following approximating operator for $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$

$$P_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} \omega_\beta(k,nx) f\left(\frac{k}{n}\right), \ f \in C[0,\infty) \tag{1.1}$$

where $\omega_\beta(k,nx)$ is known as Poisson type distribution defined as follows:

$$\omega_\beta(k,nx) = nx(nx+k\beta)^{k-1}e^{-(nx+k\beta)}/k!, \ k \in \mathbb{N}_0 \text{ and } \beta \in [0,1).$$

So, the operators (1.1) are called as Jain operators.

Jain([34], Lemma 1) showed that

$$\sum_{k=0}^{\infty} \omega_\beta(k,\alpha) = 1$$

for $0 < \alpha < \infty$ and $\beta \in [0,1)$.

Let us denote the monomials $e_r(x) = x^r$, $r \in \mathbb{N}_0$. First three monomials are also called as Korovkin test functions.

The following identities for first three monomials of $P_n^{[\beta]}$ were obtained by Jain [34]:

$$P_n^{[\beta]}(e_0;x) = 1, \tag{1.2}$$

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(1.3) $$P_n^{[\beta]}(e_1;x) = \frac{x}{1-\beta},$$

(1.4) $$P_n^{[\beta]}(e_2;x) = \frac{x^2}{(1-\beta)^2} + \frac{x}{n(1-\beta)^3}.$$

Third and fourth monomials were given by Gupta and Greubel ([29], Lemma 1) as follows:

(1.5) $$P_n^{[\beta]}(e_3;x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2}{n(1-\beta)^4} + \frac{(2\beta+1)x}{n^2(1-\beta)^5},$$

(1.6) $$P_n^{[\beta]}(e_4;x) = \frac{x^4}{(1-\beta)^4} + \frac{6x^3}{n(1-\beta)^5} + \frac{(8\beta+7)x^2}{n^2(1-\beta)^6} + \frac{(6\beta^2+8\beta+1)x}{n^3(1-\beta)^7}.$$

In order to obtain Korovkin type approximation results of the Jain operators, we should write β_n instead of β which is satisfies the following condition

$$\lim_n \beta_n = 0.$$

Some degrees of local statistical approximation of the operators $P_n^{[\beta]}(f;x)$ to the function f by means of the moduli of smoothness are investigated by Agratini [4]. Agratini also obtained remarkable results on a compact interval for the statistical convergence of the sequence $P_n^{[\beta_n]}$, under the condition

$$st - \lim_n \beta_n = 0.$$

Some useful results on the asymptotic behavior of the Jain operators are also given by Abel and Agratini [1].

Many well-known positive linear operators preserve the monomials e_0 and e_1.But we see that the Jain operators $P_n^{[\beta]}(f;x)$ do not preserve the monomial e_1. Therefore, in [14], we introduced the following variant of Jain operators jointly with Mohapatra and Örkcü:

(1.7) $$D_n^{[\beta]}(f;x) = \sum_{k=0}^{\infty} \omega_\beta(k, nu_\beta(x)) f\left(\frac{k}{n}),\ f \in C[0,\infty\right),$$

where

$$u_\beta(x) := x(1-\beta),\ x \geq 0$$

and

$$\omega_\beta(k, nu_\beta(x)) = nu_\beta(x)(nu_\beta(x)+k\beta)^{k-1}e^{-(nu_\beta(x)+k\beta)}/k!,\ k \in \mathbb{N}_0, \beta \in [0,1).$$

From the Lemma 2.1 in [14], for the operators $D_n^{[\beta]}(f;x)$, we have the following identities

(1.8) $$D_n^{[\beta]}(e_0;x) = 1,$$

(1.9) $$D_n^{[\beta]}(e_1;x) = x,$$

(1.10) $$D_n^{[\beta]}(e_2;x) = x^2 + \frac{x}{n(1-\beta)^2}.$$

After simple calculations, we can also write

$$D_n^{[\beta]}(e_3;x) = x^3 + \frac{3x^2}{n(1-\beta)^2} + \frac{(2\beta+1)x}{n^2(1-\beta)^4} \tag{1.11}$$

and

$$D_n^{[\beta]}(e_4;x) = x^4 + \frac{6x^3}{n(1-\beta)^2} + \frac{(8\beta+7)x^2}{n^2(1-\beta)^4} + \frac{(6\beta^2+8\beta+1)x}{n^3(1-\beta)^6}. \tag{1.12}$$

Positive linear operators that preserving first and second monomials are also called as Markov type. So, because of (1.8) and (1.9), the operators $D_n^{[\beta]}$ may be called as Jain-Markov operators.

Notice that, if we choose $\beta = 0$, then the operators $P_n^{[0]}$ and $D_n^{[0]}$ reduce to the classical Szász-Mirakyan operators [40], [37].

In this study, for abbreviation, we set

$$\phi_{n,k}(x) := P_n^{[\beta]}((e_1(t)-e_1(x))^k;x) \tag{1.13}$$

and

$$\psi_{n,k}(x) := D_n^{[\beta]}((e_1(t)-e_1(x))^k;x), \tag{1.14}$$

the $k-$th order central moments of the $P_n^{[\beta]}$ and $D_n^{[\beta]}$ respectively.

2. The concept of statistical convergence

At this point, let us recall some notations and definitions on the concept of statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by

$$\delta(K) := \lim_n \frac{1}{n} \sum_{k=1}^{n} \chi_K(k)$$

provided the limits exists, where χ_K is the characteristic function of K.This means that

$$\delta(K) = \lim_n \frac{1}{n} \{\text{the number } k \leq n : k \in K\}.$$

A sequence $x := (x_k)$ is statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\delta\{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\} = 0$$

[17] (see also [20]). For instance,

$$\delta(\mathbb{N}) = 1, \delta\{2k : k \in \mathbb{N}\} = \frac{1}{2} \text{ and } \delta\left\{k^2 : k \in \mathbb{N}\right\} = 0.$$

Notice that any convergent sequence is statistically convergent but not conversely. For example, the sequence

$$x_k = \begin{cases} L_1, & n = m^2, \\ L_2, & n \neq m^2 \end{cases} \qquad (m = 1, 2, 3, ...)$$

is statistically convergent to L_2 but not convergent in ordinary sense when $L_1 \neq L_2$.

Let $A := (a_{nk}), n, k = 1, 2, ..$, be an infinite summability matrix. For a given sequence $x := (x_k)$, the $A-$transform of x, denoted by $Ax := ((Ax)_n)$, is given by $(Ax)_n := \sum_{k=1}^{\infty} a_{nk}x_k$, provided by the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim x = L$ [30].Assume that A is non-negative regular summability matrix, then x is a $A-$statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_n \sum_{k:|x_{k-L}|\geq\varepsilon} a_{nk} = 0$. In this case we denote $st_A-\lim x = L$ [18], [21], [35], [36]. The case in which A is the Cesáro matrix of order one, reduces to the statistical convergence [17], [20], [22]. Also if A is the identity matrix, then it reduces to the ordinary convergence. So if $\lim_n \max_k\{a_{nk}\} = 0$,then $A-$statistically convergence is stronger than ordinary means [35].

3. Korovkin type weighted statistical approximation theorems

There are some Korovkin type statistical approximating theorems for the sequence of positive linear operators.

In approximation theory by linear positive operators, the concept of statistical convergence for $f \in C[a, b]$ endowed with the usual norm

$$\|f\|_{C[a,b]} = \max_{a\leq x\leq b} |f(x)|$$

has been examined for the first time by Gadjiev and Orhan [23].

Theorem A ([23]). *If the sequence of positive linear operators*

$$L_n : C_M[a, b] \to C[a, b]$$

satisfies the conditions

$$st - \lim_n \|L_n(e_\nu) - e_\nu\|_{C[a,b]} = 0, \text{ for } \nu = 0, 1, 2,$$

then, for any function $f \in C_M[a, b]$, *we have*

$$st - \lim_n \|L_n(f) - f\|_{C[a,b]} = 0.$$

Notice that, the space of all functions f which are continuous in $[a, b]$ and bounded all positive axis is denoted by $C_M[a, b]$.

We recall that Theorem A is given for statistical convergence but the proof also works for $A-$statistical convergence. (See [23], [15]).

We see that Theorem A works for finite intervals. But if we consider the approximation on infinite intervals, then we need weighted Korovkin theorems.

It is known that, Korovkin type theorems, related to weighted spaces, were given by Gadjiev [24] and [25].

Before giving these theorems, let us recall the following spaces and norm:

$$B_\rho(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}| \text{ a constant } M_f \text{ depending on } f \text{ exists such that } |f| \leq M_f\rho\},$$

$$C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R})| \ f \text{ continuous on } \mathbb{R}\},$$

endowed with the norm:

$$\|f\|_\rho = \sup_{x\geq 0\geq} \frac{|f(x)|}{\rho(x)}.$$

In ([16], Theorem 3) Duman and Orhan proved the following weighted Korovkin type theorem via $A-$statistical convergence:

Theorem B ([16]). *Let $A = (a_{nk})$ be a nonnegative regular summability matrix and $\rho_1(x)$ and $\rho_2(x)$ be weight functions satisfying*

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0.$$

Assume that L_n is a sequence of linear positive operators acting from C_{ρ_1} to B_{ρ_1}. Then for all $f \in C_{\rho_1}$,

$$st_A - \lim_n \|L_n(f\ ) - f\|_{\rho_2} = 0$$

if and only if

$$st_A - \lim_n \|L_n(F_\nu) - F_\nu\|_{\rho_1} = 0,$$

where $F_\nu(x) = \frac{x^\nu \rho_1(x)}{1+x^2}$, $\nu = 0, 1, 2$.

By choosing the pair of weight functions

$$\rho_0(x) = 1 + x^2,\ \rho_\lambda(x) = 1 + x^{2+\lambda},\ x \in \mathbb{R}_+ := [0, \infty) \tag{3.1}$$

with the help of Theorem B, we indicated the following Korovkin type theorem jointly with Agratini ([3], Corollary 3.1):

Theorem C ([3]). *Assume that L_n is a sequence of linear positive operators acting from $C_{\rho_0}(\mathbb{R}_+)$ to $C_{\rho_\lambda}(\mathbb{R}_+)$, $\lambda > 0$, one has*

$$st - \lim_n \|L_n(f\ ) - f\|_{\rho_\lambda} = 0,\ f \in C_{\rho_0}(\mathbb{R}_+),$$

if and only if

$$st - \lim_n \|L_n(e_\nu) - e_\nu\|_{\rho_0} = 0,\ \nu = 0, 1, 2.$$

So, we can give the first result of this section as follows:

Theorem 3.1. *Let the sequence $0 \leq \beta_n < 1$ be given such that*

$$st - \lim_n \beta_n = 0. \tag{3.2}$$

Let the operators $P_n^{[\beta_n]}$ be defined as in (1.1). Then, under the definitions in (3.1), for each $f \in C_{\rho_0}(\mathbb{R}_+)$, one has

$$st - \lim_n \left\| P_n^{[\beta_n]}(f\ ) - f \right\|_{\rho_\lambda} = 0, \tag{3.3}$$

where $\lambda > 0$.

Proof. From (1.2), it is clear that

$$st - \lim_n \left\| P_n^{[\beta_n]}(e_0) - e_0 \right\|_{\rho_0} = 0.$$

From the expressions in (1.2) and (1.3), we have

$$P_n^{[\beta_n]}(e_1(t)) - e_1(x) = \frac{x\beta_n}{1-\beta_n}$$

and from (1.4), we obtain

$$P_n^{[\beta_n]}(e_2(t)) - e_2(x) = \frac{x^2\beta_n(2-\beta_n)}{(1-\beta_n)^2} + \frac{x}{n(1-\beta_n)^3}.$$

Since

$$\frac{x}{\rho_0(x)} \text{ and } \frac{x^2}{\rho_0(x)}$$

are bounded, then we get

$$st - \lim_n \left\| P_n^{[\beta_n]}(e_i) - e_i \right\|_{\rho_0} = 0, \text{for } i = 1, 2.$$

So, in the light of Theorem C, we have (3.3) immediately. □

Theorem 3.2. *Let the operators* $D_n^{[\beta_n]}$ *be defined as in (1.7). Then, under the definitions in (3.1), for each* $f \in C_{\rho_0}(\mathbb{R}_+)$, *one has*

$$st - \lim_n \left\| D_n^{[\beta_n]}(f\ ) - f \right\|_{\rho_\lambda} = 0, \tag{3.4}$$

where $\lambda > 0$. *Where the sequence the sequence* $0 \le \beta_n < 1$ *has the condition in (3.2).*

Proof. From the expressions in (1.8) and (1.9), we have

$$st - \lim_n \left\| D_n^{[\beta_n]}(e_i) - e_i \right\|_{\rho_0} = 0,\ i = 0, 1.$$

And from (1.10), we obtain

$$D_n^{[\beta_n]}(e_2(t)) - e_2(x) = \frac{x}{n(1-\beta_n)^2}.$$

Since

$$\frac{x}{\rho_0(x)}$$

is bounded, then we get

$$st - \lim_n \left\| D_n^{[\beta_n]}(e_2) - e_2 \right\|_{\rho_0} = 0.$$

So, in the light of Theorem C, we have (3.4) immediately. □

Remark 3.3. If we take

$$st_A - \lim_n \beta_n = 0, \tag{3.5}$$

instead of the condition (3.2) then these theorems are valid for $A-$statistically convergence.

4. Weighted modulus of continuities

Let $f \in C(I)$, the first and second order classical modulus of continuities denoted by $\omega(f;\delta)$ and $\omega_2(f;\delta)$ are defined as

$$\omega(f;\delta) = \sup\{|f(t) - f(x)|\,; t, x \in I, |t-x| \le \delta\}$$

and

$$\omega_2(f;\delta) = \sup\{|f(x+h) - 2f(x) + f(x-h)|\,; x \mp h \in I, |h| \le \delta\}$$

respectively, where I is a compact finite interval.

Remarkable properties about these type of modulus of continuities can be found in [11].

In order to obtain rate of weighted approximation of the positive linear operators defined on infinite intervals, various weighted modulus of continuities are introduced. Some of them include term h in the denominator of the supremum expression. In the chronological order, let us refer to some related papers as [2], [19], [8], [27], [12], [38], [26], [31].

The weighted modulus defined in [2], in order to obtain weighted approximation properties of Szasz-Mirakyan operators on $\mathbb{R}_+$.

Jointly with Gadjieva [27], we introduced the following modulus of continuity:

$$\Omega(f;\delta) = \sup_{0 \le x, |h| \le \delta} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}. \tag{4.1}$$

There are some studies including rates of weighted approximation with the help of $\Omega(f;\delta)$. (see, for instance, [5], [10] [13], [32] and [33]).

And then in [12], we defined the following modulus of continuity:

$$\omega_\rho(f;\delta) = \sup_{0 \le x, |h| \le \delta} \frac{|f(x+h) - f(x)|}{\rho(x+h)} \tag{4.2}$$

where $\rho(x) \ge \max(1, x)$.

In [12], we introduced a generalization of the Gadjiev-Ibragimov operators which includes many well-known operators and obtain its rate of weighted convergence with the help of $\omega_\rho(f;\delta)$ defined in (4.2).

In [38], Moreno introduced another type of modulus of continuity in (4.2) as follows

$$\overline{\Omega}_\alpha(f;\delta) = \sup_{0 \le x, |h| \le \delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^\alpha}.$$

In [26], Gadjiev and Aral defined the following modulus of continuity:

$$\widetilde{\Omega}_\rho(f;\delta) = \sup_{x,t \in \mathbb{R}_+, |\rho(t) - \rho(x)| \le \delta} \frac{|f(t) - f(x)|}{(|\rho(t) - \rho(x)| + 1)\,\rho(x)}$$

where $\rho(0) = 1$ and $\inf_{x \ge 0} \rho(x) \ge 1$.

It is obvious that by choosing $\alpha = 2$, in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain $\overline{\Omega}_2(f;\delta) = \omega_{\rho_0}(f;\delta)$ for $\rho_0(x) = 1 + x^2$ defined as in (3.1), and if we choose $\alpha = 2 + \lambda$

in the definition of $\overline{\Omega}_\alpha(f;\delta)$, then we obtain

$$\widehat{\Omega}_{\rho\lambda}(f;\delta)=\sup_{0\leq x,|h|\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^{2+\lambda}}$$

(see [3]).

Finally, in [31], Holhoş defined a more general weighted modulus of continuity as

$$\omega_\varphi(f;\delta)=\sup_{0\leq x\leq y,|\varphi(y)-\varphi(x)|\leq\delta}\frac{|f(x)-f(y)|}{\rho(x)+\rho(y)}$$

such that, for $\varphi(x)=x$, this modulus of continuity is equivalent to $\Omega(f;\delta)$ defined in (4.1).

Also, let $C_\rho^0(\mathbb{R})$ be the subspace of all functions in $C_\rho(\mathbb{R})$ such that $\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$ exists finitely.

5. Weighted statistical rate of convergence of $P_n^{[\beta]}$ and $D_n^{[\beta]}$

In the light of the definitions in Section 4, we can give the following theorems:

Theorem 5.1. *Let $P_n^{[\beta]}(f;x)$ be the Jain operators defined in (1.1). Then for each $f\in C_{\rho_0}^0(\mathbb{R}_+)$ and $\overline{\rho}_\lambda(x)=(\rho_0(x))^{1+\lambda},\lambda\geq 1$ we have*

$$\frac{\left|P_n^{[\beta]}(f;x)-f(x)\right|}{\overline{\rho}_\lambda(x)}\leq 2\left(1+\frac{7}{(1-\beta)^2}+\frac{1}{n(1-\beta)^3}\right)\omega_{\rho_0}(f;\delta_n(\beta))\tag{5.1}$$

where

$$\delta_n(\beta)=\frac{1}{1-\beta}\sqrt{\beta^2+\frac{1}{n(1-\beta)}}\tag{5.2}$$

and $\omega_{\rho_0}(f;\delta)$ is the modulus of continuity defined by (4.2) .

Proof. By using the properties of $\omega_{\rho_0}(f;\delta)$,(see [38]), we can write

$$|f(t)-f(x)|\leq\left(1+(2x+t)^2\right)\left(\frac{|t-x|}{\delta}+1\right)\omega_{\rho_0}(f;\delta).\tag{5.3}$$

By applying the operators $P_n^{[\beta]}$ to (5.3) and using the (1.2), positivity and linearity of $P_n^{[\beta]}$ and Cauchy-Schwarz inequality then we obtain

$$\begin{aligned}\frac{\left|P_n^{[\beta]}(f;x)-f(x)\right|}{\overline{\rho}_\lambda(x)}\leq\ &\frac{1}{\overline{\rho}_\lambda(x)}(1+4x^2+2xP_n^{[\beta]}(e_1;x)+P_n^{[\beta]}(e_2;x))\\ &\times\left(\frac{\sqrt{\phi_{n,2}(x)}}{\delta}+1\right)\omega_{\rho_0}(f;\delta).\end{aligned}\tag{5.4}$$

Where, as indicated above, $\phi_{n,2}(x)$ is the second central moment of $P_n^{[\beta]}$ defined in (1.13).

Using the identities (1.2), (1.3) and (1.4) in (1.13), we have

$$\phi_{n,2}(x)=x^2\left(\frac{\beta}{1-\beta}\right)^2+\frac{x}{n(1-\beta)^3}.\tag{5.5}$$

On the other hand, since $\beta \in [0,1)$, after simple calculations, we get

$$\frac{(1+4x^2+2xP_n^{[\beta]}(e_1;x)+P_n^{[\beta]}(e_2;x))}{\rho_0(x)} \le \left(1+\frac{7}{(1-\beta)^2}+\frac{1}{n(1-\beta)^3}\right) \tag{5.6}$$

and

$$\frac{\sqrt{\varphi_{n,2}(x)}}{(\rho_0(x))^\lambda} \le \sqrt{\left(\frac{\beta}{1-\beta}\right)^2+\frac{1}{n(1-\beta)^3}} \tag{5.7}$$

immediately. Since $\overline{\rho}_\lambda(x)=\rho_0(x)(\rho_0(x))^\lambda$, by choosing $\delta=\delta_n(\beta)$ as in (5.2) and using (5.6) and (5.7) in (5.4), we have (5.1) which gives the proof. □

Similarly to this result, let us give following theorem.

Theorem 5.2. *Let $D_n^{[\beta]}(f;x)$ be the Jain-Markov operators defined in (1.7). Then for each $f \in C^0_{\rho_0}(\mathbb{R}_+)$ and $\overline{\rho}_\lambda(x)=(\rho_0(x))^{1+\lambda}$, $\lambda \ge 1$ we have*

$$\frac{\left|D_n^{[\beta]}(f;x)-f(x)\right|}{\overline{\rho}_\lambda(x)} \le 2\left(8+\frac{1}{n(1-\beta)^2}\right)\omega_{\rho_0}(f;\delta_n(\beta)) \tag{5.8}$$

where

$$\delta_n(\beta)=\sqrt{\frac{1}{n(1-\beta)^2}} \tag{5.9}$$

and $\omega_{\rho_0}(f;\delta)$ is the modulus of continuity defined by (4.2) .

Proof. If we use the identities (1.8), (1.9) and (1.10) in (1.14) then we have

$$\psi_{n,2}(x)=\frac{x}{n(1-\beta)^2}. \tag{5.10}$$

From (1.9), (1.10) and (5.10), we have

$$\frac{(1+4x^2+2xD_n^{[\beta]}(e_1;x)+D_n^{[\beta]}(e_2;x))}{\rho_0(x)} \le \left(8+\frac{1}{n(1-\beta)^2}\right) \tag{5.11}$$

and

$$\frac{\sqrt{\psi_{n,2}(x)}}{(\rho_0(x))^\lambda} \le \sqrt{\frac{1}{n(1-\beta)^2}}. \tag{5.12}$$

By using inequalities (5.11) and (5.12) in the following

$$\begin{aligned}\frac{\left|D_n^{[\beta]}(f;x)-f(x)\right|}{\overline{\rho}_\lambda(x)} \le\ & \frac{1}{\overline{\rho}_\lambda(x)}(1+4x^2+2xD_n^{[\beta]}(e_1;x)+D_n^{[\beta]}(e_2;x)) \\ & \times\left(\frac{\sqrt{\psi_{n,2}(x)}}{\delta}+1\right)\omega_{\rho_0}(f;\delta),\end{aligned}$$

and choosing $\delta=\delta_n(\beta)$ as in (5.9) we have desired result. □

Now, using Theorem 5.1 and Theorem 5.2, under the condition (3.2) let us give the following results including the weighted rate of statistical convergence of $P_n^{[\beta_n]}(f;x)$ to $f(x)$ and $D_n^{[\beta_n]}(f;x)$ to $f(x)$ respectively by means of $\omega_{\rho_0}(f;\delta_n(\beta_n))$.

Corollary 5.3. *Let $P_n^{[\beta_n]}(f;x)$ be the Jain operators defined in (1.1) for $\beta = \beta_n$ satisfying the condition (3.2). Then for each $f \in C_\rho^0(\mathbb{R}_+)$ and $\overline{\rho}_\lambda(x) = (\rho_0(x))^{1+\lambda}, \lambda \geq 1$ we have*

$$\left\| P_n^{[\beta_n]}(f;.) - f(.) \right\|_{\overline{\rho}_\lambda(x)} \leq 2\left(1 + \frac{7}{(1-\beta_n)^2} + \frac{1}{n(1-\beta_n)^3}\right)\omega_{\rho_0}(f;\delta_n(\beta_n))$$

where

$$\delta_n(\beta_n) = \frac{1}{1-\beta_n}\sqrt{\beta_n^2 + \frac{1}{n(1-\beta_n)}}$$

and $\omega_{\rho_0}(f;\delta_n(\beta_n))$ is the modulus of continuity defined by (4.2)

Proof. If we choose $\beta = \beta_n$ satisfying the condition (3.2) in the Theorem 5.1, the proof is obvious. ☐

Corollary 5.4. *Let $D_n^{[\beta_n]}(f;x)$ be the Jain-Markov operators defined in (1.7) for $\beta = \beta_n$ satisfying the condition (3.2). Then for each $f \in C_\rho^0(\mathbb{R}_+)$ and $\overline{\rho}_\lambda(x) = (\rho_0(x))^{1+\lambda}, \lambda \geq 1$ we have*

$$\left\| D_n^{[\beta_n]}(f;.) - f(.) \right\|_{\overline{\rho}_\lambda(x)} \leq 2\left(8 + \frac{1}{n(1-\beta)^2}\right)\omega_{\rho_0}(f;\delta_n(\beta_n))$$

where

$$\delta_n(\beta_n) = \sqrt{\frac{1}{n(1-\beta)^2}}$$

and $\omega_{\rho_0}(f;\delta_n(\beta_n))$ is the modulus of continuity defined by (4.2)

Remark 5.5. Notice that, under the condition (3.2), since

$$st - \lim_n \delta_n(\beta_n) = 0,$$

Corollary 5.1 and Corollary 5.2 give us a weighted uniform rate of statistical convergence of $P_n^{[\beta_n]}(f;x)$ to $f(x)$ and $D_n^{[\beta_n]}(f;x)$ to $f(x)$ by means of $\omega_{\rho_0}(f;\delta_n(\beta_n))$.

6. Construction of bivariate operators

In [39], using the technique of Barbosu [9], we introduced a bivariate extension of the classical Jain operators and investigated some approximation properties of it for $f \in C(I^2)$, on a compact subinterval $I \subset \mathbb{R}_+$.

Firstly, using the technique of Barbosu [9], we define the parametric extensions of the operator (1.7) for $\beta, \gamma \in [0,1)$ as follows:

$$D_n^{[\beta],x}(f;x,y) = \sum_{k=0}^{\infty} \omega_\beta(k, nu_\beta(x)) f\left(\frac{k}{n}, y\right) \tag{6.1}$$

and

$$D_m^{[\gamma],y}(f;x,y) = \sum_{l=0}^{\infty} \omega_\gamma(l, mu_\gamma(y)) f\left(x, \frac{l}{m}\right). \tag{6.2}$$

Using these extensions, let us construct a bivariate extension of the Jain-Markov operators defined by (1.7) as follows:

$$D_{n,m}^{[\beta,\gamma]}(f;x,y)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\omega_\beta(k,nu_\beta(x))\omega_\gamma(l,mu_\gamma(y))f\left(\frac{k}{n},\frac{l}{m}\right). \tag{6.3}$$

Then we have

(6.4)

$$D_{n,m}^{[\beta,\gamma]}(f;x,y)=D_n^{[\beta],x}(f;x,y)\,D_m^{[\gamma],y}(f;x,y)=D_m^{[\gamma],y}(f;x,y)\,D_n^{[\beta],x}(f;x,y).$$

Lemma 6.1. *The bivariate operator in (6.3) satisfies the following equalities:*

$$\begin{aligned} D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^1(t);x,y\right)&=0,\\ D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_y^1(s);x,y\right)&=0,\end{aligned} \tag{6.5}$$

$$\begin{aligned} D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^2(t);x,y\right)&=\tfrac{x}{n(1-\beta_n)^2},\\ D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_y^2(s);x,y\right)&=\tfrac{y}{m(1-\gamma_m)^2},\end{aligned} \tag{6.6}$$

$$\begin{aligned} D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^4(t);x,y\right)&=\tfrac{8x^3}{n(1-\beta_n)^2}+\tfrac{3x^2}{n^2(1-\beta_n)^4}+\tfrac{(6\beta_n^2+8\beta_n+1)x}{n^3(1-\beta_n)^6},\\ D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_y^4(s);x,y\right)&=\tfrac{8y^3}{m(1-\gamma_m)^2}+\tfrac{3y^2}{m^2(1-\gamma_m)^4}+\tfrac{(6\gamma_m^2+8\gamma_m+1)y}{m^3(1-\gamma_m)^6},\end{aligned} \tag{6.7}$$

where $\varphi_u^r(v):=(v-u)^r$.

Proof. Using (1.8)-(1.12), after some computations, we obtain desired results. □

7. Weighted bivariate modulus of continuities

Some popular definitions of bivariate modulus of continuities can be found in [7].

Let $B_\rho(\mathbb{R}_+^2)$ be the space of all functions $f(x,y)$ satisfying the property

$$|f(x,y)|\leq M_f\ \rho(x,y),$$

where M_f is a positive constant depending only on f and $\rho(x,y)=1+x^2+y^2$.

Let $C_\rho(\mathbb{R}_+^2)$ be the subspace of $B_\rho(\mathbb{R}_+^2)$ of all continuous functions endowed with the norm:

$$\|f\|_\rho=\sup_{(x,y)\in\mathbb{R}_+^2}\frac{|f(x,y)|}{\rho(x,y)}.$$

Let $C_\rho^0(\mathbb{R}_+^2)$ be the subspace of $C_\rho(\mathbb{R}_+^2)$ satisfying

$$\lim_{\sqrt{x^2+y^2}\to\infty}\frac{|f(x,y)|}{\rho(x,y)}$$

exists finitely.

In [33], İspir and Atakut also introduced the bivariate version of the weighted modulus of continuity defined in (4.1) as follows:

$$\Omega_\rho(f;\delta_1,\delta_2)=\sup_{(x,y)\in\mathbb{R}_+^2}\ \sup_{|k_1|\leq\delta_1,|k_2|\leq\delta_2}\frac{|f(x+k_1,y+k_2)-f(x,y)|}{\rho(x,y)\rho(k_1,k_2)}. \tag{7.1}$$

From (7.1), we have

$$\Omega_\rho(f;p_1\delta_1,p_2\delta_2)\leq 4(1+p_1)(1+p_2)(1+\delta_1^2)(1+\delta_2^2)\Omega_\rho(f;\delta_1,\delta_2) \tag{7.2}$$

for $p_1,p_2>0$.

8. Weighted approximation of $D_{n,m}^{[\beta,\gamma]}(f;x,y)$

Recently, in [28], Gark et.al. obtained rate of weighted approximation for Kantorovich variant of a combination of Bernstein–Chlodowsky and Szász type bivariate operators by means of $\Omega_\rho(f;\delta_1,\delta_2)$.

Since $\rho(x,y)=1+x^2+y^2$,we can write

$$\begin{aligned}|f(t,s)-f(x,y)|\leq\ & (1+x^2+y^2)(1+(t-x)^2)(1+(s-y)^2)\\ & \times\Omega_\rho(f;|t-x|,|s-y|).\end{aligned} \tag{8.1}$$

The proof of Theorem 1 in [28], shows that, the rate of weighted convergence for Barbosu type bivariate positive linear operators $L_{n,m}(f;x,y)$ by means of $\Omega_\rho(f;\delta_n,\delta_m)$ can be obtain from the following:

$$\begin{aligned}&|L_{n,m}(f;x,y)-f(x,y)|\leq 4(1+x^2+y^2)\\ &\times\Big[1+\tfrac{1}{\delta_n}\sqrt{L_{n,m}(\varphi_x^2(t);x,y)}+L_{n,m}(\varphi_x^2(t);x,y)\\ &+\tfrac{1}{\delta_n}\sqrt{L_{n,m}(\varphi_x^2(t);x,y)}\sqrt{L_{n,m}(\varphi_x^4(t);x,y)}\Big]\\ &\times\Big[1+\tfrac{1}{\delta_m}\sqrt{L_{n,m}(\varphi_y^2(s);x,y)}+L_{n,m}(\varphi_y^2(s);x,y)\\ &+\tfrac{1}{\delta_m}\sqrt{L_{n,m}(\varphi_y^2(s);x,y)}\sqrt{L_{n,m}(\varphi_y^4(s);x,y)}\Big]\\ &\times\Omega_\rho(f;\delta_n,\delta_m)(1+\delta_n^2)(1+\delta_m^2)\end{aligned} \tag{8.2}$$

(see [28]).

Lemma 8.1. *For the bivariate operator in (6.3), for*

$$a_n=\frac{1}{n(1-\beta_n)^2} \tag{8.3}$$

we have

$$D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^2(t);x,y\right)=xa_n \tag{8.4}$$

and

$$D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^4(t);x,y\right)\leq 15(a_n+a_n^2+a_n^3)(x+x^2+x^3). \tag{8.5}$$

Proof. If we use (8.3) in (6.6), then we have (8.4) immediately. On the other hand, using (8.3) in (6.7), we have

$$\begin{aligned}D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_x^4(t);x,y\right) &= 8x^3a_n+3x^2a_n^2+15xa_n^3\\ &\leq (a_n+a_n^2+a_n^3)(8x^3+3x^2+15x)\\ &\leq 15(a_n+a_n^2+a_n^3)(x^3+x^2+x).\end{aligned}$$

□

Lemma 8.2. *For the bivariate operator in (6.3), for*

$$b_m = \frac{1}{m\left(1-\gamma_m\right)^2} \tag{8.6}$$

we have

$$D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_y^2(s);x,y\right) = yb_m \tag{8.7}$$

and

$$D_{n,m}^{[\beta_n,\gamma_m]}\left(\varphi_y^4(s);x,y\right) \leq 15(b_m+b_m^2+b_m^3)(y+y^2+y^3). \tag{8.8}$$

Since the proof is similar to the proof of previous lemma, we will omit it.

Theorem 8.3. *If $f \in C_\rho^0(\mathbb{R}_+^2)$, then for the bivariate operator in (6.3) we have*

$$\sup_{(x,y)\in\mathbb{R}_+^2} \frac{\left|D_{n,m}^{[\beta_n,\gamma_m]}\left(f;x,y\right)-f(x,y)\right|}{(\rho(x,y))^3} \leq C\Omega_\rho(f;\sqrt{a_n},\sqrt{b_m}), \tag{8.9}$$

where the sequence (a_n) and (b_m) as in the (8.3) and (8.6) respectively.

Proof. If we apply the bivariate operator in (6.3) to the inequality (8.2), then we have

$$\begin{aligned}\left|D_{n,m}^{[\beta_n,\gamma_m]}\left(f;x,y\right)-f(x,y)\right| &\leq 4(1+x^2+y^2)\\ \times\Bigg[1+\tfrac{1}{\delta_{n,\beta_n}}&\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_x^2(t);x,y)}+D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_x^2(t);x,y)\\ &+\tfrac{1}{\delta_{n,\beta_n}}\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_x^2(t);x,y)}\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_x^4(t);x,y)}\Bigg]\\ \times\Bigg[1+\tfrac{1}{\delta_{m,\gamma_m}}&\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_y^2(s);x,y)}+D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_y^2(s);x,y)\\ &+\tfrac{1}{\delta_{m,\gamma_m}}\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_y^2(s);x,y)}\sqrt{D_{n,m}^{[\beta_n,\gamma_m]}(\varphi_y^4(s);x,y)}\Bigg]\\ &\times\Omega_\rho(f;\delta_{n,\beta_n},\delta_{m,\gamma_m})(1+\delta_{n,\beta_n}^2)(1+\delta_{m,\gamma_m}^2).\end{aligned} \tag{8.10}$$

By using Lemma 8.1, Lemma 8.2 and (8.2), we can write

$$\begin{aligned}&\left|D_{n,m}^{[\beta_n,\gamma_m]}\left(f;x,y\right)-f(x,y)\right| \leq 4(1+x^2+y^2)\\ &\times\left[1+\sqrt{x}+a_nx+\sqrt{x}\sqrt{15(a_n+a_n^2+a_n^3)(x^3+x^2+x)}\right]\\ &\times\left[1+\sqrt{y}+b_ny+\sqrt{y}\sqrt{15(b_m+b_m^2+b_m^3)(y^3+y^2+y)}\right]\\ &\times\Omega_\rho(f;\sqrt{a_n},\sqrt{b_m})(1+a_n)(1+b_m).\end{aligned} \tag{8.11}$$

Since $a_n > 0$ and $\lim_n a_n = 0$, there exists a positive number c,such that $a_n \leq c$. Again for the same reason there exists a positive number d,such that $b_m \leq d$. Using these and dividing two hand side of (8.11) by $(\rho(x,y))^3 = (1+x^2+y^2)^3$, we obtain

$$\begin{aligned}\tfrac{\left|D_{n,m}^{[\beta_n,\gamma_m]}(f;x,y)-f(x,y)\right|}{(\rho(x,y))^3} &\leq 4(1+1+c+\sqrt{15(c+c^2+c^3)})\\ &\times(1+1+d+\sqrt{15(d+d^2+d^3)})\\ &\times\Omega_\rho(f;\sqrt{a_n},\sqrt{b_m})(1+c)(1+d).\end{aligned} \tag{8.12}$$

If

$$C := 4(1+1+c+\sqrt{15(c+c^2+c^3)})(1+1+d+\sqrt{15(d+d^2+d^3)})(1+c)(1+d)$$

is substituted in (8.12), the proof is completed. □

Remark 8.4. Since $\lim_n a_n = 0$ and $\lim_m b_m = 0$, Theorem 8.1 gives the rate of weighted convergence of $D_{n,m}^{[\beta_n,\gamma_m]}(f;x,y)$ to the function $f(x,y)$ by means of $\Omega_\rho(f;\sqrt{a_n},\sqrt{b_m})$.

Remark 8.5. If $st-\lim_n \beta_n = st-\lim_m \gamma_m = 0$ then Theorem 8.1 gives the rate of weighted statistical convergence of $D_{n,m}^{[\beta_n,\gamma_m]}(f;x,y)$ to the function $f(x,y)$ by means of $\Omega_\rho(f;\sqrt{a_n},\sqrt{b_m})$.

References

[1] U. Abel and O. Agratini, *Asymptotic behavior of Jain operators*, Numer. Algor. **71** (2016), 553–565.

[2] N. I. Achieser, *Vorlesungen uber approximationstheorie*, Akademic-Verlag, Berlin, 1967.

[3] O. Agratini and O. Doğru, *Weighted approximation by q−Szász-King type operators*, Taiwanese J. of Math. **14**(4) (2010), 1283–1296.

[4] O. Agratini, *Approximation properties of a class of linear operators*, Math. Meth. Appl. Sci. **36** (2013), 2353–2358.

[5] P. N. Agrawal, H. Karslı and M. Goyal, *Szász-Baskakov type operators based on q−integers*, Journal of Inequalities and Applications **441** (2014), https://doi.org/10.1186/1029–242X–2014–441.

[6] F. Altomore and M. Campiti, *Korovkin type approximation theory and its applications*, Berlin, Walter de Gruyter, 1994.

[7] G. A. Anastassiou and S. Gal, *Approximation theory: moduli of continuity and global smoothness preservation*, Birkhäuser, Boston, 2000.

[8] N. T. Amanov, *On the weighted approximation by Szász-Mirakyan operators*, Anal. Math. **18** (1992), 167–184.

[9] D. Barbosu, *Aproximarea functiilor de mai multe variabile prin sume booleene de operatori de tip interpolator*, Ed. Risoprint, Cluj-Napoca, 2002 (in Romanian).

[10] E. Deniz, *Quantitative estimates for Jain-Kantorovich operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1, Math. and Stat. **65**(2) (2016), 121–132.

[11] R. A. DeVore and G. G. Lorentz, *Constructive approximation*, Grundlehren der mathema tischen Wissenschaften **303**, Springer-Verlag, 1993.

[12] O. Doğru, *Weighted approximation of continuous functions on the all positive axis by modifed linear positive operators*, Int. J. Comput. Numer. Anal. Appl. **1**(2) (2002), 135–147.

[13] O. Doğru, *Weighted approximation properties of Szász-type operators*, Int. Math. Journal **2**(9) (2002), 889–895.

[14] O. Doğru, R. N. Mohapatra and M. Örkcü, *Approximation properties of generalized Jain operators*, Filomat **30**(9) (2016), 2359–2366.

[15] O. Duman, M. K. Khan and C. Orhan, *A−Statistical convergence of approximating operators*, Math. Inequal. Appl. **6**(4) (2003), 689–699.

[16] O. Duman and C. Orhan, *Statistical approximation by positive linear operators*, Studia Math. **161**(2) (2004), 187–197.

[17] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[18] A. R. Freedman and J. Sember, *Densities and summability*, Pacific J. Math. **95** (1981), 293–305.

[19] G. Freud, *Investigations on weighted approximation by polynomials*, Studia Sci. Math. Hungar. **8** (1973), 285–305.

[20] J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.

[21] J. A. Fridy and H. I. Miller, *A matrix characterization of statistical convergence*, Analysis **11** (1991), 59–66.
[22] J. A. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), 3625–3631.
[23] A. D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32** (2002), 129–138.
[24] A. D. Gadjiev, *The convergence problem for a sequences of positive lnear operators on unbounded sets, and theorems analogous to that of P. P. Korovkin*, Soviet Math. Dokl. **15**(5) (1974), 1433–1436.
[25] A. D. Gadjiev, *On Korovkin type theorems*, Math. Zametki **20** (1976), 781–786 (in Russian).
[26] A. D. Gadjiev and A. Aral, *The estimates of approximation by using a new type of weighted modulus of continuity*, Computers and Mathematics with Applications **54** (2007), 127–135.
[27] E. A. Gadjieva and O. Doğru, *Weighted approximation properties of Szász operators to continuous functions*, II. Kizilirmak Int. Sci. Conference Proc., Kirikkale Univ. Turkey (1998), 29–37 (in Turkish).
[28] T. Gark, A. M. Acu and P. N. Agrawal, *Weighted approximation and GBS of Chlodowsky-Szász-Kantorovich type operators*, Analysis and Mathematical Physics **9**(3) (2019), 1429–1448.
[29] V. Gupta and G. C. Greubel, *Moment estimations of new Szász-Mirakyan-Durrmeyer operators*, Applied Mathematics and Computation, **271** (2015), 540–547.
[30] G. H. Hardy, *Divergent series*, Oxford Univ. Press, London, 1949.
[31] A. Holhoş, *Quantitative estimates for positive linear operators in weighted spaces*, General Mathematics **16**(4) (2008), 99–110.
[32] N. Ispir, *On modifed Baskakov operators on weighted spaces*, Turk. J. Math., **26**(3) (2001), 355–365.
[33] N. Ispir and C. Atakut, *Approximation by modified Szász-Mirakjan operators on weighted spaces*, Proc. Indian Acad. Sci. Math. Sci. **112**(4) (2002), 571–578.
[34] G. C. Jain, *Approximation of functions by a new class of linear operators*, J. Austral. Math. Soc. **13**(3) (1972), 271–276.
[35] E. Kolk, *Matrix summability of statistically convergent sequences*, Analysis **13** (1993), 77–83.
[36] H. I. Miller, *A measure theoretical subsequence characterization of statistical convergence*, Trans. Amer. Math. Soc. **347** (1995), 1811–1819.
[37] G. M. Mirakyan, *Approximation of continuous functions with the aid of polynomials*. Doklady Akademii Nauk SSSR **31** (1941), 201–205 (in Russian).
[38] A. J. Lopez-Moreno, *Weighted simultaneous approximation with Baskakov type operators*, Acta Math. Hungar. **104**(1-2) (2004), 143–151.
[39] G. Soylu, O. Doğru and R. N. Mohapatra, *Approximation properties of bivariate Jain operators and their Kantorovich variant*, Submitted for publication.
[40] O. Szász, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bureau Stand. **45** (1950), 239–245.

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