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FAST CHANGE-OF-BASES IN POLYNOMIAL INTERPOLATION

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Abstract. We investigate change-of-basis matrices between specific polynomial bases. They allow to quickly map polynomial interpolation formulas in a given basis to another. We derive fast procedures, in essence similar to Fast Fourier Transform, for change-of-bases matrices between canonical, Lagrange, Newton, and Chebyshev type bases. The results are particularly relevant for applying adaptive Newton hierarchical interpolation in the approximation/reconstruction/integration of univariate and multivariate functions.

1. Introduction

Let Ω be a general compact of $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $z_0, \dots, z_k$ distinct points in Ω and denote by I_k associated Lagrange interpolation operator, i.e. given a target function $f : \Omega \to \mathcal{V}$ (with $\mathcal{V}$ a vector space over $\mathbb{K}$) $I_k[f]$ is the unique polynomial of degree less than k with coefficient in $\mathcal{V}$ interpolating f at $z_0, \dots, z_k$. Given $\mathcal{B}^{(p)} := \{p_0, \dots, p_k\}$ any basis of $\mathbb{P}_k[X] := \mathrm{span}_{\mathbb{K}}\{1, X, \dots, X^k\}$,

$$I_k[f] = \sum_{j=0}^{k} c_j[f]\; p_j, \tag{1.1}$$

where $c_0[f], \dots, c_k[f]$ are $\mathbb{K}$-linear combinations of $f(z_0), \dots, f(z_k)$, uniquely determined from $I_k[f](z_j) = f(z_j)$ for $j = 0, \dots, k$. We shall accordingly view the c_j as elements of $\mathrm{span}_{\mathbb{K}}\{\delta_{z_0}, \dots, \delta_{z_k}\}$ where $\delta_{z_j} : f \mapsto f(z_j)$.

We denote by $l_{k,j}, \dots, l_{k,k}$ Lagrange polynomials associated with $z_0, \dots, z_k$,

$$l_{k,j}(z) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{z - z_i}{z_j - z_i}, \qquad j = 0, \dots, k. \tag{1.2}$$

We introduce the family of monic Newton polynomials $(w_j)_j$ by $w_0(z) = \mathbf{1}$,

$$w_j(z) = (z - z_0)\dots(z - z_{j-1}), \qquad j \geq 1. \tag{1.3}$$

For these families, the $c_j[f]$ have plain formulas. More precisely,

- with Lagrange basis $\{l_{k,0}, \dots, l_{k,k}\}$, $c_j[f] = f(z_j)$ for $j = 0, \dots, k$,
- with Newton basis $\{w_0\ , \dots, w_k\ \}$, $c_j[f] = f[z_0, \dots, z_j]$ for $j = 0, \dots, k$,

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where $f[z_0,\dots,z_j]$ are the so-called Newton divided differences, and which can be computed by recursion via Newton tableau of divided differences, see e.g. [1,7,9]. The plain formula $I_k[f]=\sum_{j=0}^k f(z_j)l_{k,j}$ is the so-called *Lagrange interpolation formula* while hierarchical procedure $I_0[f]\equiv f(z_0)$ and $I_i[f]=I_{i-1}[f]+f[z_0,\dots,z_i]w_i$, for $i=1,\dots,k$ is the so-called *Newton interpolation formula.* We note that the iterates $I_i[f]$ interpolate f at $z_0,\dots,z_i$. We also note that the computational merit of Newton procedure depends inherently on the chosen ordering on $z_0,z_1,\dots,z_k$.

For the sake of expedience, we write general interpolation formulas (1.1) as $I_k=\langle[c],[p]\rangle$, where $[c]:=(c_0,\dots,c_k)^\top$ and $[p]:=(p_0,\dots,p_k)^\top$, thus reflecting the identity $I_k[f](z)=\langle(c_0[f],\dots,c_k[f])^\top,(p_0(z),\dots,p_k(z))^\top\rangle:=\sum_{j=0}^k c_j[f]\;p_j(z)$. This allow us to employ "inner product" reasoning without the need to indicate f and z.

Given $[c]$ and $[p]$ as explained and $\mathcal{B}^{(q)}:=\{q_0,\dots,q_k\}$ another basis of $\mathbb{P}_k[X]$, we let $V^{[p\to q]}$ be the change-of-basis matrix from basis $\mathcal{B}^{(p)}$ into basis $\mathcal{B}^{(q)}$, i.e. $V^{[p\to q]}$ is a $(k+1)\times(k+1)$ matrix where the $(j+1)^{th}$ column consists in q_j coordinates in basis $\mathcal{B}^{(p)}$, hence $[q]=(V^{[p\to q]})^\top\;[p]$. By a simple adjoint argument, $I_k=\langle[c],[p]\rangle=\langle[c],(V^{[p\to q]})^{-\top}[q]\rangle=\langle(V^{[p\to q]})^{-1}[c],[q]\rangle$, hence $(V^{[p\to q]})^{-1}=V^{[q\to p]}$ maps $(c_0[f],\dots,c_k[f])^\top$ into the coefficients $(c_0'[f],\dots,c_k'[f])^\top$ associated with $\mathcal{B}^{(q)}$, i.e. $I_k[f]=\sum_{j=0}^k c_j'[f]\;q_j$. In the natural setting where $\mathcal{B}^{(p)}$ is $\{l_{k,0},\dots,l_{k,k}\}$, $V^{[p\to q]}=(q_j(z_i))_{0\le i,j\le k}$ is a Vandermonde matrix. If in addition $\mathcal{B}^{(q)}$ is the Newton basis $\{w_0,\dots,w_k\}$, the matrix $V^{[p\to q]}$ is lower triangular with diagonal entries $w_i(z_i)\neq 0$ for $i=0,\dots,k$. The inverse matrix $(V^{[p\to q]})^{-1}$ is lower triangular as well. It consists of *barycentric coefficients* as we explain further.

For the sake of *numerical practice*, other procedures not necessarily conforming to (1.1) are prescribed when evaluating $I_k[f](z)$, for instance through barycentric formulas, see e.g. [1]. The so-called *first form of barycentric interpolation formula* writes

$$I_k[f](z)=w_{k+1}(z)\sum_{j=0}^k\frac{\tau_{k,j}}{z-z_j}f(z_j),\tag{1.4}$$

where $\tau_{k,j}=1/w_{k+1}'(z_j)$ are the so-called barycentric coefficients. It is easily verified from (1.2) and (1.3) since $l_{k,j}(z)=\frac{w_{k+1}(z)}{z-z_j}\tau_{k,j}$ for $j=0,\dots,k$. By inspecting the leading coefficient of $I_k[1]$ ($\equiv 1$, since $1\in\mathbb{P}_k[X]$) we see that $\sum_{j=0}^k\tau_{k,j}=0$, hence the naming barycentric coefficients. The so-called *true form of barycentric interpolation formula* consists in eliminating w_{k+1} in (1.4) using $I_k[1]\equiv 1$, i.e.

$$I_k[f](z)=\sum_{j=0}^k\frac{\tau_{k,j}}{z-z_j}f(z_j)\Big/\sum_{j=0}^k\frac{\tau_{k,j}}{z-z_j}.\tag{1.5}$$

We refer to [1] and the many references their-in for more details.

Expedient interpolation formulas can be derived if interpolation is cast in a least squares setting. More precisely, given weights $\kappa_0,\dots,\kappa_k>0$ and semi-definite inner product $\langle f,g\rangle_k=\sum_{j=0}^k\kappa_j f(z_j)\overline{g(z_j)}$ [1], we can view $I_k[f]$ as a solution to the

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[1] here $f,g:\Omega\to\mathcal{V}$ with $\mathcal{V}=\mathbb{K}$.

least squares problem $\min_{p\in\mathbb{P}_k[X]}\|f-p\|_k^2$ (=0). Since $\langle\cdot,\cdot\rangle_k$ is definite over $\mathbb{P}_k[X]$ (equivalent to $z_0,\dots,z_k$ pairwise distinct), this solution is unique, and for any basis $\mathcal{B}^{(p)}=\{p_0,\dots,p_k\}$ of $\mathbb{P}_k[X]$ which is orthonormal w.r.t $\langle\cdot,\cdot\rangle_k$ there holds

$$I_k[f]=\sum_{i=0}^{k}\langle f,p_i\rangle_k\ p_i=\sum_{j=0}^{k}\kappa_j f(z_j)\left(\sum_{i=0}^{k}\overline{p_i(z_j)}\ p_i\right). \tag{1.6}$$

Lagrange formula corresponds actually to the particular orthonormal basis $\{l_{k,0}/\sqrt{\kappa_0},\dots,l_{k,k}/\sqrt{\kappa_k}\}$. With any other orthonormal basis $\mathcal{B}^{(p)}$, one has $l_{k,j}=\kappa_j\sum_{i=0}^{k}\overline{p_i(z_j)}p_i$ for $j=0,\dots,k$. In general, given any other basis $\mathcal{B}^{(q)}=\{q_0,\dots,q_k\}$ of $\mathbb{P}_k[X]$, the associated change-of-basis matrix from orthonormal basis $\mathcal{B}^{(p)}$ is given by $V^{[p\to q]}=(\langle q_j,p_i\rangle_k)_{0\le i,j\le k}$.

In the least squares framework, a hierarchical interpolation formula can be derived by computing $p_0,p_1,\dots,p_k$ from $1,X,\dots,X^k$ via Gram-Schmidt process with respect to $\langle\cdot,\cdot\rangle_k$. In the most illustrative setting of families $\{z_0,\dots,z_k\}$ of Gauss abscissas in $[-1,1]$ and $\kappa_0,\dots,\kappa_k$ associated Gauss weights, $p_0,\dots,p_k$ are simply orthogonal polynomials.

The quality of Lagrange interpolation can be quantified through general *Lebesgue-type inequalities* or via more specialized theorems such as *Cauchy remainder* and *Walsh equi-convergence* theorems, [7,11]. Namely, using the plain trick $f-I_k[f]=(f-p)-I_k[f-p]$ valid for any $p\in\mathbb{P}_k[X]$ and its implications on $\|f-I_k[f]\|$, or deriving remainders $f(z)-I_k[f](z)$ when f is smooth, by means of real or complex function arguments.

Lagrange interpolation is naturally disposed to approximation of linear functional of f plainly through $\mathcal{Q}[f]\simeq\sum_{j=0}^{k}c_j[f]\mathcal{Q}[p_j]$ (where c_j as in (1.1)). Point-wise evaluation $\delta_z:f\mapsto f(z)$ is merely the simplest example. Numerical integration is another major application, i.e.

$$\int f(z)d\varrho(z)\simeq\sum_{j=0}^{k}c_j[f]\int p_j(z)d\varrho(z), \tag{1.7}$$

where we assume integrals are well defined. Implied quadratures are called *interpolatory* quadratures. For example, Côtes quadratures are defined for any real interval $\Omega=[a,b]$ and $d\varrho(z)=dz$ the Lebesgue measure, by setting $z_j=a+(b-a)j/k$ for $j=0,\dots,k$ and integrating the Lagrange interpolation formula, see e.g. [8,15].

We are mainly interested in Newton hierarchical procedure. We recall that $I_k[f]=\sum_{j=0}^{k}c_j[f]w_j$, where each $c_j[f]=f[z_0,\dots,z_j]$ depend only on $f(z_0),\dots,f(z_j)$. We shall describe this dependence in more details. To this end, we introduce triangular arrays $\mathcal{W}=(w_{i,j})$, $\mathcal{T}=(\tau_{i,j})$ as in (1.9). We denote by $\mathcal{W}_l$ and $\mathcal{T}_l$ the leading principal matrices of such arrays. These are the $l\times l$ lower triangular matrices extracted by keeping only the first l rows and columns. Actually, $\mathcal{T}_l$ is the inverse of $\mathcal{W}_l$ for any $l\ge 1$. We note that $\mathcal{W}_{k+1}=V^{[l\to w]}$ is the change-of-basis matrix from Lagrange basis $\{l_{k,0},\dots,l_{k,k}\}$ into Newton basis $\{w_0,\dots,w_k\}$. Hence, $[c]:=(c_0,\dots,c_k)^\top$ as a vector of elements in $\mathrm{span}_{\mathbb{K}}\{\delta_{z_0},\dots,\delta_{z_k}\}$ is given by $[c]=\mathcal{W}_{k+1}^{-1}\times[\delta]=\mathcal{T}_{k+1}\times[\delta]$ where $[\delta]:=(\delta_{z_0},\dots,\delta_{z_k})^\top$. The c_i can thus be expressed using barycentric weights, i.e.

$$c_i : f \mapsto \sum_{j=0}^{i} \tau_{i,j} f(z_j), \qquad i \geq 0. \tag{1.8}$$

$$\begin{Bmatrix} w_{0,0} & & & & \\ w_{1,0} & w_{1,1} & & & \\ \vdots & & \ddots & & \\ w_{i,0} & w_{i,1} & \cdots & w_{i,i} & \\ \vdots & & & & \ddots \end{Bmatrix}, \begin{Bmatrix} \tau_{0,0} & & & & \\ \tau_{1,0} & \tau_{1,1} & & & \\ \vdots & & \ddots & & \\ \tau_{i,0} & \tau_{i,1} & \cdots & \tau_{i,i} & \\ \vdots & & & & \ddots \end{Bmatrix}, \tag{1.9}$$

$$w_{i,j} = w_j(z_i), \qquad \tau_{i,j} = 1/(w'_{i+1}(z_j)), \qquad i \geq 0, \quad j = 0, \dots, i.$$

It is customary to describe Newton formulas using monic Newton polynomials. In practice, appropriate normalizations has to be considered in order to prevent numerical instabilities. For a plain illustration, let $\rho_0, \dots, \rho_{2n}$ be $\{2n+1\}$-roots of unity, $R > 0$, and consider $z_j = R\rho_j$ for $j = 0, \dots, 2n$. The Newton polynomial w_{2n} is given by $w_{2n}(z) = (z^{2n+1} - R^{2n+1})/(z - z_{2n})$, hence $w_{2n}(-z_{2n}) = R^{2n+1}/z_{2n} = R^{2n}/\rho_{2n}$ has modulus R^{2n}. In addition, $(2n+1)\tau_{2n,j} = \rho_j/R^{2n}$ has modulus $1/R^{2n}$ for any $j = 0, \dots, 2n$. For n large, unless $R = 1$, loss of precision will occur while due to overflow/underflow while computing with and handling such polynomial.

For reliable interpolation schemes, e.g. by means of Chebyshev-type abscissas on real intervals or Fejér points on complex domains with smooth boundaries, one can enforce stability by enforcing prescribed orderings on points $z_0, \dots, z_k$ and considering normalized polynomials $\widetilde{w}_j(z) = w_j(z)/c^j$ where c is the logarithmic capacity of Ω or $\widetilde{w}_j(z) = w_j(z)/w_j(z_j)$ assuming c is not necessarily known, see [10, 14] for details. Associated barycentric weights become $\widetilde{\tau}_{i,j} = c^i/(w'_{i+1}(z_j))$ or $\widetilde{\tau}_{i,j} = w_i(z_i)/(w'_{i+1}(z_j))$ and are immune to overflow/underflow.

We consider pure hierarchical interpolation, $Z := (z_j)_{j\geq 0}$ is a sequence of mutually distinct points in Ω and refer to tuples $Z_k := (z_0, \dots, z_{k-1})$ as k-sections of Z, hence operator $I_k[\cdot]$ is associated with Z_{k+1}. Newton procedure is most cost effective for computing approximations $\sum_{j=0}^{k} c_j[f]w_j$ to f (or $\sum_{j=0}^{k} c_j[f]\mathcal{Q}[w_j]$ to $\mathcal{Q}[f]$). We simply query f at one node z_j at a time, compute $c_j[f]$ (or $c_j[f]$ and $\mathcal{Q}[w_j]$), then update the approximation.

Hierarchical interpolation is naturally disposed to generalization to multi-dimension. For instance, through plain cartesian (tensor) product constructions, detailed in [6]. In a nutshell, for $d \geq 1$ arbitrary integer, we consider approximation of functions defined over Ω^d by means of d-variates polynomials. Monomial are now indexed by multi-indices $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{N}^d$ and defined by $\mathbf{x}^\nu = x_1^{\nu_1} \times \cdots \times x_d^{\nu_d}$ for $\mathbf{x} = (x_1, \dots, x_d) \in \Omega^d$.

We let $\otimes_d Z := \{\mathbf{z}_\nu := (z_{\nu_1}, \dots, z_{\nu_d}) : \nu \in \mathbb{N}^d\} \subset \Omega^d$ be the d-cartesian product of sequence Z. The family of associated monic Newton polynomials is now $(w_\nu)_{\nu\in\mathbb{N}^d}$ defined by

$$w_\nu(\mathbf{x}) = w_{\nu_1}(x_1) \dots w_{\nu_d}(x_d), \qquad \mathbf{x} = (x_1, \dots, x_d) \in \Omega^d. \tag{1.10}$$

For $\nu = (\nu_1, \ldots, \nu_d), \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ with $\mu \leq \nu$ in coordinate-wise sense, we introduce $\tau_{\nu,\mu} := \prod_{j=1}^d \tau_{\nu_j,\mu_j}$ and define $\mathbf{c}_\nu$ by

$$c_\nu : f \mapsto \sum_{\mu \leq \nu} \tau_{\nu,\mu}\, f(z_\mu), \qquad \nu \in \mathbb{N}^d. \tag{1.11}$$

They are to be compared with the c_i defined in (1.8).

For $\Lambda \subset \mathbb{N}^d$ a *lower* set of indices[2], we define operator $\mathcal{I}_\Lambda$ by

$$\mathcal{I}_\Lambda[f] := \sum_{\nu \in \Lambda} c_\nu[f] w_\nu. \tag{1.12}$$

It is an interpolation operator, $\mathcal{I}_\Lambda[f]$ is the unique d-variate polynomial, belonging to $\mathbb{P}_\Lambda := \mathrm{span}_{\mathbb{K}}\{\mathbf{x}^\nu : \nu \in \Lambda\}$ and interpolating f over the grid $\Gamma_\Lambda := \{\mathbf{z}_\nu : \nu \in \Lambda\}$.

The interpolation process is hierarchical. For Λ lower and $\nu \notin \Lambda$ such that $\Lambda' = \Lambda \cup \{\nu\}$ is lower, one has $\mathbb{P}_{\Lambda'} = \mathbb{P}_\Lambda \oplus \mathrm{span}_{\mathbb{K}}\{\mathbf{x}^\nu\}$, $\Gamma_{\Lambda'} = \Gamma_\Lambda \cup \{\mathbf{z}_\nu\}$ and

$$\mathcal{I}_{\Lambda'}[f] = \mathcal{I}_\Lambda[f] + c_\nu[f] w_\nu. \tag{1.13}$$

Unlike the univariate setting, there are many candidates ν admissible in Λ,[3] hence richer approximation potential. For instance, through adaptivity

$$\Lambda_0 = \{\nu^{(0)}\} \longrightarrow \Lambda_1 = \{\nu^{(0)}, \nu^{(1)}\} \longrightarrow \ldots, \tag{1.14}$$

where $\nu^{(0)} = \mathbf{0}$ and $\nu^{(i)}$ are admitted in Λ_{i-1} according to some criterion.

The hierarchical adaptive scheme is well disposed for reconstruction and integration purposes. More precisely:

- Having, fast generated or tabulated, expansions of w_j in the canonical basis $1, z, z^2, \ldots$ allows us to produce approximations $\sum_{\nu\in\Lambda} \widehat{c}_\nu \mathbf{z}^\nu$ to f. In other words reconstruction of Fourier series if Ω is a disc. If Ω is a real interval, such as $[0, 1]$ or $[-1, 1]$, expansions of w_j in cosine or Chebyshev type bases, allows us to produce cosine or Chebyshev series.
- As in the univariate setting, hierarchical sums $\sum_{\nu\in\Lambda} c_\nu[f]\mathcal{Q}[w_\nu]$ can be used to approximate $\mathcal{Q}[f]$, e.g. $\mathcal{Q}[f] = \int_{\Omega^d} f(\mathbf{z}) d\varrho_d(\mathbf{z})$. If $\varrho_d = \otimes_{j=1}^d \varrho_1$ is a tensor product measure, $\mathcal{Q}[w_\nu] = \prod_{i=1}^d \gamma_{\nu_i}$ with $\gamma_k := \int_\Omega w_k d\varrho_1$. Having γ_k known or tabulated thus yields fast hierarchical quadratures.

Hierarchical approximation schemes (or reconstruction/integration) have a unified implementation. At every iteration, $\Lambda_k = \{\nu^{(0)}, \ldots, \nu^{(k)}\}$ is lower and $\mathcal{P}_q(\Lambda_k)$ is a priority queue of admissible indices ($\nu \in N(\Lambda_k)$). The index $\nu^{(k+1)}$ with highest priority get admitted into Λ_k, i.e. $\Lambda_k \longrightarrow \Lambda_{k+1}$. Then for $\nu \in N(\Lambda_{k+1}) - N(\Lambda_k)$, we compute $c_\nu[f]$ (and any needed quantity) and insert ν in the priority queue. Of course, priority criterion or heuristic depends on the approximation purpose.

For more insights on approximation settings of interest, we refer to recent paper [12] describing and addressing these (approximation/reconstruction /integration) objectives by means of rank-1 lattices quadratures.

The present paper is mainly concerned with fast change-of-basis in highly relevant interpolation settings, namely those involving roots of unity and Chebyshev type

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[2] also called downward closed, i.e. $\nu \in \Lambda$ and $\mu \leq \nu$ implies necessarily that $\mu \in \Lambda$.

[3] $N(\Lambda) := \{\nu \notin \Lambda : \Lambda \cup \{\nu\} \text{ is lower}\}$ contains at least d multi-indices.

abscissas and their sequences alternatives for pure hierarchical interpolation, the so-called Leja sequences on the unit disk and $\Re$-Leja on the unit interval $[-1, 1]$.

The organization of the paper is as follows. In §2, we recall properties of Chebyshev polynomials of the first kind $(T_j)_j$ and introduce a new family $(H_j)_j$ of Chebyshev-type polynomials that is relevant in our analysis. This section is not concerned, per se, with interpolation, but rather with how to identify fast recurrences on change-of-basis matrices. The main idea is to use the formulas relating (T_{2j}, T_{2j+1}) to T_j and (H_{2j}, H_{2j+1}) to H_j in conjunction with suitable "basis indexing" in order to draw fast block matrices recurrences. This modus operandi is used throughout the paper.

In §3, §4 and §5, we provide fast recurrences on change-of-bases matrices within the frameworks of interpolation with roots of unity, Chebyshev abscissas of first and second kind. The first recurrence (3.5) in §3 is an instance of Fast Fourier Transform (radix-2 FFT), and thus reflects optimal computational complexity. The same can be said for all other identified recurrences. The findings of §4 involving the new basis $(H_j)_j$ replicate to perfection FFT expediency. Sections §3, §4 and §5 are also concerned with hierarchical interpolation. We describe appropriate ordering of interpolation nodes leading more expedience and stability in Newton interpolation formulas.

In section §6, we introduce a new sequence of abscissas in $[-1, 1]$ that is very relevant for hierarchical interpolation and study its properties. The sequence can be seen as an alternative to non nested Chebyshev abscissas discussed in §4. In §7, few numerical experiments are presented.

As far as numerical stability is concerned, all the matrices studied in this paper have moderate entries and are well-conditioned. This however will not be investigated in details.

Notation. Any integer $k \geq 1$ has a unique binary representation

$$k = \sum_{j=0}^{n} a_j 2^j, \qquad \begin{array}{c} a_j \in \{0, 1\} \\ a_n = 1 \end{array}, \tag{1.15}$$

where $n = \lfloor \log_2(k) \rfloor$. The notation $\sigma_1(k) := \sum_{j=0}^{n} a_j$ stands for the number of ones in the expansion of k. We denote by $(\varepsilon_k)_{k\geq 0}$ the "bit-reversed" Van der Corput sequence: $\varepsilon_0 = 0$ and

$$\varepsilon_k = \frac{1}{2} \sum_{j=0}^{n} \frac{a_j}{2^j}, \tag{1.16}$$

for $k \geq 1$ as above. We use notation $M^{-\top}$ for the transpose of the inverse $(M^{-1})^\top$ of a non singular matrix. Hadamard product $\odot$ is defined for two matrices of same dimensions are the entrywise product.

2. Preliminaries

2.1. **Chebyshev polynomials.** We let $(T_k)_{k\geq 0}$ be the family of Chebyshev polynomials of the first kind, e.g. defined by $T_k(\cos(\theta)) = \cos(k\theta)$. The family satisfies a three-term recurrence: $T_0(x) = 1$, $T_1(x) = x$, and

$$T_{k+1}(x) + T_{k-1}(x) = 2xT_k(x), \qquad k \geq 1. \tag{2.1}$$

Polynomials $T_0, T_2, \ldots$ are even functions while polynomials $T_1, T_3, \ldots$ are odd functions. Moreover, the family satisfies a *parity recurrence*:

$$\begin{aligned} T_{2k}(x) &= \quad T_k(T_2(x)), \\ T_{2k+1}(x) + T_{2k-1}(x) &= 2xT_k(T_2(x)), \end{aligned} \qquad k \geq 1. \tag{2.2}$$

We let $(H_k)_{k\geq 0}$ be the family of polynomials defined by: $H_0(x) = 1$, and

$$H_k(x) = \prod_{\substack{j=0 \\ a_j=1}}^{n} (2T_{2^j}(x)), \qquad k = \sum_{j=0}^{n} a_j 2^j. \tag{2.3}$$

The previous is the binary representation of $k \geq 1$. As for the T_k, these polynomials have integer coefficients, every H_k has degree k, and $H_0, H_2, \ldots$ are even functions while $H_1, H_3, \ldots$ are odd functions. In addition, in view of $T_1(x) = x$ and $T_{2m}(x) = T_m(T_2(x))$ for any $m \geq 0$, a simple parity recurrence holds:

$$\begin{aligned} H_{2k}(x) &= \quad H_k(T_2(x)), \\ H_{2k+1}(x) &= 2xH_k(T_2(x)), \end{aligned} \qquad k \geq 0. \tag{2.4}$$

Every H_k has a supremum $2^{\sigma_1(k)} (\leq k+1)$ attained at $x = 1$, where $\sigma_1(k)$ is the number of ones in the binary representation of k.

The family $(\widetilde{H}_j)_{j\geq 0}$, defined by $\widetilde{H}_j(x) = H_j(x/2)$, is a *polynomial sequence* over $\mathbb{Z}[X]$. Namely, every $\widetilde{H}_k$ has degree k, integer coefficients, and leading coefficient 1. We have introduced and used this family for the purpose of explaining user-friendly generating matrices of orthogonal Frolov-Chebyshev lattices, see [5].

The linear decomposition of polynomials H_k in basis $T_0, T_1, \ldots$ can be explicitly expressed and computed by induction on k. On the one hand, in view of $2T_m \times 2T_l = 2(T_{m+l} + T_{m-l})$ for any $l \leq m$, we can expand the product in (2.3) giving H_k. In particular, we obtain the following identity

$$H_k(x) = 2 \sum_{i \in \mathcal{S}_k} T_i(x), \qquad k = 2^n + \sum_{j=0}^{n-1} a_j 2^j, \tag{2.5}$$

where $\mathcal{S}_k := \{2^n + \sum_{j=0}^{n-1} \epsilon_j a_j 2^j : \epsilon_j = \pm 1\}$, a set which consists in $2^{\sigma_1(k)-1}$ integers within $\{1, \ldots, k\}$. On the other hand, if we write $H_j = \sum_{i=0}^{j} \alpha_{i,j} T_i$ and use convention $\alpha_{j+1,j} = 0$, then $\alpha_{0,0} = 1$, $(\alpha_{0,1}, \alpha_{1,1}) = (0, 2)$ and

$$\begin{array}{lll} \alpha_{2i,2j} = \alpha_{i,j}, & \alpha_{2i+1,2j} = 0, & \\ \alpha_{2i,2j+1} = 0, & \alpha_{2i+1,2j+1} = \alpha_{i,j} + \alpha_{i+1,j}, & j \geq 1, i = 0, \ldots, j. \\ & \text{(except for } \alpha_{1,2j+1} = 2\alpha_{0,j} + \alpha_{1,j}) & \end{array} \tag{2.6}$$

We have used the parity of polynomials T_j and H_j in order to infer that $\alpha_{2i+1,2j} = \alpha_{2i,2j+1} = 0$ for any i, j, and have derived the other identities in view of recurrences (2.2) and (2.4).

An alternative and cleaner induction is given by: $\alpha_{0,0} = 1$ then

$$\alpha_{i,k} = \begin{cases} 2\alpha_{0,k-2^n} & \text{if } i = 2^n, \\ \alpha_{|i-2^n|,k-2^n} & \text{otherwise,} \end{cases} \qquad n \geq 0, 2^n \leq k < 2^{n+1}. \tag{2.7}$$

Indeed, $k = 2^n + l$ with $l = k - 2^n < 2^n$, so that $H_k = 2T_{2^n} H_l$ which in turn implies $H_k = \sum_{i=0}^{l} \alpha_{i,l} 2T_{2^n} T_i = \sum_{i=0}^{l} \alpha_{i,l} (T_{2^n+i} + T_{2^n-i})$ hence the above.

In what follows, we derive recurrences on whole matrices $(\alpha_{i,j})_{0\le i,j\le k-1}$. We will basically recast the recurrences identified for coefficients $\alpha_{i,j}$ as recurrences on these change-of-basis matrices. This is more important from a computational perspective for implementing fast transforms. Namely, mapping a linear decomposition in basis (H_j) into basis (T_j) and vice versa.

In view of (2.6), by simply reordering basis $T_0, T_1, \dots, T_{k-1}$ and basis $H_0, H_1, \dots, H_{k-1}$ by parity of indices, first even indices then odd indices, the resulting $k \times k$ change-of-basis matrix becomes block diagonal. We describe next the most suitable orderings of indices for promoting fast recurrences.

2.2. **Coordinates-permuted systems.** We let $(\mathcal{J}_n)_{n\ge 0}$ be the "ordered" sets of indices defined by: $\mathcal{J}_0 = \{0\}$,

$$\mathcal{J}_{n+1} = 2\mathcal{J}_n \wedge \{2\mathcal{J}_n + 1\}, \qquad n \ge 0, \tag{2.8}$$

where $2\mathcal{J}_n := \{2j : j \in \mathcal{J}_n\}$, $2\mathcal{J}_n + 1 := \{2j+1 : j \in \mathcal{J}_n\}$ and $\wedge$ is the concatenation operation. The sets $\mathcal{J}_0 \subset \mathcal{J}_1 \subset \dots$, reflect a specific way of re-ordering nested sets of indices $\{0, \dots, 2^n - 1\}, n \ge 0$. On that account, each order $\mathcal{J}_n$ is best described by a permutation π_n of $\{0, \dots, 2^n - 1\}$, i.e. $\mathcal{J}_n = \{\pi_n(j) : j = 0, \dots, 2^n - 1\}$. The recurrence (2.8) is reflected on these permutations as follows: π_0 is the identity over $\{0\}$, then

$$\begin{array}{rcl} \pi_{n+1}(j) & = & 2\pi_n(j), \\ \pi_{n+1}(2^n + j) & = & 2\pi_n(j) + 1, \end{array} \qquad j = 0, \dots, 2^n - 1. \tag{2.9}$$

Permutations π_n are related to the Van der Corput sequence $(\varepsilon_k)_{k\ge 0}$ given in (1.16). Indeed, The following can be verified by induction

$$\pi_n(k) = 2^n \varepsilon_k, \qquad n \ge 0, \qquad 0 \le k \le 2^n - 1. \tag{2.10}$$

This identification shows in particular that π_n have order 2, i.e.

$$\pi_n \circ \pi_n(j) = j, \qquad n \ge 0, \ \ j = 0, \dots, 2^n - 1. \tag{2.11}$$

We let $P_n \in \{0,1\}^{2^n \times 2^n}$ be the permutation matrices associated with the π_n (i.e. $P_n = (\delta_{i,\pi_n(j)})_{0\le i,j\le 2^n-1}$). We have $P_0 = [1]$, and in light of (2.11) every P_n is symmetric and satisfies $P_n \times P_n = I_{2^n}$ where I_{2^n} is the $2^n \times 2^n$ identity matrix, i.e.

$$P_n = P_n^\top = P_n^{-1} \qquad n \ge 0. \tag{2.12}$$

Given a $2^n \times 2^n$ matrix $A = (a_{i,j})_{0\le i,j\le 2^n-1}$, then $A' = (a_{\pi_n(i),\pi_n(j)})_{0\le i,j\le 2^n-1}$, which can simply be formulated as $(a_{i,j})_{i,j\in\mathcal{J}_n}$, is equal to A having its rows/columns permuted with π_n. In particular A and A' are similar matrices with $A' = P_n^{-1} A P_n$. We put forward two settings of interest to us:

- if A is the change-of-basis matrix from a basis $\{p_0, \dots, p_{2^n-1}\}$ into a basis $\{q_0, \dots, q_{2^n-1}\}$, then A' is the change-of-basis matrix from permuted basis $\{p_j\}_{j\in\mathcal{J}_{2^n}}$ into permuted basis $\{q_j\}_{j\in\mathcal{J}_{2^n}}$. The matrix $(A')^{-1} = P_n^{-1} A^{-1} P_n$ is its reverse change-of-basis matrix.
- if A is a Vandermonde matrix associated with a polynomial basis $\{q_0, \dots, q_{2^n-1}\}$ and numbers $\{z_0, \dots, z_{2^n-1}\}$ (in $\mathbb{R}$ or $\mathbb{C}$), i.e. $A = (q_j(z_i))_{0\le i,j\le 2^n-1}$, then $A' = (q_j(z_i))_{i,j\in\mathcal{J}_n}$ is the Vandermonde type matrix associated with permuted basis $\{q_j\}_{j\in\mathcal{J}_{2^n}}$ and permuted numbers $\{z_i\}_{i\in\mathcal{J}_{2^n}}$.

It is worth noting that $A = P_n^{-1} A' P_n$ (since $P_n = P_n^{-1}$). As a result, if one is given $A' = (a'_{i,j})_{0 \le i,j \le 2^n - 1}$, then $A = (a'_{\pi_n(i), \pi_n(j)})_{0 \le i,j \le 2^n - 1}$. Overall, computing A' knowing A or A knowing A' is immediate as soon as π_n is computed.

For the remainder of this paper, we will persistently derive recurrences for change-of-basis matrices $V_n^{[p \to q]}$ and $V_n^{[q \to p]} = (V_n^{[p \to q]})^{-1}$ between permuted bases $\mathcal{B}_n^{(p)} := \{p_j\}_{j \in \mathcal{J}_{2^n}}$ and $\mathcal{B}_n^{(q)} := \{q_j\}_{j \in \mathcal{J}_{2^n}}$. As explained above, formulating such matrices between the non-permuted bases is straightforward. For instance from $\{p_0, \ldots, p_{2^n - 1}\}$ into $\{q_0, \ldots, q_{2^n - 1}\}$, the matrix is given by $P_n^{-1} V_n^{[p \to q]} P_n$.

2.3. **Fast change-of-basis transforms for Chebyshev bases.** In this section, we describe fast transforms between bases (T_j) and (H_j). To this end, we first introduce $2^n \times 2^n$ matrices J_{2^n}, $\widetilde{J}_{2^n}$, Q_n, and $\widetilde{Q}_n$ for $n \ge 0$ by

$$J_{2^n} := \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \qquad \widetilde{J}_{2^n} := \begin{bmatrix} 2 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \tag{2.13}$$

$$Q_n := P_n^{-1} J_{2^n} P_n, \qquad \widetilde{Q}_n := P_n^{-1} \widetilde{J}_{2^n} P_n,$$

with $J_1 = Q_0 = [1]$, $\widetilde{J}_1 = \widetilde{Q}_0 = [2]$. We note that $\widetilde{J}_{2^n} = \widetilde{I}_{2^n} J_{2^n} = J_{2^n} + E_{1,1}$, where $\widetilde{I}_{2^n} := \operatorname{diag}[2, 1, 1, \ldots]$ and $E_{1,1} = e_1 e_1^\top$, $e_1 = (1, 0, \ldots, 0)^\top$ considered in $\mathbb{R}^{2^n}$. Since the leading row/column of P_n is e_1 (implied from $\pi_n(0) = 0$), we also have $\widetilde{Q}_n = \widetilde{I}_{2^n} Q_n = Q_n + E_{1,1}$ for any $n \ge 1$.

For $n \ge 0$, we let $V_n^{[t \to h]}$ be the $2^n \times 2^n$ change-of-basis matrix from permuted basis $\mathcal{B}_n^{(t)} := \{T_j\}_{j \in \mathcal{J}_{2^n}}$ into permuted basis $\mathcal{B}_n^{(h)} := \{H_j\}_{j \in \mathcal{J}_{2^n}}$, and $V_n^{[h \to t]}$ be the reverse change-of-basis matrix. Plain recurrences can be derived for such matrices.

Proposition 2.1. *There holds* $V_0^{[t \to h]} = V_0^{[h \to t]} = [1]$, *and for* $n \ge 0$

$$V_{n+1}^{[t \to h]} = \begin{bmatrix} V_n^{[t \to h]} & \mathbf{0} \\ \mathbf{0} & \widetilde{Q}_n^\top V_n^{[t \to h]} \end{bmatrix}, \tag{2.14}$$

$$V_{n+1}^{[h \to t]} = \begin{bmatrix} V_n^{[h \to t]} & \mathbf{0} \\ \mathbf{0} & V_n^{[h \to t]} \widetilde{Q}_n^{-\top} \end{bmatrix}. \tag{2.15}$$

Proof. We recall that $\mathcal{J}_{n+1} = 2\mathcal{J}_n \wedge \{2\mathcal{J}_n + 1\}$. The zero blocks are inferred from the parity of polynomials T_j and H_j. Given that $H_j = \sum_i \alpha_{i,j} T_i$, then $H_{2j}(x) = H_j(T_2(x)) = \sum_i \alpha_{i,j} T_i(T_2(x)) = \sum_i \alpha_{i,j} T_{2i}(x)$. The leading block $V_n^{[t \to h]}$ follows. Then $H_{2j+1}(x) = 2x H_j(T_2(x)) = 2x \sum_i \alpha_{i,j} T_{2i}(x) = 2\alpha_{0,j} T_1(x) + \sum_{i \ne 0} \alpha_{i,j} (T_{2i-1}(x) + T_{2i+1}(x))$ hence $H_{2j+1} = (2\alpha_{0,j} + \alpha_{1,j}) T_1 + \sum_{i=1}^{j} (\alpha_{i,j} + \alpha_{i+1,j}) T_{2i+1}$. As a result, if $A = (\alpha_{i,j})_{0 \le i,j \le 2^n - 1}$ then $\widetilde{J}_{2^n}^\top A$ is the change-of-basis matrix from $T_1, T_3, \ldots, T_{2^{n+1} - 1}$ into $H_1, H_3, \ldots, H_{2^{n+1} - 1}$. From permuted basis $(T_{2i+1})_{i \in \mathcal{J}_n}$ into permuted basis $(H_{2j+1})_{j \in \mathcal{J}_n}$, it is thus equal to $P_n^{-1} (\widetilde{J}_{2^n}^\top A) P_n = (P_n^{-1} \widetilde{J}_{2^n}^\top P_n)(P_n^{-1} A P_n) = \widetilde{Q}_n^\top V_n^{[t \to h]}$. The proof of (2.14) is complete. That for (2.15) follows by inversion, since $V_n^{[h \to t]} = (V_n^{[h \to t]})^{-1}$, for any $n \ge 0$. ☐

The above recurrences are relatively simple. As far as implementation is concerned, difficulties can only arise on computing with and handling the matrices $\widetilde{Q}_n$ and $\widetilde{Q}_n^{-1}$, or rather Q_n and Q_n^{-1}, as already noted.[4]

Matrices $Q_n = P_n^{-1} J_{2^n} P_n$ and Q_n^{-1} indexed by $i, j \in \{0, \dots, 2^n - 1\}$ have explicit forms. The entries of Q_n are $(Q_n)_{i,j} = (J_{2^n})_{\pi_n(i), \pi_n(j)}$ hence equal to 1 if $\pi_n(i) = \pi_n(j)$ or $\pi_n(i) = \pi_n(j) + 1$, and to 0 otherwise. It is easily verified $J_{2^n}^{-1}$ is lower with entries $(-1)^{i-j}$, the entries of Q_n^{-1} are thus equal to $(-1)^{\pi_n(i) - \pi_n(j)}$ if $\pi_n(i) \geq \pi_n(j)$ and to 0 otherwise. Given that $\mathcal{J}_n = \{\pi_n(j) : j = 0, \dots, 2^n - 1\}$ were generated for $n = 0, \dots, N$, assembling $Q_0, \dots, Q_N$ and $Q_0^{-1}, \dots, Q_N^{-1}$ is straightforward.

Matrices Q_n and $\widetilde{Q}_n$ can be computed differently. In view of (2.13), $R_n := \widetilde{Q}_n - \widetilde{I}_{2^n} = Q_n - I_{2^n}$ are given by $R_n = (\delta_{\pi_n(i), \pi_n(j)+1})_{0 \leq i,j \leq 2^n - 1}$. In particular, matrices R_n satisfy a plain recurrence: $R_0 = [0]$ and

$$R_{n+1} = \begin{bmatrix} \mathbf{0} & R_n \\ I_{2^n} & \mathbf{0} \end{bmatrix}, \qquad n \geq 0. \tag{2.16}$$

The proof uses induction on n and (2.9), see [5]. We note that the actions of matrices R_n and $R_n^\top$ are explicit. For $\boldsymbol{x} = (x_0, \dots, x_{2^n - 1})^\top$,

- $\boldsymbol{y} = R_n \boldsymbol{x}$ is given by $y_i = x_{\pi_n(\pi_n(i) - 1)}$ for $i = 1, \dots, 2^n - 1$ and $y_0 = 0$;
- $\boldsymbol{y} = R_n^\top \boldsymbol{x}$ is given by $y_i = x_{\pi_n(\pi_n(i) + 1)}$ for $i = 0, \dots, 2^n - 2$ and $y_{2^n - 1} = 0$.

Accordingly, the actions of Q_n, $Q_n^\top$, $\widetilde{Q}_n$, and $\widetilde{Q}_n^\top$ are straighforward as well.

We can now outline fast transforms involving matrices $V_N^{[t \to h]}$ and $V_N^{[h \to t]}$. In view of (2.14), given $\boldsymbol{z} \in \mathbb{R}^{2^{n+1}}$ vertical concatenation of $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^{2^n}$, $\boldsymbol{w}_1 = V_n^{[t \to h]} \boldsymbol{z}_1$, and $\boldsymbol{w}_2 = V_n^{[t \to h]} \boldsymbol{z}_2$, then $\boldsymbol{w} = V_{n+1}^{[t \to h]} \boldsymbol{z}$ is the vertical concatenation of $\boldsymbol{w}_1$ and $\widetilde{Q}_n^\top \boldsymbol{w}_2 (= \boldsymbol{w}_2 + \widetilde{R}_n^\top \boldsymbol{w}_2$ where $\widetilde{R}_n = R_n + E_{1,1})$. Inversely, given $\boldsymbol{w}$ vertical concatenation of $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathbb{R}^{2^n}$, then $\boldsymbol{z} = V_{n+1}^{[h \to t]} \boldsymbol{w}$ is the vertical concatenation of $\boldsymbol{z}_1 = V_n^{[h \to t]} \boldsymbol{w}_1$ and $\boldsymbol{z}_2 = V_n^{[h \to t]} (\widetilde{Q}_n^{-\top} \boldsymbol{w}_2)$. Given that $\mathcal{J}_n$ were generated (hence π_n are known), and the auxiliary actions of $\widetilde{Q}_n^\top$ and $\widetilde{Q}_n^{-\top}$ were implemented for $n = 0, \dots, N$, computing transforms $V_N^{[t \to h]} \boldsymbol{x}$ or $V_N^{[h \to t]} \boldsymbol{x}$ is straightforward. In number of operations, the complexity is $\mathcal{O}(M \log(M))$ with $M = 2^N$.

For mapping between bases (T_j) and (H_j), we are naturally inclined to use identified change-of-basis recurrences. For example, given a decomposition $P = \sum_{j=0}^k b_j H_j$, one can be interested in coefficients c_j s.t. $P = \sum_{j=0}^k c_j T_j$. It is immediate that $\boldsymbol{c} = A \boldsymbol{b}$ where $\boldsymbol{b} = (b_0, \dots, b_k)^\top$, $\boldsymbol{c} = (c_0, \dots, c_k)^\top$, and $A = (\alpha_{i,j})_{0 \leq i,j \leq k}$ is the change-of-basis matrix from $T_0, T_1, \dots, T_k$ into $H_0, H_1, \dots, H_k$, the entries of which are discussed in (2.6) and (2.7). We can also rely on fast transforms. More precisely, let n be s.t. $2^n \leq k + 1 < 2^{n+1}$, $\boldsymbol{b} = (b_0, \dots, b_k, 0, \dots 0)^\top$ and $\boldsymbol{c} = (c_0, \dots, c_k, 0, \dots 0)^\top$ both considered in $\mathbb{R}^{2^{n+1}}$, and $A = (\alpha_{i,j})_{0 \leq i,j \leq 2^{n+1} - 1}$. We have $\boldsymbol{c} = A \boldsymbol{b}$ and this now implies $P_{n+1}^{-1} \boldsymbol{c} = V_{n+1}^{[t \to h]} (P_{n+1}^{-1} \boldsymbol{b})$. In order to compute $c_0, \dots, c_k$, we proceed as follows:

- compute $\boldsymbol{z} = P_{n+1}^{-1} \boldsymbol{b}$, i.e. $z_j = b_{\pi_{n+1}(j)}$, $j = 0, \dots, 2^{n+1} - 1$;
- compute $\boldsymbol{w} = V_{n+1}^{[t \to h]} \boldsymbol{z}$, using fast transforms as explained above;

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[4] $\widetilde{Q}_n = Q_n + E_{1,1}$ and $\widetilde{Q}_n^{-1} = Q_n^{-1} \widetilde{I}_{2^n}^{-1}$, hence equal to Q_n^{-1} with it first column halved.

- compute $\boldsymbol{c} = P_{n+1}\boldsymbol{w}$, i.e. $c_j = w_{\pi_{n+1}(j)}$, $j = 0, \dots, k$.

The latter step is justified by $P_{n+1} = P_{n+1}^{-1}$, see (2.12).

3. Roots of unity and bit-reversed sequence

In this section, Ω is the closed unit disk of the complex domain. For $k \geq 1$ fixed, we consider $\mathcal{U}_k := \{\rho_0, \dots, \rho_{k-1}\}$ with $\rho_j = (e^{\mathbf{i}2\pi/k})^j$, the set of k-roots of unity. We then consider the semi-definite hermitian product $\langle f, g\rangle_k = (1/k)\sum_{j=0}^{k-1} f(\rho_j)\overline{g(\rho_j)}$. This product is definite over $\mathbb{P}_{k-1}[X]$ for which the canonical basis $1, z, z^2, \dots, z^{k-1}$ is orthonormal. In view of (1.6), we have $I_{\mathcal{U}_k}[f] = \sum_{j=0}^{k-1}\langle f, z^j\rangle_k \; z^j$, hence

$$I_{\mathcal{U}_k}[f] = \frac{1}{k}\sum_{l=0}^{k-1} f(\rho_l)\sum_{j=0}^{k-1}(z/\rho_l)^j = \sum_{l=0}^{k-1} f(\rho_l)\frac{1}{k}\frac{(z/\rho_l)^k - 1}{z/\rho_l - 1}, \tag{3.1}$$

is the interpolation operator associated with $\mathcal{U}_k$. The Vandermonde matrix $F := ((\rho_i)^j)_{0\leq i,j\leq k-1}$, which also the change-of-basis from the Lagrange basis into the canonical basis, is ($\sqrt{k}$ times) the $k \times k$ DFT matrix. Its inverse is F^*/k where F^* is the conjugate transpose of F.

The above is applicable as is with sets $\mathcal{U}_{2^n}$ for $n \geq 0$. Such sets are in addition nested, i.e. $\mathcal{U}_{2^n} \subset \mathcal{U}_{2^{n+1}}$, symmetric with respect to 0, and satisfy $\mathcal{U}_{2^n} = \{z^2 : z \in \mathcal{U}_{2^{n+1}}\}$. This implies convenient properties (in the line of radix-2 FFT) best described using a sequential framework.

We consider the bit-reversed sequence $E = (e_k)_{k\geq 0}$, defined by

$$e_k := \exp\left(\mathbf{i}\pi\sum_{j=0}^{n}\frac{a_j}{2^j}\right), \qquad k = \sum_{j=0}^{n} a_j 2^j. \tag{3.2}$$

We have $E = (1, -1, \mathbf{i}, -\mathbf{i}, e^{\mathbf{i}\pi/4}, -e^{\mathbf{i}\pi/4}, e^{\mathbf{i}3\pi/4}, -e^{\mathbf{i}3\pi/4}, \dots)$. The sequence E is a Leja sequence over the closed unit disk. Every 2^n-section of E is equal to $\mathcal{U}_{2^n}$ in the set sense. Observe that $e_{2j+1} = -e_{2j}$ and $e_{2j}^2 = e_j$, hence $(z - e_{2j})(z - e_{2j+1}) = (z^2 - e_j)$ for any j. Newton polynomials associated with E can thus be factorized. Induction yields that for any $2^n \leq k < 2^{n+1}$ as above,

$$w_k(z) = \prod_{\substack{j=0 \\ a_j=1}}^{n}(z^{2^j} + e_k^{2^j}). \tag{3.3}$$

By developing the product, $w_k(z) = \sum_{l\preceq k}(e_k)^{k-l}z^l$ where $l \preceq k$ in the sense of binary expansions, if $l = \sum_{j=0}^{n} b_j 2^j$, then $\{j : b_j = 1\} \subset \{j : a_j = 1\}$.

We propose to derive change-of-basis matrices between Lagrange basis (associated with 2^n-section E_{2^n}) and hierarchical bases, the canonical basis $(z^j)_{j\geq 0}$ and the Newton basis $(w_j(z))_{j\geq 0}$. This is ideally described in the permuted-coordinate systems associated with orderings $\mathcal{J}_n$. To this end, we introduce permuted Vandermonde $2^n \times 2^n$ matrices

$$V_n := \big((e_i)^j\big)_{i\in\mathcal{J}_n, j\in\mathcal{J}_n}, \qquad V_n^{[w]} := \big(w_j(e_i)\big)_{i\in\mathcal{J}_n, j\in\mathcal{J}_n}, \tag{3.4}$$

and diagonal $2^n \times 2^n$ matrices $D_n = \mathrm{diag}[(e_{2i})_{i\in\mathcal{J}_n}]$ for any $n \geq 0$.

Proposition 3.1. *There holds* $V_0 = [1]$, *and for* $n \geq 0$

$$V_{n+1} = \begin{bmatrix} V_n & +D_n V_n \\ V_n & -D_n V_n \end{bmatrix}. \tag{3.5}$$

Proof. The recurrence on $\mathcal{J}_n$ implies the block representation

$$\mathcal{J}_{n+1}\left\{\vphantom{\begin{array}{c}a\\b\\c\\d\end{array}}\right. \overbrace{\qquad V_{n+1} \qquad}^{\mathcal{J}_{n+1}} \quad = \quad \begin{array}{ccc} & \overbrace{\quad}^{2\mathcal{J}_n} & \overbrace{\quad}^{2\mathcal{J}_n+1} \\ 2\mathcal{J}_n \quad \{ & X_1 & Y_1 \\ 2\mathcal{J}_n+1 \quad \{ & X_2 & Y_2 \end{array} .$$

Since $e_{2i+1} = -e_{2i}$ for any i, then $X_2 = X_1$ and $Y_2 = -Y_1$. Since $(e_{2i})^2 = e_i$ for any i, then $X_1 = V_n$ and $Y_1 = D_n V_n$. The proof is complete. ☐

By inverting, it is immediate to derive a recurrence for matrices V_n^{-1}. For the sake of consistency, we formulate it for $V_n^{-\top} = (V_n^{-1})^\top$.

Proposition 3.2. *There holds* $V_0^{-\top} = [1]$, *and for* $n \geq 0$

$$V_{n+1}^{-\top} = \frac{1}{2}\begin{bmatrix} V_n^{-\top} & +D_n^{-1} V_n^{-\top} \\ V_n^{-\top} & -D_n^{-1} V_n^{-\top} \end{bmatrix}. \tag{3.6}$$

Recurrences (3.5) and (3.6) are similar up to a factor 1/2 and the change of matrices D_n into D_n^{-1} which are also diagonal with $D_n^{-1} = \operatorname{diag}[(1/e_{2i})_{i\in\mathcal{J}_n}]$. In light of this observation, matrices $V_n^{-\top}$ satisfy

$$V_n^{-\top} = \frac{1}{2^n}\big((1/e_i)^j\big)_{\substack{i\in\mathcal{J}_n \\ j\in\mathcal{J}_n}} = \overline{V_n}/2^n, \qquad n \geq 0. \tag{3.7}$$

We therefore recover $V_n^{-1} = V_n^*/2^n$ for any $n \geq 0$.

We now turn to matrices $V_n^{[w]}$. We introduce $2^n \times 2^n$ matrices D_n^+ and D_n^- by $D_n^\pm = (\pm e_{2i} - e_{2j})_{i,j\in\mathcal{J}_n}$. Since $e_{2i+1} = -e_{2i}$, $(e_{2i})^2 = e_i$ for any i, and $w_{2j}(z) = w_j(z^2)$, $w_{2j+1}(z) = (z - e_{2j})w_k(z^2)$ for any j, the same arguments used in proving Proposition 3.1 yield the following.

Proposition 3.3. *There holds* $V_0^{[w]} = [1]$, *and for* $n \geq 0$

$$V_{n+1}^{[w]} = \begin{bmatrix} V_n^{[w]} & D_n^+ \odot V_n^{[w]} \\ V_n^{[w]} & D_n^- \odot V_n^{[w]} \end{bmatrix}. \tag{3.8}$$

The Hadamard products $\odot$ can be further simplified. Indeed, there holds

$$D_n^\pm \odot V_n^{[w]} = \pm D_n V_n^{[w]} - V_n^{[w]} D_n. \tag{3.9}$$

Although not straightforward as V_n^{-1}, it is within reach to derive simple recurrences for matrices $(V_n^{[w]})^{-1}$, also formulated for transposes.

Proposition 3.4. *There holds* $(V_0^{[w]})^{-\top} = [1]$, *and for* $n \geq 0$

$$(V_{n+1}^{[w]})^{-\top} = \frac{1}{2}\left[\begin{array}{cc} (V_n^{[w]})^{-\top} + V_n' & +D_n^{-1}(V_n^{[w]})^{-\top} \\ (V_n^{[w]})^{-\top} - V_n' & -D_n^{-1}(V_n^{[w]})^{-\top} \end{array}\right], \tag{3.10}$$

where $V_n' = D_n^{-1}(V_n^{[w]})^{-\top}D_n$.

Remark 3.5. We note that the entries of matrices $V_n^{[w]}$ and $(V_n^{[w]})^{-1}$ are the collocation and barycentric coefficients as in lower triangular matrices $\mathcal{W}_{2^n}$ and $\mathcal{T}_{2^n}$ (see (1.9)) associated with E, except for the rows and columns being permuted according to π_n, i.e. $V_n^{[w]} = (w_{i,j})_{i,j\in\mathcal{J}_n}$ and $(V_n^{[w]})^{-1} = (\tau_{i,j})_{i,j\in\mathcal{J}_n}$. Recurrences (3.8) and (3.10) can thus be viewed as fast procedures for computing such coefficients.

We introduce notation $\mathcal{B}_n^{(z)} := \{z^j\}_{j\in\mathcal{J}_{2^n}}$, $\mathcal{B}_n^{(l)} := \{l_{2^n,j}(z)\}_{j\in\mathcal{J}_{2^n}}$, and $\mathcal{B}_n^{(w)} := \{w_j(z)\}_{j\in\mathcal{J}_{2^n}}$ for the canonical basis and Lagrange/Newton bases associated with $E_{2^n} = (e_0, \ldots, e_{2^n-1})$ all permuted according to π_n. We have derived the recurrences for the change-of-basis matrices between $\mathcal{B}_n^{(l)}$ and $\mathcal{B}_n^{(z)}$ (i.e. V_n and inverse) and between $\mathcal{B}_n^{(l)}$ and $\mathcal{B}_n^{(w)}$ (i.e. $V_n^{[w]}$ and inverse). Those for change-of-basis matrices $V_n^{[z\to w]}$ and $V_n^{[w\to z]}$ between $\mathcal{B}_n^{(z)}$ and $\mathcal{B}_n^{(w)}$ can also be easily derived.

Proposition 3.6. *There holds* $V_0^{[z\to w]} = V_0^{[w\to z]} = [1]$, *and for* $n \geq 0$

$$V_{n+1}^{[z\to w]} = \left[\begin{array}{cc} V_n^{[z\to w]} & -V_n^{[z\to w]}D_n \\ \mathbf{0} & V_n^{[z\to w]} \end{array}\right], \tag{3.11}$$

$$V_{n+1}^{[w\to z]} = \left[\begin{array}{cc} V_n^{[w\to z]} & D_nV_n^{[w\to z]} \\ \mathbf{0} & V_n^{[w\to z]} \end{array}\right]. \tag{3.12}$$

Proof. We use the block representation as in the proof of Proposition 3.1. Given $j \in \mathcal{J}_n$, $w_{2j}(z) = w_j(z^2)$ and $w_{2j+1}(z) = (z - e_{2j})w_j(z^2)$, hence $w_{2j}(z) = \sum_{i\in\mathcal{J}_n} \alpha_{i,j}z^{2i}$ and $w_{2j+1}(z) = \sum_{i\in\mathcal{J}_n} \alpha_{i,j}z^{2i+1} - e_{2j}\sum_{i\in\mathcal{J}_n} \alpha_{i,j}z^{2i}$ given that $V_n^{[z\to w]} = (\alpha_{i,j})$. We imply the first recurrence. The recurrence for $V_n^{[w\to z]} = (V_n^{[z\to w]})^{-1}$ is a simple verification. ☐

We can recapitulate all the previous in the following table.

(3.13)

	$\mathcal{B}_n^{(l)}$	$\mathcal{B}_n^{(z)}$	$\mathcal{B}_n^{(w)}$
$\mathcal{B}_n^{(l)}$	I_{2^n}	V_n	$V_n^{[w]}$
$\mathcal{B}_n^{(z)}$	V_n^{-1}	I_{2^n}	$V_n^{[z\to w]}$
$\mathcal{B}_n^{(w)}$	$(V_n^{[w]})^{-1}$	$V_n^{[w\to z]}$	I_{2^n}

Remark 3.7. Implementing recurrences in real arithmetics is immediate for the matrices $V_n^{[\cdot]}$ having only entries of modulus 1 or 0 such as V_n, V_n^{-1}, $V_n^{[z\to w]}$, and $V_n^{[w\to z]}$. We simply need to explicit and implement the recurrences implied for matrices $|V_n^{[\cdot]}|$ and $\varphi(V_n^{[\cdot]})$ of entry-wise moduli and arguments. For example, $|V_n|$

is the $2^n \times 2^n$ all-ones matrix while $\varphi(V_n)$ satisfies the recurrence: $\varphi(V_0) = [0]$ and for $n \geq 0$

$$\varphi(V_{n+1}) = \begin{bmatrix} \varphi(V_n) & \varphi(V_n) + u_n \mathbf{1}^\top \\ \varphi(V_n) & \varphi(V_n) + u_n \mathbf{1}^\top + \pi \end{bmatrix}, \tag{3.14}$$

where $u_n = (\varphi(e_{2i}))_{i \in \mathcal{J}_n} \in \mathbb{R}^{2^n}$ and $\mathbf{1}$ is the all-ones vector in $\mathbb{R}^{2^n}$. Having computed $\varphi(V_N)$, entries of V_N are obtained via the polar form.

We now can outline how the results of the present section can be used for fast computations in polynomial interpolation.

Interpolation at roots of unity: for a target function $f : \Omega \to \mathbb{C}$ we compute $f(e_0), \dots, f(e_{2^n-1})$ and stack them in a vector $\boldsymbol{b} = (b_0, \dots, b_{2^n-1})^\top$. The vector $\boldsymbol{c} = (c_0, \dots, c_{2^n-1})^\top$ such that $I_{\mathcal{U}_{2^n}}[f] = \sum_{k=0}^{2^n-1} c_k z^k$ satisfies $V_n(P_n^{-1}\boldsymbol{c}) = P_n^{-1}\boldsymbol{b}$. In order to compute $\boldsymbol{c}$, we proceed as follows

- compute $\boldsymbol{y} = P_n^{-1}\boldsymbol{b}$, i.e. $y_j = b_{\pi_n(j)} = f(e_{\pi_n(j)})$, $j = 0, \dots, 2^n - 1$;
- compute $\boldsymbol{w} = V_n^{-1}\boldsymbol{y}$, using fast transforms based in (3.6);
- compute $\boldsymbol{c} = P_n\boldsymbol{w}$, i.e. $c_j = w_{\pi_n(j)}$, $j = 0, \dots, 2^n - 1$.

Computing $\boldsymbol{w}$ using (3.6) can be rapidly performed. In number of operations, the complexity is $\mathcal{O}(2^n \log(2^n))$. In the same way, if $\boldsymbol{c}$ is such that $I_{\mathcal{U}_{2^n}}[f] = \sum_{k=0}^{2^n-1} c_k w_k$, i.e. coefficients in the Newton basis, then $V_n^{[w]}(P_n^{-1}\boldsymbol{c}) = P_n^{-1}\boldsymbol{b}$. The above can be applied, with the only difference that $\boldsymbol{w} = (V_n^{[w]})^{-1}\boldsymbol{y}$. We note in view of (3.10) that computing $(V_n^{[w]})^{-1}\boldsymbol{y}$ is clearly more involved than that of computing $V_n^{-1}\boldsymbol{y}$.

Hierarchical interpolation using E: As far as Newton formulas are concerned, it is not imperative to rely on fast transforms. Such formulas are better suited to hierarchical computations. They can be implemented as follows: first, we generate $e_0, \dots, e_N$ for N big enough and compute associated barycentric coefficients $\{\tau_{i,j}\}_{0 \leq i,j \leq N}$ ($= \mathcal{T}_{N+1}$, see (1.9)). We then let $I_{-1}[f] \equiv 0$ and proceed one index k at a time (i.e. $k = 0, 1, \dots$)

- query the target function f at e_k;
- compute the new Newton coefficient $c_k = \sum_{j=0}^{k} \tau_{k,j} f(e_j)$;
- update $I_k[f] = I_{k-1}[f] + c_k w_k$.

Polynomials $I_k[f] = \sum_{j=0}^{k} c_j w_j$ are the hierarchical approximations to f.

Computing barycentric coefficients $\tau_{i,j} = 1/w'_{i+1}(e_j)$ can be carried out via plain recurrences. Indeed, using $w_{2i+1}(z) = w_{i+1}(z^2)/(z + e_{2i})$ and $w_{2(i+1)}(z) = w_{i+1}(z^2)$ for any i, deriving with respect to z, and using that $e_{2j+1} = -e_{2j}$ and $e_{2j}^2 = e_j$ for any j, we draw the following recurrence: $\tau_{0,0} = 1$ and for $i \geq 0$

$$\begin{aligned} \tau_{2i,2j} &= (1 + \gamma_{i,j})\tau_{i,j}/2 \\ \tau_{2i,2j+1} &= (1 - \gamma_{i,j})\tau_{i,j}/2 \end{aligned}, \quad \gamma_{i,j} = e_{2i}/e_{2j}, \quad j = 0, \dots, i, \tag{3.15}$$

and

$$\begin{aligned} \tau_{2i+1,2j} &= +\gamma_j \ \tau_{i,j}/2 \\ \tau_{2i+1,2j+1} &= -\gamma_j \ \tau_{i,j}/2 \end{aligned}, \quad \gamma_j = 1/e_{2j}, \quad j = 0, \dots, i. \tag{3.16}$$

Mapping to canonical basis: hierarchical Newton scheme can also be used if the primary goal is formulating $I_N[f]$ in the canonical basis. For instance, having computed $c_0, c_1, \dots, c_N$ s.t. $I_N[f] = \sum_{j=0}^{N} c_j w_j$, then $I_N[f] = \sum_{j=0}^{k} b_j z^j$, where $\boldsymbol{b} = A\boldsymbol{c}$ with $\boldsymbol{b} = (b_0, \dots, b_N)^\top$, $\boldsymbol{c} = (c_0, \dots, c_N)^\top$, and $A = (\alpha_{i,j})_{0 \le i,j \le N}$ is the change-of-basis matrix from $\{1, z, \dots, z^N\}$ into $\{w_0, w_1, \dots, w_N\}$. The mapping can also be carried hierarchically while computing Newton formulas. We simply add a step where we read the decomposition of w_k in basis $1, \dots, z^k$ and use it in order to distribute $c_k w_k$ over $1, \dots, z^k$.

We note that coefficients $\alpha_{i,j}$ satisfy a plain recurrence. In view of $w_{2j}(z) = w_j(z^2)$ and $w_{2j+1}(z) = w_j(z^2)(z + e_{2j+1})$ for any $j \ge 0$, we have $\alpha_{0,0} = 1$, $(\alpha_{0,1}, \alpha_{1,1}) = (-1, 1)$ and

$$\begin{aligned} \alpha_{2i,2j} &= \alpha_{i,j}, & \alpha_{2i+1,2j} &= 0, \\ \alpha_{2i,2j+1} &= \alpha_{i,j} e_{2j+1}, & \alpha_{2i+1,2j+1} &= \alpha_{i,j}, \end{aligned} \qquad j \ge 1, i = 0, \dots, j, \tag{3.17}$$

with the convention $\alpha_{k+1,k} = 0$.

Remark 3.8. It is easily verified that the recurrences and computations identified in this section are unchanged if $E = (e_j)_{j \ge 0}$ is any sequence defined by $(e_0, e_1) = (1, -1)$ and $(e_{2j}, e_{2j+1}) = (\sqrt{e_j}, -\sqrt{e_j})$ for $j \ge 1$ (where $\sqrt{e_j}$ is either of the square roots of e_j). All such sequences are instance of Leja sequences on the unit disk.

The bit-reversed sequence $E = (e_k)_{k \ge 0}$, defined by (3.2), has a particular property. In terms of the Van der Corput sequence $(\varepsilon_j)_{j \ge 0}$, we have that $e_k = e^{\mathbf{i}2\pi\varepsilon_k}$ for any k. In view of (2.10), this implies that $e_k = e^{\mathbf{i}2\pi \times \pi_n(k)/2^n}$ if $k < 2^n$, hence

$$e_{\pi_n(k)} = \exp\Big(\mathbf{i}\frac{2\pi k}{2^n}\Big), \qquad n \ge 0, \qquad 0 \le k \le 2^n - 1. \tag{3.18}$$

For $n \ge 0$ fixed, the permuted set $\{e_i\}_{i \in \mathcal{J}_n}$ is simply equal to $\{e^{\mathbf{i}2\pi k/2^n}\}_{k=0}^{2^n-1}$, the set of regular "non-permuted" 2^n-roots of unity in this order. Matrices $\big(p_j(e_i)\big)_{i,j \in \mathcal{J}_n}$ are merely regular Vandermonde type matrices $(p_j(\rho_i))_{0 \le i,j \le 2^n-1}$ but having only their columns permuted according to π_n. In the 3-steps procedure implementing interpolation at roots of unity, we simply have $y_j = f(e^{\mathbf{i}2\pi j/2^n})$. Also $\{e_{2i}\}_{i \in \mathcal{J}_n}$ is equal to $\{e^{\mathbf{i}2\pi k/2^{n+1}}\}_{k=0}^{2^n-1}$ since it is the first half of $\{e_i\}_{i \in \mathcal{J}_{n+1}}$. Computations involving the bit-reversed sequence are of course better outlined using non-permuted indexing and cast in a classical FFT framework. However, for the sake of generality, see Remark 3.8, we opted for permuted indexing.

4. CHEBYSHEV ABSCISSAS OF FIRST KIND

In this section, Ω is the unit interval $[-1, 1]$. For $k \ge 1$ fixed, we consider the set of k roots of Chebyshev polynomial T_k, i.e. $\Xi_k := \{\xi_0, \dots, \xi_{k-1}\}$ with $\xi_i = \cos(\theta_i)$, $\theta_i := \frac{2i+1}{2k}\pi$. We then consider the semi-definite inner product $\langle f, g \rangle_k = (1/k) \sum_{i=0}^{k-1} f(\xi_i) g(\xi_i)$. This product is definite over $\mathbb{P}_{k-1}[X]$ for which $T_0, \sqrt{2}T_1, \dots, \sqrt{2}T_{k-1}$ form an orthonormal basis. In view of (1.6),

$$I_{\Xi_k}[f] = \langle f, T_0 \rangle_k + 2 \sum_{j=1}^{k-1} \langle f, T_j \rangle_k T_j, \tag{4.1}$$

is the interpolation operator associated with Ξ_k. The Vandermonde matrix $C = (T_j(\xi_i))_{0 \le i,j \le k-1}$, which is the change-of-basis matrix from Lagrange basis into the

Chebyshev basis, is up to normalizing of columns the DCT-III matrix, see [13]. Its inverse is $\widetilde{C}^\top/k$ where $\widetilde{C} = C \times \mathrm{diag}[1, 2, \dots, 2]$, i.e. the columns of C, except the first one, get multiplied by 2.

Remark 4.1. In view of (4.1), Lagrange polynomials have plain formulas. Indeed, $I_{\Xi_k}[f] = \sum_{i=0}^{k-1} f(\xi_i) l_i(x)$ with $l_i(x) := (1+2\sum_{j=1}^{k-1} T_j(\xi_i)T_j)/k$. Moreover, by simple trigonometry, $l_i(x) = (d_{k-1}(\theta-\theta_i) + d_{k-1}(\theta+\theta_i))/(2k)$ for $x = \cos(\theta)$, where d_{k-1} is the Dirichlet kernel of order $k-1$, i.e. $d_{k-1}(\theta) = 1 + 2\sum_{j=1}^{k-1}\cos(j\theta) = \sin((2k-1)\theta/2)/\sin(\theta/2)$.

The previous is of course applicable with sets of roots Ξ_{2^n}. Such sets are in addition symmetric with respect to 0 and related by a recurrence, i.e. $\Xi_{2^n} = \{2\xi^2 - 1 : \xi \in \Xi_{2^{n+1}}\}$. We will be able, as in the complex setting, to derive fast change-of-basis matrices.

In order not to overload proofs by reproducing the arguments used in §2, we make the remark below. We recall that matrices J_{2^n}, $\widetilde{J}_{2^n}$ and their permuted variant Q_n and $\widetilde{Q}_n$ are introduced in (2.13).

Remark 4.2. For $x_0, \dots, x_{2^n-1}$ arbitrary, matrices $X = (T_{2j}(x_i))_{0\le i,j\le 2^n-1}$, $Y = (T_{2j+1}(x_i))_{0\le i,j\le 2^n-1}$ and $D = \mathrm{diag}[(2x_i)_{0\le i\le 2^n-1}]$ are related in two ways. First, since $T_1(x) = xT_0(x)$ and $T_{2j-1}(x) + T_{2j+1}(x) = 2xT_{2j}(x)$ for any $j \ge 1$, then $Y\widetilde{J}_{2^n}^\top = DX$. Also, since $2xT_{2j+1}(x) = T_{2j}(x) + T_{2(j+1)}(x)$ for any $j \ge 0$ then $DY - XJ_{2^n}$ consists only in zero columns except the last one which is equal to $(T_{2^{n+1}}(x_0), \dots, T_{2^{n+1}}(x_{2^n-1}))^\top$. If now we consider permuted matrices $X = (T_{2j}(x_i))_{i,j\in\mathcal{J}_n}$, $Y = (T_{2j+1}(x_i))_{i,j\in\mathcal{J}_n}$ and $D = \mathrm{diag}[(2x_i)_{i\in\mathcal{J}_n}]$, then $Y\widetilde{Q}_n^\top = DX$ and $DY - XQ_n$ consists only in zero columns except for the last which is equal to $(T_{2^{n+1}}(x_j))_{j\in\mathcal{J}_n}$. The latter is justified by the fact that the last element in $\mathcal{J}_n$ is $2^n - 1$ for any $n \ge 0$. We note in particular that if $x_0, \dots, x_{2^n-1}$ are all roots of $T_{2^{n+1}}$, then $DY = XQ_n$.

In order to fully exploit the recurrence identified on the sets Ξ_{2^n}, we re-define them via: $\Xi_1 = \{0\}$, and

$$\Xi_{2^{n+1}} = \left\{\sqrt{\frac{\xi+1}{2}}, -\sqrt{\frac{\xi+1}{2}} : \xi \in \Xi_{2^n}\right\}, \qquad n \ge 0. \tag{4.2}$$

We will subsequenly write $\Xi_{2^m} = \{\xi_{m,0}, \dots, \xi_{m,2^m-1}\}$ taking this ordering into account. In particular, there holds

$$\begin{array}{c} \xi_{n+1,2i+1} = -\xi_{n+1,2i} \\ T_2(\xi_{n+1,2i+1}) = T_2(\xi_{n+1,2i}) = \xi_{n,i} \end{array}, \qquad n \ge 0, i = 0, \dots, 2^n - 1. \tag{4.3}$$

For $n \ge 0$ fixed, we introduce $2^n \times 2^n$ matrices

$$V_n := \big(T_j(\xi_{n,i})\big)_{i\in\mathcal{J}_n, j\in\mathcal{J}_n} \qquad V_n^{[h]} := \big(H_j(\xi_{n,i})\big)_{i\in\mathcal{J}_n, j\in\mathcal{J}_n}, \tag{4.4}$$

and $2^n \times 2^n$ diagonal matrices $D_n = \mathrm{diag}[(2\xi_{n+1,2i})_{i\in\mathcal{J}_n}]$.

Proposition 4.3. *There holds $V_0 = [1]$, and for $n \ge 0$*

$$V_{n+1} = \begin{bmatrix} V_n & +D_n^{-1}V_nQ_n \\ V_n & -D_n^{-1}V_nQ_n \end{bmatrix}, \quad V_{n+1} = \begin{bmatrix} V_n & +D_nV_n\widetilde{Q}_n^{-\top} \\ V_n & -D_nV_n\widetilde{Q}_n^{-\top} \end{bmatrix}. \tag{4.5}$$

Proof. We use the blocks representation as in the proof of Proposition 3.1. Identities $T_{2j}(-x) = T_{2j}(x) = T_j(T_2(x))$ and $T_{2j+1}(-x) = -T_{2j+1}(x)$ for any j, combined with (4.3) imply $X_2 = X_1 = V_n$ and $Y_2 = -Y_1$. We conclude using Remark 4.2 which implies $Y_1 \widetilde{Q}_n^\top = D_n X_1$ and $D_n Y_1 = X_1 Q_n$. $\square$

The matrix V_n is equal to $(T_j(\xi_{n,i}))_{0 \le i,j \le 2^n-1}$ having its rows/columns permuted according to π_n. Since the leading row/column are not permuted in the process, then $V_n^{-1} = \widetilde{V}_n^\top / 2^n$ where $\widetilde{V}_n = V_n \times \mathrm{diag}[1, 2, \dots, 2]$. Using this and $\widetilde{Q}_n = \mathrm{diag}[2, 1, 1, \dots] Q_n$ for any $n \ge 1$, or by inverting the recurrence identified above, it is immediate to derive a recurrence for matrices V_n^{-1}. For the sake of consistency, we formulate it for $V_n^{-\top} = (V_n^{-1})^\top$.

Proposition 4.4. *There holds* $V_0^{-\top} = [1]$, *and for* $n \ge 0$

$$V_{n+1}^{-\top} = \frac{1}{2} \begin{bmatrix} V_n^{-\top} & +D_n V_n^{-\top} Q_n^{-\top} \\ V_n^{-\top} & -D_n V_n^{-\top} Q_n^{-\top} \end{bmatrix}, \tag{4.6}$$

$$V_{n+1}^{-\top} = \frac{1}{2} \begin{bmatrix} V_n^{-\top} & +D_n^{-1} V_n^{-\top} \widetilde{Q}_n \\ V_n^{-\top} & -D_n^{-1} V_n^{-\top} \widetilde{Q}_n \end{bmatrix}. \tag{4.7}$$

The same arguments used in order to derive (4.5) apply to matrices $V_n^{[h]}$. Having said that, the plain recurrence (2.4) on the hierarchical basis $(H_j)_{j \ge 0}$ implies simpler recurrences.

Proposition 4.5. *There holds* $V_0^{[h]} = (V_0^{[h]})^{-\top} = [1]$, *and for* $n \ge 0$

$$V_{n+1}^{[h]} = \begin{bmatrix} V_n^{[h]} & +D_n V_n^{[h]} \\ V_n^{[h]} & -D_n V_n^{[h]} \end{bmatrix}, \tag{4.8}$$

$$(V_{n+1}^{[h]})^{-\top} = \frac{1}{2} \begin{bmatrix} (V_n^{[h]})^{-\top} & +D_n^{-1} (V_n^{[h]})^{-\top} \\ (V_n^{[h]})^{-\top} & -D_n^{-1} (V_n^{[h]})^{-\top} \end{bmatrix}. \tag{4.9}$$

The remark following Proposition 3.2 applies here too. For any $n \ge 0$,

$$(V_n^{[h]})^{-\top} = \frac{1}{2^n} \big(1/H_j(\xi_{n,i})\big)_{\substack{i \in \mathcal{J}_n \\ j \in \mathcal{J}_n}}. \tag{4.10}$$

Remark 4.6. Identity (4.10) also holds if indexing $i, j \in \mathcal{J}_n$ is reversed back to $0 \le i, j \le 2^n - 1$ for both matrices. Also, if we consider non permuted abscissas $\xi_i = \cos(\theta_i)$, $\theta_i := \frac{2i+1}{2 \times 2^n} \pi$, the above shows that the inverse of $(H_j(\xi_i))_{0 \le i,j \le 2^n}$ is equal to the transpose of $\frac{1}{2^n} (1/H_j(\xi_i))_{0 \le i,j \le 2^n}$.

Now, for n fixed, we let $W_{n,0}, \dots, W_{n,2^n-1}$ be Newton polynomials associated with Ξ_{2^n} according to $W_{n,0} \equiv 1$ and $W_{n,j}(x) = \prod_{i=0}^{j-1} 2(x - \xi_{n,i})$. Here we have multiplied monic Newton polynomials w_j by 2^j. This yields more notational clarity and grants numerical stability since $1/2$ is the capacity of $[-1, 1]$. The recurrence in (4.3) combined with $T_2(x) = 2x^2 - 1$, yields

$$\begin{aligned} W_{n+1,2j} \quad (x) &= W_{n,j}(T_2(x)) \\ W_{n+1,2j+1}(x) &= W_{n,j}(T_2(x)) \times 2(x - \xi_{n+1,2j}) \end{aligned}, \tag{4.11}$$

for $j = 0, \dots, 2^n - 1$. In other words, plain recurrences are implied across orders 2^n for Newton polynomials associated with the permuted sets Ξ_{2^n}.

Similarly to the complex setting, we can derive factorizations as (3.3). One can verify by induction that for any $n \geq 0$ and any $k = \sum_{j=0}^{n-1} a_j 2^j$,

$$W_{n,k}(x) = \prod_{\substack{j=0 \\ a_j=1}}^{n-1} 2\big(T_{2^j}(x) + T_{2^j}(\xi_{n,k})\big). \tag{4.12}$$

By developing the product, we write $W_{n,k}(x) = \sum_{l \preceq k} H_{k-l}(\xi_{n,k}) H_l(x)$ where $l \preceq k$ in the sense of binary expansions, as explained following (3.3).

We introduce $2^n \times 2^n$ matrices

$$V_n^{[w]} := \big(W_{n,j}(\xi_{n,i})\big)_{\substack{i \in \mathcal{J}_n, \\ j \in \mathcal{J}_n}} \tag{4.13}$$

and matrices D_n^+ and D_n^- by $D_n^{\pm} = (\pm 2\xi_{n+1,2i} - 2\xi_{n+1,2j})_{i,j \in \mathcal{J}_n}$. Combining (4.3) and (4.11), then proceeding as in complex setting, we derive recurrences similar to Proposition 3.3 and Proposition 3.4. We recall that $\odot$ is the Hadamard product hence $D_n^{\pm} \odot V_n^{[w]} = \pm D_n V_n^{[w]} - V_n^{[w]} D_n$.

Proposition 4.7. *There holds* $V_0^{[w]} = (V_0^{[w]})^{-\top} = [1]$, *and for* $n \geq 0$,

$$V_{n+1}^{[w]} = \begin{bmatrix} V_n^{[w]} & D_n^+ \odot V_n^{[w]} \\ V_n^{[w]} & D_n^- \odot V_n^{[w]} \end{bmatrix}. \tag{4.14}$$

$$(V_{n+1}^{[w]})^{-\top} = \frac{1}{2} \begin{bmatrix} (V_n^{[w]})^{-\top} + V_n' & +D_n^{-1}(V_n^{[w]})^{-\top} \\ (V_n^{[w]})^{-\top} - V_n' & -D_n^{-1}(V_n^{[w]})^{-\top} \end{bmatrix}, \tag{4.15}$$

where $V_n' = D_n^{-1}(V_n^{[w]})^{-\top} D_n$.

With $\mathcal{B}_n^{(t)} := \{T_j\}_{j \in \mathcal{J}_{2^n}}$, $\mathcal{B}_n^{(h)} := \{H_j\}_{j \in \mathcal{J}_{2^n}}$, permuted Chebyshev bases and $\mathcal{B}_n^{(l)} := \{l_{2^n,j}(x)\}_{j \in \mathcal{J}_{2^n}}$, $\mathcal{B}_n^{(w)} := \{W_{2^n,j}(z)\}_{j \in \mathcal{J}_{2^n}}$, Lagrange/Newton bases associated with Ξ_{2^n} (Ξ_{2^n} ordered according to construction (4.2)) then also permuted, we have basically derived the recurrences for change-of-basis matrices between $\mathcal{B}_n^{(l)}$ and each basis $\mathcal{B}_n^{(t)}$, $\mathcal{B}_n^{(h)}$, $\mathcal{B}_n^{(w)}$. The recurrences for change-of-basis matrices between $\mathcal{B}_n^{(t)}$ and $\mathcal{B}_n^{(w)}$ and between $\mathcal{B}_n^{(h)}$ and $\mathcal{B}_n^{(w)}$ can also be easily derived.

Proposition 4.8. *There holds:* $V_0^{[t \to w]} = V_0^{[w \to t]} = [1]$, *and for* $n \geq 0$

$$V_{n+1}^{[t \to w]} = \begin{bmatrix} V_n^{[t \to w]} & -V_n^{[t \to w]} D_n \\ \mathbf{0} & \widetilde{Q}_n^{\top} V_n^{[t \to w]} \end{bmatrix}, \tag{4.16}$$

$$V_{n+1}^{[w \to t]} = \begin{bmatrix} V_n^{[w \to t]} & D_n V_n^{[w \to t]} \widetilde{Q}_n^{-\top} \\ \mathbf{0} & V_n^{[w \to t]} \widetilde{Q}_n^{-\top} \end{bmatrix}. \tag{4.17}$$

Proof. $W_{n+1,2j}(x) = W_{n,j}(T_2(x))$ and $W_{n+1,2j+1}(x) = 2(x - \xi_{n+1,2j}) W_{n,j}(T_2(x))$ for any $j \in \mathcal{J}_n$. Using the exact same arguments used to prove (2.14), we derive the first recurrence. The second is a direct verification. □

Using the same arguments in combination with $H_{2i}(x) = H_i(T_2(x))$ and $H_{2i+1}(x) = 2xH_i(T_2(x))$ for any $i \geq 0$ or simply that $V_n^{[h\to w]} = V_n^{[h\to t]}V_n^{[t\to w]}$, we derive the recurrences for change-of-basis between bases $\mathcal{B}_n^{(h)}$ and $\mathcal{B}_n^{(w)}$.

Proposition 4.9. *There holds* $V_0^{[h\to w]} = V_0^{[w\to h]} = [1]$, *and for* $n \geq 0$

$$V_{n+1}^{[h\to w]} = \begin{bmatrix} V_n^{[h\to w]} & -V_n^{[h\to w]}D_n \\ \mathbf{0} & V_n^{[h\to w]} \end{bmatrix}, \tag{4.18}$$

$$V_{n+1}^{[w\to h]} = \begin{bmatrix} V_n^{[w\to h]} & +D_nV_n^{[w\to h]} \\ \mathbf{0} & V_n^{[w\to h]} \end{bmatrix}. \tag{4.19}$$

For the sake of completeness, we sketch out how the numerous results in this section can be implemented. We let $n \geq 0$ fixed, and consider the set $\Xi_{2^n} = \{\xi_j = \cos(\theta_j) : j = 0, \ldots, 2^n - 1\}$, where $\theta_j = (2j+1)\pi/2^{n+1}$. We let $f : [-1,1] \to \mathbb{R}$ be a function and denote by $I_{\Xi_{2^n}}[f]$ the polynomial of degree $\leq 2^n - 1$ interpolating f over Ξ_{2^n}. We distinguish:

Lagrange interpolation formula: In view of Remark 4.1 pertaining to Dirichlet kernel d_{2^n-1}, for $x = \cos(\theta)$

$$I_{\Xi_{2^n}}[f](x) = \frac{1}{2}\Big(K_{2^n}(\theta) + K_{2^n}(-\theta)\Big), \tag{4.20}$$

where $K_{2^n}(\theta) = \dfrac{1}{2^n}\displaystyle\sum_{i=0}^{2^n-1} f(\xi_i)d_{2^n-1}(\theta - \theta_i)$.

Interpolation formula in Chebyshev bases: We let $\langle\cdot,\cdot\rangle_{2^n}$ be the semi-definite inner product associated with Ξ_{2^n}. We have

$$\begin{aligned} I_{\Xi_{2^n}}[f] &= \langle f, 1\rangle_{2^n} + 2\sum_{j=1}^{2^n-1}\langle f, T_j\rangle_{2^n}\ T_j \\ &= \langle f, 1\rangle_{2^n} + \sum_{j=1}^{2^n-1}\langle f, \frac{1}{H_j}\rangle_{2^n}\ H_j. \end{aligned} \tag{4.21}$$

The first formula is (4.1) while the second is implied from Remark 4.6. The coefficients $\langle f, T_j\rangle_{2^n}$, for $j = 0, \ldots, 2^n - 1$, are the coordinates of $A^\top \boldsymbol{y}/2^n$ if we consider $A = (T_j(\xi_i))_{0\leq i,j\leq 2^n-1}$ and $\boldsymbol{y} = (f(\xi_0), \ldots, f(\xi_{2^n-1}))^\top$. The same can be said for coefficients $\langle f, 1/H_j\rangle_{2^n}$ with $A = (1/H_j(\xi_i))_{0\leq i,j\leq 2^n-1}$. Having "optimally" computed matrix $A^\top$, computing the desired coefficients is merely a matrix-vector product.

Fast formulas in permuted bases: In general, $I_{\Xi_{2^n}}[f] = \sum_{j=0}^{2^n-1} c_jp_j$ where $p_0, \ldots, p_{2^n-1}$ is any basis of $\mathbb{P}_{2^n-1}[X]$ and $\boldsymbol{c} = (c_0, \ldots, c_{2^n-1})^\top$ given by $\boldsymbol{c} = ((p_j(\xi_i))_{0\leq i,j\leq 2^n})^{-1}\boldsymbol{y}$ with $\boldsymbol{y}$ as above. If we rather consider that Ξ_{2^n} is ordered $\{\xi_{n,0}, \ldots, \xi_{n,2^n-1}\}$ as in (4.2) and $\boldsymbol{y} = (f(\xi_{n,0}), \ldots, f(\xi_{n,2^n-1}))^\top$ then $\boldsymbol{c} = ((p_j(\xi_{n,i}))_{0\leq i,j\leq 2^n})^{-1}\boldsymbol{y}$. In particular $\boldsymbol{c}' = P_n^{-1}\boldsymbol{c}$ and $\boldsymbol{y}' = P_n^{-1}\boldsymbol{y}$ satisfy:

- if $p_j = T_j$, then $\boldsymbol{c}' = V_n^{-1}\boldsymbol{y}'$. Recurrences in (4.7) can be used.
- if $p_j = H_j$, then $\boldsymbol{c}' = (V_n^{[h]})^{-1}\boldsymbol{y}'$. The recurrence in (4.9) is used.

- if $p_j = W_{n,j}$, then $\boldsymbol{c}' = (V_n^{[w]})^{-1}\boldsymbol{y}$. The recurrence in (4.15) is used.

The second recurrence in (4.7) is most appropriate in case of Chebyshev basis as it involves $\widetilde{Q}_n$ not its inverse. Obtaining $\boldsymbol{y}'$ from $\boldsymbol{y}$ and $\boldsymbol{c}$ from $\boldsymbol{c}'$ is immediate, $y'_j = y_{\pi_n(j)}$ and $c_j = c_{\pi_n(j)}$ for $j = 0, \dots, 2^n - 1$.

Newton interpolation formulas: We let again $\Xi_{2^n} = \{\xi_{n,0}, \dots, \xi_{n,2^n-1}\}$ be ordered as in (4.2) and I_j for $j = 0, \dots, 2^n - 1$ be Lagrange interpolation operators associated with the $\{j+1\}$-sections of Ξ_n (hence $I_{\Xi_{2^n}} = I_{2^n-1}$). We will consider $f(\xi_{n,0}), \dots, f(\xi_{n,2^n-1})$ as sequential queries of f. Assuming we have computed barycentric coefficients $\tau_{i,j}^{(n)} := 2/W'_{n,i+1}(\xi_{n,j})$ for all i, j with $i = 0, \dots, 2^n - 1$ and $j = 0, \dots, i$, we proceed as detailed in the complex setting. We query f at $\xi_{n,k}$, compute a new Newton coefficient $c_k = \sum_{j=0}^{k} \tau_{k,j}^{(n)} f(\xi_{n,j})$, and update $I_k[f] = I_{k-1}[f] + c_k W_{n,k}$, one query at a time. We may be able to early stop the approximation process at some k as soon as a prescribed convergence criterion is satisfied.

Unlike the complex setting, barycentric coefficients are now in addition indexed by n. However, their computation is similar. Using recurrences (4.11), deriving with respect to x, and then using recurrences (4.3), we draw the following recurrence: for any $m \geq 0$, $i = 0, \dots, 2^m - 1$ and $j = 0, \dots, i$,

$$\begin{array}{c} \tau_{2i,2j}^{(m+1)} = (1+\gamma_{i,j})\ \tau_{i,j}^{(m)}/2 \\ \tau_{2i,2j+1}^{(m+1)} = (1-\gamma_{i,j})\ \tau_{i,j}^{(m)}/2 \end{array}, \qquad \gamma_{i,j} = \frac{\xi_{m+1,2i}}{\xi_{m+1,2j}}, \tag{4.22}$$

$$\begin{array}{c} \tau_{2i+1,2j}^{(m+1)} = +\gamma_j\ \tau_{i,j}^{(m)}/2 \\ \tau_{2i+1,2j+1}^{(m+1)} = -\gamma_j\ \tau_{i,j}^{(m)}/2 \end{array}, \qquad \gamma_j = \frac{1}{2\xi_{m+1,2j}}, \tag{4.23}$$

with $\tau_{0,0}^{(m)} = 1$ for any $m \geq 0$.

As we have already seen in §2 and §3, mapping a final approximation $I_N[f]$ or hierarchical approximations $I_k[f]$ to Chebyshev basis can be carried without difficulty if the change-of-basis matrix from $T_0, T_1, \dots, T_{2^n-1}$ into $W_{n,0}, W_{n,j}, \dots, W_{n,2^n-1}$ is already precomputed. A recurrence for the coefficients $\beta_{i,j}^{(n)}$ s.t. $W_{n,j} = \sum_{i=0}^{j} \beta_{i,j}^{(n)} T_i$ is not difficult to derive.

Remark 4.10. The recurrences identified in this section are unchanged if in recurrence (4.2) we had $\Xi_{2^{m+1}} = \{\epsilon_\xi\sqrt{(\xi+1)/2}, -\epsilon_\xi\sqrt{(\xi+1)/2} : \xi \in \Xi_{2^m}\}$ with $\epsilon_\xi = \pm 1$. Indeed, if $\Xi_{2^n} = \{\xi_{n,0}, \dots, \xi_{n,2^n-1}\}$ taking into account such an ordering, then (4.3) and (4.11) stay valid. Changes are mainly reflected in diagonal matrices $D_n = \mathrm{diag}[(2\xi_{n+1,2i})_{i\in\mathcal{J}_n}]$ and are propagated to all matrices of interest.

5. Chebyshev abscissas of second kind and $\Re$-Leja sequences

In this section, Ω is the unit interval $[-1, 1]$. For $k \geq 2$ fixed, we consider the set of k roots of polynomial $T_k - T_{k-2}$, i.e. $\widetilde{\Xi}_k := \{\widetilde{\xi}_0, \dots, \widetilde{\xi}_{k-1}\}$ with $\widetilde{\xi}_j := \cos(\frac{j\pi}{k-1})$. We then consider the semi-definite inner product $\langle f, g\rangle_k = \sum_{i=0}^{\prime\prime k-1} f(\widetilde{\xi}_j) g(\widetilde{\xi}_j)/(k-1)$, with $\sum''$ meaning that $f(1)g(1)$ and $f(-1)g(-1)$ are halved. This product is definite over $\mathbb{P}_{k-1}[X]$ for which $T_0, \sqrt{2}T_1, \dots, \sqrt{2}T_{k-2}, T_{k-1}$ form an orthonormal basis. In particular, in view of (1.6)

$$I_{\widetilde{\Xi}_k}[f] = \langle f, T_0\rangle_k + \Big(2\sum_{j=1}^{k-2}\langle f, T_j\rangle_k T_j\Big) + \langle f, T_{k-1}\rangle_k T_{k-1}, \tag{5.1}$$

is the interpolation operator associated with $\widetilde{\Xi}_k$. The Vandermonde matrix $C = (T_j(\widetilde{\xi}_i))_{0\le i,j\le k-1}$ is up to normalizing of columns the DCT-I matrix, see [13]. Similarly to Remark 4.1, here also Lagrange polynomials have plain formulas and can be formulated using Dirichlet kernel.

We observe that the sets $\widetilde{\Xi}_k$ are symmetric w.r.t 0, all contain $+1$ and -1 (and 0 if k odd), and $\widetilde{\Xi}_k \subset \widetilde{\Xi}_{2k-1}$ with $\widetilde{\Xi}_k = \{2\xi^2 - 1 : \xi \in \widetilde{\Xi}_{2k-1}\}$ for any $k \ge 2$. This holds in particular with values $k = 2^n + 1$ for $n \ge 0$. Using the adequate re-ordering of sets $\widetilde{\Xi}_{2^n+1}$, one is able to enforce convenient recurrences on associated Newton polynomials. This is in fact immediate by re-defining via a recurrence: $\widetilde{\Xi}_2 = \{+1, -1\}$, and $\widetilde{\Xi}_{2^{n+1}+1} = \widetilde{\Xi}_{2^n+1} \wedge \Xi_{2^n}$ with Ξ_{2^n} consists in Chebyshev abscissas of order 2^n ordered according to construction (4.2). The sets implied by this construction are the sections of a fixed infinite sequence, a typical instance of the so-called $\Re$-*Leja sequences.*

The $\Re$-Leja sequences are defined by sequential projection into $[-1, 1]$, with repetition ruled out, of Leja sequences over the unit disk $\mathcal{U}$ initiated at 1. The process is detailed in its generality in [2, 3]. The bit-reversed sequence E defined in (3.2), when sequentially projected, yields a specific $\Re$-Leja sequence $(\cos(\theta_j))_{j\ge0}$ where angles are θ_j defined by recurrence: $(\theta_0, \theta_1, \theta_2) = (0, \pi, \pi/2)$ and

$$\theta_{2j-1} = \theta_j/2, \qquad \theta_{2j} = \theta_{2j-1} + \pi, \qquad\qquad j \ge 2. \tag{5.2}$$

The analysis in the present section applies to any sequence R defined by $R = \{+1, -1\} \wedge \Xi_1 \wedge \Xi_2 \wedge \dots$ where $\Xi_{2^{n+1}}$ is related to Ξ_{2^n} as described in Remark 4.10. In other words, any sequence $R = (r_j)_{j\ge0}$ generated by $(r_0, r_1, r_2) = (1, -1, 0)$, then $r_{2i-1} = \pm\sqrt{(1 + r_i)/2}$, $r_{2i} = -r_{2i-1}$, $i \ge 2$. All such sequences (comprising the two described above) are particular instances of $\Re$-Leja sequences and they all satisfy

$$\begin{array}{c} r_{2i-1} = -r_{2i} \\ T_2(r_{2i-1}) = T_2(r_{2i}) = r_i \end{array}, \qquad i \ge 2. \tag{5.3}$$

The first property is also shared by all $\Re$-Leja sequences. The second is specific to the present context and is more relevant as it will promotes fast recurrences.

We let $R = (r_j)_{j\ge0}$ be any sequence as discussed, and introduce associated normalized Newton polynomials W_k by $W_0 \equiv 1$ and $W_k(x) = \prod_{i=0}^{k-1} 2(x - r_i)$ for $k \ge 1$. In particular, $W_1(x) = 2(x - 1)$, $W_2(x) = 4(x^2 - 1)$ and $W_3(x) = 8(x^3 - x)$. In view of (5.3), the following recurrence hold

$$\begin{array}{c} W_{2N-1}(x) = W_N(2x^2 - 1)/(2x) \\ W_{2N}(x) = W_{2N-1}(x) \times 2(x + r_{2N}) \end{array}, \qquad N \ge 2. \tag{5.4}$$

Vandermonde matrices of interest are $(T_j(r_i))$, $(H_j(r_i))$, and $(W_j(r_i))$ for $i, j \in \{0, \dots, k\}$. For values $k = 2^n$, by permuting such matrices considering $i \in \mathcal{I}'_n$, $j \in \mathcal{J}'_n$ with $\mathcal{I}'_n$, $\mathcal{J}'_n$ adequate re-ordering of $\{0, \dots, 2^n\}$, we can derive block-type recurrences. The latter can not however be as plain as the ones derived in previous sections. We choose not to address this.

We shall only address the problem of decomposing polynomials W_k in Chebyshev bases $(T_j)_j$ (and $(H_j)_j$). First, $W_1 = 2T_1 - 2T_0 = H_1 - 2H_0$, $W_2 = 2T_2 - 2T_0 = H_2 - 2H_0$, and $W_3 = 2T_3 - 2T_1 = H_3 - 2H_1$. In general, we can use a reproducing formula, e.g. the fact that $W_k = I_{\Xi_{2^m}}[W_k]$ for any $k < 2^m$ or $W_k = I_{\tilde{\Xi}_{2^m}}[W_k]$ for any $k < 2^m + 1$, combined with fast analytical computations of these interpolations operators, which we have already identified. For example, in view of (4.21), for any $k \geq 3$

$$W_k = \langle W_k, 1\rangle + 2\sum_{j=1}^{k}\langle W_k, T_j\rangle T_j = \langle W_k, 1\rangle + \sum_{j=1}^{k}\langle W_k, 1/H_j\rangle H_j, \tag{5.5}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\Xi_{2^m}}$ and m is any integer such that $k < 2^m$. The sums are stopped at k since W_k has degree k. The sums are of course independent of m. The first sum is also valid with $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(\pi\sqrt{1-x^2})^{-1}dx$ for which $T_0, \sqrt{2}T_1, \sqrt{2}T_2, \dots$ is orthonormal. We propose to derive a recurrence on the first sum coefficients.

We denote by $\beta_{i,k}$ the coefficients s.t. $W_k = \sum_{i=0}^{k}\beta_{i,k}T_i$. In particular $\beta_{0,0} = 1$, $(\beta_{0,1}, \beta_{1,1}) = (-2, 2)$, and $(\beta_{0,2}, \beta_{1,2}, \beta_{2,2}) = (-2, 0, 2)$. In order to identify a recurrence, we will simply make use of (5.4), in particular the fact that W_{2N-1} are odd polynomials.

Proposition 5.1. *We have* $(\beta_{0,2}, \beta_{1,2}, \beta_{2,2}) = (-2, 0, 2)$*, then given* $N \geq 2$

- *for* $k = 2N - 1$*:* $\beta_{2i,k} = 0$ *and* $\beta_{2i+1,k} = \beta_{i,N} - \beta_{2i-1,k}$*,*
- *for* $k = 2N$*:* $\beta_{2i,k} = \beta_{i,N}$ *and* $\beta_{2i+1,k} = 2r_k\ \beta_{2i+1,k-1}$*,*

for any $i = 0, \dots, N$*. We use the convention* $\beta_{l,k} = 0$ *for* $l \notin \{0, \dots, k\}$*.*

Proof. We let $N \geq 2$ and $k = 2N - 1$. Since W_k is an odd polynomial, then $W_k = \sum_{i=0}^{N-1}\beta_{2i+1,k}\ T_{2i+1}$, hence $2xW_k(x) = \sum_{i=0}^{N-1}\beta_{2i+1,k}(T_{2i} + T_{2i+2})$. Identifying with $W_N(2x^2-1) = \sum_{i=0}^{N}\beta_{i,N}T_i(2x^2-1) = \sum_{i=0}^{N}\beta_{i,N}T_{2i}(x)$ yields the recurrence for $\beta_{i,k}$. As for coefficients $\beta_{i,2N}$, we simply use that $W_{2N}(x) = W_N(2x^2-1) + 2r_{2N}W_{2N-1}(x)$ and identification. □

As far as Chebyshev basis is concerned, we can use a different approach. Namely, $W_{2^n+1} = 2(T_{2^n+1} - T_{2^n-1})$, then for $k = 2^n + 1, \dots, 2^{n+1}$, we use $W_k = W_{2^n+1}W_{n,k-(2^n+1)} = 2(T_{2^n+1} - T_{2^n-1})W_{n,k-(2^n+1)}$ where $W_{n,j}$ are Normalized Newton polynomials associated with $(r_{2^n+1}, \dots, r_{2^{n+1}})$. Having the decomposition of $W_{n,k-(2^n+1)}$ in Chebyshev basis allows us to deduce that of W_k, by virtue of identity $2T_i(x)T_j(x) = T_{i+j}(x) + T_{|i-j|}(x)$. Having said that, the recurrences identified in Proposition 5 are already adequate and fast enough for our needs.

The computation and stability of hierarchical Newton formulas using the prescribed sequences R is discussed in details [4]. Having that $(\beta_{i,j})_{0\leq i,j\leq N}$ is already computed for N big enough, mapping hierarchical approximation $I_k[f]$ into Chebyshev basis is straightforward.

6. New type of $\Re$-Leja sequence

We introduce a new sequence R, enforcing "recurrence convenience" as in §3 and §4. We define $R = (r_j)_{j\geq 0}$ by $r_0 = \cos(2\pi/3) = -1/2$, $r_1 = -r_0$, then

$$r_{2i} = \sqrt{\frac{r_i+1}{2}}, \qquad r_{2i+1} = -r_{2i}, \qquad i \geq 1. \tag{6.1}$$

The choice $r_0 = -1/2$ is not arbitrary. It is the solution of $2x^2 - 1 = x$, other than 1. This simple construction insures the following identities

$$\begin{array}{c} r_{2i+1} = -r_{2i} \\ T_2(r_{2i+1}) = T_2(r_{2i}) = r_i \end{array}, \qquad i \geq 0, \tag{6.2}$$

which are similar to identities (4.3) but holds here for any $i \geq 0$.

We introduce normalized Newton polynomials $(W_k)_{k\geq 0}$ by $W_0 \equiv 1$ and $W_k(x) = \prod_{i=0}^{k-1} 2(x - r_i)$ for $k \geq 1$. We have,

$$\begin{array}{c} W_{2j}(x) = W_j(T_2(x)) \\ W_{2j+1}(x) = W_j(T_2(x)) \times 2(x - r_{2j}) \end{array}, \qquad j \geq 0. \tag{6.3}$$

Using induction, one can verify that for any $k = \sum_{j=0}^{n} a_j 2^j$,

$$W_k(x) = \prod_{\substack{j=0 \\ a_j=1}}^{n} 2\big(T_{2^j}(x) + T_{2^j}(r_k)\big). \tag{6.4}$$

By developing the product, we write $W_k(x) = \sum_{l \preceq k} H_{k-l}(r_k) H_l(x)$ where $l \preceq k$ in the sense of binary expansions as explained in previous sections. We note that since $T_{2^n}(r_{2^n}) = T_{2^{n-1}}(r_{2^{n-1}}) = \cdots = T_1(r_1) = 1/2$, then $W_{2^n}(x) = 2(T_{2^n}(x) + T_{2^n}(r_{2^n})) = 2T_{2^n}(x) + 1$. For $n \geq 0$ fixed, R_{2^n} the 2^n-section of R consists in the roots of $2T_{2^n} + 1$ permuted in some way.

As far a change-of-bases matrices are concerned, we are able to reproduce the analysis of §4 with the sequence R. We adopt the same notation and introduce for every $n \geq 0$ the $2^n \times 2^n$ matrices

$$V_n := \big(T_j(r_i)\big)_{\substack{i\in\mathcal{J}_n \\ j\in\mathcal{J}_n}}, \quad V_n^{[h]} := \big(H_j(r_i)\big)_{\substack{i\in\mathcal{J}_n \\ j\in\mathcal{J}_n}}, \quad V_n^{[w]} := \big(W_j(r_i)\big)_{\substack{i\in\mathcal{J}_n \\ j\in\mathcal{J}_n}}, \tag{6.5}$$

and the $2^n \times 2^n$ matrices

$$D_n = \operatorname{diag}[(2r_{2i})_{i\in\mathcal{J}_n}], \qquad D_n^{\pm} = (\pm 2r_{2i} - 2r_{2j})_{i\in\mathcal{J}_n, j\in\mathcal{J}_n}. \tag{6.6}$$

By inspection of §4, we see that the recurrences identified in propositions 4.3, 4.4 and 4.5 hold for the introduced matrices V_n and $V_n^{[h]}$. In particular, we infer that

$$(V_n^{[h]})^{-\top} = \frac{1}{2^n}\big(1/H_j(r_i)\big)_{\substack{i\in\mathcal{J}_n \\ j\in\mathcal{J}_n}}. \tag{6.7}$$

We note that the identity still holds if indexing $i, j \in \mathcal{J}_n$ is reversed back to $i, j \in \{0, \ldots, 2^n - 1\}$ for both $V_n^{[h]}$ and the matrix on the right hand side.

We have already noted that $W_{2^n}(x) = 2T_{2^n}(x) + 1$. In particular, the section R_{2^n} of R viewed as a set consists in the roots of $2T_{2^n} + 1$. In view of (6.7), the associated interpolation operator $I_{R_{2^n}}$ can be formulated in basis $H_0, H_1, \ldots$ as in (4.21) with now $\langle f, g\rangle_{2^n} = (1/2^n)\sum_{i=0}^{2^n-1} f(r_i)g(r_i)$.

The recurrences identified in propositions 4.7, 4.8, and 4.9 hold as well. In particular, they can be used in order to map Newton interpolation formulas into Chebyshev bases.

As far as hierarchical interpolation is concerned, the same analysis as in §3 can be invoked. First, plain recurrences are available for barycentric coefficients $\tau_{i,j} := 2/W'_{i+1}(r_j)$. Indeed, since $W_{2i+2}(x) = W_{i+1}(T_2(x))$ and $W_{2i+1}(x) = W_{i+1}(T_2(x))/(2x + 2r_{2i})$ for any $i \geq 0$, then deriving with respect to x, and using (6.2) we draw the following recurrence: $\tau_{0,0} = 1$, then for $i \geq 0$ and $j = 0, \dots, i$

$$\begin{array}{ll} \tau_{2i,2j} & = (1+\gamma_{i,j})\tau_{i,j}/2 \\ \tau_{2i,2j+1} & = (1-\gamma_{i,j})\tau_{i,j}/2 \end{array}, \qquad \gamma_{i,j} = r_{2i}/r_{2j}, \tag{6.8}$$

and

$$\begin{array}{ll} \tau_{2i+1,2j} & = +\gamma_j \ \tau_{i,j}/2 \\ \tau_{2i+1,2j+1} & = -\gamma_j \ \tau_{i,j}/2 \end{array}, \qquad \gamma_j = 1/(2r_{2j}). \tag{6.9}$$

Having the lower triangular matrix $\mathcal{T}_N = (\tau_{i,j})_{0 \leq i,j \leq N-1}$ for N large enough, we can query the target function f at r_k, compute a new Newton coefficient $c_k = \sum_{j=0}^{k} \tau_{k,j} f(r_j)$ and update $I_k[f] = I_{k-1}[f] + c_k W_k$ (with $I_{k-1}[f] \equiv 0$) one query at a time.

Decompositions $W_j = \sum_{i=0}^{j} \beta_{i,j} T_i$ are easily computed. Combining (6.3) with the arguments used in order to prove (2.6), we draw the following recurrence: $\beta_{0,0} = 1$, $(\beta_{0,1}, \beta_{1,1}) = (1, 2)$ and

$$\begin{array}{llr} \beta_{2i,2j} & = \beta_{i,j}, & \beta_{2i+1,2j} = 0, \\ \beta_{2i,2j+1} & = 2r_{2j+1}\beta_{i,j}, & \beta_{2i+1,2j+1} = \beta_{i,j} + \beta_{i+1,j}, \\ & & \text{(except for } \beta_{1,2j+1} = 2\beta_{0,j} + \beta_{1,j}) \end{array} \tag{6.10}$$

for $j \geq 1$ and $i = 0, \dots, j$. We use convention that $\beta_{k+1,k} = 0$.

As noted in all other sections, all the results are unchanged if in definition (6.1) we have considered $(r_0, r_1) = (-1/2, 1/2)$ and $r_{2i} = \epsilon_i \sqrt{(r_i + 1)/2}$, $r_{2i+1} = -r_{2i}$ for $i \geq 1$, where $\epsilon_i = \pm 1$.

7. Numerical experiment

We consider the ordinary generating function for Chebyshev polynomials, i.e. $\gamma(t, x) = \sum_{n=0}^{\infty} T_n(x)\, t^n$, defined for any $|t| < 1$ and explicitly given by $\gamma(t, x) = (1 - tx)/(1 - 2tx + t^2)$. We note that $\partial_t \gamma(t, x) = \sum_{n=1}^{\infty} T_n(x)\, n t^{n-1}$, and is explicitly given by

$$\partial_t \gamma(t, y) = \frac{-y}{1 - 2ty + t^2} + \frac{-(1 - ty)(2t - 2y)}{(1 - 2ty + t^2)^2}. \tag{7.1}$$

For ρ fixed $\partial_t \gamma(\rho, \cdot)$ has a slower converging Chebyshev series than $\gamma(\rho, \cdot)$.

We let $\rho = 0.9$ and consider $f(x) = \partial_t \gamma(\rho, T_3(x))$, in other words

$$f(x) = \sum_{n=1}^{\infty} n \rho^{n-1} \ T_{3n}(x), \tag{7.2}$$

which showcases a sparse Chebyshev series. We rewrite $f = \sum_{j=0}^{\infty} c_j T_j$, hence $c_{3n} = n\rho_1^{n-1}$, $c_{3n+1} = c_{3n+2} = 0$. We shall use formula (7.1) with $t = \rho$ and $y = T_{3n}(x) = 4x^3 - 3x$ for querying f.

We compute approximations $I_0[f], I_1[f], \ldots$ to f by virtue of hierarchical Newton interpolation scheme, which we map into a Chebyshev series. The sequence R from §6 is used. Basically, for increasing k we compute coefficients $b_{0,0}, \ldots, b_{0,k}$ such that $I_k[f] = \sum_{j=0}^{k} b_{k,j} T_j$.

We let $S_k[f]$ be the best polynomial approximation of degree $\leq k$ to f in $\mathcal{H} := L^2([-1,1], dx/(\pi\sqrt{1-x^2}))$, i.e. $S_k[f] = \sum_{n:3n\leq k} n\rho_1^{n-1} T_{3n}$. Since $T_0, \sqrt{2}T_1, \sqrt{2}T_2, \ldots$ is an orthonormal basis of $\mathcal{H}$, truncation errors $\|f - S_k[f]\|_{\mathcal{H}} = (\sum_{n:3n>k}(n\rho^{n-1})^2/2)^{1/2}$ can be explicitly formulated or computed to high precision. We will compare $\delta_k^t := \|f - S_k[f]\|_{\mathcal{H}}$ and $\delta_k^n := \|f - I_k[f]\|_{\mathcal{H}}$. We note that $(\delta_k^n)^2 = (\delta_k^t)^2 + \|I_k[f] - S_k[f]\|_{\mathcal{H}}^2$ and $\|I_k[f] - S_k[f]\|_{\mathcal{H}}^2 = \lambda_{0,k}^2 + (\lambda_{1,k}^2 + \cdots + \lambda_{k,k}^2)/2$ where $\lambda_{j,k} = c_j - b_{j,k}$. We plot below $\log_2(\delta_k^t)$ and $\log_2(\delta_k^n)$ versus $\log_2(k)$.

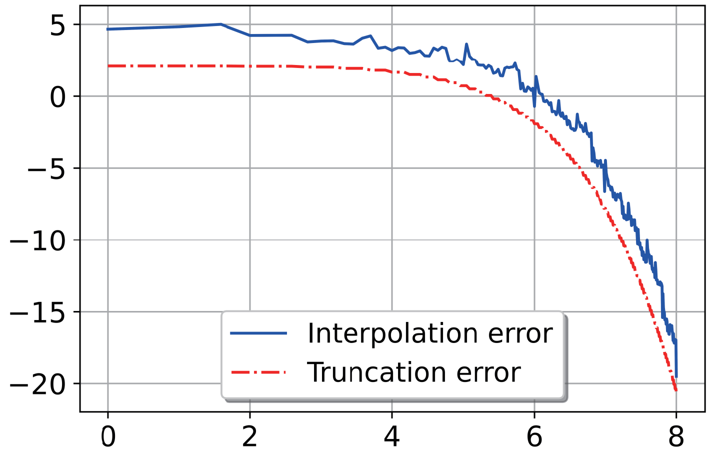

FIGURE 1. log-log plot of δ_k^t and δ_k^n.

We now let $\rho_i = \frac{0.8}{i}$ for $i = 1, \ldots, 4$, let $\rho = (\rho_1, \ldots, \rho_4)$ and define the function f for $\boldsymbol{y} = (y_1, \ldots, y_4) \in [-1,1]^4$ by

$$f(\boldsymbol{y}) = \prod_{j=1}^{4} \gamma(\rho_j, y_j) = \sum_{\nu \in \mathbb{N}^4} T_\nu(\boldsymbol{y}) \rho^\nu. \tag{7.3}$$

Notation $T_\nu(\boldsymbol{y}) = \prod_{j=1}^{4} T_{\nu_j}(y_j)$ and $\rho^\nu = \prod_{j=1}^{4} \rho_j^{\nu_j}$ for $\nu = (\nu_1, \ldots, \nu_4) \in \mathbb{N}^4$ is standard. The function f has an anisotropic dependance in the y_j reflected by the Chebyshev series. Queries of f are easily obtained since γ is explicit.

We implement sparse hierarchical interpolation as schematized in (1.14) in order to approximate f. The multi-index $\nu^{(i)}$ admitted in Λ_{i-1} is the multi-index with the largest Newton increment $\Delta_\nu[f] = c_\nu W_\nu$ in L_∞-norm, i.e. $|c_\nu| \prod_{j=1}^{4} \|W_{\nu_j}\|_{L_\infty}$. The sequence R defined in (6.1) is used. Univariate polynomials W_k are computed according to (6.3). Associated barycentric coefficients $\tau_{i,j}$ are computed according to (6.8) and (6.9). The Newton coefficients c_ν are computed as in (1.11). The approximation errors are $\delta_k^n := \|f - I_{\Lambda_{k-1}}[f]\|_\infty$ for $k \geq 0$.

The truncated series $\sum_{\nu\in\Lambda} \rho^\nu T_\nu$ of f yield

$$\|f - \sum_{\nu\in\Lambda} \rho^\nu T_\nu\|_\infty = \sum_{\nu\notin\Lambda} \rho^\nu = f(\mathbf{1}) - \sum_{\nu\in\mathbf{\Lambda}} \rho^\nu, \tag{7.4}$$

with the supremum attained at $\mathbf{1} = (1,1,1,1)$. We let $\Lambda_0^t \subset \Lambda_1^t \subset \dots$ be the nested lower sets associated with the k largest ρ^ν for $k = 1, 2, \dots$. The sets are obtained by exploring $\mathbb{N}^d$ adaptively starting from $\Lambda_0^t = \{\mathbf{0}\}$ and iteratively admitting in Λ_{k-1}^t the multi-index in its reduced margin with the largest ρ^ν. This is also as schematized in (1.14) except the admission criterion is straightforward. Formula (7.4) allows us to inductively compute the decreasing sequence $(\delta_k^t)_{k\geq 1}$ with $\delta_k^t := \|f - \sum_{\nu\in\Lambda_{k-1}^t} \rho^\nu T_\nu\|_\infty$.

We plot and compare δ_k^n and δ_k^t versus $\#(\Lambda_{k-1}) = \#(\Lambda_{k-1}^t) = k$. The norm $\|\cdot\|_\infty$ in $\delta_k^n = \|f - I_{\Lambda_{k-1}}[f]\|_\infty$ is approximated by a maximum over 10^4 points randomly chosen in $[-1,1]^4$ prior to the execution of the interpolation algorithm.

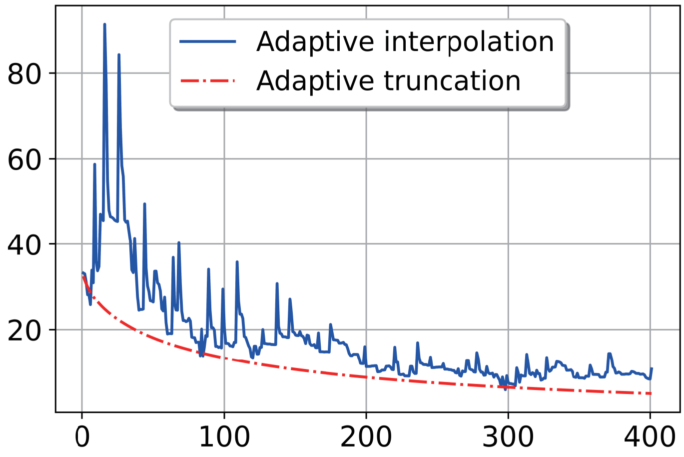

FIGURE 2. comparaison of δ_k^n and δ_k^t in k.

In both tests, Newton interpolation formulas yield a wiggly yet steady convergence. The scheme is challenged by dimension, hence the need for more refined admission criteria in procedure (1.14). One alternative can be to compute and exploit Chebyshev series produced by interpolation, i.e. $I_\Lambda[f] = \sum_{\nu\in\Lambda}^k b_{\Lambda,\nu}\, T_\nu$ in order to refine analysis. For instance, admitting in Λ the index ν yielding the largest change in $I_{\Lambda\cup\{\nu\}}[f]$ over $I_\Lambda[f]$ in L_2-norm, w.r.t Chebyshev measure $\prod_j dx_j/(\pi\sqrt{1-x_j^2})$.

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