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ON THE EXISTENCE OF WEAK SOLUTIONS FOR AN UNSTEADY ROTATIONAL SMAGORINSKY MODEL

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Abstract. In this paper we show that the rotational Smagorinsky model for turbulent flows, can be put, for a wide range of parameters in the setting of Bochner pseudo-monotone evolution equations. This allows to prove existence of weak solutions a) identifying a proper functional setting in weighted spaces and b) checking some easily verifiable assumptions, at fixed time. We also will discuss the critical role of the exponents present in the model (power of the distance function and power of the curl) for what concerns the application of the theory of pseudo-monotone operators.

1. Introduction

In this paper we study the unsteady rotational Smagorinsky model for incompressible turbulence

$$
\begin{aligned}
\partial_t \overline{\mathbf{v}} + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \operatorname{curl}\big(C_\alpha \ell^\alpha |\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}}\big) + \nabla \overline{q} &= \mathbf{f} && \text{in } (0,T)\times\Omega,\\
\overline{\boldsymbol{\omega}} &= \operatorname{curl}\overline{\mathbf{v}} && \text{in } (0,T)\times\Omega,\\
\operatorname{div}\overline{\mathbf{v}} &= 0 && \text{in } (0,T)\times\Omega,\\
\overline{\mathbf{v}} &= \mathbf{0} && \text{on } (0,T)\times\partial\Omega,\\
\overline{\mathbf{v}}(0) &= \overline{\mathbf{v}_0} && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, ℓ is the Prandtl mixing length, $\alpha > 0$ is a given exponent, $C_\alpha > 0$ is a calibration constant, $\overline{\mathbf{v}}$ is the mean velocity, $\overline{\boldsymbol{\omega}}$ is the mean vorticity, and $\overline{q}$ is the sum of the Bernoulli pressure of the fluid and certain potentials such as the turbulent kinetic energy and others. Here and in the sequel, for each smooth vector $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ we define as curl the vector

$$
(\operatorname{curl}\mathbf{u})_i := \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial u_j}{\partial x_k},
$$

where ϵ_{ijk} is the Levi–Civita totally anti-symmetric tensor.

Note that in the equations (1.1) the linear dissipative term $-\nu\Delta\overline{\mathbf{v}}$ is not present, since we are considering flows at very high Reynolds number, and then viscous effects are negligible compared to the Reynolds stresses. Note that the presence of the linear dissipative term, for any $\nu > 0$ will allow for some simplifications of

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the proofs. Nevertheless, to obtain results and estimates independent of $\nu > 0$, a treatment as the one we provide is requested.

According to standard assumptions (see (2.7) and (2.8) in Section 2.3), we will assume that ℓ behaves as the distance to the boundary. This means that $\ell(\mathbf{x}) \approx d(\mathbf{x})$ when $\mathbf{x} \in \Omega$ and $\mathbf{x}$ is close to the boundary $\partial\Omega$ (see (2.9) and (2.10) in what follows). As it is common in turbulence modeling, we assume that the flow fields are stochastic processes, and the bar operator stands for the expectation in the Reynolds decomposition $\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}'$, $\pi = \overline{\pi} + \pi'$, where π denotes the pressure and $\overline{\pi}$ the mean pressure (see Section 2.1 below, even if other choices are possible, as for instance denoting by the bar operator the long time-averaging).

The natural value of the parameter α is equal to 2 and this model is similar to the widely used Smagorinsky model, but the term $\operatorname{div}(\mathrm{C_s}\ell^2|\mathrm{D}\overline{\mathbf{v}}|\mathrm{D}\overline{\mathbf{v}})$ is replaced here by $\operatorname{curl}(C_2\ell^2|\overline{\boldsymbol{\omega}}|\overline{\boldsymbol{\omega}})$. The equivalence between both models can be understood for homogeneous isotropic turbulence, by the equality of the enstrophy $\overline{|\boldsymbol{\omega}|^2}$ to the total mean deformation $2\overline{|D\mathbf{v}|^2}$. The equivalence can be obtained by a straightforward generalization of [9, Lemma 4.7]. Then, according to this equality, in [9, Section 5.5.1] it is proved that the $-5/3$ Kolmogorov law yields to express the eddy viscosity as $\nu_T = C_2\ell^2|\overline{\boldsymbol{\omega}}|$. The rotational structure of the eddy diffusion is a peculiarity of the model which is suitable for high-speed flows with thin attached boundary-layers. The mathematical treatment of rotational models is one of the main theoretical contribution of this paper.

The numerical performance of this model in the steady state case has been initially tested by Baldwin and Lomax [2], so that this model is also known as the Baldwin–Lomax model. Numerical analysis foundations also in the statistical non-equilibrium setting can be found in [23]. It is important here to underline the fact that this model is a URANS (Unsteady Reynolds Averaged Navier-Stokes) model (see in [27] and the modeling carried out in Section 2 below).

The analytical properties of a steady version of this model have been recently studied in [3] in the setting of weighted Sobolev spaces. Some unsteady versions, with the presence of a dispersive term –which allows for a more classical treatment– have been recently studied in [5, 21].

The steady version can be treated within the standard theory of monotone operators, plus a localization argument, while the unsteady one requires a more delicate argument to deal with the precise choice of spaces and formulation of the problem. As we will prove, a proper definition of the functional setting will make system (1.1) to fit into the framework of evolution problems with Bochner pseudo-monotone operators, for which the theory has been recently developed by two of the authors in [13]. The theory developed in [13] represents an extension and an adaption to unsteady problems of the classical theory of pseudo-monotone operators from Brézis [6], [7], already described in the classical monograph of Lions [19]. Our main result is the following, which covers all possible positive powers of the distance function which are strictly smaller than the critical value $\alpha = 2$.

Theorem 1.1. *Let us suppose that $\ell(\mathbf{x}) = d(\mathbf{x}, \partial\Omega)$ and let $\alpha \in [0, 2)$, $0 < T < \infty$, $\overline{\mathbf{v}_0} \in L^2_\sigma(\Omega)$, and $\mathbf{f} \in L^{3/2}(0, T; (W^{1,3}_0(\Omega, d^\alpha)^*)$. Then, there exists a weak solution* *to the initial boundary value problem* (1.1) *such that*

$$\overline{\mathbf{v}} \in C([0,T];L^2_\sigma(\Omega)) \cap L^3(0,T;W^{1,3}_{0,\sigma}(\Omega,d^\alpha)),$$

and for all $t \in [0,T]$

$$\frac{1}{2}\|\overline{\mathbf{v}}(t)\|^2 + \int_0^t \int_\Omega C_\alpha d^\alpha(\mathbf{x})|\overline{\boldsymbol{\omega}}(s,\mathbf{x})|^3\,d\mathbf{x}\,ds = \frac{1}{2}\|\overline{\mathbf{v}_0}\|^2 + \int_0^t \langle \mathbf{f}, \overline{\mathbf{v}}\rangle_{W_0^{1,3}(\Omega,d^\alpha)} ds.$$

The limitation $\alpha < 2$ seems to be intrinsic to the problem due to the fact that d^α is not anymore a Muckenhoupt weight for $\alpha \geq 2$ (cf. Definition 4.4). Hence, for $\alpha \geq 2$ most of the analytical properties may fail, since we cannot ensure that the quantity from the energy estimates controls the (weighted) full gradient of the solution. For values of α larger or equal than 2, even the weak formulation, the density of smooth functions, and the meaning of the boundary conditions may fail; the solution of the problem, if possible, would pass through the introduction of a more general setting, of very weak solutions.

In the last section we will also consider the existence for a family of problems with different powers of the vorticity in the turbulent stress tensor, still with the distance function raised to any exponent smaller than the critical one, cf. Thm. 5.5.

Plan of the paper. In Section 2 we derive the rotational Smagorinsky model from a classical turbulence modeling process, in Section 3 we define the notion of Bochner pseudo-monotone operators and we recall the main result for general evolutionary problems, in Section 4 we recall the main results on weighted spaces, which will be used to properly formulate the problem. Next in the final Section 5 we show how the hypotheses apply to problem (1.1), for relevant choices of the weight functions and discuss generalization and critical values of the parameters.

2. Modeling

2.1. Reynolds decomposition. Let us consider the Navier–Stokes equations (NSE in the sequel) written with the convective term in the rotational formulation:

$$\begin{aligned}
\mathbf{v}_t + \boldsymbol{\omega}\times\mathbf{v} - \nu\Delta\mathbf{v} + \nabla\left(\pi + \frac{|\mathbf{v}|^2}{2}\right) &= \mathbf{f} && \text{in } (0,T)\times\Omega,\\
\boldsymbol{\omega} &= \operatorname{curl}\mathbf{v} && \text{in } (0,T)\times\Omega,\\
\operatorname{div}\mathbf{v} &= 0 && \text{in } (0,T)\times\Omega,\\
\mathbf{v} &= \mathbf{0} && \text{on } (0,T)\times\partial\Omega,\\
\mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega,
\end{aligned} \tag{2.1}$$

where $\mathbf{v} = \mathbf{v}(t,\mathbf{x},\omega)$ is the velocity field, $\pi = \pi(t,\mathbf{x},\omega)$ the pressure, $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ the vorticity, $(t,\mathbf{x}) \in \mathbb{R}_+ \times \Omega$, $\omega \in X(\mathcal{B},P)$, where $X(\mathcal{B},P)$ is a given probability space on the space of initial data.

For instance, if $\mathbf{f} = \mathbf{0}$ (the argument can be adapted also to include a smooth enough external force) it holds that for each divergence-free element of $\mathbf{v}_0 \in H^{1/2}(\Omega)$ there exists a lower bound $T = T(\|\mathbf{v}_0\|_{1/2}) > 0$ for the life-span of the unique Fujita–Kato mild solution. Since the life-span can be estimated with the norm of the initial datum, by fixing $X = \overline{B(0,R)} \subseteq H^{1/2}(\Omega) \cap \{\nabla\cdot\mathbf{v} = 0\}$ for some $R > 0$, then the life-span is bounded from below by some $T_X > 0$. This means that for each $\mathbf{v}_0 \in X$, there exists a unique $\mathbf{v} \in C(0, T_X; H^{1/2}(\Omega))$ solution of the NSE.

We then introduce P, which is a probability measure on the Borel sets of X. More specifically, P can be constructed as limit of averages of Dirac measures as in [9] or the renormalized Lebesgue measure constructed from the Borel sets of X. The final result does not depend on the choice of P. Let us denote the expectation with a bar, hence

$$\overline{\mathbf{v}_0} = \int_X \mathbf{v}_0 \, dP(\mathbf{v}_0),$$

and

$$\overline{\mathbf{v}}(t, \mathbf{x}) = \int_X \mathbf{v}(t, \mathbf{x}, \mathbf{v}_0) \, dP(\mathbf{v}_0) \qquad \overline{\pi}(t, \mathbf{x}) = \int_X \pi(t, \mathbf{x}, \mathbf{v}_0) \, dP(\mathbf{v}_0).$$

More generally, for any field $\Psi = \partial_t \mathbf{v}, \boldsymbol{\omega}, \nabla \mathbf{v}, \Delta \mathbf{v}, \frac{|\mathbf{v}|^2}{2}...$, we can define the statistical mean as

$$\overline{\Psi}(t, \mathbf{x}) = \int_X \Psi(t, \mathbf{x}, \mathbf{v}_0) \, dP(\mathbf{v}_0),$$

and consequently we can perform the usual decomposition of Ψ as

$$\Psi = \overline{\Psi} + \Psi',$$

which is known as the Reynolds decomposition. The properties of the statistical averaging process imply (Reynolds rules) that for all $\Psi, \Theta \in X$

$$\partial_t \overline{\Psi} = \overline{\partial_t \Psi}, \qquad \nabla \overline{\Psi} = \overline{\nabla \Psi}, \qquad \overline{\Psi'} = 0, \qquad \overline{\overline{\Psi}\, \Theta} = \overline{\Psi}\, \overline{\Theta},$$

hence, taking the expectation of the NSE (2.1) yields

$$\begin{aligned} \overline{\mathbf{v}}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \overline{\boldsymbol{\omega}' \times \mathbf{v}'} - \nu \Delta \overline{\mathbf{v}} + \nabla \left(\overline{\pi} + \frac{|\overline{\mathbf{v}}|^2}{2} + \frac{\overline{|\mathbf{v}'|^2}}{2} \right) &= \overline{\mathbf{f}}, \\ \overline{\boldsymbol{\omega}} &= \operatorname{curl} \overline{\mathbf{v}}, \\ \operatorname{div} \overline{\mathbf{v}} &= 0, \\ \overline{\mathbf{v}}|_{\partial\Omega} &= \mathbf{0}, \\ \overline{\mathbf{v}}|_{t=0} &= \overline{\mathbf{v}_0}. \end{aligned} \tag{2.2}$$

The basic closure and modeling problems concern expressing $\overline{\boldsymbol{\omega}' \times \mathbf{v}'}$ in terms of averaged variables.

2.2. **Rotational Reynolds stress.** When taking the expectation of the NSE with the convective term written in the usual form, we get the term $\operatorname{div}(\overline{\mathbf{v}' \otimes \mathbf{v}'})$. The quantity $\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}$ is called the Reynolds stress and the Boussinesq assumption consists in assuming that

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_T D \overline{\mathbf{v}},$$

where $\nu_T \geq 0$ is an eddy viscosity which remains to be determined and modeled in terms of $\overline{\mathbf{v}}$. If we want to use such a Boussinesq assumption, we must express the turbulent stress (which is a vector in the rotational formulation)

$$\mathbf{s} := \overline{\boldsymbol{\omega}' \times \mathbf{v}'}$$

in terms of derivatives of mean quantities. This is similar to the approach used when modeling the more standard Reynolds stress tensor. We prove in what follows the following theorem

Theorem 2.1. *Assume that Ω is connected and of class C^1. Then, there exists a vector $\mathbf{a}^{(R)} = \mathbf{a}^{(R)}(t, \mathbf{x})$ and a scalar potential $\Phi = \Phi(t, \mathbf{x})$ such that*

$$\begin{aligned} \overline{\mathbf{v}}_t + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \operatorname{curl} \mathbf{a}^{(R)} - \nu \Delta \overline{\mathbf{v}} + \nabla(\overline{\pi} + \tfrac{1}{2}|\overline{\mathbf{v}}|^2 + k - \Phi) &= \overline{\mathbf{f}}, \\ \operatorname{div} \overline{\mathbf{v}} &= 0, \\ \overline{\mathbf{v}}|_{\partial\Omega} &= \mathbf{0}, \\ \overline{\mathbf{v}}|_{t=0} &= \overline{\mathbf{v}_0}, \end{aligned} \tag{2.3}$$

where $k = \frac{1}{2}\overline{|\mathbf{v}'|^2}$ is the turbulent kinetic energy.

Proof. Let $\mathbf{a}^{(\mathrm{R})}$ and Φ be given by:

$$\begin{aligned} \mathbf{a}^{(\mathrm{R})}(t, \mathbf{x}) &= \frac{1}{4\pi} \int_\Omega \frac{\operatorname{curl} \mathbf{s}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' + \frac{1}{4\pi} \int_{\partial\Omega} \frac{\mathbf{s}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \times d\sigma(\mathbf{x}'), \\ \Phi(t, \mathbf{x}) &= \frac{1}{4\pi} \int_\Omega \frac{\operatorname{div} \mathbf{s}(\mathrm{t}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' - \frac{1}{4\pi} \int_{\partial\Omega} \frac{\mathbf{s}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \cdot d\sigma(\mathbf{x}'). \end{aligned} \tag{2.4}$$

Therefore, by the Helmholtz–Hodge theorem, we have the relation

$$\overline{\boldsymbol{\omega}' \times \mathbf{v}'} = \operatorname{curl} \mathbf{a}^{(\mathrm{R})} - \nabla \Phi. \tag{2.5}$$

Inserting (2.5) into (2.2) gives (2.3). □

The vector $\mathbf{a}^{(\mathrm{R})}$, which is continuously and uniquely determined by formula (2.4), is called the rotational Reynolds stress tensor. From now on we write $\bar{q}$ instead of $\overline{\pi} + \frac{1}{2}|\overline{\mathbf{v}}|^2 + k - \Phi$.

2.3. **Closure assumption: Rotational Smagorinsky model.** In order to finish the modeling of turbulent quantities, it remains to link $\mathbf{a}^{(\mathrm{R})}$ to the mean vorticity $\overline{\boldsymbol{\omega}}$. Notice that $\mathbf{a}^{(\mathrm{R})}$ has the dimension of a squared velocity, while $\overline{\boldsymbol{\omega}}$ those of a frequency. Therefore, adapting the Boussinesq assumption to this case yields to assume

$$\mathbf{a}^{(\mathrm{R})} = \nu_T \, \overline{\boldsymbol{\omega}},$$

in which $\nu_T \geq 0$ is a quantity with the dimensions of a viscosity. According to the $-5/3$ Kolmogorov law and following [9, Section 5.5.1], we can assume (for an homogeneous and isotropic flow, in the limit $\nu \to 0$)

$$\nu_T = \nu_T(\ell, |\overline{\boldsymbol{\omega}}|),$$

where ℓ is the Prandtl mixing length. The dimensional analysis of the expression shows that a consistent expression is

$$\nu_T = C_\omega \ell^2 |\overline{\boldsymbol{\omega}}|, \tag{2.6}$$

with C_ω a dimensionless constant. This raises the question of the determination of ℓ. In the case of a flow over a plate, one finds in Obukhov [22] the following classical law:

$$\ell = \ell(z) = \kappa z, \tag{2.7}$$

where $z \geq 0$ is the distance from the plate and κ the von Kármán constant. The Van Driest formula [26] defines ℓ by:

$$\ell(z) := \kappa\, z\,(1 - \mathrm{e}^{-z/A}); \tag{2.8}$$

here A depends on the oscillations of the plate and on the kinematic viscosity ν, while $z \geq 0$ is again the distance from the plate.

According to these formula, we shall assume throughout the rest of the paper that the function $\ell : \overline{\Omega} \to \mathbb{R}^+$ is of class C^2 and satisfies the two following properties:

$$a)\ \ell(\mathbf{x}) \approx d(\mathbf{x}, \partial\Omega) \qquad \text{for } \mathbf{x} \text{ close to } \partial\Omega; \tag{2.9}$$

$$b)\ \forall\, K \subset\subset \Omega,\ \exists\, \ell_K > 0 \quad \text{s.t.} \quad \ell(\mathbf{x}) \geq \ell_K > 0 \quad \forall\, \mathbf{x} \in K, \tag{2.10}$$

where $d(\mathbf{x}, \partial\Omega)$ denotes the distance from the boundary. In practice, we could have directly assumed $\ell(\mathbf{x}) = d(\mathbf{x})$, i.e.,

$$\nu_T = C_\omega d^2 |\overline{\boldsymbol{\omega}}|. \tag{2.11}$$

2.4. **Generalised Rotational Smagorinsky models by dimensional analysis.** The analysis of the previous section can be put also in a more general framework of Large Eddy Simulation (LES) models, looking also at possible modifications of the parameters present in the expression of the turbulent (rotational) stress vector. Let $\ell_0 > 0$ be a typical length scale of the motion. For instance, in the case of a flow over a plate, one can take

$$\ell_0 = \frac{\nu}{v_*},$$

where ν is the kinematic viscosity and v_* is the so-called friction velocity (cf. [1]).

We consider (modulo introducing an appropriate non-dimensionalization of the equations) the following operator

$$\operatorname{curl}\left(\ell_0^{2-\alpha} \ell^\alpha |\overline{\boldsymbol{\omega}}| \overline{\boldsymbol{\omega}}\right) \tag{2.12}$$

with $\alpha \in [0, 2]$, which is degenerate at the boundary and for which the natural treatment is through scales of weighted Banach spaces.

We report some discussion about the relationships between the scaling of the weight and that of the power of the curl. In the framework of LES methods we show that even starting with

$$\nu_T = \ell_0^{2-\alpha} \ell^\alpha |\overline{\boldsymbol{\omega}}|^{p-2} \tag{2.13}$$

this determines a link between powers α and p. Nevertheless, in the last section we will also point out the limiting behavior of the exponent $p = 3$ present in model (1.1), when $p = \alpha + 1$.

If one thinks of a flow as composed of eddies of different sizes in different places, then in a region of large eddies the changes of velocity and its curl are both $\mathcal{O}(1)$ of the typical distance. In a region of smaller eddies the velocity changes over a distance of $\mathcal{O}$(eddy length scale), so the local deformation is $\mathcal{O}$(1/eddy length scale), cf. [4, § 3.3.2]. Hence, the rotational Smagorinsky model introduces a turbulent viscosity $\nu_T = (C\delta)^2 |\overline{\boldsymbol{\omega}}|$, where δ is the (local) smallest resolved scale, such that

$$\nu_T = \begin{cases} \mathcal{O}(\delta^2) & \text{in regions where } |\overline{\boldsymbol{\omega}}| = \mathcal{O}(1), \\ \mathcal{O}(\delta) & \text{in the smallest resolved scale where } |\overline{\boldsymbol{\omega}}| = \mathcal{O}(\delta^{-1}). \end{cases}$$

By extrapolation, motivated by experiments with central difference approximations to linear convection diffusion problems, the following alternate scaling has also been proposed (cf. [4] and Layton [17]) $\nu_T = (C\delta)^{p-1}|\mathbf{D}\overline{\mathbf{v}}|^{p-2}$, and we consider here the rotational counterpart

$$\nu_T = (C\delta)^{p-1}|\overline{\boldsymbol{\omega}}|^{p-2}, \qquad 1 < p < \infty,$$

which resembles general power laws for non-Newtonian fluids. The above choice of ν_T satisfies

$$\nu_T = \begin{cases} \mathcal{O}(\delta^p) & \text{in regions where } |\overline{\boldsymbol{\omega}}| = \mathcal{O}(1), \\ \mathcal{O}(\delta) & \text{in the smallest resolved scale where } |\overline{\boldsymbol{\omega}}| = \mathcal{O}(\delta^{-1}). \end{cases}$$

The justification of the presence of the critical value $p-1$ as power of the distance function can be done directly by dimensional arguments as in [3]. In fact, recall that both $\nabla\overline{\mathbf{v}}$ and $\overline{\boldsymbol{\omega}}$ have dimensions T^{-1}, where T is a time, and in (2.6) the turbulent viscosity $\nu_T = d^2|\overline{\boldsymbol{\omega}}| \sim L^2T^{-1}$ (where L is a length) has the dimensions of a viscosity. This is the only way to identify (by using just a typical length and the vorticity) a quantity with the dimensions of a viscosity. Introducing as third parameter as the friction velocity $v_* \sim LT^{-1}$, one can consider more general combinations. The outcome is to find a turbulent eddy viscosity of the following form

$$\nu_T = v_*^{\theta} d^{\alpha}|\overline{\boldsymbol{\omega}}|^{p-2},$$

for some constants θ, α, p. It turns out (cf. [3]) that the dimensions of this quantity are $\nu_T \sim L^{\theta+\alpha}T^{2-\theta-p}$, and to respect dimensions of the viscosity one has to fix

$$\theta = 3 - p \qquad \text{and} \qquad \alpha = p - 1.$$

A sound generalization of the rotational Smagorinsky model is then the one with rotational stress

$$\mathbf{S}(v_*, d, \overline{\boldsymbol{\omega}}) = C v_*^{3-p} d^{p-1}|\overline{\boldsymbol{\omega}}|^{p-2}\overline{\boldsymbol{\omega}},$$

and, after re-scaling, one can assume $Cv_*^{3-p} = 1$. Note that, even for different values of p, the power of the distance is always the critical one (in terms of analytical properties of the weight functions), since $d^{p-1} \notin A_p$, cf. Lemma 4.5.

In the last section we will show that from the point of view of mathematical properties, the turbulent eddy viscosity

$$\nu_T = d^{p-1}|\overline{\boldsymbol{\omega}}|^{p-2},$$

can be handled in terms of an existence theory by (pseudo)monotone operators only for $p \geq 3$. Hence, the exponent $p = 3$ plays for the weighted rotational operators, the same role that the exponent $p = 11/5$ plays for the usual p-NSE with stress tensor $S(\mathbf{D}\overline{\mathbf{v}}) = c|\mathbf{D}\overline{\mathbf{v}}|^{p-2}$.

From now and so far no risk of confusion occurs, we do not write the bar anymore.

3. Evolution equations in an abstract setting

As already claimed in the introduction, a proper setting to the rotational Smagorinsky model is that of pseudo-monotone evolution problems so we briefly recall the abstract existence result we will use on the sequel.

For the convenience of the reader, we recall the following definition.

Definition 3.1. Let X, Y be Banach spaces. An operator $A : X \to Y$ is called

(i) **bounded**, if for all bounded $M \subseteq X$, the image $A(M) \subseteq Y$ is bounded.
(ii) **coercive**, if $Y = X^*$ and $\lim\limits_{\|x\|_X \to \infty} \frac{\langle Ax, x\rangle_X}{\|x\|_X} = \infty$.
(iii) **pseudo-monotone**, if $Y = X^*$ and for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ from
$$x_n \overset{n\to\infty}{\rightharpoonup} x \text{ in } X,$$
$$\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle_X \le 0,$$
it follows that $\langle Ax, x - y\rangle_X \le \liminf_{n\to\infty} \langle Ax_n, x_n - y\rangle_X$ for all $y \in X$.

It is well-known that for each $f \in X^*$, the steady problem $Ax = f$ admits a solution if A is bounded, coercive and pseudo-monotone, see [6], [7]. A typical example of a pseudo-monotone operator is the sum of a hemi-continuous monotone and a compact operator. Recently, two of the authors in [13] developed an abstract framework for evolution problems, by using the concepts of Bochner pseudo-monotone and Bochner coercive operators to generalize the ideas of [19, Sec. 2.5], [16], [11], [12] and [24]. We want to access this theory for our concrete example. Therefore, for the remainder of this section, we assume that (V, H, id) is an evolution triple, i.e., V is a separable, reflexive Banach space, H a separable Hilbert space and V embeds densely into H. For $I := (0, T)$, $T \in (0, \infty)$, and $p \in (1, \infty)$, we set
$$\mathcal{X} := L^p(I, V) \qquad \text{and} \qquad \mathcal{Y} := L^\infty(I, H).$$
In this framework we have the following notion of a time derivative.

Definition 3.2. A function $\mathbf{u} \in \mathcal{X}$ has a **generalized time derivative** if there exists a function $\mathbf{w} \in L^{p'}(I, V^*)$ such that
$$-\int_I (\mathbf{u}(s), v)_H \varphi'(s)\, ds = \int_I \langle \mathbf{w}(s), v\rangle_V \varphi(s)\, ds$$
for every $v \in V$ and $\varphi \in C_0^\infty(I)$. Since such a function is unique, $\frac{d\mathbf{u}}{dt} := \mathbf{w}$ is well-defined. By
$$\mathcal{W} := W^{1,p,p'}(I, V, V^*) := \big\{\mathbf{u} \in \mathcal{X} \mid \exists\, \tfrac{d\mathbf{u}}{dt} \in L^{p'}(I, V^*)\big\},$$
we denote the **Bochner–Sobolev space** with respect to the evolution triple (V, H, id).

In the context of evolutionary problems, the following generalized notions of pseudo-monotonicity and coercivity (cf. Definition 3.1) are particularly relevant and useful.

Definition 3.3 (Bochner pseudo-monotonicity)**.** An operator $\mathcal{A} : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$ is said to be **Bochner pseudo-monotone** if for a sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subseteq \mathcal{X} \cap \mathcal{Y}$ from
$$\begin{aligned} \mathbf{u}_n &\overset{n\to\infty}{\rightharpoonup} \mathbf{u} && \text{in } \mathcal{X}. \\ \mathbf{u}_n &\overset{*}{\rightharpoonup} \mathbf{u} && \text{in } \mathcal{Y} \quad (n \to \infty), \\ \mathbf{u}_n(t) &\overset{n\to\infty}{\rightharpoonup} \mathbf{u}(t) && \text{in } H \quad \text{for a.e. } t \in I, \end{aligned}$$
and
$$\limsup_{n\to\infty} \langle \mathcal{A}\mathbf{u}_n, \mathbf{u}_n - \mathbf{u}\rangle_{\mathcal{X}} \le 0,$$

it follows that $\langle \mathcal{A}\mathbf{u}, \mathbf{u}-\mathbf{v}\rangle_{\mathcal{X}} \le \liminf_{n\to\infty} \langle \mathcal{A}\mathbf{u}_n, \mathbf{u}_n - \mathbf{v}\rangle_{\mathcal{X}}$ for every $\mathbf{v} \in \mathcal{X}$.

Definition 3.4 (Bochner coercivity)**.** An operator $\mathcal{A} : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$ is called:

(i) **Bochner coercive with respect to $\mathbf{f} \in \mathcal{X}^*$ and** $u_0 \in H$ if there is a constant $M := M(\mathbf{f}, u_0, \mathcal{A}) > 0$ such that for every $\mathbf{u} \in \mathcal{X} \cap \mathcal{Y}$ from

$$\tfrac{1}{2}\|\mathbf{u}(t)\|_H^2 + \langle \mathcal{A}\mathbf{u} - \mathbf{f}, \mathbf{u}\chi_{[0,t]}\rangle_{\mathcal{X}} \le \tfrac{1}{2}\|u_0\|_H^2 \quad \text{for a.e. } t \in I,$$

it follows that $\|\mathbf{u}\|_{\mathcal{X}\cap\mathcal{Y}} = \|\mathbf{u}\|_{\mathcal{X}} + \|\mathbf{u}\|_{\mathcal{Y}} \le M$.

(ii) **Bochner coercive** if it is Bochner coercive with respect to $\mathbf{f}$ and u_0, for every $\mathbf{f} \in \mathcal{X}^*$ and $u_0 \in H$.

The critical role of the above definitions is that they identify a vast class of problems for which existence can be established. In fact, if $\mathcal{A} : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$ is bounded, Bochner pseudo-monotone, and Bochner coercive, then the corresponding evolution problem $\frac{d\mathbf{u}}{dt} + \mathcal{A}\mathbf{u} = \mathbf{f}$ is solvable for any initial datum $u_0 \in H$. This result was recently obtained in [13, Thm. 4.1].

This result is particularly relevant since the difficulty is then shifted to the verification of the properties of induced operators, which can be performed time-by-time in the known steady setting. We will not describe the full result, but we propose a particular, simplified setting sufficient to solve (1.1).

The existence result is mainly based on the following proposition giving sufficient conditions which have to be checked at any fixed time slice $t \in I$ and which is a particular case of [13, Prop. 3.13].

Proposition 3.5. *Let $A : V \to V^*$ be an operator. Assume that there exists a number $p \in (1, \infty)$ and constants $c_0, c_1 > 0$ such that*[1]*:*

(C.1): *For every $v \in V$ there holds*

$$\|Av\|_{V^*} \le c_0 \|v\|_V^{p-1}.$$

(C.2): *$A : V \to V^*$ is pseudo-monotone.*

(C.3): *For every $v \in V$ there holds*

$$\langle Av, v\rangle_V \ge c_1 \|v\|_V^p.$$

Then, the induced operator $\mathcal{A} : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$, for all $\mathbf{u} \in \mathcal{X} \cap \mathcal{Y}$ and $\mathbf{v} \in \mathcal{X}$ defined by

$$\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle_{\mathcal{X}} := \int_I \langle A(\mathbf{u}(t)), \mathbf{v}(t)\rangle_V \, dt,$$

is well-defined, bounded, Bochner pseudo-monotone, and Bochner coercive.

On the basis of Proposition 3.5, we immediately obtain the following existence result, which will be used to study the families of rotational models just checking that the conditions (C.1)–(C.3) are satisfied, after a proper choice of the functional setting.

[1]For a pseudo-monotone operator $A : X \to X^*$ (local) boundedness implies demi-continuity, i.e., $x_n \to x$ in X $(n \to \infty)$ implies $Ax_n \rightharpoonup Ax$ in X^* $(n \to \infty)$, hence we do not need here to make any further assumptions of demi-continuity.

Theorem 3.6. *Let $A : V \to V^*$ be an operator satisfying* (C.1)–(C.3)*. Then, for arbitrary $u_0 \in H$ and $\mathbf{f} \in L^{p'}(I, V^*)$, there exists a solution $\mathbf{u} \in \mathcal{W}$ of the evolution equation*

$$\int_I \Big\langle \frac{d\mathbf{u}}{dt}(t) + A(\mathbf{u}(t)), \mathbf{v}(t) \Big\rangle_V = \int_I \langle \mathbf{f}(t), \mathbf{v}(t) \rangle_V \, dt \quad \forall \mathbf{v} \in \mathcal{X},$$
$$\mathbf{u}_c(0) = u_0 \quad \text{in } H.$$

Here, the initial condition has to be understood in the sense of the unique continuous representation $\mathbf{u}_c \in C^0(\overline{I}, H)$ of $\mathbf{u} \in \mathcal{W}$ (cf. [28, Prop. 23.23]).

4. Weighted spaces

Since (1.1) is a boundary value problem with the principal part given by a space dependent (and degenerate at the boundary) operator, a natural functional setting would be that of weighted Sobolev spaces. Apart from classical Lebesgue and Sobolev spaces, we will use their weighted counterparts. We follow the notation from the classical book of Kufner et al. [14].

A weight ϱ on $\mathbb{R}^n$ is a locally integrable function satisfying almost everywhere $0 < \varrho(\mathbf{x}) < \infty$. The weighted space $L^p(\Omega, \varrho)$, $1 < p < \infty$, is defined as follows

$$L^p(\Omega, \varrho) := \Big\{ \mathbf{f} : \ \Omega \to \mathbb{R}^n \text{ measurable } \mid \int_\Omega |\mathbf{f}(\mathbf{x})|^p \, \varrho(\mathbf{x}) \, \mathrm{d}\mathbf{x} < \infty \Big\}.$$

For $p > 1$ we have by using Hölder's inequality that

$$\varrho^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega) \quad \Rightarrow \quad L^p(\Omega, \varrho) \subset L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega),$$

allowing to work in the standard setting of distributions. It turns out that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega, \varrho)$ if the weight satisfies $\varrho^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\mathbb{R}^n)$, see [14]. In addition, $L^p(\Omega, \varrho)$ is a Banach space when equipped with the norm

$$\|\mathbf{f}\|_{p,\varrho} := \Big(\int_\Omega |\mathbf{f}(\mathbf{x})|^p \varrho(\mathbf{x}) \, \mathrm{d}\mathbf{x} \Big)^{1/p}.$$

Next, we define weighted Sobolev spaces

$$W^{k,p}(\Omega, \varrho) := \{ \mathbf{f} : \ \Omega \to \mathbb{R}^n \mid D^\alpha \mathbf{f} \in L^p(\Omega, \varrho) \text{ for all } \alpha \text{ s.t. } |\alpha| \leq k \},$$

equipped with the norm

$$\|\mathbf{f}\|_{k,p,\varrho} := \Big(\sum_{|\alpha| \leq k} \|D^\alpha \mathbf{f}\|_{p,\varrho}^p \Big)^{1/p},$$

and, as usual, we define $W_0^{k,p}(\Omega, \varrho)$ as follows

$$W_0^{k,p}(\Omega, \varrho) := \overline{\{\phi \in C_0^\infty(\Omega)\}}^{\| \cdot \|_{k,p,\varrho}}.$$

In our application the weight $\varrho(\mathbf{x})$ will be a power of the distance $d(\mathbf{x}) \geq 0$ of the point $\mathbf{x} \in \Omega$ from the boundary $\partial\Omega$. Consequently, we specialize to this setting and give specific notions regarding these so-called *power-type weights*, see Kufner [14]. First, it turns out that $W^{k,p}(\Omega, d^\alpha)$ is a separable Banach space provided $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $1 \leq p < \infty$. In this special setting, since $d(\mathbf{x}) \geq C_K > 0$ for each compact $K \subset\subset \Omega$, several results are stronger or more precise due to the inclusion $L^p(\Omega, d^\alpha) \subset L^p_{\text{loc}}(\Omega)$, valid for all $\alpha \in \mathbb{R}$.

We recall the following classical result about the distance function (cf. [14]).

Lemma 4.1. *Let Ω be a domain of class $C^{0,1}$, which means that in a small enough neighborhood Ω_P, for $P \in \partial\Omega$, the boundary $\partial\Omega \cap \Omega_P$ can be expressed (after a rigid rotation) as $x_3 = a(x_1, x_2)$ for Lipschitz continuous a. Then, there exist constants $0 < c_0, c_1 \in \mathbb{R}$ such that*

$$c_0\, d(\mathbf{x}) \leq |a(x') - x_3| \leq c_1\, d(\mathbf{x}) \qquad \forall\, \mathbf{x} = (x', x_3) \in \Omega_P.$$

One of the most relevant properties of the distance function is that the following embedding holds true

$$L^p(\Omega, d^\alpha) \subset L^1(\Omega) \quad \text{if} \quad \alpha < p - 1. \tag{4.1}$$

It follows directly from Hölder's inequality

$$\int_\Omega |f|\, \mathrm{d}\mathbf{x} = \int_\Omega d^{\alpha/p} |f| d^{-\alpha/p} d\mathbf{x} \leq \Big(\int_\Omega d^\alpha |f|^p d\mathbf{x} \Big)^{1/p} \Big(\int_\Omega d^{-\alpha p'/p} d\mathbf{x} \Big)^{1/p'},$$

and using Lemma 4.1 the latter integral is finite if and only if

$$\frac{\alpha\, p'}{p} = \frac{\alpha}{p-1} < 1.$$

In the same way we have also that

$$\forall\, \alpha \in [0, p-1[\qquad L^p(\Omega, d^\alpha) \subset L^q(\Omega) \qquad \forall\, q \in \big[1, \frac{p}{1+\alpha}\big[. \tag{4.2}$$

As in [14, Prop. 9.10] it can be shown that:

Lemma 4.2. *The quantity $\big(\int_\Omega d^\alpha |\nabla \mathbf{f}|^p \, \mathrm{d}\mathbf{x} \big)^{\frac{1}{p}}$ is an equivalent norm in $W_0^{1,p}(\Omega, d^\alpha)$, provided that $0 \leq \alpha < p - 1$.*

In this case functions from $W_0^{1,p}(\Omega, d^\alpha)$ are zero on $\partial\Omega$ in the sense that they can be approximated by smooth functions with compact support. In the sequel we will use certain Hardy–Sobolev inequalities. Note that inequalities of this kind, when d is replaced by $|\mathbf{x}| = d(\mathbf{x}, 0)$ are known as Caffarelli–Kohn–Nirenberg inequalities [8].

Lemma 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. For $p \in [1, n)$, $\alpha \neq p - 1$ and $q \in [p, \frac{np}{n-p}]$ there exists a constant $c > 0$ such that for all $f \in W_0^{1,p}(\Omega, d^\alpha)$ there holds*

$$\left(\int_\Omega d^{\frac{q}{p}(n-p+\alpha)-n} |f|^q \, d\mathbf{x} \right)^{\frac{1}{q}} \leq c \left(\int_\Omega d^\alpha |\nabla f|^p \, d\mathbf{x} \right)^{\frac{1}{p}}. \tag{4.3}$$

Proof. This follows from the definition of the space $W_0^{1,p}(\Omega, d^\alpha)$, [18, Theorem 2.1] and the classical (p, α) Hardy inequality

$$\left(\int_\Omega d^{\alpha-p} |f|^p \, d\mathbf{x} \right)^{\frac{1}{p}} \leq c \left(\int_\Omega d^\alpha |\nabla f|^p \, d\mathbf{x} \right)^{\frac{1}{p}}, \tag{4.4}$$

which is valid for all $p \in (1, \infty)$ and $\alpha \neq p - 1$, for functions in $W_0^{1,p}(\Omega, d^\alpha)$ (cf. [20], [15, Theorem 8.10.14]). □

In addition to (4.1) and its role in Hardy-type inequalities, the critical nature of the power $\alpha = p - 1$ also occurs in the notion of Muckenhoupt weights and their relation with the maximal function.

Definition 4.4. We say that a weight $\varrho \in L^1_{\text{loc}}(\mathbb{R}^3)$ belongs to the Muckenhoupt class A_p, for $1 < p < \infty$, if there exists C such that

$$\sup_{Q \subset \mathbb{R}^n} \left(\fint_Q \varrho(\mathbf{x}) \,\mathrm{d}\mathbf{x} \right) \left(\fint_Q \varrho(\mathbf{x})^{1/(1-p)} \,\mathrm{d}\mathbf{x} \right)^{p-1} \leq C,$$

where Q denotes a cube in $\mathbb{R}^3$.

The powers of the distance function belong to the class A_p according to the following well-known result for general domains (say it is enough that $\partial\Omega$ is a $n-1$-dimensional closed set, see [10]). Here and in the sequel the boundary will be at least locally Lipschitz to have the outward unit vector properly defined.

Lemma 4.5. *The function $\varrho(\mathbf{x}) = \big(d(\mathbf{x})\big)^\alpha$ is a Muckenhoupt weight of class A_p if and only if $-1 < \alpha < p - 1$.*

4.1. **Solenoidal spaces.** A standard approach in fluid mechanics, is to incorporate the divergence-free constraint directly in the function spaces. These spaces are built upon completing the space of solenoidal smooth vector fields with compact support, denoted as $\phi \in C^\infty_{0,\sigma}(\Omega)$. For $\alpha \in \mathbb{R}$ define

$$L^p_\sigma(\Omega, d^\alpha) := \overline{\Big\{ \phi \in C^\infty_{0,\sigma}(\Omega) \Big\}}^{\|\cdot\|_{p,d^\alpha}},$$

$$W^{1,p}_{0,\sigma}(\Omega, d^\alpha) := \overline{\Big\{ \phi \in C^\infty_{0,\sigma}(\Omega) \Big\}}^{\|\cdot\|_{1,p,d^\alpha}}.$$

For $\alpha = 0$ they reduce to the classical spaces $L^p_\sigma(\Omega)$ and $W^{1,p}_{0,\sigma}(\Omega)$. Next, we will extensively use the following extension of classical inequalities linking curl/divergence and full gradient estimates (cf. [3]).

Lemma 4.6. *Let $1 < p < \infty$ and assume that the weight ϱ belongs to the class A_p. Then, there exists a constant C, depending on the domain Ω and on the weight $\varrho \in A_p$, such that*

$$\|\nabla \mathbf{u}\|_{p,\varrho} \leq C(\|\operatorname{div} \mathbf{u}\|_{p,\varrho} + \|\operatorname{curl} \mathbf{u}\|_{p,\varrho}) \qquad \forall\, \mathbf{u} \in W^{1,p}_0(\Omega, \varrho).$$

In particular, we will use the latter result in the following special form

Corollary 4.7. *For $-1 < \alpha < p - 1$ there exists a constant $C = C(\Omega, \alpha, p)$ such that*

$$\int_\Omega d^\alpha |\nabla \mathbf{v}|^p \,\mathrm{d}\mathbf{x} \leq C \int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}|^p \,\mathrm{d}\mathbf{x} \qquad \forall\, \mathbf{v} \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha). \tag{4.5}$$

5. Application to the rotational turbulence models: the proof of Theorem 1.1

In this section we verify that the initial boundary value problem (1.1), after a proper selection of parameters, and definition of both the operators and functional spaces, can be put in the framework of the abstract Theorem 3.6. This will be enough to give a proof of the main result of this paper, that is the existence of weak solutions in Theorem 1.1.

In our setting the choice of the natural spaces is determined by the problem itself which yields, by the a priori estimate obtained by testing with the velocity $\overline{\mathbf{v}}$, that the integral

$$\int_0^T \int_\Omega d^\alpha |\operatorname{curl} \overline{\mathbf{v}}|^3 \, d\mathbf{x} dt$$

is finite. Hence, for almost all $t \in [0,T]$ the integral $\int_\Omega d^\alpha |\operatorname{curl} \overline{\mathbf{v}}|^3 \, d\mathbf{x}$ will be finite, determines the choice for the Banach space V.

In order to identify the evolution triple to be used for the proper formulation, we need to clarify the relationship with the $L^2(\Omega)$ norm. We have the following result which immediately derives from the basic results on weighted spaces of the previous section.

Lemma 5.1. *Let $\mathbf{u} \in C^\infty_{0,\sigma}(\Omega)$ and $\alpha \in [0,2)$. Then, there exists $C = C(\alpha,\Omega)$ such that*

$$\left(\int_\Omega |\mathbf{u}|^2 \, d\mathbf{x}\right)^{1/2} \leq C \left(\int_\Omega d^\alpha |\operatorname{curl} \mathbf{u}|^3 \, d\mathbf{x}\right)^{1/3}. \tag{5.1}$$

Proof. For $\alpha < 2$, combining (4.3) with $q = \frac{3p}{3-p+\alpha}$ and (4.5), it follows for every $p \in (\alpha+1,3)$

$$\int_\Omega |\mathbf{u}|^{\frac{3p}{3-p+\alpha}} \, \mathrm{d}\mathbf{x} \leq c \left(\int_\Omega d^\alpha |\nabla \mathbf{u}|^p \, \mathrm{d}\mathbf{x}\right)^{\frac{q}{p}} \leq c \left(\int_\Omega d^\alpha |\operatorname{curl} \mathbf{u}|^p \, \mathrm{d}\mathbf{x}\right)^{\frac{q}{p}},$$

for all $\mathbf{v} \in C^\infty_{0,\sigma}(\Omega)$. Since $2 \leq \frac{3p}{3-p+\alpha}$ the assertion follows from Hölder's inequality as Ω is bounded. □

Lemma 5.1 shows that one can work with the following evolution triple for all $\alpha \in [0,2)$

$$(V, H, \mathrm{id}) := \big(W^{1,3}_{0,\sigma}(\Omega, d^\alpha), L^2_\sigma(\Omega), \mathrm{id}\big).$$

and as functional setting for (1.1) we use the following spaces and operators, where $0 \leq \alpha < 2$

$$V := W^{1,3}_{0,\sigma}(\Omega, d^\alpha) \qquad \|\mathbf{v}\|_V := \left(\int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}|^3 \, d\mathbf{x}\right)^{1/3}$$

$$H := L^2_\sigma(\Omega) \qquad \|\mathbf{v}\|_H := \left(\int_\Omega |\mathbf{v}|^2 \, d\mathbf{x}\right)^{1/2}$$

$$\mathcal{X} := L^3(I, V), \qquad \mathcal{Y} := L^\infty(I, H)$$

$$\mathcal{W} := \left\{\mathbf{u} \in L^3(I,V) \mid \exists \, \frac{d\mathbf{u}}{dt} \in L^{3/2}(I, V^*)\right\},$$

and define the operator $A := S + B : V \to V^*$ via

$$\langle S\mathbf{v}, \mathbf{w}\rangle_V := \int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}| \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \, d\mathbf{x},$$

$$\langle B\mathbf{v}, \mathbf{w}\rangle_V := \int_\Omega (\operatorname{curl} \mathbf{v} \times \mathbf{v}) \cdot \mathbf{w} \, d\mathbf{x}.$$

The induced operator $\mathcal{S}: \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$ inherits the properties of the operator S (cf. [29, Chapter 30]). Note that S is a strictly monotone, bounded, coercive, and continuous operator. These properties are practically the same known for the p-Laplace operator. In fact, from the definition, one obtains directly the following two inequalities:

$$\|S\mathbf{v}\|_{V^*} \leq \|\mathbf{v}\|_V^2 \qquad \forall\, \mathbf{v} \in V,$$
$$\langle S\mathbf{v}, \mathbf{v}\rangle_V = \|\mathbf{v}\|_V^3 \qquad \forall\, \mathbf{v} \in V.$$

The monotonicity of S derives from the following lemma (cf. [3, Lemma 3.3]).

Lemma 5.2. *For smooth enough vector field $\overline{\boldsymbol{\omega}}_i$ (it is actually enough that $d^{\frac{\alpha}{p}}\overline{\boldsymbol{\omega}}_i \in L^p(\Omega)$, with $1 < p < \infty$) and for $\alpha \in \mathbb{R}^+$ it holds that*

$$\int_\Omega (d^\alpha |\overline{\boldsymbol{\omega}}_1|^{p-2}\overline{\boldsymbol{\omega}}_1 - d^\alpha |\overline{\boldsymbol{\omega}}_2|^{p-2}\overline{\boldsymbol{\omega}}_2) \cdot (\overline{\boldsymbol{\omega}}_1 - \overline{\boldsymbol{\omega}}_2)\, \mathrm{d}\mathbf{x} \geq 0,$$

for any (not necessarily the distance) bounded function such that $d : \Omega \to \mathbb{R}^+$ for a.e. $\mathbf{x} \in \Omega$.

The proof of the above lemma is based on the observation that it can be proved that $d^\alpha(|\overline{\boldsymbol{\omega}}_1|^{p-2}\overline{\boldsymbol{\omega}}_1 - |\overline{\boldsymbol{\omega}}_2|^{p-2}\overline{\boldsymbol{\omega}}_2) \cdot (\overline{\boldsymbol{\omega}}_1 - \overline{\boldsymbol{\omega}}_2) \geq 0$ point-wise. Then weighted integrability of the functions is used to prove that the integral is finite.

To treat the operator B, and the induced one $\mathcal{B}: \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$, we need to properly adapt the estimates on the convective term in weighted spaces and this is mainly based on the previously Hardy-type inequalities (4.3).

Lemma 5.3 (Boundedness of B)**.** *For all $\alpha \in [0, 2)$ the operator $B : V \to V^*$ is bounded. It satisfies $\langle B\mathbf{u}, \mathbf{v}\rangle_V \leq c\|\mathbf{u}\|_V^2 \|\mathbf{v}\|_V$ and $\langle B\mathbf{u}, \mathbf{u}\rangle_V = 0$, for all $\mathbf{u}, \mathbf{v} \in V$.*

Proof. The proof is based on the estimation of the space integral, by using appropriate weighted version of classical Sobolev spaces tools. We have in fact, for all smooth functions with compact support the following inequality (obtained multiplying and dividing a.e. $\mathbf{x} \in \Omega$ by the positive function $d^{\alpha/3}$)

$$\begin{aligned}\left|\int_\Omega (\operatorname{curl}\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}\, d\mathbf{x}\right| &\leq \int_\Omega d^{-\alpha/6}|\mathbf{u}|\, d^{\alpha/3}|\operatorname{curl}\mathbf{v}|\, d^{-\alpha/6}|\mathbf{w}|\, d\mathbf{x} \\ &\leq \left(\int_\Omega d^{-\alpha/2}|\mathbf{u}|^3\, d\mathbf{x}\right)^{1/3} \left(\int_\Omega d^{\alpha}|\operatorname{curl}\mathbf{v}|^3\, d\mathbf{x}\right)^{1/3} \left(\int_\Omega d^{-\alpha/2}|\mathbf{w}|^3\, d\mathbf{x}\right)^{1/3}.\end{aligned}$$

It remains to show that all $\mathbf{u} \in V$ also belong to the weighted space $L^3(\Omega, d^{-\frac{\alpha}{2}})$, with a continuous embedding. From (4.3) it follows for all $p \in [1, 3)$, $\alpha \in [0, 2)$ and $\mathbf{u} \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha)$ that

$$\left(\int_\Omega d^{-\frac{\alpha}{2}}|\mathbf{u}|^{\frac{p(6-\alpha)}{2(3-p+\alpha)}}\, d\mathbf{x}\right)^{\frac{1}{q}} \leq c \left(\int_\Omega d^\alpha |\nabla\mathbf{u}|^p\, d\mathbf{x}\right)^{\frac{1}{p}},$$

with

$$q := \frac{p(6-\alpha)}{2(3-p+\alpha)} < p^*.$$

One easily checks that for all $\alpha \in [0,2)$ there exists a $p \in (1+\alpha, 3)$ such that $3 < q < p^*$. Since Ω is bounded we deduce from this $V \hookrightarrow L^3(\Omega, d^{-\frac{\alpha}{2}})$ by using Hölder's inequality.

Once the integral $\int_\Omega (\mathbf{u} \times \operatorname{curl} \mathbf{v}) \cdot \mathbf{w}\, d\mathbf{x}$ is well-defined for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, it immediately follows that $\langle B\mathbf{u}, \mathbf{u}\rangle_V = 0$ for all $\mathbf{u} \in V$, since a.e. in Ω it holds $(\mathbf{v} \times \operatorname{curl} \mathbf{v}) \cdot \mathbf{v} = 0$. □

This is enough for what concerns the growth and coercivity. We need now to show compactness for B in order to prove pseudo-monotonicity.

Lemma 5.4 (Compactness of B)**.** *Let $\alpha \in [0,2)$. Then, the weak convergence $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in V implies (up to a sub-sequence) that*

$$B\mathbf{u}_n \to B\mathbf{u} \qquad \text{in } V^*,$$

i.e., the operator B is compact.

Proof. By the boundedness of the weakly converging sequence $(\mathbf{u}_n)_{n\in\mathbb{N}} \subseteq V$ and by (4.2) we get that

$$\|\mathbf{u}_n\|_{W^{1,r}(\Omega)} \le C \qquad \forall\, r \in \big[1, \frac{3}{1+\alpha}\big[.$$

Hence, by the usual (unweighted) compact Sobolev embedding $W^{1,r}(\Omega) \hookrightarrow\hookrightarrow L^{\tilde r}(\Omega)$, valid for all $\tilde r < (\frac{3}{1+\alpha})^* = \frac{3}{\alpha}$ we get also that (up to a sub-sequence)

$$\mathbf{u}_n \to \mathbf{u} \qquad \text{a.e. and in } L^{\tilde r}(\Omega).$$

By using the definition of B, the properties of the curl (with summation over repeated indices), and integration by parts, we have that for all $\mathbf{u}, \mathbf{v} \in V$

$$\begin{aligned} \langle B\mathbf{u}, \mathbf{v}\rangle &= \int_\Omega \epsilon_{jkl}\epsilon_{jlm} u_k v_i \frac{\partial u_m}{\partial x_l}\, d\mathbf{x} = \int_\Omega (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) u_k v_i \frac{\partial u_m}{\partial x_l}\, d\mathbf{x} \\ &= -\int_\Omega u_k u_i \frac{\partial v_i}{\partial x_k}\, d\mathbf{x} = -\int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \langle B\mathbf{u}_n, \mathbf{v}\rangle - \langle B\mathbf{u}, \mathbf{v}\rangle &= -\int_\Omega (\mathbf{u}_n \otimes \mathbf{u}_n) : \nabla \mathbf{v} - (\mathbf{u} \otimes \mathbf{u}) : \nabla \mathbf{v}\, d\mathbf{x} \\ &= -\int_\Omega \big((\mathbf{u}_n - \mathbf{u}) \otimes \mathbf{u}_n\big) : \nabla \mathbf{v} + \big(\mathbf{u} \otimes (\mathbf{u}_n - \mathbf{u})\big) : \nabla \mathbf{v}\, d\mathbf{x}. \end{aligned}$$

By Hölder inequality we get, as in the proof of Lemma 5.3,

$$\begin{aligned} &\left| \int_\Omega \big((\mathbf{u}_n - \mathbf{u}) \otimes \mathbf{u}_n\big) : \nabla \mathbf{v}\, d\mathbf{x} \right| \\ &\le \left(\int_\Omega d^{-\alpha/2} |\mathbf{u}_n - \mathbf{u}|^3\, d\mathbf{x} \right)^{1/3} \left(\int_\Omega d^{\alpha} |\nabla \mathbf{v}|^3\, d\mathbf{x} \right)^{1/3} \left(\int_\Omega d^{-\alpha/2} |\mathbf{u}_n|^3\, d\mathbf{x} \right)^{1/3} \\ &\le \left(\int_\Omega d^{-\alpha/2} |\mathbf{u}_n - \mathbf{u}|^3\, d\mathbf{x} \right)^{1/3} \|\mathbf{v}\|_V \|\mathbf{u}_n\|_V. \end{aligned}$$

We now observe that the last two terms are uniformly bounded, while

$$d^{-\alpha/2} |\mathbf{u}_n - \mathbf{u}|^3 \to 0 \qquad \text{a.e. } \mathbf{x} \in \Omega.$$

Consequently, to show that the integral vanishes it is enough to prove that for some $q > 3$ there holds

$$\|\mathbf{u}_n - \mathbf{u}\|_{L^q(\Omega, d^{-\alpha/2})} \leq C$$

uniformly in $n \in \mathbb{N}$, which permits to apply the Vitali theorem in the weighted space $L^3(\Omega, d^{-\alpha/2})$. However, this was already obtained in the proof of Lemma 5.3. The other term in the decomposition of $\langle B\mathbf{u}_n, \mathbf{v}\rangle - \langle B\mathbf{u}, \mathbf{v}\rangle$ can be treated in the same way. □

Proof of Theorem 1.1. In the previous lemmas we have proved that $A = S + B$ is continuous and pseudo-monotone since it is the sum of a monotone continuous and a compact operator. Collecting the estimates we have that in particular that the boundedness and coercivity are as follows

$$\|A\mathbf{v}\|_{V^*} \leq c_0 \|\mathbf{v}\|_V^2,$$
$$\langle A\mathbf{v}, \mathbf{v}\rangle_V \geq \|\mathbf{v}\|_V^3,$$

since $\langle B\mathbf{v}, \mathbf{v}\rangle_V = 0$. Hence all hypotheses from (**C.1**) to (**C.3**) are satisfied.

This shows that the induced operator $\mathcal{A}$ is Bochner pseudo-monotone and coercive, hence all the hypotheses of the abstract existence Theorem 3.6 are satisfied. This proves the main result of this paper, that is the existence of weak solution in Theorem 1.1. □

5.1. **The case $p > 3$.** In this section we show that most of the results of the previous section can be extended (even with easier proofs) to the system with the following operator

$$\langle S_p \mathbf{v}, \mathbf{w}\rangle_V := \int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}|^{p-2} \operatorname{curl} \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \, d\mathbf{x} \qquad \text{with } p > 3, \ 0 \leq \alpha < p - 1,$$

while the use of the tools typical of pseudo-monotone operators fails for $p < 3$. We can then prove the following result

Theorem 5.5. *Let $p > 3$, $\alpha \in [0, p-1)$, $0 < T < \infty$, $\overline{\mathbf{v}_0} \in L^2_\sigma(\Omega)$, and $\mathbf{f} \in L^{p'}(0, T; (W_0^{1,p}(\Omega, d^\alpha))^*)$. Then, there exists a weak solution to the initial boundary value problem*

$$\begin{aligned}
\partial_t \overline{\mathbf{v}} + \overline{\boldsymbol{\omega}} \times \overline{\mathbf{v}} + \operatorname{curl}\left(d^\alpha |\overline{\boldsymbol{\omega}}|^{p-2} \overline{\boldsymbol{\omega}}\right) + \nabla \overline{q} &= \mathbf{f} && \text{in } (0,T) \times \Omega, \\
\overline{\boldsymbol{\omega}} &= \operatorname{curl} \overline{\mathbf{v}} && \text{in } (0,T) \times \Omega, \\
\operatorname{div} \overline{\mathbf{v}} &= 0 && \text{in } (0,T) \times \Omega, \\
\overline{\mathbf{v}} &= \mathbf{0} && \text{on } (0,T) \times \partial\Omega, \\
\overline{\mathbf{v}}(0) &= \overline{\mathbf{v}_0} && \text{in } \Omega,
\end{aligned}$$

such that

$$\overline{\mathbf{v}} \in C([0,T]; L^2_\sigma(\Omega)) \cap L^p(0, T; W^{1,p}_{0,\sigma}(\Omega, d^\alpha)),$$

and for all $t \in [0,T]$

$$\begin{aligned}&\frac{1}{2}\|\overline{\mathbf{v}}(t)\|^2 + \int_0^t \int_\Omega C_\alpha d^\alpha(\mathbf{x})|\overline{\boldsymbol{\omega}}(s,\mathbf{x})|^p \, d\mathbf{x}\, ds\\ &= \frac{1}{2}\|\overline{\mathbf{v}_0}\|^2 + \int_0^t \langle \mathbf{f}(s), \overline{\mathbf{v}}(s)\rangle_{W_0^{1,p}(\Omega,d^\alpha)} ds.\end{aligned}$$

The proof of this result is again just a verification that the hypotheses of the abstract theorem are satisfied, but we will highlight the critical role of the parameters.

First note that for $p > 3$ and $0 \le \alpha < p-1$ the inclusion $W_0^{1,p}(\Omega, d^\alpha) \subset L^2(\Omega)$ holds true. Directly by Hölder's inequality with $\delta = p/2$, $\delta' = p/(p-2)$, and Hardy inequality (4.4) we get

$$\begin{aligned}\int_\Omega |\mathbf{u}|^2 \, d\mathbf{x} &= \int_\Omega d^{\frac{2}{p}(\alpha-p)}|\mathbf{u}|^2 d^{\frac{2}{p}(p-\alpha)} \, d\mathbf{x}\\ &\le \left(\int_\Omega d^{\alpha-p}|\mathbf{u}|^p \, d\mathbf{x}\right)^{\frac{2}{p}} \left(\int_\Omega d^{2\frac{p-\alpha}{p-2}} \, d\mathbf{x}\right)^{\frac{p-2}{p}}\\ &\le C(p,\alpha)\left(\int_\Omega d^\alpha |\nabla \mathbf{u}|^p \, d\mathbf{x}\right)^{\frac{2}{p}},\end{aligned}$$

since $p - \alpha > 0$. This shows that one can work with the evolution triple

$$(V, H, \mathrm{id}) := \big(W_{0,\sigma}^{1,p}(\Omega, d^\alpha), L_\sigma^2(\Omega), \mathrm{id}\big) \qquad \text{with} \quad p > 3,\ 0 \le \alpha < p-1.$$

The properties of the operator S_p are practically the same as those of the operator S, hence one can directly show

$$\begin{aligned}\|S_p \mathbf{v}\|_{V^*} &\le \|\mathbf{v}\|_V^{p-1} \qquad \forall\, \mathbf{v} \in V,\\ \langle S_p \mathbf{v}, \mathbf{v}\rangle_V &= \|\mathbf{v}\|_V^p \qquad \forall\, \mathbf{v} \in V.\end{aligned}$$

On the other hand, the properties of B are to be checked. The operator is the same as before, but the functional setting is different.

To show the boundedness of $B : V \to V^*$ for $0 \le \alpha < p-1$, we proceed as in Lemma 5.3 and we get

$$\begin{aligned}&\left|\int_\Omega (\operatorname{curl} \mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} \, d\mathbf{x}\right| \le \int_\Omega d^{-\alpha/2p}|\mathbf{u}|\, d^{\alpha/p}|\operatorname{curl} \mathbf{v}|\, d^{-\alpha/2p}|\mathbf{w}| \, d\mathbf{x}\\ &\le \left(\int_\Omega d^{-\alpha p'/p}|\mathbf{u}|^{2p'} \, d\mathbf{x}\right)^{\frac{1}{2p'}} \left(\int_\Omega d^\alpha |\nabla \mathbf{v}|^p \, d\mathbf{x}\right)^{\frac{1}{p}} \left(\int_\Omega d^{-\alpha p'/p}|\mathbf{w}|^{2p'} \, d\mathbf{x}\right)^{\frac{1}{2p'}}.\end{aligned}$$

Next, observe that $2p' = 2p/(p-1) < p$ is satisfied for $p > 3$. Consequently, in this case we can directly apply Hölder inequality with exponents $\delta = (p-1)/2$ and $\delta' = (p-1)/(p-3)$ to bound the first and third integrals as follows:

$$\begin{aligned}\int_\Omega d^{-\alpha/(p-1)}|\mathbf{u}|^{2p'} \, d\mathbf{x} &= \int_\Omega d^{(\alpha-p)2/(p-1)}|\mathbf{u}|^{2p'} d^{(2p-3\alpha)/(p-1)} \, d\mathbf{x}\\ &\le \left(\int_\Omega d^{\alpha-p}|\mathbf{u}|^p \, d\mathbf{x}\right)^{\frac{2}{p-1}} \left(\int_\Omega d^{(2p-3\alpha)/(p-3)} \, d\mathbf{x}\right)^{\frac{p-3}{p-1}}.\end{aligned}$$

The first term from the right-hand side is bounded with $\left(\int_\Omega d^\alpha |\nabla \mathbf{u}|^p \, d\mathbf{x}\right)^{\frac{2}{p-1}}$ by using (4.4), while the second is finite if

$$\frac{2p-3\alpha}{p-3} > -1 \quad \Longleftrightarrow \quad \alpha < p-1.$$

This shows that

$$\langle B\mathbf{u}, \overline{\mathbf{w}}\rangle \leq C(\Omega, \alpha, p)\|\mathbf{u}\|_V^2 \|\overline{\mathbf{w}}\|_V,$$

and the compactness of B follows with the same arguments used in the previous section (almost everywhere convergence and Vitali theorem).

Remark 5.6. The case $1 < p < 3$ does not fit with the theory for the reasons we now explain. The argument with Hardy inequality as in the previous lemma requires $p > 3$. If we try to apply the same argument used for $p = 3$ with Hardy–Sobolev inequality (4.3), we can write

$$\begin{aligned}&\left|\int_\Omega (\operatorname{curl} \mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} \, d\mathbf{x}\right| \\ &\leq \left(\int_\Omega d^{-\alpha p'/p} |\mathbf{u}|^{2p'} \, d\mathbf{x}\right)^{\frac{1}{2p'}} \left(\int_\Omega d^\alpha |\operatorname{curl} \mathbf{v}|^p \, d\mathbf{x}\right)^{\frac{1}{p}} \left(\int_\Omega d^{-\alpha p'/p} |\mathbf{w}|^{2p'} \, d\mathbf{x}\right)^{\frac{1}{2p'}},\end{aligned}$$

and then estimate the first and third integral with (4.3) for $q = 2p' < p^*$, which holds for $p > \frac{9}{5}$. Hence, to apply (4.3) the precise exponent will be $q = \frac{p}{p-1}\frac{3p-3-\alpha}{3-p+\alpha}$, and since $q \geq 2p'$ this implies that we have to request for

$$\alpha \leq \frac{5p-9}{3}.$$

Since we would like to treat cases with α smaller but "arbitrarily close" to $p-1$, the inequality

$$p-1 \leq \frac{5p-9}{3},$$

should be correct. On the other hand the latter can be satisfied only for $p \geq 3$. Since we are out of the range of permitted p this shows that the estimate can not be used. Being the inequalities Hardy–Sobolev inequalities sharp, this proves that operator B is not bounded for $\frac{9}{5} < p < 3$, when α is close to $p-1$, hence the basic assumptions to use the pseudo-monotone methods are not satisfied. The existence of weak solutions, if possible, should be obtained with different methods and possibly considering different weak formulations of the problem.

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