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NONLINEAR DIFFERENTIAL EQUATIONS IN A BANACH SUBSPACE OF CONTINUOUS FUNCTIONS

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Abstract. We prove results of existence of a solution (resp. existence and uniqness of a solution) for nonlinear differential equations of type $x' + G(x)x = F(x)$, (where x' denotes the derivative function of $x \in X$) in an abstract Banach subspace X of the space of bounded real-valued continuous functions, satisfying some general and natural property. In our work, the functions F and G are (not necessarily linear) continuous functions from X into itself. Several examples will be given, in various function spaces, to illustrate our results.

1. Introduction

Let X be a Banach subspace of the Banach space $(BC(\mathbb{R}), \|\cdot\|_\infty)$ of all real-valued bounded continuous functions equipped with the sup-norm and let $F, G : X \to X$ be two functions. The goal of this paper is to give existence of solutions $x \in X$ (under some conditions on the space X and the functions F and G) to the differential equations of type:

$$(E) \quad x' + G(x)x = F(x),$$

where, $x' : t \mapsto x'(t)$ denotes the derivative function of $x \in X$ with respect the variable $t \in \mathbb{R}$. We prove in our main result (Theorem 2.1) an existence result by using the Schauder's fixed point theorem and another result of existence and uniqueness by using the Banach-Picard theorem. Our formalism encompasses several examples of spaces X of $BC(\mathbb{R})$ in particular the space $PAP(\mathbb{R})$ of all pseudo almost periodic functions (we will give the precise definition in Section 2.2) which has attracted the interest of many authors in recent years. For references in this aera, we refer for example to the non-exhaustive list of works [1, 2, 4, 10, 11, 13–16]. Our contributions in this paper are axed in the following directions:

— The equations treated in the litterature in the $PAP(\mathbb{R})$ space (see for instance the above references), are of the form $x'(t) + a(t)x(t) = H(x(t), t)$, where a is a function depending only on t, that is, the map $x \mapsto a(\cdot)x$ is a linear operator, and moreover the function H is a function defined on $\mathbb{R} \times \mathbb{R}$. In our results, we deal with operators not necessarily linear, that is, we replace the map $x \mapsto a(\cdot)x$ with a more

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general mapping of the form $x \mapsto G(x)x$. In fact, the case studied in the literature corresponds to the constant function $G(x) = a(\cdot)$, for all $x \in X = PAP(\mathbb{R})$. In our work, G is a very general continuous function. Moreover, the Nemytskii operator $\mathcal{N}_H$ (where, $\mathcal{N}_H(x)(t) = H(x(t), t)$) is also replaced by a more general operator F. For example for any $\alpha, \beta \in L^1(\mathbb{R})$ with $0 < \|\alpha\|_1 \leq 1$ (where $\|x\|_1 = \int_{-\infty}^{+\infty} |x(s)|ds$ and $x * \alpha(t) = \int_{-\infty}^{+\infty} x(s)\alpha(s-t)ds$ denotes the convolution of x and α), the continuous operators $F, G : PAP(\mathbb{R}) \to PAP(\mathbb{R})$ defined by: $\forall (x,t) \in PAP(\mathbb{R}) \times \mathbb{R}$,

$$\begin{aligned} F(x)(t) &= \frac{1}{3}\left(\sin(t) + \sin(\sqrt{2}t) + (1+t^2)^{-1} x * \alpha(t)\right) \\ G(x)(t) &= 3 + \sin(2t) + (1+t^2)^{-1} \cos\left(x * \beta(t)\right), \end{aligned}$$

can not be writen in the form $H(x(t), t)$ for some function H defined on $\mathbb{R} \times \mathbb{R}$. However, with these functions, our results apply and give at last one pseudo almost periodic solution to the equation (E) (see Example 2.15 in Section 2.2 for details).

— Our approach unifies several function spaces. We deal with abstract Banach subspace X of $BC(\mathbb{R})$ including some classical spaces as the space of all continuous w-periodic functions, the space of all almost periodic functions or the space all pseudo almost periodic functions (see Proposition 2.10). Thus, depending on the type of functions F and G, the solutions will exist in the adequat corresponding type of space (see examples in Section 2.2). It is the interest of working in an abstract subspace X of $BC(\mathbb{R})$ satisfying a natural condition that we call (H_0) in this paper.

— The proof of our main result (Theorem 2.1) is subdivided into several simple and natural lemmas, in order to make the reading more pleasant and less computational. Among the lemmas proven in this paper, some of them can possibly have utility independent of this article.

This paper is organized as follows. In section 2, we give our first main result of existence of solutions of the equation (E) under some general assumption (Theorem 2.1) and we will then give several examples to illustrate this result. In Section 3 we give our second main theorem consisting on the attractivity of solutions (Theorem 3.3).

2. The main result

Let X be a Banach subspace of $BC(\mathbb{R})$. The set $B_X(0, r)$ denotes the closed ball of X centred at 0 with radious $r > 0$. For each $l, r > 0$, we define the following closed convex subsets of X:

$$B_{[l,r]} := \{x \in X : x(t) \in [l, r], \forall t \in \mathbb{R}\},$$

$$X_{[l,+\infty[} := \{x \in X : x(t) \in [l, +\infty[, \forall t \in \mathbb{R}\}.$$

We need to introduce, the following well defined operator for each $l > 0$ (see Lemma 2.7):

$$\begin{aligned} T : BC(\mathbb{R}) \times BC(\mathbb{R})_{[l,+\infty[} &\to BC(\mathbb{R}) \\ (f, g) &\mapsto [t \mapsto \int_{-\infty}^{t} e^{-\int_s^t g(u)du} f(s)ds], \end{aligned}$$

It is classical and easy to see that for every $(f,g) \in BC(\mathbb{R}) \times BC(\mathbb{R})_{[l,+\infty[}$, the function $T(f,g)$ is differentiable and satisfies:

$$T(f,g)'(t) \quad = \quad -g(t)T(f,g)(t) + f(t), \;\; \forall t \in \mathbb{R}. \tag{2.1}$$

We consider the following conditions (H_0), (H_1) and $(\tilde{H}_1)$:

(H_0) the subspace X is invariant under T in the sense that for each $l > 0$, $T(X \times X_{[l,+\infty[}) \subset X$.

The property (H_0) is satifed by several classical Banach subspaces of $BC(\mathbb{R})$ (see Proposition 2.10 in Section 2.2).

(H_1) The functions $F, G : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|_\infty)$ are continuous and satisfies:

• $\inf_{x\in X, t\in\mathbb{R}} G(x)(t) > 0$ and there exists $k, M \in \mathbb{R}$ such that F and G are bounded on $B_{[k,M]}$ and that $k \le \frac{F(x)(t)}{G(x)(t)} \le M$ for all $(x,t) \in B_{[k,M]} \times \mathbb{R}$.

• for every sequence $(x_n) \subset X$, if (x_n) converges uniformly on each compact of $\mathbb{R}$, then $(F(x_n))$ and $(G(x_n))$ are relatively compact in $(X, \|\cdot\|_\infty)$.

Notice that if we assume that F and G are bounded on the whole space X and $\inf_{x\in X, t\in\mathbb{R}} G(x)(t) > 0$, then we can take in the assumption (H_1)

$$k := \inf_{(x,t)\in X\times\mathbb{R}} \frac{F(x)(t)}{G(x)(t)} \text{ and } M := \sup_{(x,t)\in X\times\mathbb{R}} \frac{F(x)(t)}{G(x)(t)}.$$

Notice also that the second point of (H_1) is crucial and ensures that the Schauder fixed point theorem applies. This condition is automaticaly satisfied in the subspace $X = C_w(\mathbb{R})$ of all w-periodic continuous functions since in this case the uniform convergence on each compact of $\mathbb{R}$ is equivalent to the uniform convergence on $\mathbb{R}$. It is also satisfied on other spaces, in general and various situations (see Section 2.2 for some examples) and has already been used in the literature (see for instance the condition (H_3) in [11] and the condition $(E5)$, page 248 in [9]).

$(\tilde{H}_1)$ The functions $F, G : (X, \|\cdot\|_\infty) \to (X, \|\cdot\|_\infty)$ are Lipschitz,

$$l := \inf_{x\in X, t\in\mathbb{R}} G(x)(t) > 0,$$

and there exists $k, M \in \mathbb{R}$ such that, $k \le \frac{F(x)(t)}{G(x)(t)} \le M$ for all $(x,t) \in B_{[k,M]} \times \mathbb{R}$,

$$r := \sup_{x\in B_{[k,M]}} \|F(x)\|_\infty < +\infty$$

and

$$\max(\frac{r}{l^2}, \frac{1}{l})(L_F + L_G) < 1,$$

where L_F and L_G denotes the constant of Lipschitz of F and G respectively.

Theorem 2.1. *Under the assumption (H_0) and (H_1) (resp. (H_0) and $(\tilde{H}_1)$), the equation (E) has at least one solution x^* in $B_{[k,M]}$ (resp. has a unique solution x^* in $B_{[k,M]}$), where k and M are given by the assumption (H_1) (resp. by the assumption $(\tilde{H}_1)$).*

The proof of the above theorem will be given in the following section.

Remark 2.2. If moreover the function F is assumed to be a positive function, then we can take $k \geq 0$ and in this case there exists at last one positive solution $x^* \in X$.

Remark 2.3. Theorem 2.1 is also true under the same assumption for the following equation ($\inf \widetilde{G} > 0$):

$$(\widetilde{E}) \quad x' - \widetilde{G}(x)x = \widetilde{F}(x).$$

since, with $G(x)(t) = \widetilde{G}(x)(-t) > 0$ and $F(x)(t) := -\widetilde{F}(x)(-t)$, we see easily that x is a solution of $(\widetilde{E})$ if and only if $t \mapsto y(t) := x(-t)$ is a solution of (E).

Remark 2.4. The assumptions (H_1) and $(\tilde{H}_1)$ also apply to equations with delays τ and σ, of the form: $x' + G_\tau(x)x = F_\sigma(x)$, where $G_\tau(x) : t \mapsto G(x)(t - \tau(t))$ and $F_\sigma(x) : t \mapsto F(x)(t - \sigma(t))$.

2.1. **The proof of Theorem 2.1.** In order to prove Theorem 2.1 we need some intermediate results, the proof will be given at the end of this section. Let us start with the following elementary lemma.

Lemma 2.5. *Let $g \in BC(\mathbb{R})$ such that $\inf_{t\in\mathbb{R}} g(t) > 0$. Then, we have that*

$$\int_{-\infty}^{t} g(s)e^{-\int_s^t g(u)du} ds = 1, \ \forall t \in \mathbb{R}.$$

Proof. Since $\inf_{t\in\mathbb{R}} g(t) > 0$, then clearly we have that

$$\int_{-\infty}^{t} g(u)du = +\infty.$$

It follows that

$$\begin{aligned} \int_{-\infty}^{t} g(s)e^{-\int_s^t g(u)du} ds &= \int_{-\infty}^{t} g(s)e^{\int_t^s g(u)du} ds \\ &= [e^{\int_t^s g(u)du}]_{-\infty}^{t} \\ &= 1. \end{aligned}$$

□

Lemma 2.6. *Let $l > 0$ and let $g_1, g_2 \in BC(\mathbb{R})$ such that $\inf_{t\in\mathbb{R}} g_i(t) \geq l$ for $i \in \{1, 2\}$. Then, for every $t, s \in \mathbb{R}$ such that $s \leq t$, we have*

$$\left| e^{-\int_s^t g_1(u)du} - e^{-\int_s^t g_2(u)du} \right| \leq e^{-l(t-s)} \left| \int_s^t g_1(u)du - \int_s^t g_2(u)du \right|,$$

and consequently,

$$\left| e^{-\int_s^t g_1(u)du} - e^{-\int_s^t g_2(u)du} \right| \leq (t-s)e^{-l(t-s)} \|g_1 - g_2\|_\infty.$$

Proof. First, recall that the function $x \mapsto e^x$ is e^b-Lipschitz on any interval $]-\infty, b]$, by the mean value theorem. Since $g_1, g_2 \geq l$, we have that $-\int_s^t g_1(u)du \leq -l(t-s)$ and $-\int_s^t g_2(u)du \leq -l(t-s)$, for every $s \leq t$. It follows from the fact that $x \mapsto e^x$ is $e^{-l(t-s)}$-Lipschitz on the interval $]-\infty, -l(t-s)]$ that,

$$\left| e^{-\int_s^t g_1(u)du} - e^{-\int_s^t g_2(u)du} \right| \leq e^{-l(t-s)} \left| \int_s^t g_1(u)du - \int_s^t g_2(u)du \right|.$$

Thus, for every $t, s \in \mathbb{R}$ such that $s \leq t$, we have that

$$\left|e^{-\int_s^t g_1(u)du} - e^{-\int_s^t g_2(u)du}\right| \leq (t-s)e^{-l(t-s)}\|g_1 - g_2\|_\infty.$$

□

Lemma 2.7. *Let $l, r, r' > 0$ be positive real numbers. Then, the following assertions hold.*

(i) For every $(f, g) \in BC(\mathbb{R}) \times BC(\mathbb{R})_{[l,+\infty[}$, the function $T(f, g)$ is Lipschitz on $\mathbb{R}$ with a constant of Lipschitz less than $\|f\|_\infty(\frac{\|g\|_\infty}{l} + 1)$. Consequently, the familly $\mathcal{F} := \{T(f, g) : (f, g) \in B_{BC(\mathbb{R})}(0, r) \times BC(\mathbb{R})_{[l,r']}\}$ is uniformly equi-continuous on $\mathbb{R}$.

(ii) The operator T is Lipschitz on $B_{BC(\mathbb{R})}(0, r) \times BC(\mathbb{R})_{[l,+\infty[}$, that is, for every $(f_1, g_1), (f_2, g_2) \in B_{BC(\mathbb{R})}(0, r) \times BC(\mathbb{R})_{[l,+\infty[}$, we have that

$$\|T(f_1, g_1) - T(f_2, g_2)\|_\infty \leq \max(\frac{r}{l^2}, \frac{1}{l})(\|g_1 - g_2\|_\infty + \|f_1 - f_2\|_\infty).$$

Proof. (i) Let us prove that $T(f, g)$ is Lipschitz on $\mathbb{R}$. First, it is easy to see that since $\inf_{t\in\mathbb{R}} g(t) \geq l$, then $\|T(f, g)\|_\infty \leq \frac{\|f\|_\infty}{l}$. On the other hand, from the formula (2.1), we have that for every $(f, g) \in BC(\mathbb{R}) \times BC(\mathbb{R})_{[l,+\infty[}$:

$$T(f, g)'(t) = -g(t)T(f, g)(t) + f(t), \ \forall t \in \mathbb{R}.$$

Thus, we have that

$$\|T(f, g)'\|_\infty \leq \|g\|_\infty \|T(f, g)\|_\infty + \|f\|_\infty \leq \|f\|_\infty(\frac{\|g\|_\infty}{l} + 1).$$

Hence, by the mean value theorem, $T(f, g)$ is Lipschitz with a constant of Lipschitz less that $\|f\|_\infty(\frac{\|g\|_\infty}{l} + 1)$. It follows that the familly $\mathcal{F} := \{T(f, g) : (f, g) \in B_{BC(\mathbb{R})}(0, r) \times BC(\mathbb{R})_{[l,r']}\}$ is uniformly equi-continuous on $\mathbb{R}$.

(ii) Using Lemma 2.6, we have that

$$\begin{aligned}
|T(f_1, g_1)(t) - T(f_2, g_2)(t)| &\leq \int_{-\infty}^t \left|e^{-\int_s^t g_1(u)du} - e^{-\int_s^t g_2(u)du}\right| |f_1(s)|ds \\
&\quad + \int_{-\infty}^t e^{-\int_s^t g_2(u)du}|f_1(s) - f_2(s)|ds \\
&\leq r\|g_1 - g_2\|_\infty \int_{-\infty}^t (t-s)e^{-l(t-s)}ds \\
&\quad + \|f_1 - f_2\|_\infty \int_{-\infty}^t e^{-l(t-s)}ds \\
&= \frac{r}{l^2}\|g_1 - g_2\|_\infty + \frac{1}{l}\|f_1 - f_2\|_\infty.
\end{aligned}$$

Hence, $\|T(f_1, g_1) - T(f_2, g_2)\|_\infty \leq \max(\frac{r}{l^2}, \frac{1}{l})(\|f_1 - f_2\|_\infty + \|g_1 - g_2\|_\infty)$.

□

Lemma 2.8. *Under the assumptions* (H_0) *and* (H_1) *(resp. the assumptions* (H_0) *and* $(\tilde{H}_1)$*), the operators* Γ *defined for all* $x \in B_{[k,M]} \subset X$ *by*

$$\Gamma(x) := T(F(x), G(x)) = [t \mapsto \int_{-\infty}^{t} e^{-\int_s^t G(x)(u)du} F(x)(s)ds],$$

satisfies the following assertions:

(a) Γ *maps* $B_{[k,M]}$ *into* $B_{[k,M]}$.

(b) Γ *is norm-to-norm continuous and satisfies: for every sequence* $(x_n) \subset B_{[k,M]}$, *if* (x_n) *converges uniformly on each compact of* $\mathbb{R}$ *to some point of* $BC(\mathbb{R})$, *then* $(\Gamma(x_n))$ *is relatively compact in* $(B_{[k,M]}, \|\cdot\|_\infty)$ *(resp.* Γ *is contraction).*

(c) $\Gamma(B_{[k,M]})$ *is equi-continuous at each point of* $\mathbb{R}$. *Moreover, the set* $\Gamma(B_{[k,M]})(t) := \{\Gamma(x)(t) : x \in B_{[k,M]}\} \subset [k, M]$ *is relatively compact in* $\mathbb{R}$.

Proof. Assume (H_0) and (H_1) and let us set $r := \sup_{x \in B_{[k,M]}} \|F(x)\|_\infty$, $l := \inf_{x \in X, t \in \mathbb{R}} G(x)(t) > 0$ and $r' := \sup_{x \in B_{[k,M]}, t \in \mathbb{R}} G(x)(t) > 0$. Since T satisfies (H_0), the operator Γ maps $B_{[k,M]}$ into X as follows:

$$\begin{array}{rcl} \Gamma : B_{[k,M]} & \to & B_X(0,r) \times X_{[l,+\infty[} \to X \\ x & \mapsto & (F(x), G(x)) \mapsto \Gamma(x) = T(F(x), G(x)). \end{array}$$

(a) Let us prove that Γ maps $B_{[k,M]}$ into $B_{[k,M]}$. Indeed, by assumption we have that

$$k \le \frac{F(x)(s)}{G(x)(s)} \le M, \forall (x,s) \in B_{[k,M]} \times \mathbb{R},$$

we get using Lemma 2.5 that for every $x \in B_{[k,M]}$ and every $t \in \mathbb{R}$

$$\begin{aligned} k &= \int_{-\infty}^{t} kG(x)(s) e^{-\int_s^t G(x)(u)du} ds \\ &\le \Gamma(x)(t) = \int_{-\infty}^{t} e^{-\int_s^t G(x)(u)du} F(x)(s)ds \\ &\le \int_{-\infty}^{t} MG(x)(s) e^{-\int_s^t G(x)(u)du} ds \\ &= M. \end{aligned}$$

Thus, $\Gamma(x) \in B_{[k,M]}$.

(b) Using part (ii) of lemma 2.7, we get that for every $x, y \in B_{[k,M]}$,

$$\begin{aligned} \|\Gamma(x) - \Gamma(y)\|_\infty &= \|T(F(x), G(x)) - T(F(y), G(y))\|_\infty \\ &\le \max(\frac{r}{l^2}, \frac{1}{l})(\|F(x) - F(y)\|_\infty + \|G(x) - G(y)\|_\infty). \end{aligned} \tag{2.2}$$

It follows using the assumption (H_1), that Γ is continuous and that for every sequence $(x_n) \subset B_{[k,M]}$, if (x_n) converges on each compact subset of $\mathbb{R}$ to some point of $BC(\mathbb{R})$, then $(\Gamma(x_n))$ is relatively compact in $(B_{[k,M]}, \|\cdot\|_\infty)$.

(c) We obtain that $\Gamma(B_{[k,M]})(\subset B_{[k,M]})$ is equi-continuous at each point of $\mathbb{R}$ by using Lemma 2.7 with the positive real numbers $r, l, r' > 0$ defined above. Moreover, it is clear that the set $\Gamma(B_{[k,M]})(t) := \{\Gamma(x)(t) : x \in B_{[k,M]}\} \subset [k, M]$ is relatively compact in $\mathbb{R}$.

Now, if we assume that the assumption $(\tilde{H}_1)$ holds, then F and G are $\|\cdot\|_\infty$-to-$\|\cdot\|_\infty$ Lipschitz functions and so using the inequality (2.2) we get that

$$\|\Gamma(x)-\Gamma(y)\|_\infty \leq \max(\frac{r}{l^2},\frac{1}{l})(L_F+L_G)\|x-y\|_\infty,$$

which implies that Γ is a contraction by the assumption $(\tilde{H}_1)$.

$\square$

Now, we give the proof of Theorem 2.1. Let us denote, $\overline{\text{co}}^{\|\cdot\|_\infty}(\Gamma(B_{[k,M]}))$ the norm-closed convex hull of $\Gamma(B_{[k,M]})$.

Proof of Theorem 2.1. We treat two situations:

• *Under the assumptions* (H_0) *and* (H_1). By Lemma 2.8, $\Gamma(B_{[k,M]})\subset B_{[k,M]}$ and so

$$K:=\overline{\text{co}}^{\|\cdot\|_\infty}(\Gamma(B_{[k,M]}))\subset B_{[k,M]}.$$

Then, $\Gamma(K)\subset\Gamma(B_{[k,M]})\subset K$ and we have that the operator $\Gamma:K\to K$ is well defined and continuous. We are going to prove that $\Gamma(K)$ is relatively compact for the norm $\|\cdot\|_\infty$. Using part (c) of Lemma 2.8, we see that K (as a closed convex hull) is also equi-continuous at each point of $\mathbb{R}$ and that $K(t):=\{x(t):x\in K\}\subset[k,M]$ is relatively compact in $\mathbb{R}$. Thus, from the Arzela-Ascoli theorem, the restriction of K to any interval $[-m,m]$ of $\mathbb{R}$ $(m\in\mathbb{N})$ is relatively compact in the space $(C([-m,m]),\|\cdot\|_\infty)$ of all continuous functions on $[-m,m]$. Now, let (x_n) be any sequence of K. Then, we have that the restriction of (x_n) to each interval $[-m,m]$ has a subsequence $(x_{\mu_m(n)})$ converging uniformly on this interval. Using the Cantor diagonal process, there exists a subsequence $(x_{\mu(n)})$ converging uniformly on each compact subset of $\mathbb{R}$. Then, by Lemma 2.8 (see part (b)), there exists a subsequence that we will denote again $(x_{\mu(n)})$ such that $(\Gamma(x_{\mu(n)}))$ converges in $(BC(\mathbb{R}),\|\cdot\|_\infty)$. Thus, $(\Gamma(K),\|\cdot\|_\infty)$ is relatively compact. Using the Schauder fixed point theorem we get a fixed point $x^*\in K\subset B_{[k,M]}$ for Γ, which satisfies the equation (E) by the formula (2.1).

• *Under the assumptions* (H_0) *and* $(\tilde{H}_1)$. In this situation the operator Γ is contraction by Lemma 2.8, so the Banach-Picard theorem applies and gives a unique fixed point $x^*\in B_{[k,M]}$ for Γ, which is the unique solution of the equation (E) in the set $B_{[k,M]}$ by the formula (2.1). $\square$

2.2. **Examples and properties.** In this section, we give examples satisfying our results. The assumption (H_0) is satified for several classical subspace of $BC(\mathbb{R})$. We give in Proposition 2.10 (see bellow) some examples of classical spaces satisfying this property. We need to introduce some definitions.

For a fixed $w\in\mathbb{R}\setminus\{0\}$, we denote $C_w(\mathbb{R})$ the Banach subspace of $BC(\mathbb{R})$ consisting on all continuous w-periodic functions.

Definition 2.9. A continuous function $f:\mathbb{R}\to\mathbb{R}$ is called (Bohr) almost periodic if for each $\varepsilon>0$, there exists $l_\varepsilon>0$ such that every interval of length l_ε contains at least a number τ with the following property:

$$\sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| < \varepsilon.$$

The number τ is then called an ε-period of f. The collection of all almost periodic functions $f : \mathbb{R} \to \mathbb{R}$ will be denoted by $AP(\mathbb{R})$. It is known that the space $AP(\mathbb{R})$ is a Banach subspace of $BC(\mathbb{R})$ (see for instance [9]). Clearly, for every $w \in \mathbb{R}$, we have that $C_w(\mathbb{R}) \hookrightarrow AP(\mathbb{R})$ (a Banach subspace). A classical example of an almost periodic function which is not periodic is given by the following function

$$f(t) = \sin(t) + \sin(\sqrt{2}t).$$

The space of continuous ergodic functions is defined as follows:

$$PAP_0(\mathbb{R}) := \{g \in BC(\mathbb{R}) : \lim_{r\to+\infty} \frac{1}{2r} \int_{-r}^{r} |g(t)|dt = 0\}.$$

Clearly, $PAP_0(\mathbb{R})$ is a Banach subspace of $BC(\mathbb{R})$. It is easy to see that $AP(\mathbb{R}) \cap PAP_0(\mathbb{R}) = \{0\}$ (see for instace [9]). Then, we define the Banach subspace of $BC(\mathbb{R})$ of all pseudo almost periodic function denoted $PAP(\mathbb{R})$, as follows:

$$PAP(\mathbb{R}) := AP(\mathbb{R}) \oplus PAP_0(\mathbb{R}).$$

Finally, we introduce the following space of all pseudo w-periodic functions denoted $PP_w(\mathbb{R})$ by

$$PP_w(\mathbb{R}) := C_w(\mathbb{R}) \oplus PAP_0(\mathbb{R}).$$

Clearly, $PP_w(\mathbb{R})$ is a Banach subspace of $PAP(\mathbb{R})$ for every $w \in \mathbb{R}$. Finally, $BC_U(\mathbb{R})$ denotes the Banach subspace of $BC(\mathbb{R})$ of uniformly continuous functions.

Proposition 2.10. *The following spaces of continuous functions, satisfy the assumption (H_0): $X = BC(\mathbb{R})$, $BC_U(\mathbb{R})$, $C_w(\mathbb{R})$, $AP(\mathbb{R})$, $PAP_0(\mathbb{R})$, $PAP(\mathbb{R})$ and $PP_w(\mathbb{R})$.*

Proof. The result is clear and easy for $X = BC(\mathbb{R})$, $C_w(\mathbb{R})$. For $X = BC_U(\mathbb{R})$, we use the point (i) of Lemma 2.7. The proof for $X = AP(\mathbb{R})$ can be found in [10, Lemma 1.3]. For $X = PAP_0(\mathbb{R})$, just follow the proof of [6, Lemma 1.3, Ch2. p. 90]. The proof for $X = PAP(\mathbb{R})$ is given in step 2 of the proof of [3, Theorem 1] and finally the proof for $X = PP_w(\mathbb{R})$ can be given in the same way. $\square$

Given two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, their convolution $f * g$, if it exists, is defined by:

$$f * g(t) = \int_{-\infty}^{+\infty} f(s)g(s-t)ds.$$

One can generate various types of almost periodic functions using the convolution.

Proposition 2.11. (*see [9, Proposition 3.4 and Proposition 5.3]*) *The following assertions hold.*

(i) Let $x \in AP(\mathbb{R})$ and $\alpha \in L^1(\mathbb{R})$. Then $x * \alpha \in AP(\mathbb{R})$.

(ii) Let $x \in PAP_0(\mathbb{R})$ and $\alpha \in L^1(\mathbb{R})$. Then $x * \alpha \in PAP_0(\mathbb{R})$.

(iii) Let $x \in PAP(\mathbb{R})$ and $\alpha \in L^1(\mathbb{R})$. Then $x * \alpha \in PAP(\mathbb{R})$.

Now, we give a general way to construct Lipschitz functions $F : PAP(\mathbb{R}) \to PAP(\mathbb{R})$.

Definition 2.12. (see [6]) A continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called pseudo almost periodic in t uniformly with respect to $x \in \mathbb{R}$, if the two following conditions are satisfied:

(i) $\forall x \in \mathbb{R}$, $f(x, \cdot) \in PAP(\mathbb{R})$.

(ii) for all compact set $K \subset \mathbb{R}$, we have that: $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall t \in \mathbb{R}$, $\forall x, y \in K$:

$$|x - y| \leq \delta \Longrightarrow |f(x, t) - f(y, t)| \leq \varepsilon.$$

The set of all such functions will be denoted $PAP_U(\mathbb{R} \times \mathbb{R})$.

For the study of Nemytskii operators we refer to [5, 7, 8].

Lemma 2.13. *(see [5, 8],) Let $f \in PAP_U(\mathbb{R} \times \mathbb{R})$ and $x \in PAP(\mathbb{R})$. Suppose that for some bounded subset B of $\mathbb{R}$, f is bounded on $B \times \mathbb{R}$. Then, the function $[t \mapsto f(x(t), t)] \in PAP(\mathbb{R})$.*

Using the above lemma, we deduce easily the following proposition.

Proposition 2.14. *Let $f \in PAP_U(\mathbb{R} \times \mathbb{R})$ be a Lipschitz function with respect to the first variable, that is, there exists $L_f \geq 0$ such that:*

$$|f(s, t) - f(s', t)| \leq L_f |s - s'|, \ \ \forall s, s', t \in \mathbb{R}.$$

Then, the function F defined by $F(x) := f(x(\cdot), \cdot)$ is a Lipschitz function for the norm $\| \cdot \|_\infty$ that maps $(PAP(\mathbb{R}), \| \cdot \|_\infty)$ into $(PAP(\mathbb{R}), \| \cdot \|_\infty)$.

The above proposition says that each Lipschitz function $f \in PAP_U(\mathbb{R} \times \mathbb{R})$ induce a Lipschitz function $F : (PAP(\mathbb{R}), \| \cdot \|_\infty) \to (PAP(\mathbb{R}), \| \cdot \|_\infty)$. However, the converse is not true in general. Indeed, for any $\alpha \in L^1(\mathbb{R}) \setminus \{0\}$, the $\|\alpha\|_1$-Lipschitz function $F : (PAP(\mathbb{R}), \| \cdot \|_\infty) \to (PAP(\mathbb{R}), \| \cdot \|_\infty)$, defined by

$$F(x)(t) := \sin(t) + \sin(\sqrt{2}t) + (1 + t^2)^{-1} x * \alpha(t)$$

cannot, under any circumstances, be written in the form $f(x(t), t)$ for some $f \in PAP_U(\mathbb{R} \times \mathbb{R})$. This prove that there are many more continuous (Lipschitz) functions $F : (PAP(\mathbb{R}), \| \cdot \|_\infty) \to (PAP(\mathbb{R}), \| \cdot \|_\infty)$, than those which come from the functions $f \in PAP_U(\mathbb{R} \times \mathbb{R})$ as in Proposition 2.14.

Now, we give simple examples satisfying our theorems. We start with the example announced in the introduction.

Example 2.15. Let $\alpha, \beta \in L^1(\mathbb{R})$ be such that $0 < \|\alpha\|_1 \leq 1$.

$$\begin{aligned} F(x)(t) &= \frac{1}{3} \Big(\sin(t) + \sin(\sqrt{2}t) + (1 + t^2)^{-1} x * \alpha(t) \Big) \\ G(x)(t) &= 3 + \sin(2t) + (1 + t^2)^{-1} \cos(x * \beta(t)). \end{aligned}$$

The assumption (H_0) is satisfied by Proposition 2.10. We are going to prove that the assumption (H_1) is also satisfied. Indeed, clearly, $F, G : PAP(\mathbb{R}) \to PAP(\mathbb{R})$ are $\|\alpha\|_1$-Lipschitz and $\|\beta\|_1$-Lipschitz respectively (Notice that F is not bounded but F and G are bounded on bounded sets). We have that $G(x)(t) \geq 1$, for every $(x, t) \in PAP(\mathbb{R}) \times \mathbb{R}$. On the other hand, we have that $|F(x)(t)| \leq \frac{1}{3}(2 + \|x\|_\infty \|\alpha\|_1)$

for every $(x,t) \in PAP(\mathbb{R})\times\mathbb{R}$. Then, for every $x \in PAP(\mathbb{R})$ such that $\|x\|_\infty \leq \frac{1}{\|\alpha\|_1}$, we have that

$$\left|\frac{F(x)(t)}{G(x)(t)}\right| \leq \frac{1}{3}(2+\|x\|_\infty\|\alpha\|_1) \leq 1 \leq \frac{1}{\|\alpha\|_1}.$$

Thus,

$$-\frac{1}{\|\alpha\|_1} \leq \frac{F(x)(t)}{G(x)(t)} \leq \frac{1}{\|\alpha\|_1}, \;\; \forall (x,t) \in B_{[-\frac{1}{\|\alpha\|_1},\frac{1}{\|\alpha\|_1}]} \times \mathbb{R}.$$

Now, let $(x_n) \subset PAP(\mathbb{R})$ be a sequence converging on each compact of $\mathbb{R}$ to some $x \in BC(\mathbb{R})$. Then, there exists a constant $M \geq \|x\|_\infty$ such that $\|x_n\|_\infty \leq M$ for all $n \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there exists $A_\varepsilon > 0$ such that for every $|t| \geq A_\varepsilon$, we have that $(1+t^2)^{-1} \leq \frac{\varepsilon}{2M\|\alpha\|_1}$. Thus,

$$(1+t^2)^{-1}|x_n * \alpha(t) - x * \alpha(t)| \leq 2M\|\alpha\|_1(1+t^2)^{-1} < \varepsilon, \;\; \forall |t| \geq A_\varepsilon.$$

On the other hand, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have that $\sup_{t\in[-A_\varepsilon,A_\varepsilon]} |x_n(t) - x(t)| < \varepsilon$. Hence,

$$\sup_{t\in\mathbb{R}}(1+t^2)^{-1}|x_n * \alpha(t) - x * \alpha(t)| \leq 2\varepsilon, \;\; \forall n \geq N.$$

Hence, we get that the sequence $(F(x_n))$ converges in $(BC(\mathbb{R}), \|\cdot\|_\infty)$ and so it is relatively compact in $PAP(\mathbb{R})$. The same argument hold for $(G(x_n))$. Thus, the assumption (H_0) and (H_1) are satisfied, so Theorem 2.1 gives at last one solution $x^* \in B_{[-\frac{1}{\|\alpha\|_1},\frac{1}{\|\alpha\|_1}]} \subset PAP(\mathbb{R})$ for the equation (E).

Example 2.16. (**Example in the space $PP_w(\mathbb{R}) \subset PAP(\mathbb{R})$ under the assumption (H_1)**) As a simple consequence of Theorem 2.1, we obtain that the following equation has a nonnegative solution $x^* \in PP_w(\mathbb{R})$

$$x' + G(x)x = F(x)$$

where $F, G : PP_w(\mathbb{R}) \to PP_w(\mathbb{R})$ are defined by $G(x)(t) = e^{\|x\|_{0,1}+\frac{1}{1+t^2}\sin(x(t))}$ and $F(x)(t) := 3 + \sin(\|x\|_{0,1}) + \frac{1}{1+t^2}\cos(x(t))$, where $\|x\|_{0,1} := \int_0^1 |x(s)|ds \leq \|x\|_\infty$.

- The space $PP_w(\mathbb{R})$ satisfies (H_0) by Proposition 2.10.
- The assumption (H_1) is satisfied. Indeed, clearly we have that

$$\inf_{x\in X, t\in\mathbb{R}} G(x)(t) \geq e^{-1} > 0.$$

Let us set $G_0(x,t) := \|x\|_{0,1} + \frac{1}{1+t^2}\sin(x(t))$, we have that

$$\begin{aligned}|G_0(x,t) - G_0(y,t)| &\leq \|x-y\|_{0,1} + \frac{1}{1+t^2}|\sin(x(t)) - \sin(y(t))| \\ &\leq \|x-y\|_{0,1} + \frac{1}{1+t^2}|x(t) - y(t)| \\ &\leq 2\|x-y\|_\infty.\end{aligned}$$

Then, G_0 is Lipschitz from $PP_w(\mathbb{R})$ into $PP_w(\mathbb{R})$. Since $t \mapsto e^t$ is continuous, it follows that G is continuous from $PP_w(\mathbb{R})$ into $PP_w(\mathbb{R})$. On the other hand, it is clear that G is bounded on bounded sets. Finally, let $(x_n) \subset X$ be a sequence converging on each compact of $\mathbb{R}$. Notice that $|\|x_n\|_{0,1} - \|x_m\|_{0,1}| \leq \|x_n - x_m\|_{0,1} \leq$

$\sup_{t\in[0,1]}|x_n(t)-x_m(t)|$. On the other hand, we see that for every $A\geq 1$, we have that

$$\begin{aligned}\sup_{t\in\mathbb{R}}|\frac{1}{1+t^2}\sin(x_n(t))-\frac{1}{1+t^2}\sin(x_m(t))| &\leq \\ \max(\sup_{t\in[-A,A]}|\frac{1}{1+t^2}(\sin(x_n(t))-\sin(x_m(t)))|,\frac{2}{1+A^2}) &\leq \\ \max(\sup_{t\in[-A,A]}|x_n(t)-x_m(t)|,\frac{2}{1+A^2}).&\end{aligned}$$

It follows that

$$\begin{aligned}\|G_0(x_n)-G_0(x_m)\|_\infty &\leq \sup_{t\in[0,1]}|x_n(t)-x_m(t)| \\ &\quad +\max(\sup_{t\in[-A,A]}|x_n(t)-x_m(t)|,\frac{2}{1+A^2}).\end{aligned}$$

So, $\limsup_{n,m\to+\infty}\|G_0(x_n)-G_0(x_m)\|_\infty\leq\frac{2}{1+A^2}$ for every $A\geq 1$. Sending A to $+\infty$, we get that $(G_0(x_n))$ converges in $(BC(\mathbb{R}),\|\cdot\|_\infty)$ and by the continuity of the function $t\mapsto e^t$, we have that $(G(x_n))$ converges in $(BC(\mathbb{R}),\|\cdot\|_\infty)$.

As above, we see that $(F(x_n))$ converges in $(BC(\mathbb{R}),\|\cdot\|_\infty)$. Now, it is clear that $0\leq F(x)(t)\leq 5$ for every $x\in PP_w(\mathbb{R})$ and every $t\in\mathbb{R}$. Thus, we can take

$$0\leq k:=\inf_{(x,t)\in X\times\mathbb{R}}\frac{F(x)(t)}{G(x)(t)}\text{ and }M:=\sup_{(x,t)\in X\times\mathbb{R}}\frac{F(x)(t)}{G(x)(t)},$$

and so the assumption (H_1) is satisfied. Using Theorem 2.1 we get that there exists at least one nonnegative solution of the equation (E).

Example 2.17. (**Example in the space $C_{2\pi}(\mathbb{R})$ under the assumption** $(\tilde{H}_1)$.) We have the following simple example.

- $G:C_{2\pi}(\mathbb{R})\to C_{2\pi}(\mathbb{R})$, defined by

$$G(x)=4+(1+\|x\|_{0,1})(1+\sin(\cdot)),$$

is a 2-Lipschitz and satisfies $\inf_{x\in X,t\in\mathbb{R}}G(x)(t)\geq l=4$ (where $\|x\|_{0,1}:=\int_0^1|x(s)|ds\leq\|x\|_\infty$)

- $F:C_{2\pi}(\mathbb{R})\to C_{2\pi}(\mathbb{R})$, defined by $F(x):=2+\sin(\cdot)+\cos(x)$, is nonnegative, 1-Lipschitz and bounded by $r=4$.

Since $\max(\frac{r}{l^2},\frac{1}{l})(L_F+L_G)=\frac{3}{4}<1$, then $(\tilde{H}_1)$ is satisfied. This permits to give thanks to Theorem 2.1 a continuous 2π-periodic solution to the equation (E). Moreover, there exists a unique solution in $B_{[0,1]}$.

3. Global attractivity of solutions

In this section, we give a result on the attractivity of solutions of the equation (E).

Definition 3.1. A solution x^* of the equation (E), is said to be globally attractive if for any other solution x of (E), we have that $\lim_{t\to+\infty}|x(t)-x^*(t)|=0$.

Consider the following condition:

(C) the functions $F, G : X \to X$ satisfies (H_1) and moreover: there exists $L_F, L_G \geq 0$ such that

(i) $|F(x)(t) - F(y)(t)| \leq L_F|x(t) - y(t)|$ and $|G(x)(t) - G(y)(t)| \leq L_G|x(t) - y(t)|$, $\forall x, y \in X$ and $t \in \mathbb{R}$.

(ii) $l > L_G \max(M, -k) + L_F$, where $l := \inf_{x\in X, t\in\mathbb{R}} G(x)(t) > 0$, k and M ($k \leq M$) are given by the assumption (H_1).

Notice that general examples of functions satisfying the point (i) above, are given by Proposition 2.14 (see also Example 3.4).

Recall that the upper Dini derivative of a function $W : \mathbb{R} \to \mathbb{R}$ is given by the following formula:

$$D^+W(t) = \limsup_{h\to 0^+} \frac{W(t+h) - W(t)}{h}.$$

We need the following result from [12].

Theorem 3.2. (*[12, Theorem 3.]*) *If W is a continuous function that has a finite Dini derivative $D^+W(t)$ at every point t of $\mathbb{R}$, then*

$$\underline{\int_a^b} D^+W(t)dt \leq W(b) - W(a) \leq \overline{\int_a^b} D^+W(t)dt,$$

for each interval $[a, b]$, where the integrals are the lower and upper Riemann integrals, respectively.

The previous theorem applies in particular to $W(t) = |f(t)|$, for all $t \in \mathbb{R}$, where f is a continuously differentiable function on $\mathbb{R}$. Recall that in this case, we have: W is a continuous function and $D^+W(t) = sgn(f(t))f'(t)$ for all $t \in \mathbb{R}$, where $sgn(t)$ denotes the number which is equal to 1 if $t \geq 0$ and -1 otherwize. Thus, the above theorem applies in this case, this is what we will need in the prove the following theorem.

Theorem 3.3. *Under the assumption (H_0) and (C), there exists at last one solution $x^* \in X$ for the equation (E), which is globally attractive.*

Proof. By Theorem 2.1, there exists at last one solution $x^* \in B_{[k,M]}$, this implies that $\|x^*\|_\infty \leq \max(M, -k)$. Suppose that x is another solution of (E). In this case, we have

$$x' = -G(x)x + F(x),$$

$$(x^*)' = -G(x^*)x^* + F(x^*).$$

In particular, x, x^* are continuously differentiable functions. On the other hand, we have $|x'(t)| \leq \|G(x)\|_\infty \|x\|_\infty + \|F(x)\|_\infty < +\infty$ for every $t \in \mathbb{R}$ and so x is a Lipschitz function on $\mathbb{R}$ by the mean value theorem. Similarily, x^* is a Lipschitz function on $\mathbb{R}$. Consider the following Lyapunov functional:

$$W(t) = |x(t) - x^*(t)|, \quad \forall t \in \mathbb{R}.$$

After calculating the Dini derivative of W, we get,

$$\begin{aligned}
D^+W(t) &= sgn(x(t)-x^*(t))\left(x'(t)-(x^*)'(t)\right) \\
&= sgn(x(t)-x^*(t))[-G(x)(t)x(t)+G(x^*)(t)x^*(t)+F(x)(t)-F(x^*)(t)] \\
&= sgn(x(t)-x^*(t))[-(G(x)(t)(x(t)-x^*(t))+ \\
&\quad (G(x^*)(t)-G(x)(t))x^*(t)+F(x)(t)-F(x^*)(t)] \\
&= -G(x)(t)|x(t)-x^*(t)|+sgn(x(t)-x^*(t))[G(x^*)(t)-G(x)(t)]x^*(t)+ \\
&\quad sgn(x(t)-x^*(t))[F(x)(t)-F(x^*)(t)] \\
&\leq -l|x(t)-x^*(t)|+(L_G\|x^*\|_\infty+L_F)|x(t)-x^*(t)| \\
&\leq -(l-(L_G\max(M,-k)+L_F))|x(t)-x^*(t)|.
\end{aligned}$$

By using the right hand inequality in Theorem 3.2, we get

$$W(t)+\int_0^t(l-(L_G\max(M,-k)+L_F))|x(t)-x^*(t)|ds \quad\leq\quad W(0).$$

Since $W(t)\geq 0$ for every $t\in\mathbb{R}$, it follows that

$$\limsup_{t\to+\infty}\int_0^t|x(t)-x^*(t)|ds \quad\leq\quad \frac{W(0)}{l-(L_G\max(M,-k)+L_F)}<+\infty.$$

Since $x,x^*\in X$ are uniformly continuous on $\mathbb{R}$ (in fact Lipschitz), we obtain that $\lim_{t\to+\infty}|x(t)-x^*(t)|=0$. □

Example 3.4. (**Example of globally attractive solution in the space** $PAP(\mathbb{R})$ **under the assumption** (C).)

• $G: PAP(\mathbb{R})\to PAP(\mathbb{R})$, defined by

$$G(x)(t)=4+\sin(t)+\sin(\sqrt{2}t)+\frac{1}{1+t^2}\cos(x(t)),$$

is 1-Lipschitz (on the variable x) and $l:=\inf_{x\in X,t\in\mathbb{R}}G(x)(t)\geq 1>0$.

• $F: PAP(\mathbb{R})\to PAP(\mathbb{R})$, defined by

$$F(x)(t):=\frac{1}{10}[2+\cos(t)+\frac{1}{1+t^2}\sin(x(t))],$$

is 1/10-Lipschitz (on the variable x) and $0\leq F(x)(t)\leq 4/10$.

We have that

$$0=k\leq\frac{F(x)(t)}{G(x)(t)}\leq M=4/10.$$

As in the Example 2.16 the assumption (H_1) is satisfied. Since $\max(M,-k)=4/10$, $L_G\leq 1$, $L_F\leq 1/10$ we have $l\geq 1>4/10+1/10\geq\max(M,-k)L_G+L_F$, then the condition (C) is satisfied and so by Theorem 3.3 there exists at last one positive solution in $PAP(\mathbb{R})$ which is globally attractive.

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