Pure and Applied Functional Analysis
Volume 8, Number 1, 2023, 49–66

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

DERIVED HEAT TRACE ASYMPTOTICS FOR THE DE RHAM AND DOLBEAULT COMPLEXES

J. ÁLVAREZ LÓPEZ AND P. GILKEY

Abstract. We examine the derived heat trace asymptotics in both the real and the complex settings for a generalized Witten perturbation. If the dimension is even, in the real context we show the integral of the local density for the derived heat trace asymptotics is half the Euler characteristic of the underlying manifold. In the complex context, we assume the underlying geometry is Kähler and show the integral of the local density for the derived heat trace asymptotics defined by the Dolbeault complex is a characteristic number of the complex tangent bundle and the twisting vector bundle. We identify this characteristic number if the real dimension is 2 or 4. In both the real and complex settings, the local density differs from the corresponding characteristic class by a divergence term.

1. Introduction

1.1. **Motivation.** Let h be a smooth function on a compact Riemannian manifold $\mathcal{M} = (M, g)$ without boundary of dimension m. In this context, Witten [27] defined a perturbed de Rham differential $d_h := d + \operatorname{ext}(dh)$ where $\operatorname{ext}(\cdot)$ denotes left exterior multiplication. Since $d_h = e^{-h} \circ d \circ e^h$, d_h is gauge equivalent to d and consequently the Betti numbers are unchanged. Let $\operatorname{int}(\cdot)$ denote left interior multiplication. The perturbed de Rham co-differential is given by $\delta_h := \delta + \operatorname{int}(dh)$ and the perturbed Laplacian is given by $\Delta_h := d_h\delta_h + \delta_h d_h$. For a generic metric g, Δ_h and Δ will not be gauge equivalent and their spectra will be different. Witten obtained the Morse inequalities by analyzing the spectral asymptotics of the family of operators Δ_{sh} as $s \to \infty$ when h is a Morse function; we also refer to subsequent work of Bismut and Zhang [6] and of Helffer and Sjöstrand [17]. One can replace dh by a real closed 1-form ω to define $d_\omega := d + \operatorname{ext}(\omega)$. Since d_ω need not be gauge equivalent to d, the twisted Betti numbers β^p_ω can be different. However the numbers $\beta^p_{s\omega}$ defined by the family $s \to s\omega$ for $s \in \mathbb{R}$ have well defined ground values (common values of the numbers $\beta^p_{s\omega}$ for all but finitely many s, where these numbers may jump), which are called the Novikov numbers and only depend upon the cohomology class defined by ω in de Rham cohomology. If ω is of Morse type, these Novikov numbers satisfy a generalized Morse inequality; we refer to work of Braverman and Farber [7], Pazhitnov [22], and other authors [9, 10, 16, 20].

2020 *Mathematics Subject Classification.* 58J20.

Key words and phrases. Witten deformation, local index density, de Rham complex, Dolbeault complex, derived heat trace asymptotics.

Research partially supported by grants: PID2019-105138GB-C21 (Spain), MTM2017-89686-P (AEI/FEDER, UE).

We studied the local index density for Witten deformation of the de Rham and Dolbeault complexes in two previous papers. In [1], we used invariance theory to show that the local index density of the twisted de Rham complex is the Euler form if m is even, and vanishes if m is odd; in particular, it does not depend on ω. A different proof of this result was given previously by the first author, Kordyukov, and Leichtnam [3], where it was applied to study certain trace formulas for foliated flows. Let (E,h) be a Hermitian vector bundle over a Kähler manifold (M,g,J). In analogy to the real setting, one may introduce the perturbed Dolbeault operator $\bar\partial_{\bar\omega}=\bar\partial+\operatorname{ext}(\bar\omega)$ mapping $C^\infty(E\otimes\Lambda^{p,q}(M))\to C^\infty(E\otimes\Lambda^{p,q+1}(M))$ where ω is a ∂ closed form of type $(1,0)$. If $\omega=0$, it follows from the work of Atiyah, Bott, and Patodi [5] and the work the second author [11, 12] that the local index density is $\{\operatorname{Td}(M,g,J)\wedge\operatorname{ch}(E,h)\}_m$ as we shall explain presently; this result can fail if the metric is not assumed to be Kähler. In [2], we showed that the index density of $\bar\partial_{\bar\omega}$ exhibits non-trivial dependence on ω; we summarize the results of [1, 2] below in Theorem 1.1.

Günther and Schimming [15] defined a sequence of secondary heat invariants for the de Rham complex of which the first is called the derived heat invariant. In a different form, this invariant was used by Ray and Singer [25] to study the analytic torsion. In the perturbed setting, define $\mathfrak{a}^{\mathrm{deR}}_{m,n}:=\sum_p(-1)^p p\cdot a_{m,n}(x,\Delta^p_\omega)$, where Δ^p_ω is the restriction of Δ_ω to forms of degree p, and $a_{m,n}(x,\Delta^p_\omega)$ is its heat trace invariant of order n. Another version of this invariant, $\mathfrak{a}^{\mathrm{Dol}}_{m,n}$, is similarly defined for the Witten-Novikov pertubation of the Dolbeault complex in the Kähler setting. Our results for these invariants are given below in Theorem 1.3. We summarize those results as follows. Consider first the Riemannian setting. Let $\mathcal{E}_{m,2k}$ denote the Euler invariant of order $2k$ (see Section 1.3). We prove that $\mathfrak{a}^{\mathrm{deR}}_{m,n}$ vanishes if $n<m-1$, and exhibits a nontrivial dependence on ω for even $n\geq m$. We show that $\mathfrak{a}^{\mathrm{deR}}_{m,m-1}=\mathcal{E}_{m,m-1}$ if m is odd, and that $\int_M\mathfrak{a}^{\mathrm{deR}}_{m,m}\operatorname{dvol}=\frac{m}{2}\int_M\mathcal{E}_{m,m}\operatorname{dvol}$ if m is even. In particular, $\int_M\mathfrak{a}^{\mathrm{deR}}_{m,m}\operatorname{dvol}$ is independent of ω, which is relevant in [4] to study certain zeta invariants associated to closed 1-forms (our original motivation). In the Kähler setting, the situation is more complicated. We show that $\mathfrak{a}^{\mathrm{Dol}}_{m,n}=0$ if $n<m-2$, and that $\mathfrak{a}^{\mathrm{Dol}}_{m,n}$ exhibits a nontrivial dependence on ω for even $n\geq m-2$. We describe $\mathfrak{a}^{\mathrm{Dol}}_{m,m-2}$ as a perturbation of $\frac{1}{(\mathfrak{m}-1)!}g(\operatorname{Td}(M,g,J)\wedge\operatorname{ch}(E,h),\Omega^{\mathfrak{m}-1})$. We prove that $\int_M\mathfrak{a}^{\mathrm{Dol}}_{m,m}\operatorname{dvol}$ is a characteristic invariant independent of ω. We determine $\int_M\mathfrak{a}^{\mathrm{Dol}}_{m,m}\operatorname{dvol}$ in general if $m=2$ or $m=4$ in terms of characteristic classes of the complex tangent bundle of M and the twisting bundle E. This is a global result as the local invariants $\mathfrak{a}^{\mathrm{Dol}}_{m,m}$ exhibit non-trivial dependence on the twisting $(1,0)$ form ω.

1.2. **The real setting.** Let $\mathcal{M}:=(M,g,\omega)$ where (M,g) is an m-dimensional Riemannian manifold without boundary which is equipped with an auxiliary real closed 1-form ω. Let dvol be the associated Riemannian measure on M. Let $\mathfrak{J}^p_{m,n}$ be the space of smooth p-form valued invariants which are homogeneous of weight n in the derivatives of the metric and ω; we refer to Section 2 for a precise definition. These spaces vanish for $p+n$ odd. The First Theorem of Invariants of Weyl [26] shows that such invariants can be expressed in terms of contractions of indices in pairs where the indices range from 1 to m. We illustrate this result as follows. Let R_{ijkl} be the components of the curvature tensor and let ω_i be the components of ω. The scalar curvature $\tau \in \mathfrak{J}^0_{m,2}$, the norm squared $\|\omega\|^2 \in \mathfrak{J}^0_{m,2}$, and the first Pontryagin form $p_1 \in \mathfrak{J}^4_{m,4}$ are given by

$$\tau = \sum_{i,j,k,\ell=1}^{m} g^{i\ell} g^{jk} R_{ijkl}, \quad \|\omega\|^2 = \sum_{j,k=1}^{m} g^{jk}\omega_j\omega_k,$$
$$p_1 = -\frac{1}{8\pi^2} g^{ad} g^{bc} R_{ijab} R_{klcd} dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell .$$

The restriction map $r : \mathfrak{J}^p_{m,n} \to \mathfrak{J}^p_{m-1,n}$ is defined by restricting the range of summation to range from 1 to $m-1$ rather than from 1 to m. We shall discuss the restriction map further in Section 2.

1.3. **The Euler form.** Let $\mathcal{E}_{m,2k}$ be the integrand of the Chern-Gauss-Bonnet Theorem [8]:

$$\mathcal{E}_{m,2k} := \sum_{i_1,\dots,i_k,j_1,\dots,j_k=1}^{m} \frac{(-1)^k}{8^k\pi^k k!} g(e^{i_1} \wedge \dots \wedge e^{i_k}, e^{j_1} \wedge \dots \wedge e^{j_k})$$
$$R_{i_1 i_2 j_1 j_2} \dots R_{i_{k-1} i_k j_{k-1} j_k} \in \mathfrak{J}^0_{m,2k} .$$

This element is universal; $r(\mathcal{E}_{m,2k}) = \mathcal{E}_{m-1,2k}$ for any m. One has, for example,

$$\mathcal{E}_{m,2} := (4\pi)^{-1}\tau \text{ and } \mathcal{E}_{m,4} := (32\pi^2)^{-1}\{\tau^2 - 4\|\rho\|^2 + \|R\|^2\}$$

where ρ is the Ricci tensor. Similar formulas for $\mathcal{E}_{m,6}$ and $\mathcal{E}_{m,8}$ can be found in Pekonen [23]. Let δ be the co-derivative

$$\delta : \mathfrak{J}^1_{m,n-1} \to \mathfrak{J}^0_{m,n} .$$

1.4. **The Kähler setting.** Let $\mathcal{K} := (M, g, J, E, h, \omega)$ where (M, g, J) is a Kähler manifold of real dimension $m = 2\mathfrak{m}$, (E, h) is a holomorphic vector bundle over M which is equipped with a Hermitian fiber metric h, and ω is a ∂ closed $(1, 0)$ form. Let Ω be the Kähler form of $\mathcal{K}$; $\operatorname{dvol} = \frac{1}{\mathfrak{m}!}\Omega^{\mathfrak{m}}$. Let $\mathfrak{K}^k_{m,n}$ be the corresponding space of invariants in the complex setting; we refer to Section 2 for precise details. These spaces vanish for $k + n$ odd. Again, we have a natural restriction map $r : \mathfrak{K}^k_{m,n} \to \mathfrak{K}^k_{m-2,n}$. We can pair form valued invariants with an appropriate power of the Kähler form to obtain scalar invariants; if $P \in \mathfrak{K}^{2k}_{m,2k}$ then $g(P, \Omega^k) \in \mathfrak{K}^0_{m,2k}$. For example, if $k = \mathfrak{m}$, then the Hodge $\star$ operator takes the form $\star P = \frac{1}{\mathfrak{m}!} g(P, \Omega^{\mathfrak{m}})$.

1.5. **The ring of characteristic forms.** Let c_k, ch_k, and Td_k be the k^{th} Chern class, Chern character, and Todd class, respectively (see [18]). For example, $\operatorname{ch}_0 = \dim(E)$, $\operatorname{Td}_0 = 1$,

$$\begin{array}{lll} \operatorname{ch}_1 = c_1, & \operatorname{ch}_2 = \frac{1}{2}(c_1^2 - 2c_2), & \operatorname{ch}_3 = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3), \\ \operatorname{Td}_1 = \frac{c_1}{2}, & \operatorname{Td}_2 = \frac{c_1^2+c_2}{12}, & \operatorname{Td}_3 = \frac{c_1c_2}{24} . \end{array}$$

Let $T_cM := (TM, J)$ be the associated complex tangent bundle. We decompose the graded ring of characteristic forms

$$\mathfrak{C}_m := \mathbb{C}[\operatorname{ch}_1(T_cM), \dots, \operatorname{ch}_{\mathfrak{m}}(T_cM), \operatorname{ch}_1(E, h), \dots, \operatorname{ch}_{\mathfrak{m}}(E, h)]$$

into homogeneous components $\mathfrak{C}_m = \oplus_k \mathfrak{C}_m^{2k}$ where $\mathfrak{C}_m^{2k} \subset \mathfrak{K}_{m,2k}^{2k}$.

1.6. **The Witten deformation.** In the real setting, let $d_\omega := d + \operatorname{ext}(\omega)$ be the Witten deformation of the exterior derivative; the adjoint is then given by $\delta_\omega := \delta + \operatorname{int}(\omega)$. Decompose the associated Laplacian

$$\Delta_{\mathcal{M}} := d_\omega \delta_\omega + \delta_\omega d_\omega = \oplus_p \Delta_{\mathcal{M}}^p$$

where $\Delta_{\mathcal{M}}^p$ is a self-adjoint elliptic operator of Laplace type on $C^\infty(\Lambda^p(M))$. Similarly, in the complex setting, let $\bar\partial_\omega = \bar\partial + \operatorname{ext}(\bar\omega)$ define the deformation of the Dolbeault operator; the adjoint is then $\delta''_\omega = \delta'' + \operatorname{int}(\omega)$. Decompose the associated Laplacian

$$\Delta_{\mathcal{K}} := 2(\bar\partial_{\bar\omega}\delta''_\omega + \delta''_\omega\bar\partial_{\bar\omega}) = \oplus_{p,q}\Delta_{\mathcal{K}}^{p,q}$$

where $\Delta_{\mathcal{K}}^{p,q}$ is a self-adjoint operator of Laplace type on $C^\infty(\Lambda^{p,q}(M)\otimes E)$.

1.7. **The local index density.** If Δ is an operator of Laplace type on a compact Riemannian manifold of dimension m, let $a_{m,n}(x,\Delta)$ be the local heat trace asymptotics. If $f \in C^\infty(M)$, then

$$\operatorname{Tr}_{L^2}\{fe^{-t\Delta}\} \sim \sum_{n=0}^\infty t^{(n-m)/2}\int_M f(x)a_{m,n}(x,\Delta)\,\mathrm{dvol} \quad \text{as} \quad t \downarrow 0\,.$$

The invariants $a_{m,n}(x,\Delta)$ vanish for n odd. We introduce the local index densities for the de Rham complex and Dolbeault complex by setting:

$$\begin{aligned} a_{m,n}^{\mathrm{deR}}(x) &:= \sum_p(-1)^p a_{m,n}(x,\Delta_{\mathcal{M}}^p) \in \mathfrak{J}_{m,n}^0,\\ a_{m,n}^{\mathrm{Dol}}(x) &:= \sum_p(-1)^p a_{m,n}(x,\Delta_{\mathcal{K}}^{0,p}) \in \mathfrak{K}_{m,n}^0\,. \end{aligned}$$

A cancellation argument due to Bott yields:

$$\begin{aligned} &\int_M a_{m,n}^{\mathrm{deR}}(x)\,\mathrm{dvol}(g) = \left\{\begin{array}{ll} 0 & \text{for } n \neq m\\ \text{Euler characteristic}(M) & \text{for } n = m\end{array}\right\},\\ &\int_M a_{m,n}^{\mathrm{Dol}}(x)\,\mathrm{dvol}(g) = \left\{\begin{array}{ll} 0 & \text{for } n \neq m\\ \text{arithmetic genus}(M,E) & \text{for } n = m\end{array}\right\}. \end{aligned} \tag{1.1}$$

Let $\Im(\omega)$ be the imaginary part of ω. Set $\Theta := \sum_k \frac{1}{k!\pi^k}\{d\Im(\omega)\}^k$. We have the following previous results [1,2] of the authors.

Theorem 1.1. $a_{m,n}^{\mathrm{deR}} = \left\{\begin{array}{ll} 0 & \text{if } n < m\\ \mathcal{E}_{m,m} & \text{if } n = m\end{array}\right\},$

$$a_{m,n}^{\mathrm{Dol}} = \left\{\begin{array}{ll} 0 & \text{if } n < m\\ \frac{1}{\mathfrak{m}!}g(\Omega^{\mathfrak{m}},\{\mathrm{Td}(T_cM)\wedge\mathrm{ch}(E)\wedge\Theta\}_m) & \text{if } n = m\end{array}\right\}.$$

Remark 1.2. We note that this result for $a_{m,n}^{\mathrm{Dol}}$ can fail in the Hermitian setting. Although Equation (1.1) continues to hold if (M,g,J) is only assumed Hermitian, it is necessary to restrict to the context of Kähler geometry to identify the local index density with a characteristic form even if E is trivial and $\omega = 0$. We refer to Gilkey, Nikčević, and Pohjanpelto [14] for details.

1.8. **Derived invariants.** The following is the first in a sequence of invariants introduced by Günther and Schimming [15] in the real category (see the discussion on page 181 of Gilkey [13]); a related invariant appears in the discussion by Ray and Singer [25] of analytic torsion. Define

$$\mathfrak{a}^{\text{deR}}_{m,n} := \sum_p (-1)^p p \cdot a_{m,n}(x, \Delta^p_{\mathcal{M}}) \in \mathfrak{I}^0_{m,n}\,,$$

$$\mathfrak{a}^{\text{Dol}}_{m,n} := \sum_p (-1)^p p \cdot a_{m,n}(x, \Delta^{0,p}_{\mathcal{K}}) \in \mathfrak{K}^0_{m,n}\,.$$

If we define

$$\begin{array}{c}\mathfrak{A}^{\text{Der}}_{m,n}(s) := \sum_p (-1)^p s^p a_{m,n}(x, \Delta^p),\\ \mathfrak{A}^{\text{Dol}}_{m,n}(s) := \sum_p (-1)^p s^p a_{m,n}(x, \Delta^{0,p}),\end{array} \tag{1.2}$$

we then have

$$\begin{array}{ll} a^{\text{deR}}_{m,n} = \mathfrak{A}^{\text{Der}}_{m,n}(s)|_{s=1}, & \mathfrak{a}^{\text{deR}}_{m,n} = \partial_s \mathfrak{A}^{\text{Der}}_{m,n}(s)|_{s=1},\\ a^{\text{deR}}_{m,n} = \mathfrak{A}^{\text{Dol}}_{m,n}(s)|_{s=1}, & \mathfrak{a}^{\text{deR}}_{m,n} = \partial_s \mathfrak{A}^{\text{Dol}}_{m,n}(s)|_{s=1}\,.\end{array} \tag{1.3}$$

This formalism will play a useful role subsequently in Section 7 and perhaps justifies the words "derived" in discussing these invariants. The terminology "derived", is of course, not ours – see for example Bismut and Zhang [6]. Ramachandran [24] notes concerning the invariant $\sum_p (-1)^p p \dim H^p(M)$ that "...this secondary invariant dates back to 1848 when it was introduced by A. Cayley...".

Theorem 1.3.

(1) *Let $\mathcal{M} = (M, g, \omega)$ be a Riemannian manifold which is equipped with an auxiliary closed 1-form ω.*
  (a) *If $n < m-1$, then $\mathfrak{a}^{\text{deR}}_{m,n} = 0$.*
  (b) *If m is odd, then $\mathfrak{a}^{\text{deR}}_{m,m-1} = \mathcal{E}_{m,m-1}$ is independent of ω.*
  (c) *If m is even, then $\mathfrak{a}^{\text{deR}}_{m,m} = \frac{m}{2}\mathcal{E}_{m,m} + \delta Q^1_{m,m-1}$ where $Q^1_{m,n-1} \in \mathfrak{I}^1_{m,n-1}$ is 1-form valued.*

(2) *Let $\mathcal{K} = (M, g, J, E, h, \omega)$ where (M, g, J) is a Kähler manifold, (E, h) is a holomorphic Hermitian vector bundle over M, and ω is an auxiliary ∂ closed form of type $(1,0)$.*
  (a) *If $n < m-2$, $\mathfrak{a}^{\text{Dol}}_{m,n}(\mathcal{K}) = 0$.*
  (b) *If $n = m-2$,*
  $\mathfrak{a}^{\text{Dol}}_{m,m-2} = \frac{1}{(\mathfrak{m}-1)!} g(\text{Td}(M,g,J) \wedge \text{ch}(E,h) \wedge \Theta, \Omega^{\mathfrak{m}-1})$.
  Moreover, there exists $Q^1_{m,m-3} \in \mathfrak{K}^1_{m,m-3}$ so
  $\mathfrak{a}^{\text{Dol}}_{m,m-2} = \frac{1}{(\mathfrak{m}-1)!} g(\text{Td}(M,g,J) \wedge \text{ch}(E,h), \Omega^{\mathfrak{m}-1}) + \delta Q^1_{m,m-3}$.
  (c) $\mathfrak{a}^{\text{Dol}}_{m,m} = \frac{1}{\mathfrak{m}!} g(R^m_m, \Omega^{\mathfrak{m}}) + \delta Q^1_{m,m-1}$ *where $Q_{m,m-1} \in \mathfrak{K}^1_{m,m-1}$ is 1-form valued and where $R^m_m \in \mathfrak{C}^m_m$ is a characteristic form.*

Assertion (1c) and Assertion (2c) show that $\int_M \mathfrak{a}^{\text{deR}}_{m,m}\,\text{dvol}$ is a characteristic number which is independent of ω in the Riemannian setting and $\int_M \mathfrak{a}^{\text{Dol}}_{m,m}\,\text{dvol}$ is a characteristic number which is independent of the structures in the Kähler setting. In Theorem 1.3 (2b), we may eliminate Θ at the cost of introducing an additional divergence term. The divergence terms are in general present. We have, for example, $Q^1_{2,1} = (2\pi)^{-1}\Theta$ in Assertion (2b). We will establish the following result in Section 7. It shows, in particular, that $\mathfrak{a}^{\mathrm{Dol}}_{m,m}$ is not a multiple of $a^{\mathrm{Dol}}_{m,m}$.

Theorem 1.4. *Use Theorem 1.3 to express* $\mathfrak{a}^{\mathrm{Dol}}_{m,m} = \frac{1}{\mathfrak{m}!}g(R^m_m, \Omega^{\mathfrak{m}}) + \delta Q^1_{m,m-1}$ *for* $Q^1_{m,m-1} \in \mathfrak{K}^1_{m,m-1}$ *and* $R^m_m \in \mathfrak{C}^m_m$.

(1) *Let* $\mathcal{M} = (M, g, J, E, h)$ *where* (M, g, J) *is a Kähler manifold and* (E, h) *is a Hermitian vector bundle bundle.*
  (a) *If* $m = 2$, $R^2_2 = \frac{1}{3}c_1(T_cM)\operatorname{ch}_0(E) + \frac{1}{2}c_1(E)$.
  (b) *If* $m = 4$,
  $R^4_4 = (\operatorname{Td}_2 + \frac{1}{24}c_1^2)(T_cM)\operatorname{ch}_0(E) + \frac{7}{12}c_1(T_cM)c_1(E) + \operatorname{ch}_2(E)$.

(2) *Let* $\mathcal{M} = \mathcal{M}_1 \times \cdots \times \mathcal{M}_{\mathfrak{m}}$ *where* $\mathcal{M}_i = (M_i, g_i, J_i)$ *are Riemann surfaces and* (E_i, h_i) *are Hermitian line bundles.*
  (a) *If* (M_i, g_i, J_i) *are flat tori,*
  $R^m_m(\mathcal{M}) = \frac{1}{2}\mathfrak{m}a^{\mathrm{Dol}}_{m,m}(\mathcal{M}) = \frac{1}{2}\mathfrak{m}\operatorname{ch}_{\mathfrak{m}}(E)$.
  (b) *If* (E_i, h_i) *is trivial for all* i,
  $R^m_m(\mathcal{M}) = \frac{2}{3}\mathfrak{m}a^{\mathrm{Dol}}_{m,m}(\mathcal{M}) = \frac{2}{3}\mathfrak{m}\operatorname{Td}_{\mathfrak{m}}(T_cM)$.

2. Spaces of invariants

2.1. **Spaces of local formula in the real setting.** It is necessary to be a bit careful concerning what we mean by a local formula. Introduce local coordinates $x = (x^1, \ldots, x^m)$ which are centered at a point P of M. Let $U = (i_1, \ldots, i_a)$ be a collection of indices. Decompose $\omega = \omega_i dx^i$. Set $|U| := a$, $\partial_i := \frac{\partial}{\partial x^i}$,

$$\begin{aligned}
&\partial^U_x := \partial_{i_1}\ldots\partial_{i_a}, && g_{ij} := g(\partial_i, \partial_j), && g_{ij/U} := \partial^U_x g_{ij},\\
&\omega_{i/U} := \partial^U_x\omega_i, && \mathrm{weight}\{g_{ij/U}\} = |U|, && \mathrm{weight}\{\omega_{i/U}\} = 1 + |U|.
\end{aligned}$$

Let $S_m(\mathbb{R})$ be the vector space of symmetric $m \times m$ real matrices. Let $\mathcal{O}_m \subset S_m(\mathbb{R})$ be the subset of matrices g so that g defines a positive definite inner product; $\mathcal{O}_m$ is an open subset of $S_m(\mathbb{R})$. Let

$$\mathfrak{R} := C^\infty(\mathcal{O}_m)[g_{ij/U}, \omega_i, \omega_{i/U}]_{|U|>0}\,.$$

These are the local formulas in the derivatives of the metric and of ω with coefficients which are smooth functions of the Riemannian metric g which we will be considering. The weight induces a natural grading on $\mathfrak{R}$ and we may decompose $\mathfrak{R}_m = \oplus_n\mathfrak{R}_{m,n}$ into the polynomials which are homogeneous of weight n. If $P \in \mathfrak{R}_m$, we can evaluate P in a coordinate system in the obvious fashion; we say P is *invariant* if the value of P is independent of the particular local coordinate system chosen and we let $\mathfrak{I}_m \subset \mathfrak{R}_m$ be the ring of invariant local formulae; we may decompose $\mathfrak{I}_m = \oplus_n\mathfrak{I}_{m,n}$ where $\mathfrak{I}_{m,n} \subset \mathfrak{R}_{m,n}$. The space of p-form valued invariants $\mathfrak{I}^p_{m,n}$ is defined similarly.

Clearly we can express the curvature tensor R and its covariant derivatives in terms of the derivatives of the metric. Conversely, in geodesic coordinates, we can express the derivatives of the metric in terms of the covariant derivatives of the curvature tensor. The *weight* of the curvature tensor R is 2 since it is linear in the second derivatives of the metric and quadratic in the first derivatives of the metric.

We increase the weight by 1 for every explicit covariant derivative which appears. Thus, for example, the scalar curvature τ and $\|\omega\|^2$ have weight 2 while the square of the norms $\|\rho\|^2$ and $\|R\|^2$ of the Ricci tensor and full curvature tensor, respectively, have weight 4. Similarly, $\|\nabla R\|^2$ has weight 6 and $\mathcal{E}_{m,2k}$ has weight $2k$.

2.2. **Spaces of local formulae in the complex setting.** The situation is considerably more delicate here and we must proceed with some care. Let (M, g, J) be a Hermitian manifold; we assume $J^*g = g$ but do not impose the Kähler condition. Let (E, h) be a holomorphic vector bundle over M of dimension ℓ. Fix a point P of M. Choose local holomorphic coordinates $\vec{z} = (z^1, \ldots, z^{\mathfrak{m}})$ centered at P and a local holomorphic frame $\vec{s} = (s_1, \ldots, s_\ell)$ for E. Let $U = (\alpha_1, \ldots, \alpha_a)$ and $V = (\beta_1, \ldots, \beta_b)$ be collections of indices. Expand $\omega = \omega_\alpha dz^\alpha$ and $\bar{\omega} = \bar{\omega}_{\bar{\beta}} d\bar{z}^{\bar{\beta}}$. Set

$$\begin{aligned} &|U| := a, \quad |V| := b, \quad \partial_\alpha := \tfrac{\partial}{\partial z^\alpha}, \quad \partial_{z;U} = \partial_{\alpha_1} \ldots \partial_{\alpha_a}, \\ &\partial_{\bar{\beta}} := \tfrac{\partial}{\partial \bar{z}^{\bar{\beta}}}, \quad \partial_{\bar{z};\bar{V}} = \partial_{\bar{\beta}_1} \ldots \partial_{\bar{\beta}_b}, \quad g_{\alpha\bar{\beta}} := g(\partial_\alpha, \partial_{\bar{\beta}}), \quad h_{p\bar{q}} := h(s_p, s_q)\,. \end{aligned}$$

Introduce the following notation for the derivatives of the structures involved:

$$\begin{aligned} g_{\alpha\bar{\beta}/U\bar{V}} &:= \partial_{z;U} \partial_{\bar{z};\bar{V}} g(\partial_\alpha, \partial_{\bar{\beta}}), \quad & h_{p\bar{q}/U\bar{V}} &:= \partial_{z;U} \partial_{\bar{z};\bar{V}} h(s_p, s_q), \\ \omega_{\alpha/U\bar{V}} &:= \partial_{z;U} \partial_{\bar{z};\bar{V}} \omega_\alpha, & \bar{\omega}_{\bar{\beta}/U\bar{V}} &:= \partial_{z;U} \partial_{\bar{z};\bar{V}} \bar{\omega}_{\bar{\beta}}. \end{aligned} \tag{2.1}$$

If $|U| = 0$, there are no holomorphic derivatives, if $|V| = 0$, there are no anti-holomorphic derivatives, and if $|U| + |V| = 0$, there are no derivatives at all. We have

$$\bar{\partial}\omega = -\omega_{\alpha/\bar{\beta}} dz^\alpha \wedge d\bar{z}^{\bar{\beta}} \text{ and } \partial\bar{\omega} = \bar{\omega}_{\bar{\beta}/\alpha} dz^\alpha \wedge d\bar{z}^{\bar{\beta}}\,.$$

Consequently, the variables $\{\omega_\alpha, \omega_{\alpha/\bar{\beta}}, \bar{\omega}_{\bar{\beta}}, \bar{\omega}_{\bar{\beta}/\alpha}\}$ are tensorial unlike the remainder of the variables defined in Equation (2.1). Define

$$\begin{aligned} &\text{weight}\{g_{\alpha\bar{\beta}/U\bar{V}}\} = \text{weight}\{h_{p\bar{q}/U\bar{V}}\} = |U| + |V|, \\ &\text{weight}\{\omega_{\alpha;U\bar{V}}\} = \text{weight}\{\bar{\omega}_{\bar{\beta};U\bar{V}}\} = 1 + |U| + |V|. \end{aligned}$$

Let $S_k(\mathbb{C})$ be the vector space of Hermitian $k \times k$ complex matrices. Let $\mathcal{U}_{\mathfrak{m}} \subset S_{\mathfrak{m}}(\mathbb{C}) \otimes S_\ell(\mathbb{C})$ be the subset of matrices (g, h) so that g and h define positive definite inner products; $\mathcal{U}_{\mathfrak{m}}$ is an open subset of $S_{\mathfrak{m}}(\mathbb{C}) \otimes S_\ell(\mathbb{C})$. We suppress the dependence upon ℓ in the interests of notational simplicity as it will play no role in our further development whereas the dependence on $\mathfrak{m}$ will be crucial. We consider the polynomial ring

$$\mathcal{P}_m := C^\infty(\mathcal{U}_{\mathfrak{m}})[g_{\alpha\bar{\beta}/U\bar{V}}, h_{p\bar{q}/U\bar{V}}, \omega_{\alpha/U\bar{V}}, \bar{\omega}_{\bar{\beta}/U\bar{V}}, \omega_\alpha, \bar{\omega}_{\bar{\beta}}]_{|U|+|V|>0}\,.$$

As in the real setting, we say that $P \in \mathcal{P}_m$ is *invariant* if the evaluation is independent of the particular coordinate system $\vec{z}$ and frame $\vec{s}$ chosen. Let $\mathfrak{K}_m$ be the ring of invariants in the Kähler context; $\mathfrak{K}_m$ is a graded ring and we may use the weight to decompose $\mathfrak{K}_m = \oplus_n \mathfrak{K}_{m,n}$ into the polynomials which are homogeneous of weight n. For example, $a_{m,n}(\Delta^{p,q}) \in \mathfrak{K}_{m,n}$. The spaces $\mathfrak{K}_{m,n}^p$ of p-form valued invariants are defined similarly. For example, $d\omega \in \mathfrak{K}_{m,2}^2$.

2.3. **Homotheties.** The weight of a polynomial describes its behaviour under homotheties. The following is immediate from the definition and could be used to give an equivalent definition of the weight.

Lemma 2.1.

(1) *Let* $P \in \mathfrak{J}^p_{m,k}$. *Then* $P(x, c^2g, \omega) = c^{p-k}P(x, g, \omega)$.
(2) *Let* $P \in \mathfrak{K}^p_{m,k}$. *Then* $P(x, c^2g, J, E, h, \omega) = c^{p-k}P(x, g, J, E, h, \omega)$.

Example 2.2. It is worth giving some examples to illustrate Lemma 2.1. We work in the real context. Let $\|\omega\|^2_g := g^{ij}\omega_i\omega_j$. Since ω_i has weight 1, $\|\omega\|^2_g$ has weight 2 so $\|\omega\|^2_g \in \mathfrak{J}^0_{m,2}$. If we rescale the metric and set $g_{c;ij} = c^2g_{ij}$, then $g_c^{ij} = c^{-2}g^{ij}$ so $\|\omega\|^2_{g_c} = c^{-2}\|\omega\|^2_g$. The components of the curvature tensor $R_{ijk}{}^{\ell}$ have weight 2 since they are linear in the 2-jets of the metric and quadratic in the 1-jets of the metric. The Levi-Civita connection is unchanged if we rescale the metric and thus $R_{g_c;ijk}{}^{\ell} = R_{g;ijk}{}^{\ell}$. Let $\tau_g \in \mathfrak{J}^0_{m,2}$ be the scalar curvature; $\tau_{g_c} = g_c^{jk}R_{ijk}{}^i = c^{-2}g^{jk}R_{ijk}{}^i = c^{-2}\tau_g$. We have $d\tau_g \in \mathfrak{J}^1_{m,3}$ and $d\tau_{g_c} = c^{-2}d\tau_g$.

2.4. **The restriction map.** As noted in the introduction, a spanning set for the space of invariants in the real setting is given by contraction of indices in pairs and alternations of indices where the indices range from 1 to m; there is an analogous result in the complex setting. If $P \in \mathfrak{J}^p_{m,n}$, $r(P) \in \mathfrak{J}^p_{m-1,n}$ is defined by restricting the range of summation to range from 1 to $m-1$. Although Weyl's First Theorem of Invariants yields a spanning set, it is not a basis as one has the Bianchi identities and higher order analogues. Thus it is not immediately obvious that the restriction map is independent of the particular expression of an invariant in terms of a Weyl spanning set. To get around this difficulty, it is convenient to use a slightly different more geometric formalism. Let $\mathbb{T}^1 = (S^1, d\theta^2, 0)$ be the circle with the flat structures. If $\mathcal{N} = (N, ds^2_N, \omega_N)$ is an $m-1$ dimensional structure, one forms the m dimensional structure

$$\mathcal{M} = \mathcal{N} \times \mathbb{T}^1 := (N \times S^1, ds^2_N + d\theta^2, \pi_1^*\omega_N)\,.$$

Let $i_\theta(x) = (x, \theta)$ be an inclusion of N in $N \times S^1$; the particular basepoint θ chosen is irrelevant since $\mathbb{T}^1$ is homogeneous. Then $r(P) \in \mathfrak{J}^p_{m-1,n}$ is characterized by the identity

$$r(P)(\mathcal{N}) := i_\theta^* P(\mathcal{N} \times \mathbb{T}^1)\,. \tag{2.2}$$

In the complex setting, the invariance theory is that of the unitary group rather than the orthogonal group and one sums over pairs of holomorphic and anti-holomorphic indices where the indices range from 1 to $\mathfrak{m}$. Instead of considering $\mathbb{T}^1$ one considers the flat 2-torus $\mathbb{T}^2$ but the remainder of the analysis is the same and one obtains:

Lemma 2.3.

(1) *r is a well defined map from $\mathfrak{J}^p_{m,n}$ onto $\mathfrak{J}^p_{m-1,n}$.*
(2) *r is a well defined map from $\mathfrak{K}^p_{m,n}$ onto $\mathfrak{K}^p_{m-2,n}$.*

We introduce the ring

$$\mathfrak{T}_m := \mathbb{C}[\mathrm{ch}_k(TM, J, g), \mathrm{ch}_k(E, h), d\omega, d\bar{\omega}, \omega, \bar{\omega}]\,.$$

We may decompose $\mathfrak{T}_m := \oplus_k \mathfrak{T}_m^k$ where $\mathfrak{T}_m^k \subset \mathfrak{K}_{m,k}^k$. We refer to [1] for the proof of Assertions (1) and to [2] for the proof of Assertions 2 in the following result; it is necessary to restrict to the Kähler setting.

Lemma 2.4.
(1) *In the Riemannian setting, we have that:*
  (a) $r : \mathfrak{I}_{m,n}^0 \to \mathfrak{I}_{m-1,n}^0$ *is injective if* $n < m$.
  (b) *If* m *is even, then* $\ker\{r : \mathfrak{I}_{m,m}^0 \to \mathfrak{I}_{m-1,m}^0\} = \mathcal{E}_{m,m} \cdot \mathbb{R}$.
(2) *In the Kähler setting, we have that:*
  (a) $r : \mathfrak{K}_{m,n}^0 \to \mathfrak{K}_{m-2,n}^0$ *is injective if* $n < m$.
  (b) $\ker\{r : \mathfrak{K}_{m,m}^0 \to \mathfrak{K}_{m-2,m}^0\} = \star\mathfrak{T}_m^m$.

We have the following useful result.

Lemma 2.5. $r(\mathfrak{a}_{m,n}^{\mathrm{deR}}) = -(4\pi)^{-1/2} a_{m-1,n}^{\mathrm{deR}}$ and $r(\mathfrak{a}_{m,n}^{\mathrm{Dol}}) = -(4\pi)^{-1} a_{m-1,n}^{\mathrm{Dol}}$.

Proof. Although the lemma in the real setting follows from work of Günther and Schimming [15], we shall give a direct proof in the interests of completeness. Let $\mathcal{M}_m = \mathcal{N}_{m-1} \times \mathbb{T}^1$. We compute:

$$\Lambda^p M = \Lambda^p N \oplus \Lambda^{p-1} N \wedge d\theta,$$

$$\Delta_{\mathcal{M}}^p = \{\Delta_{\mathcal{N}}^p \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_{\mathbb{T}^1}^0\} \oplus \{\Delta_{\mathcal{N}}^{p-1} \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_{\mathbb{T}^1}^1\},$$

$$a_{m,n}((x,\theta), \Delta_{\mathcal{M}}^p) = (4\pi)^{-1/2}\{a_{m-1,n}(x, \Delta_{\mathcal{N}}^p) + a_{m-1,n}(x, \Delta_{\mathcal{N}}^{p-1})\},$$

$$\begin{aligned}
&\sum_p (-1)^p p \cdot a_{m,n}((x,\theta), \Delta_{\mathcal{M}}^p) \\
&\quad = (4\pi)^{-1/2} \sum_p (-1)^p p \cdot \Big\{ a_{m-1,n}(x, \Delta_{\mathcal{N}}^p) + a_{m-1,n}(x, \Delta_{\mathcal{N}}^{p-1}) \Big\} \\
&\quad = (4\pi)^{-1/2} \sum_p (-1)^p a_{m-1,n}(x, \Delta_{\mathcal{N}}^p)\{p - (p+1)\}.
\end{aligned}$$

Assertion 1 now follows from Equation (2.2); the argument is the same for the Dolbeault complex. □

2.5. **Low dimensional computations.** We refer to [1] for the following result:

Lemma 2.6. $a_{1,2}^{\mathrm{deR}} = -\dfrac{\delta\omega}{\sqrt{\pi}}$ *and* $a_{2,2}^{\mathrm{Dol}} = \dfrac{\tau}{8\pi} - \dfrac{1}{\pi}\delta(\Re(\omega))$.

3. The proof of Theorem 1.3 (1a,2a)

Let $n < m - 1$. By Lemma 2.5, $r(\mathfrak{a}_{m,n}^{\mathrm{deR}}) = a_{m-1,n}^{\mathrm{deR}}$. By Theorem 1.1, $a_{m-1,n}^{\mathrm{deR}} = 0$. By Lemma 2.4, $r : \mathfrak{I}_{m,n} \to \mathfrak{I}_{m-1,n}$ is injective for $n < m$. This shows that $\mathfrak{a}_{m,n}^{\mathrm{deR}} = 0$ which establishes Theorem 1.3 (1a); the proof of Theorem 1.3 (2a) is the same in the Kähler setting. □

4. The proof of Theorem 1.3 (1b,2b)

We use Lemma 2.5 and Theorem 1.1 to see

$$r(\mathfrak{a}_{2k+1,2k}^{\mathrm{deR}}) = -(4\pi)^{-1/2} a_{2k,2k}^{\mathrm{deR}} \quad \text{and} \quad a_{2k,2k}^{\mathrm{deR}} = \mathcal{E}_{2k,2k}\,.$$

By Lemma 2.4, $r : \mathfrak{J}_{2k+1,2k} \to \mathfrak{J}_{2k,2k}$ is injective. By construction, we have that $r\mathcal{E}_{2k+1,2k} = \mathcal{E}_{2k,2k}$. Assertion (1b) follows. The same argument in the Kähler setting shows

$$\mathfrak{a}_{m,m-2}^{\mathrm{Dol}} = \frac{1}{(\mathfrak{m}-1)!} g(\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h) \wedge \Theta, \Omega^{\mathfrak{m}-1})$$

which establishes the first part of Assertion 2b. It is immediate from the definition that $\Theta = 1 + d\Phi$. Since Td and ch are closed,

$$\begin{aligned} & \{\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h) \wedge \Theta\}_{m-2} \\ = \quad & \{\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h)\}_{m-2} + d\{\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h) \wedge \Phi\}_{m-3}\,. \end{aligned}$$

Taking the inner product with $\frac{1}{(\mathfrak{m}-1)!}\Omega^{\mathfrak{m}-1}$ is just the Hodge $\star$ operator in complex dimension $\mathfrak{m}-1$. Thus

$$\begin{aligned} & \tfrac{1}{(\mathfrak{m}-1)!} g(\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h) \wedge \Theta, \Omega^{\mathfrak{m}-1}) \\ & = \tfrac{1}{(\mathfrak{m}-1)!} g(\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h), \Omega^{\mathfrak{m}-1}) + \delta Q_{m-2,m-3}^1 \end{aligned} \tag{4.1}$$

for $Q_{m-2,m-3}^1 = \star\{\mathrm{Td}(M,g,J) \wedge \mathrm{ch}(E,h) \wedge \Phi\}_{m-3}$. We have $\delta r = r\delta$. By Lemma 2.3 and Lemma 2.4, we can lift Equation (4.1) to a corresponding equation in complex dimension $\mathfrak{m}$ and thereby complete the proof of Assertion (2b). □

Remark 4.1. This gives an explicit description of the 1-form $Q_{m,m-3}^1$ of Theorem 1.3 (2b) as the unique element $Q_{m,m-3}^1 \in \mathfrak{K}_{m,m-3}^1$ such that

$$r(Q_{m,m-3}^1) = \star\{\mathrm{Td} \wedge \mathrm{ch} \wedge \Phi\}_{m-3} \in \mathfrak{K}_{m-2,m-3}^1.$$

5. The proof of Theorem 1.3 (1c)

We continue our discussion by generalizing a result of Gilkey [13] (see Assertion 4 of Lemma 2.9.1 on page 210) from the context of purely metric invariants to invariants which also depend upon the derivatives of ω.

Lemma 5.1. Let $P \in \mathfrak{J}_{m,n}$ where $n \neq m$. Suppose $\int_M P(x,g,\omega)\,\mathrm{dvol}(g)$ is independent of (g,ω) for all closed m-dimensional manifolds M. Then there exists $Q_{m,n-1} \in \mathfrak{J}_{m,n-1}^1$ so that $P = \delta Q_{m,n-1}$.

Proof. Let $f \in C^\infty(M)$ and $\varepsilon \in \mathbb{R}$. We consider the conformal variation

$$P_1(f,g,\omega)\,\mathrm{dvol}(g) := \partial_\varepsilon \left\{ P(x, e^{2\varepsilon f} g, \omega)\,\mathrm{dvol}(e^{2\varepsilon f} g) \right\}\Big|_{\varepsilon=0}\,.$$

Since P_1 is linear in the jets of f, we have that

$$P_1(f,g,\omega) = \sum_k \sum_{i_1,\dots,i_k} f_{;i_1\dots i_k} Q^{i_1\dots i_k}(g,\omega)\,.$$

We integrate by parts formally to express

$$P_1 = \delta Q + \sum_k \sum_{i_1,\dots,i_k} (-1)^k f Q^{i_1\dots i_k}{}_{;i_k\dots i_1}\,.$$

By assumption,

$$\begin{aligned} 0 &= \partial_\varepsilon\left\{\int_M P(x,e^{2\varepsilon f}g,\omega)\,\mathrm{dvol}(e^{2\varepsilon f}g)\right\}\Big|_{\varepsilon=0} = \int_M P_1(f,g,\omega)\,\mathrm{dvol}(g)\\ &= \int_M f\sum_k\sum_{i_1,\dots,i_k}(-1)^k Q^{i_1\dots i_k}{}_{;i_k\dots i_1}\,\mathrm{dvol}(g)\,. \end{aligned}$$

Since f was arbitrary,

$$\sum_k\sum_{i_1,\dots,i_k}(-1)^k Q^{i_1\dots i_k}{}_{;i_k\dots i_1}=0\,.$$

Thus $P_1(f,g,\omega)=\delta Q(f,g,\omega)$. We set $f=1$ to see

$$P_1(1,g,\omega)=\delta Q(1,g,\omega)\,. \tag{5.1}$$

We have that $\mathrm{dvol}(e^{2\varepsilon}g)=e^{m\varepsilon}\,\mathrm{dvol}(g)$. Since P is homogeneous of weight n, Lemma 2.1 yields $P(x,e^{2\varepsilon}g,\omega)\,\mathrm{dvol}(e^{2\varepsilon}g)=e^{(m-n)\varepsilon}P(x,g,\omega)\,\mathrm{dvol}(g)$. Differentiating this identity shows $P_1=(m-n)P$; the Lemma now follows from Equation (5.1). □

We now establish Assertion (1c) of Theorem 1.3. Let $P=r(\mathfrak{a}^{\mathrm{deR}}_{m,m})$ belong to $\mathfrak{I}^0_{m-1,m}$. By Lemma 2.5, P is a multiple of $a^{\mathrm{deR}}_{m-1,m}$. The hypothesis of Lemma 5.1 are satisfied by Equation (1.1). Consequently, Lemma 5.1 shows that $r(\mathfrak{a}^{\mathrm{deR}}_{m,m})=\delta Q^1_{m-1,m-1}$. By Lemma 2.4, we may choose $Q^1_{m,m-1}$ so $r(Q^1_{m,m-1})=Q^1_{m-1,m-1}$. Since $\delta r=r\delta$, $r(\mathfrak{a}^{\mathrm{deR}}_{m,m}-\delta Q^1_{m,m-1})=0$. We use Lemma 2.4 to see that

$$\mathfrak{a}^{\mathrm{deR}}_{m,m}-\delta Q_{m,m-1}=c(m)\mathcal{E}_{m,m}\,.$$

To evaluate the coefficient $c(m)$, we may take $\omega=0$ and use (local) Poincare duality to see that $a_{m,m}(\Delta^p_{\mathcal{M}})=a_{m,m}(\Delta^{m-p}_{\mathcal{M}})$. Consequently, we may show that $c(m)=\frac12 m$ by computing:

$$\begin{aligned} 2\mathfrak{a}^{\mathrm{deR}}_{m,m} &= \sum_p(-1)^p p\cdot\left\{a_{m,m}(x,\Delta^p_{\mathcal{M}})+a_{m,m}(x,\Delta^{m-p}_{\mathcal{M}})\right\}\\ &= \sum_p(-1)^p a_{m,m}(x,\Delta^p_{\mathcal{M}})\{p+m-p\}=m\cdot a^{\mathrm{deR}}_{m,m}\,. \end{aligned}$$

Suppose $m=2$. We use Lemma 2.6 to see $-(4\pi)^{-1/2}a^{\mathrm{deR}}_{1,2}=(2\pi)^{-1}\delta\omega$. Consequently, $r(Q^1_{2,1})$ is non-zero. □

6. The proof of Theorem 1.3 (2c)

There are several fundamental differences between the real and the complex settings and we must treat the variables $\{\omega_\alpha,\bar\omega_{\bar\beta}\}$ differently from the other variables. In Section 6.1, we introduce some additional notation. Section 6.2 is devoted to showing that in fact the heat trace invariants $a_{m,n}(\Delta^{p,q})$ do not involve the $\{\omega_\alpha,\bar\omega_{\bar\beta}\}$ variables but only the derivatives of positive order in ω are present; this uses a gauge renormalization. This is in marked contrast to the real setting. In Section 6.3, we complete the analysis; again, there is a subtlety in that the conformal variation of a Kähler metric need no longer be Kähler. Consequently, we pass (temporarily) to the Hermitian setting.

6.1. **Spaces of invariants.** We adopt the notation of Section 2.2. Recall that $\mathcal{U}_{\mathfrak{m}} \subset S_{\mathfrak{m}}(\mathbb{C}) \otimes S_\ell(\mathbb{C})$ is the subset of matrices (g, h) so that g and h define positive definite inner products on $\mathbb{C}^{\mathfrak{m}}$ and $\mathbb{C}^\ell$, respectively. We adjoin variables sucessively to the ground ring $C^\infty(\mathcal{U}_{\mathfrak{m}})$ to define the sub-rings:

$$\mathfrak{U}_m := C^\infty(\mathcal{U}_{\mathfrak{m}})[g_{\alpha\bar\beta/U\bar V}, h_{p\bar q/U\bar V}]_{|U|+|V|>0},$$
$$\mathfrak{V}_m := \mathfrak{U}_m[\omega_{\alpha/UV}, \bar\omega_{\bar\beta/UV}]_{|U|+|V|>0}\,.$$

We explicitly exclude the variables $\{\omega_\alpha, \bar\omega_{\bar\beta}\}$ from $\mathfrak{U}_m$ and from $\mathfrak{V}_m$. Let $\mathfrak{U}_{m,n} \subset \mathfrak{U}_m$ and $\mathfrak{V}_{m,n} \subset \mathfrak{V}_m$ be the subspaces of weight n. We impose no assumption of invariance as we are simply interested in the nature of the polynomials for the moment. Let

$$\operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{U}) \quad \text{and} \quad \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{V})$$

be the vector space of endomorphisms of $\Lambda^{p,q}(M) \otimes E$ with coefficients in the rings $\mathfrak{U}$ and $\mathfrak{V}$, respectively.

6.2. **Normalizing the gauge.** The variables $\{\omega_\alpha, \bar\omega_{\bar\beta}\}$ are troublesome and we eliminate them from consideration as follows. We work in the Hermitian setting as we have imposed no conditions on Ω.

Lemma 6.1. *$a_{m,n}(x, \Delta^{p,q}) \in \mathfrak{V}_{m,n}$, i.e., $a_{m,n}(x, \Delta^{p,q})$ does not involve the variables $\{\omega_\alpha, \bar\omega_{\bar\beta}\}$.*

Proof. We may decompose

$$\bar\partial_{\bar\omega} = \operatorname{ext}(d\bar z^{\bar\beta})\left\{\partial_{\bar\beta} + \bar\omega_{\bar\beta}\right\} \text{ and } \delta''_\omega = \operatorname{int}(dz^\alpha)\left\{-\partial_\alpha + \omega_\alpha\right\} + \mathcal{L}$$

where $\mathcal{L} \in \operatorname{End}(\Lambda(M) \otimes E; \mathfrak{U}_{m,1})$ arises by taking the adjoint of $\bar\partial$ and is linear in the first derivatives of g and of h; $\mathcal{L}$ and $\operatorname{int}(dz^\alpha)$ lower the anti-holomorphic index q by 1 and $\operatorname{ext}(d\bar z^{\bar\beta})$ raises the anti-holomorphic index q by 1. We define:

$$\begin{aligned}
&\mathcal{E}_1^{p,q} := \operatorname{ext}(d\bar z^{\bar\beta})\operatorname{int}(dz^\alpha)\omega_{\alpha/\bar\beta} \in \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{V}_{m,2}),\\
&\mathcal{E}_2^{p,q} := \operatorname{int}(dz^\alpha)\operatorname{ext}(d\bar z^{\bar\beta})\bar\omega_{\bar\beta/\alpha} \in \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{V}_{m,2}),\\
&\mathcal{L}^{p,q,\alpha} := \operatorname{ext}(d\bar z^{\bar\beta})\partial_{\bar\beta}\{\operatorname{int}(dz^\alpha)\} \in \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{U}_{m,1}),\\
&\mathcal{L}^{p,q,\bar\beta} := \operatorname{ext}(d\bar z^{\bar\beta})\mathcal{L} + \mathcal{L}\operatorname{ext}(d\bar z^{\bar\beta}) \in \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{U}_{m,1}),\\
&\mathcal{Q}^{p,q} := \operatorname{ext}(d\bar z^{\bar\beta})\partial_{\bar\beta}\{\mathcal{L}\} \in \operatorname{End}(\Lambda^{p,q}(M) \otimes E; \mathfrak{U}_{m,2}).
\end{aligned}$$

Since the matrices $\operatorname{ext}(dz^{\bar\beta})$ are constant with respect to the coordinate frame, we do not need to introduce their derivatives. Note that $\mathcal{E}_1^{p,q}$ and $\mathcal{E}_2^{p,q}$ are invariantly defined while $\mathcal{L}^{p,q,\alpha}$, $\mathcal{L}^{p,q,\bar\beta}$, and $\mathcal{Q}^{p,q}$ are not invariantly defined but rather depend on $(\vec z, \vec s)$. We use the identity

$$\operatorname{int}(dz^\alpha)\operatorname{ext}(d\bar z^{\bar\beta}) + \operatorname{ext}(d\bar z^{\bar\beta})\operatorname{int}(dz^\alpha) = g^{\alpha\bar\beta}$$

to express $\Delta^{p,q} := 2\delta''_\omega\partial_{\bar\omega} + 2\partial_{\bar\omega}\delta''_\omega$ in the form:

$$\Delta^{p,q} = 2g^{\alpha\bar\beta}\{-\partial_\alpha\partial_{\bar\beta}+\omega_\alpha\partial_{\bar\beta}-\omega_{\bar\beta}\partial_\alpha+\omega_\alpha\bar\omega_{\bar\beta}\}+2\mathcal{E}_1^{p,q}-2\mathcal{E}_2^{p,q}+2\mathcal{Q}^{p,q}$$
$$+2\mathcal{L}^{p,q,\alpha}(-\partial_\alpha+\omega_\alpha)+2\mathcal{L}^{p,q,\bar\beta}(\partial_{\bar\beta}+\bar\omega_{\bar\beta}).$$

Fix a point P of M and let $\xi := \omega_\alpha(P)z^\alpha - \bar\omega_{\bar\beta}(P)z^{\bar\beta}$. Since ξ is purely imaginary, $\Delta_\xi^{p,q} := e^{-\xi}\Delta^{p,q}e^{\xi}$ is defined by a unitary change of gauge. We compute

$$\Delta_\xi^{p,q} = 2g^{\alpha\bar\beta}\{-\partial_\alpha\partial_{\bar\beta}+(\omega_\alpha-\omega_\alpha(P))\partial_{\bar\beta}-(\bar\omega_{\bar\beta}-\bar\omega_{\bar\beta}(P))\partial_\alpha\}$$
$$+2g^{\alpha\bar\beta}(\omega_\alpha-\omega_\alpha(P))(\bar\omega_{\bar\beta}-\bar\omega_{\bar\beta}(P))+2\mathcal{E}_1^{p,q}-2\mathcal{E}_2^{p,q}+2\mathcal{Q}^{p,q}$$
$$+2\mathcal{L}^{p,q,\alpha}(-\partial_\alpha+\omega_\alpha-\omega_\alpha(P))+2\mathcal{L}^{p,q,\bar\beta}(\partial_{\bar\beta}+\bar\omega_{\bar\beta}-\bar\omega_{\bar\beta}(P)).$$

Since $\Delta^{p,q}$ and $\Delta_\xi^{p,q}$ differ by a gauge transformation,

$$a_{m,n}(\Delta^{p,q}) = a_{m,n}(\Delta_\xi^{p,q})\,.$$

It is immediate that the symbol of $\Delta_\xi^{p,q}$ and all the derivatives of the symbol of $\Delta_\xi^{p,q}$ at P do not involve the ω_α and $\bar\omega_{\bar\beta}$ variables. Consequently, $a_{m,n}(\Delta_\xi^{p,q})\in\mathfrak{V}_{m,n}$. $\square$

Remark 6.2. We have exploited a fundamental difference between the deformed real Laplacian and the deformed complex Laplacian that relates to gauge freedom. We illustrate this as follows. Suppose $m=1$. We work on the circle. Let $\omega = adx$ for $a\in\mathbb{R}$. Then $d_a=\partial_x+a$ and $\delta_a=-\partial_x+a$; $\Delta_a^0=\Delta_a^1=\delta_a d_a=-(\partial_x^2+a^2)$ is already normalized optimally since there is no first order term and, unlike in the complex setting, it is not possible to make a change of gauge that eliminates the dependence of the symbol on a.

6.3. **Conformal variations.** If g is a Kähler metric, then the conformal variation $e^{2\varepsilon f}g$ need no longer be Kähler. However, this variation remains within the class of Hermitian manifolds. Since Equation (1.1) continues to hold for Hermitian manifolds, the argument given to prove Lemma 5.1 shows that we may express $a_{m-2,m}^{\mathrm{Dol}}=\delta Q_{m-2,m-1}^1$ in the class of Hermitian and hence in the class of Kähler manifolds. Thus we may choose $Q_{m,m-1}^1$ so that

$$\mathfrak{a}_{m,m}^{\mathrm{Dol}}-\delta Q_{m,m-1}^1\in\ker(r)\,.$$

The integration by parts procedure discussed in Section 5 arises from a conformal variation of g; thus although additional derivatives of ω may be introduced, the number of derivatives of ω are not reduced. Thus Q belongs to $\mathfrak{V}_{m,m-1}^1$ and we have $\mathfrak{a}_{m,m}^{\mathrm{Dol}}-\delta Q\in\mathfrak{V}_{m,m}$. Let

$$\mathfrak{S}_m := \mathbb{C}[\mathrm{ch}_k(TM,J,g),\mathrm{ch}_k(E,h),d\omega,d\bar\omega]\subset\mathfrak{T}_m$$

be obtained by omitting the $\{\omega_\alpha,\bar\omega_{\bar\beta}\}$ variables. We restrict to the Kähler setting to obtain

$$\mathfrak{a}_{m,m}^{\mathrm{Dol}}-\delta Q\in\star\{\mathfrak{T}_m^m\}\cap\mathfrak{V}_{m,m}=\star\mathfrak{S}_m^m\,.$$

Since we have omitted the $\{\omega_\alpha,\bar\omega_{\bar\beta}\}$ variables, and since ch_k is a closed differential form, we complete the proof of Theorem 1.3 by computing

$$\begin{aligned}\mathfrak{S}_m^m &= \mathfrak{C}_m^m+d\{\omega\wedge\mathfrak{S}_m^{m-2}+\bar\omega\wedge\mathfrak{S}_m^{m-2}\},\\ \star\mathfrak{S}_m^m &= \star\mathfrak{C}_m^m+\delta\star\{\omega\wedge\mathfrak{S}_m^{m-2}+\bar\omega\wedge\mathfrak{S}_m^{m-2}\}\,.\end{aligned}$$

7. The proof of Theorem 1.4

Let $\Delta_E^{p,q}$ be the complex Laplacian with coefficients in E. Let Δ^n be the real Laplacian. If P is a local invariant, let $P[\mathcal{M}] = \int_M P(\mathcal{M})\,\mathrm{dvol}$. We will use the following resuls in our analysis. Assertion 1 is immediate from the definition, Assertions (2,3) are Serre-duality, and Assertions (4,5) follow by specializing the Hirzebruch-Riemann-Roch Theorem.

Theorem 7.1. *Let E be a holomorphic vector bundle over a Kähler manifold $\mathcal{M}$.*

(1) $\Delta_E^{p,q} = \Delta^{0,q}_{\Lambda^{p,0}\otimes E}$.
(2) *There exists a conjugate linear isomorphism intertwining $\Delta_E^{p,q}$ and $\Delta_{E^*}^{m-p,m-q}$.*
(3) *Complex conjugation intertwines $\Delta^{p,q}$ and $\Delta^{q,p}$; $\Delta^n = \oplus_{p+q=n}\Delta^{p,q}$.*
(4) *If m=2, then* $\mathrm{index}(\bar\partial) = \{\frac12 c_1(T_cM)\,\mathrm{ch}_0(E) + c_1(E)\}[M]$.
(5) *If m=4, then*
$$\mathrm{index}(\bar\partial) = \{\mathrm{Td}_2(T_cM)\,\mathrm{ch}_0(E) + \mathrm{ch}_1(T_cM)\,\mathrm{ch}_1(E) + \mathrm{ch}_2(E)\}[\mathcal{M}].$$

We use Leibnitz's formula to establish the following result.

Lemma 7.2. *Let $\mathcal{M}_j$ have real dimension m_j. Then*
$$\begin{aligned}&\mathfrak{a}^{\mathrm{Dol}}_{m_1+m_2,m_1+m_2}[\mathcal{M}_1\times\mathcal{M}_2]\\ =\;&\mathfrak{a}^{\mathrm{Dol}}_{m_1,m_1}[\mathcal{M}_1]a^{\mathrm{Dol}}_{m_2,m_2}[\mathcal{M}_2] + a^{\mathrm{Dol}}_{m_1,m_1}[\mathcal{M}_1]\mathfrak{a}^{\mathrm{Dol}}_{m_2,m_2}[\mathcal{M}_2]\,.\end{aligned}$$

Proof. We adopt the notion of Equation (1.2) and set
$$\mathfrak{A}^{\mathrm{Dol}}_{m,n}[\mathcal{M}](s) := \sum_p (-1)^p a_{m,n}(\Delta^{(0,p)})[\mathcal{M}]s^p\,.$$

We have the following product formula for the heat trace asymptotics:
$$\begin{aligned}&a_{m,n}(\Delta^{(0,p)})[\mathcal{M}_1\times\mathcal{M}_2]\\ &= \sum_{p_1+p_2=p}\sum_{n_1+n_2=n} a_{m_1,n_1}(\Delta^{(0,p_1)})[\mathcal{M}_1]\cdot a_{m_2,n_2}(\Delta^{(0,p_2)})[\mathcal{M}_2]\,.\end{aligned}$$

This then yields that
$$\mathfrak{A}^{\mathrm{Dol}}_{m,n}[\mathcal{M}_1\times\mathcal{M}_2](s) = \sum_{n_1+n_2=n}\mathfrak{A}^{\mathrm{Dol}}_{m_1,n_1}[\mathcal{M}_1](s)\cdot\mathfrak{A}^{\mathrm{Dol}}_{m_2,n_2}[\mathcal{M}_2](s)\,. \tag{7.1}$$

We differentiate Equation (7.1) and use Equation (1.3) to obtain
$$\mathfrak{a}^{\mathrm{Dol}}_{m,m}[\mathcal{M}] = \sum_{n_1+n_2=m}\mathfrak{a}^{\mathrm{Dol}}_{m_1,n_1}[\mathcal{M}_1]\cdot a^{\mathrm{Dol}}_{m_2,n_2}[\mathcal{M}_2] + a^{\mathrm{Dol}}_{m_1,n_1}[\mathcal{M}_1]\cdot\mathfrak{a}^{\mathrm{Dol}}_{m_2,n_2}[\mathcal{M}_2]\,.$$

We have that $a^{\mathrm{Dol}}_{m_1,n_1}[\mathcal{M}_1]=0$ if $m_1\neq n_1$ and $a^{\mathrm{Dol}}_{m_2,n_2}[\mathcal{M}_2]=0$ if $m_2\neq n_2$. We may therefore safely set $n_1=m_1$ and $n_2=m_2$ in the above identity to complete the proof of Lemma 7.2. □

The heat trace invariants are additive with respect to direct sums, i.e.
$$\begin{aligned}&a_{m,n}(\Delta^{p,q})(M,g,J,E_1\oplus E_2,h_1\oplus h_2)\\ =\;&a_{m,n}(\Delta^{p,q})(M,g,J,E_1,h_1) + a_{m,n}(\Delta^{p,q})(M,g,J,E_2,h_2)\,.\end{aligned}$$

Since $a_{m,m}^{\mathrm{Dol}}$ and $\mathfrak{a}_{m,m}^{\mathrm{Dol}}$ can be expressed in terms of characteristic classes, only the Chern character $\operatorname{ch}(E)$ enters. If E is a line bundle, then we have that $\operatorname{ch}_k(E,h) = \frac{1}{k!}c_1^k(E,h)$.

7.1. **The proof of Theorem 1.4 (1a).** Suppose that $m = 2$. There exist constants α^q and β^q so that $a_{2,2}(\Delta_E^{0,q}) = \alpha^q c_1(T_cM)\operatorname{ch}_0(E) + \beta^q c_1(E)$. We have $c_1(E^*) = -c_1(E)$. Since $\Lambda^{1,0}(M)$ is the dual of T_cM, we have that $c_1(\Lambda^{1,0}M) = -c_1(T_cM)$. We suppose $\dim E = 1$ and use Theorem 7.1 to compute:

$$\begin{aligned}
\{\alpha^1 c_1(T_cM) + \beta^1 c_1(E)\}[\mathcal{M}] &= a_{2,2}(\Delta_E^{0,1})[\mathcal{M}]\\
&= a_{2,2}(\Delta_{E^*}^{1,0})[\mathcal{M}] = a_{2,2}(\Delta_{\Lambda^{1,0}(M)\otimes E^*}^{0,0})[\mathcal{M}]\\
&= \{\alpha^0 c_1(T_cM) + \beta^0 c_1(\Lambda^{1,0}(M)) + \beta^0 c_1(E^*)\}[\mathcal{M}]\\
&= \big\{(\alpha^0 - \beta^0)c_1(T_cM) - \beta^0 c_1(E)\big\}\big[\mathcal{M}\big]\,.
\end{aligned}$$

Consequently, $\alpha^1 = \alpha^0 - \beta^0$ and $\beta^1 = -\beta^0$. Therefore,

$$\begin{aligned}
\operatorname{index}(\bar\partial_E) &= \tfrac{1}{2}\{c_1(T_cM) + c_1(E)\}[\mathcal{M}] = a_{2,2}^{\mathrm{Dol}}[\mathcal{M}]\\
&= \{(\alpha^0 - \alpha^1)c_1(T_cM) + (\beta^0 - \beta^1)c_1(E)\}[\mathcal{M}]\\
&= \{\beta^0 c_1(T_cM) + 2\beta^0 c_1(E)\}[\mathcal{M}]\,.
\end{aligned}$$

Consequently, $\beta^0 = \frac{1}{2}$ and $\beta^1 = -\frac{1}{2}$. Let $\mathcal{S}^2$ be the unit sphere in $\mathbb{R}^3$ with the usual metric and complex structure; $\tau = 2$ and $\operatorname{vol}(\mathcal{S}^2) = 4\pi$. We compute

$$\tfrac{1}{8\pi}\tau[\mathcal{S}^2] = 1 = \operatorname{index}(\bar\partial)[\mathcal{S}^2] = \tfrac{1}{2}c_1[\mathcal{S}^2]$$

so $c_1(\mathcal{S}^2) = \frac{1}{4\pi}\tau$. Take E trivial. McKean and Singer [19] (see also Patodi [21]) computed $a_{2,2}(\Delta^p)$. Together with Theorem 7.1, this shows

$$\begin{aligned}
&a_{2,2}(\Delta^{0,0}) = a_{2,2}(\Delta^0) = \tfrac{1}{24\pi}\tau = \tfrac{1}{6}c_1(T_c),\\
&a_{2,2}(\Delta^{0,1}) = \tfrac{1}{2}a_{2,2}(\Delta^1) = \tfrac{1}{2}\tfrac{1}{24\pi}(-4\tau) = -\tfrac{1}{3}c_1(T_c)\,.
\end{aligned}$$

Thus $\alpha^0 = \frac{1}{6}$ and $\alpha^1 = \alpha^0 - \beta^0 = \frac{1}{6} - \frac{1}{2} = -\frac{1}{3}$. □

7.2. **The proof of Theorem 1.4 (1b).** Suppose that $m = 4$ and that $\mathcal{M} = (M, g, J, E, h)$ is a complex surface. There exist constants α and β and a characteristic class $P_2(T_cM)$ so that

$$\mathfrak{a}_{4,4}^{\mathrm{Dol}}[\mathcal{M}] = \{P_2(T_cM)\operatorname{ch}_0(E) + \alpha c_1(T_cM)c_1(E) + \beta\operatorname{ch}_2(E)\}[\mathcal{M}]\,.$$

Let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Suppose first that $\mathcal{M}_i$ are flat tori and (E_i, h_i) are line bundles with $c_1(E_i)[\mathcal{M}_i] \neq 0$. Then

$$\begin{aligned}
&c_1(T_c(M))[\mathcal{M}] = 0, \qquad \{P_2(T_cM)\operatorname{ch}_0(E)\}[\mathcal{M}] = 0,\\
&\operatorname{ch}_2(E_1 \otimes E_2)[\mathcal{M}] = c_1(E_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2]\,.
\end{aligned}$$

Thus Theorem 7.1, Lemma 7.2 and Theorem 1.4 (1a) yield

$$\begin{aligned}
\beta c_1(E_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2] &= \beta\operatorname{ch}_2(E_1 \otimes E_2)[\mathcal{M}] = \mathfrak{a}_{4,4}^{\mathrm{Dol}}[\mathcal{M}]\\
&= \mathfrak{a}_{2,2}^{\mathrm{Dol}}[\mathcal{M}_1] \cdot a_{2,2}^{\mathrm{Dol}}[\mathcal{M}_2] + a_{2,2}^{\mathrm{Dol}}[\mathcal{M}_1] \cdot \mathfrak{a}_{2,2}^{\mathrm{Dol}}[\mathcal{M}_2]\\
&= \tfrac{1}{2}c_1(E_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2] + c_1(E_1)[\mathcal{M}_1] \cdot \tfrac{1}{2}c_1(E_2)[\mathcal{M}_2]\\
&= c_1(E_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2]\,.
\end{aligned}$$

This shows that $\beta = 1$. Suppose $M_1 = \mathcal{S}^2$, E_1 is trivial, M_2 is the flat torus, and $c_1(E_2)[\mathcal{M}_2] \neq 0$. Then $P_2(T_cM)\operatorname{ch}_0(E) = P_2(T_cM_1 \oplus \mathbf{1})\operatorname{ch}_0(E) = 0$ and $\operatorname{ch}_2(E) = \operatorname{ch}_2(E_2) = 0$. Thus

$$\begin{aligned}
&\alpha c_1(T_cM_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2] = \mathfrak{a}_{4,4}^{\mathrm{Dol}}[\mathcal{M}] \\
&\quad = \mathfrak{a}_{2,2}^{\mathrm{Dol}}[\mathcal{M}_1] \cdot a_{2,2}^{\mathrm{Dol}}[\mathcal{M}_2] + a_{2,2}^{\mathrm{Dol}}[\mathcal{M}_1] \cdot \mathfrak{a}_{2,2}^{\mathrm{Dol}}[\mathcal{M}_2] \\
&\quad = \tfrac{1}{3}c_1(T_cM_1)[\mathcal{M}_1] \cdot c_1(E_2)[\mathcal{M}_2] + \tfrac{1}{2}c_1(T_cM_1)[\mathcal{M}_1] \cdot \tfrac{1}{2}c_1(E_2)[\mathcal{M}_2]\,.
\end{aligned}$$

This shows that $\alpha = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Therefore, we have that

$$\mathfrak{a}_{4,4}^{\mathrm{Dol}}[\mathcal{M}] = \{P_2(T_cM)\operatorname{ch}_0(E) + \tfrac{7}{12}c_1(T_cM)c_1(E) + \operatorname{ch}_2(E)\}[\mathcal{M}]\,. \tag{7.2}$$

We now use Serre duality. Let $E = \mathbf{1} \oplus \Lambda^{2,0}(T_cM)$. We have

$$\begin{aligned}
\mathfrak{a}_{4,4}^{\mathrm{Dol}}(\mathcal{M}) &= -a_{4,4}(\Delta^{0,1}_{\mathbf{1}\oplus\Lambda^{2,0}}) + 2a_{4,4}(\Delta^{0,2}_{\mathbf{1}\oplus\Lambda^{2,0}}) \\
&= -a_{4,4}(\Delta^{2,1}_{\mathbf{1}\oplus(\Lambda^{2,0})^*}) + 2a_{4,4}(\Delta^{2,0}_{\mathbf{1}\oplus(\Lambda^{2,0})^*}) \\
&= -a_{4,4}(\Delta^{0,1}_{\Lambda^{2,0}\oplus\mathbf{1}}) + 2a_{4,4}(\Delta^{0,0}_{\Lambda^{2,0}\oplus\mathbf{1}})
\end{aligned} \tag{7.3}$$

We add the expressions of the second and third lines of Equation (7.3) to see

$$2\mathfrak{a}_{4,4}^{\mathrm{Dol}}(\mathcal{M}) = 2a_{4,4}(\Delta^{0,0}_E) - 2a_{4,4}(\Delta^{0,1}_E) + 2a_{4,4}(\Delta^{0,2}_E) = 2a_{4,4}^{\mathrm{Dol}}(\mathcal{M})\,. \tag{7.4}$$

We use Equation (7.2), Equation (7.4), and Theorem 1.4 to see

$$\begin{aligned}
&\{P_2(T_cM)\operatorname{ch}_0(E) + \tfrac{7}{12}c_1(T_cM)c_1(E) + \operatorname{ch}_2(E)\}[\mathcal{M}] \\
= \;&\{\operatorname{Td}_2(T_cM)\operatorname{ch}_0(E) + \tfrac{1}{2}c_1(T_cM)c_1(E) + \operatorname{ch}_2(E)\}[\mathcal{M}]\,.
\end{aligned}$$

We have $\operatorname{ch}_0(E) = 2$ and $\operatorname{ch}_1(E) = c_1(\Lambda^{2,0}) = -c_1(T_cM)$. Consequently

$$\begin{aligned}
P_2(T_cM) &= \operatorname{Td}_2(T_cM) + \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{7}{12})(-c_1^2(T_cM)) \\
&= \operatorname{Td}_2(T_cM) + \tfrac{1}{24}c_1^2(T_cM). \qquad \square
\end{aligned}$$

7.3. **The proof of Theorem 1.4 (2a,2b).** Let $\mathcal{M} := \mathcal{N}_1 \times \cdots \times \mathcal{N}_{\mathfrak{m}}$ where (N_i, g_i, J_i) are flat tori of real dimension 2 and (E_i, h_i) are Hermitian line bundles over N_i. Let $E = E_1 \otimes \cdots \otimes E_{\mathfrak{m}}$. Then $\operatorname{ch}_{\mathfrak{m}}(E) = c_1(E_1)\dots c_1(E_{\mathfrak{m}})$. We apply Lemma 7.2 and Theorem 1.4 to prove Theorem 1.4 (2a) by computing

$$\begin{aligned}
&a_{2,2}^{\mathrm{Dol}}[\mathcal{M}] = \textstyle\prod_i c_1(E_i)[\mathcal{N}_i], \qquad \mathfrak{A}_{2,2}^{\mathrm{Dol}}(s)[\mathcal{N}_i] = \tfrac{1}{2}(1+s)c_1(E_i)[\mathcal{N}_i], \\
&\mathfrak{a}_{m,m}^{\mathrm{Dol}}[\mathcal{M}] = \partial_s\left\{\mathfrak{A}_{m,m}^{\mathrm{Dol}}(s)[\mathcal{M}]\right\}|_{s=1} \\
&\qquad = \partial_s\{2^{-\mathfrak{m}}(1+s)^{\mathfrak{m}}\textstyle\prod_i c_1(E_i)[\mathcal{N}_i]\}|_{s=1} \\
&\qquad = \tfrac{1}{2}\mathfrak{m}\operatorname{ch}_k(E)[\mathcal{M}] = \tfrac{1}{2}\mathfrak{m}_{2,2}^{\mathrm{Dol}}[\mathcal{M}].
\end{aligned}$$

Similarly, let $\mathcal{M}_i$ be arbitrary Riemann surfaces and let (E_i, h_i) be trivial. We complete the proof of Theorem 1.4 (2b) by computing

$$\begin{aligned}
&a_{2,2}^{\mathrm{Dol}}[\mathcal{M}] = 2^{-\mathfrak{m}}\textstyle\prod_i c_1(T_c)[\mathcal{M}_i], \\
&\mathfrak{a}_{m,m}^{\mathrm{Dol}}[\mathcal{M}] = \partial_s\left\{\mathfrak{A}_{m,m}^{\mathrm{Dol}}(s)[\mathcal{M}]\right\}|_{s=1} \\
&\qquad = \partial_s\{6^{-\mathfrak{m}}(1+2s)^{\mathfrak{m}}\textstyle\prod_i c_1(T_c)[\mathcal{M}_i]\}|_{s=1} \\
&\qquad = \tfrac{1}{3}\mathfrak{m}2^{-\mathfrak{m}+1}\textstyle\prod_i c_1(T_c)[\mathcal{M}_i] = \tfrac{2}{3}\mathfrak{m}a_{m,m}^{\mathrm{Dol}}[\mathcal{M}]. \qquad \square
\end{aligned}$$

References

[1] J. Álvarez-López and P. Gilkey, *The local index density of the perturbed de Rham complex*, Czech. Math. J. **71** (2021), 901–932.

[2] J. Álvarez-López and P. Gilkey, *The Witten deformation of the Dolbeault complex*, J. Geometry **112** (2021): 25.

[3] J. A. Álvarez López, Y. A. Kordyukov, and E. Leichtnam, *A trace formula for foliated flows*, in preparation.

[4] J. A. Álvarez López, Y. A. Kordyukov, and E. Leichtnam, *Zeta invariants of Morse forms*, https://doi.org/10.48550/arXiv.2112.03191.

[5] M. F. Atiyah, R. Bott, and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **13** (1973), 279–330; errata **28** (1975), 277–280.

[6] J.-M. Bismut, and W. Zhang, *An extension of a theorem by Cheeger and Muller*, Astérisque **205**, Soc. Math. France (1992)

[7] M. Braverman, and M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros'*, Math. Proc. Camb. Phil. Soc. **122** (1997), 357–375.

[8] S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. Math. (2) **45** (1944), 741–752.

[9] D. Burghelea and S. Haller, *On the topology and analysis of a closed one form. I. (Novikov's theory revisited)*, Essays on geometry and related topics, Vol. 1, 2, Monogr. Enseign. Math., Vol. 38, Enseignement Math., Geneva, 2001, pp. 133–175.

[10] D. Burghelea, and S. Haller, *Dynamics, Laplace transform and spectral geometry*, J. Topol. **1** (2008), 115–151.

[11] P. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Adv. Math. **10** (1973), 344–382.

[12] P. Gilkey, *Curvature and the eigenvalues of Dolbeault complex for Kähler manifolds*, Adv. Math. **11** (1973), 311–325.

[13] P. Gilkey, *Invariance Theory, the heat equation, and the Atiyah-Singer index theorem* 2^{nd} *ed*, CRC Press (1995).

[14] P. Gilkey, S. Nikčević, and J. Pohjanpelto, *The local index formula for a Hermitian manifold*, Pac. J. Math. **180** (1997), 51–56.

[15] P. Günther and R. Schimming, *Curvature and spectrum of compact Riemannian manifolds*, J. Diff. Geo. **12** (1977), 599-618.

[16] F. R. Harvey and G. Minervini, *Morse Novikov theory and cohomology with forward supports*, Math. Ann. **335** (2006), 787–818.

[17] B. Helffer, and J. Sjöstrand, *Puits multiples en mecanique semi-classique IV: Edude du complexe de Witten*, Commun. Partial Differ. Equations **10** (1985), 245–340.

[18] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag (Berlin) 1966.

[19] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Diff. Geo. **1** (1967), 43–69.

[20] G. Minervini, *A current approach to Morse and Novikov theories'*, Rend. Mat. Appl., VII. Ser. **36** (2015), 95–195.

[21] V. K. Patodi, *Curvature and the eigenforms of the Laplace operator*, J. Diff. Geo. **5** (1971), 233–249.

[22] A. Pazhitnov, *An analytic proof of the real part of Novikov's inequalities*, Soviet Math., Dokl. **35** (1987), 456–457.

[23] O. Pekonen, *The Einstein field equation in a multidimensional universe*, Gen. Relativity Gravitation **20** (1988), 667–670.

[24] N. Ramachandran, *Higher Euler characteristics: variations on a theme of Euler*, Homology, Homotopy, and Applications **18** (2016), 231–246.

[25] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifold*, Adv in Math **7** (1971), 145–210.

[26] H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton (1946) (8^{th} printing).

[27] E. Witten, *Supersymmetry and Morse theory*, J. Differ. Geom. **17** (1982), 661–692.

Manuscript received October 3 2020
revised December 27 2020

J. Álvarez López
Department of Geometry and Topology, Faculty of Mathematics, University of Santiago de Compostela, 15782 Santiago de Compostela, Spain
E-mail address: jesus.alvarez@usc.es

P. Gilkey
Mathematics Department, University of Oregon, Eugene Oregon 97403, USA
E-mail address: gilkey@uoregon.edu