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TURNPIKE PHENOMENON AND ITS STABILITY FOR DISCRETE-TIME OPTIMAL CONTROL PROBLEMS WITH A LYAPUNOV FUNCTION

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Abstract. In our recent research we studied the turnpike phenomenon for discrete disperse dynamical systems introduced in 1980 by A. M. Rubinov, which have a prototype in mathematical economics. This dynamical system is generated by a set-valued mapping acting on a compact metric space. In the present paper we generalize the results obtained in that research to the case when the set-valued mapping generating the dynamical system acts on a metric space which is not compact itself but all its bounded, closed sets are compact. In particular, we show that the turnpike phenomenon is stable under small perturbations of the set-valued mapping and an objective function.

1. Introduction

In [18, 19] A. M. Rubinov introduced a discrete disperse dynamical system generated by a set-valued mapping acting on a compact metric space, which were studied in [7, 18, 19, 24, 26–28]. This disperse dynamical system has prototype in the mathematical economics [14, 18, 19, 25]. In particular, it is an abstract extension of the classical von Neumann-Gale model [14, 18, 19, 25]. This dynamical system is described by a compact metric space of states and a transition operator which is set-valued. Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. Usually in the dynamical systems theory a transition operator is single-valued. In [7, 18, 19, 24, 26–28] and in the present paper we study dynamical systems with a set-valued transition operator. Such dynamical systems correspond to certain models of economic dynamics [14, 18, 19, 25]. In our recent research we study the turnpike phenomenon for this discrete disperse dynamical system which is generated by a set-valued mapping acting on a compact metric space [30–37]. It should be mentioned that turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [21]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path and a turnpike). This

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property was further investigated for optimal trajectories of models of economic dynamics. See, for example, [14, 19, 25] and the references mentioned there. Recently it was shown that the turnpike phenomenon holds for many important classes of problems arising in various areas of research [6, 10–13, 15–17, 22, 23, 29]. For related infinite horizon problems see [1–5, 8, 9, 20, 25]. In the present paper we generalize the results obtained in our previous research to the case when the set-valued mapping generating the dynamical system acts on a metric space which is not compact itself but all its bounded, closed sets are compact. In particular, we show that the turnpike phenomenon is stable under small perturbations of the set-valued mapping and an objective function.

2. Preliminaries

Assume that (X, ρ) is a metric space and that $\mathcal{A} \subset X \times X$ is a nonempty closed subset of the metric space $X \times X$ equipped with the metric $\rho_1 : X \times X \to [0, \infty)$ defined by

$$\rho_1((x_1, x_2), (y_1, y_2)) = \rho(x_1, y_1) + \rho(x_2, y_2),\ x_1, x_2, y_1, y_2 \in X.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X :\ \rho(x, y) \le r\}.$$

Fix

$$\widehat{\theta} \in X.$$

Assume that the following assumption holds:

(A1) For each $r > 0$ the set $B(\widehat{\theta}, r)$ is compact.

Set

$$X_{\mathcal{A}} = \{x \in X :\ \{x\} \times X \cap \mathcal{A} \ne \emptyset\}. \tag{2.1}$$

For each $x \in X_{\mathcal{A}}$ set

$$a(x) = \{y \in X :\ (x, y) \in \mathcal{A}\}. \tag{2.2}$$

For each $x \in X$ and each nonempty set $B \subset X$ put

$$\rho(x, B) = \inf\{\rho(x, y) :\ y \in B\}.$$

For each $(x_1, x_2) \in X \times X$ and each nonempty set $B \subset X \times X$ put

$$\rho_1((x_1, x_2), B) = \inf\{\rho_1((x_1, x_2), (y_1, y_2)) :\ (y_1, y_2) \in B\}.$$

For each nonempty subset $D \subset X$ set

$$a(D) = \cup\{a(x) :\ x \in D\} \text{ and } a^0(E) = E.$$

We denote by $\mathrm{Card}(B)$ the cardinality of a set B and suppose that the sum over empty set is zero.

Assume that $\widehat{\phi} : X \to R^1$ satisfies

$$\lim_{\rho(x, \widehat{\theta}) \to \infty} \widehat{\phi}(x) = \infty \tag{2.3}$$

and that $\phi : X \to R^1$ is a continuous function such that

$$\phi(z) \ge \widehat{\phi}(z) \text{ for all } z \in X, \tag{2.4}$$

$$\phi(y) \le \phi(x) \text{ for each } (x, y) \in \mathcal{A}. \tag{2.5}$$

In this paper we study convergence and structure of trajectories of the perturbed dynamical system generated by the set-valued mapping a. Following [18, 19] this system is called a discrete dispersive dynamical system.

Let $T_2 > T_1$ be integers. A sequence $\{x_t\}_{t=T_1}^{T_2} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $(x_t, x_{t+1}) \in \mathcal{A}$ for all integers $t \in \{T_1, \dots, T_2 - 1\}$.

A sequence $\{x_t\}_{t=T_1}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $(x_t, x_{t+1}) \in \mathcal{A}$ for all integers $t \ge T_1$.

Denote by $Y(T_1, T_2, a)$ the set of all trajectories $\{y_t\}_{t=T_1}^{T_2}$ of a and by $Y(T_1, \infty, a)$ the set of all trajectories $\{y_t\}_{t=T_1}^{\infty}$ of a.

Evidently, the function ϕ is a Lyapunov function for the dynamical system generated by the mapping a. In economic growth theory usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space [14, 18, 25]. We study approximate solutions of the problem

$$\phi(x_T) \to \max,$$

$$\{x_t\}_{t=0}^{T} \text{ is a trajectory satisfying } x_0 = x,$$

where $x \in X$ and a natural number T are given.

In Section 3 we will prove the following result.

Proposition 2.1. *The following properties are equivalent:*

1. *There exists a trajectory $\{x_t\}_{t=0}^{\infty}$ of a.*
2. *There exists $M > 0$ and for each integer $n \ge 1$ there exist a trajectory $\{x_t\}_{t=0}^{n}$ satisfying $\rho(x_0, \widehat{\theta}) \le M$.*

In this paper we assume that property (1) of Proposition 2.1 holds.

Define

$$\Omega(a) = \{z \in X : \text{ for each } \epsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty}$$
$$\text{such that } \liminf_{t\to\infty} \rho(z, x_t) \le \epsilon\}. \tag{2.6}$$

Clearly, $\Omega(a)$ is a nonempty closed subset of (X, ρ). In the literature the set $\Omega(a)$ is called a global attractor of a. Note that in [18, 19] $\Omega(a)$ is called a turnpike set of a. This terminology is motivated by mathematical economics [14, 18, 19, 25].

In Section 3 we prove the following result.

Proposition 2.2. $\Omega(a) \neq \emptyset$ *and for every trajectory $\{x_t\}_{t=0}^{\infty}$ of a,*

$$\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0.$$

It is not difficult to see that the following proposition holds.

Proposition 2.3. *Assume that $B \subset X$ is a nonempty, closed set such that for each trajectory $\{x_t\}_{t=0}^{\infty}$ the equation*

$$\lim_{t\to\infty} \rho(x_t, B) = 0$$

is true. Then $\Omega(a) \subset B$ holds.

The following theorem will be proved in Section 4.

Theorem 2.4. *Let ϵ, M be positive real numbers. Then there is a positive integer T such that for every trajectory $\{x_t\}_{t=0}^T$ which satisfies*

$$\rho(x_0, \widehat{\theta}) \le M$$

the inequality

$$\min\{\rho(x_t, \Omega(a)) : \ t = 0, \dots, T\} \le \epsilon$$

holds.

The next theorem is proved in Section 5.

Theorem 2.5. *The following properties are equivalent:*

(1) *if a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies*

$$x_{t+1} \in a(x_t) \text{ and } \phi(x_{t+1}) = \phi(x_t)$$

for all integers t then the inclusion $x_t \in \Omega(a)$ holds for all integers t;

(2) *for each $M, \epsilon > 0$ there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^T$ of a which satisfies*

$$\rho(x_0, \widehat{\theta}) \le M \text{ and } \phi(x_0) - \phi(x_T) \le \delta$$

the inequality

$$\rho(x_t, \Omega(a)) \le \epsilon$$

holds for all integers $t = L, \dots, T - L$.

From here we assume that property (1) of Theorem 2.5 hold. This property indeed holds for models of economic dynamics which are prototypes of our dynamical system [14, 18, 25].

Denote by $\mathcal{L}$ the set of all functions $\psi : X \to R^1$ such that

$$\psi(x) \ge \widehat{\phi}(x) \text{ for all } x \in X. \tag{2.7}$$

We equip the set $\mathcal{L}$ with the uniformity which is determined by the base

$$\mathcal{E}(N, \epsilon) = \{(\psi_1, \psi_2) \in \mathcal{L} \times \mathcal{L} : \ |\psi_1(z) - \psi_2(z)| \le \epsilon \text{ for each } z \in B(\widehat{\theta}, N)\},$$

where $N, \epsilon > 0$. This uniform space is Hausdorff, has a countable base and therefore it is metrizable. The uniformity defined above induces in $\mathcal{L}$ a topology.

In Section 7 we prove the following turnpike result.

Theorem 2.6. *Let $\epsilon, M > 0$. Then there exist natural numbers L, Q, a number $\delta > 0$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each integer $T > L$, each $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ such that*

$$\rho(x_0, \widehat{\theta}) \le M$$

and that for all integers $t = 0, \dots, T - 1$,

$$\psi(x_{t+1}) \le \psi(x_t)$$

and

$$\rho_1((x_t, x_{t+1}), \mathcal{A}) \le \delta$$

there exist nonnegative integers $a_i < b_i \le T$, $i = 1, \dots, q$, *where* $q \in \{1, \dots, Q\}$ *is an integer such that*

$$a_{i+1} > b_i,\ i \in \{1, \dots, q\} \setminus \{q\},$$

for each $i \in \{1, \dots, q\}$,

$$\rho(x_t, \Omega(a)) \le \epsilon,\ t = a_i, \dots, b_i$$

and that

$$Card(\{0, \dots, T\} \setminus \cup_{i=1}^{q} \{a_i, \dots, b_i\}) \le L.$$

This theorem shows that a weak turnpike property holds for approximate trajectories of our dynamical system. Note that the constants L, Q depend only on ϵ, M.

In this paper we obtain a strong version of Theorem 2.6 assuming that the following property holds which was introduced in [28].

(P1) If $z_1, z_2 \in \Omega(a)$ satisfies $\phi(z_1) = \phi(z_2)$, then $z_1 = z_2$.

Note that for models of economic growth which are prototype of our dynamical system property (P1) holds [14, 18, 19, 25].

In Section 7 we prove the following turnpike result.

Theorem 2.7. *Assume that (P1) holds and that* $\epsilon, M > 0$. *Then there exist natural numbers* L, Q, *a number* $\delta > 0$ *and a neighborhood* $\mathcal{U}$ *of* ϕ *in* $\mathcal{L}$ *such that for each integer* $T > L$, *each* $\psi \in \mathcal{U}$ *and each sequence* $\{x_t\}_{t=0}^{T} \subset X$ *such that*

$$\rho(x_0, \widehat{\theta}) \le M$$

and that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \le \psi(x_t)$$

and

$$\rho_1((x_t, x_{t+1}), \mathcal{A}) \le \delta$$

there exist nonnegative integers $a_i < b_i \le T$, $i = 1, \dots, q$, *where* $q \in \{1, \dots, Q\}$ *is an integer and* $z_i \in \Omega(a)$, $i = 1, \dots, q$ *such that*

$$a_{i+1} > b_i,\ i \in \{1, \dots, q\} \setminus \{q\},$$

for each $i \in \{1, \dots, q\}$,

$$\rho(x_t, z_i) \le \epsilon,\ t = a_i, \dots, b_i$$

and that

$$Card(\{0, \dots, T\} \setminus \cup_{i=1}^{q} \{a_i, \dots, b_i\}) \le L.$$

3. Proofs of Propositions 2.1 and 2.2

Proof of Proposition 2.1

Clearly, (1) implies (2). Assume that (2) holds. Then there exist $M > 0$ and trajectories $\{x_t^{(n)}\}_{t=0}^{n}$, $n = 1, 2, \dots$ such that

$$\rho(x_0^{(n)}, \widehat{\theta}) \le M,\ n = 1, 2, \dots. \tag{3.1}$$

Since the set $B(\widehat{\theta}, M)$ is compact and ϕ is continuous equation (3.1) implies that there exists $M_0 > 0$ such that

$$\phi(x_0^{(n)}) \le M_0,\ n = 1, 2, \dots.$$

Together with (2.5) this implies that

$$\phi(x_t^{(n)}) \le M_0,\ t = 0, \dots, n,\ n = 1, 2, \dots.$$

Combined with (2.3) and (2.4) this implies that the set

$$\{x_t^{(n)}:\ t = 0, \dots, n,\ n = 1, 2, \dots\}$$

is bounded. By (A1), extracting subsequences and using the diagonalization process we obtain that there exist a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer $t \ge 0$ there exists

$$x_t = \lim_{i \to \infty} x_t^{(n_j)}.$$

Since the set $\mathcal{A}$ is closed the sequence $\{x_t\}_{t=0}^{\infty}$ is a trajectory of a. Proposition 2.1 is proved.

Proof of Proposition 2.2

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory of a. In view of (2.5),

$$\phi(x_{t+1}) \le \phi(x_t) \le \phi(x_0)$$

for every integer $t \ge 0$. Equations (2.3) and (2.4) imply that the sequence $\{x_t\}_{t=0}^{\infty}$ is bounded. In view of (A1) it has a convergent subsequence and its limit belongs to $\Omega(a)$. Now we show that

$$\lim_{t \to \infty} \rho(x_t, \Omega(a)) = 0.$$

Assume the contrary. Then there exist $\epsilon > 0$ and a strictly increasing sequence of natural numbers $\{t_j\}_{j=1}^{\infty}$ such that

$$\rho(x_{t_j}, \Omega(a)) > \epsilon,\ j = 1, 2, \dots. \tag{3.2}$$

Since the sequence $\{x_t\}_{t=0}^{\infty}$ is bounded the sequence $\{x_{t_j}\}_{j=1}^{\infty}$ has a convergent subsequence and its limit belongs to $\Omega(a)$. This contradicts (3.2). The contradiction we have reached completes the proof of Proposition 2.2.

4. Proof of Theorem 2.4

Assume that the theorem does not hold. Then for every natural number n there exists a trajectory $\{x_t^{(n)}\}_{t=0}^{n}$ of a such that

$$\rho(x_0^{(n)}, \widehat{\theta}) \le M, \tag{4.1}$$

$$\rho(x_t^{(n)}, \Omega(a)) > \epsilon,\ t = 0, \dots, n. \tag{4.2}$$

Assumption (A1), (4.1) and the continuity of ϕ imply that the sequence $\{\phi(x_0^{(n)})\}_{n=1}^{\infty}$ is bounded. Together with (2.5) this implies that the set

$$\{\phi(x_t^{(n)}):\ t = 0, \dots, n,\ n = 1, 2, \dots\}$$

is bounded from above. Combining with (2.3) and (2.4) this implies that the set

$$\{x_t^{(n)}:\ t = 0, \dots, n,\ n = 1, 2, \dots\}$$

is bounded. Extracting subsequences and using the diagonalization process we obtain that there exist a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer $t \geq 0$ there exists

$$x_t = \lim_{i\to\infty} x_t^{(n_j)}. \tag{4.3}$$

Since the set $\mathcal{A}$ is closed $\{x_t\}_{t=0}^{\infty}$ is a trajectory of a. It follows from (4.2) and (4.3) that for every integer $t \geq 0$,

$$\rho(x_t, \Omega(a)) \geq \epsilon.$$

This contradicts Proposition 2.2. The contradiction we have reached completes the proof of Theorem 2.4.

5. Proof of Theorem 2.5

Clearly, (2) implies (1). Assume that (1) holds and (2) does not hold. Then there exist $\epsilon, M > 0$ such that for each integer $k \geq 1$ there exist an integer $T_k > 2k$ and a trajectory $\{x_t^{(k)}\}_{t=0}^{T_k}$ such that

$$\rho(x_0^{(k)}, \widehat{\theta}) \leq M, \tag{5.1}$$

$$|\phi(x_0^{(k)}) - \phi(x_T^{(k)})| \leq k^{-1} \tag{5.2}$$

and

$$\max\{\rho(x_t^{(k)}, \Omega(a)) :\ t = k, \dots, T_k - k\} > \epsilon. \tag{5.3}$$

In view of (5.1), the sequence $\{\phi(x_0^{(k)})\}_{k=1}^{\infty}$ is bounded. Together with (2.5) this implies that the set

$$\{\phi(x_t^{(k)}) :\ t = 0, \dots, T_k,\ k = 1, 2, \dots\}$$

is bounded from above. Combined with (2.3) and (2.4) this implies that the set

$$\{x_t^{(k)} :\ t = 0, \dots, T_k,\ k = 1, 2, \dots\} \text{ is bounded.} \tag{5.4}$$

Let $p \geq 1$ be an integer. In view of (5.3), there exists an integer

$$\tau_p \in \{p, \dots, T_p - p\} \tag{5.5}$$

such that

$$\rho(x_{\tau_p}^{(p)}, \Omega(a)) > \epsilon. \tag{5.6}$$

Define

$$y_t^{(p)} = x_{t+\tau_p}^{(p)},\ t = -\tau_p, \dots, T_p - \tau_p. \tag{5.7}$$

Equations (5.6) and (5.7) imply that

$$\rho(y_0^{(p)}, \Omega(a)) = \rho(x_{\tau_p}^{(p)}, \Omega(a)) > \epsilon \tag{5.8}$$

It follows from (5.4) and (5.7) that the set

$$\{y_t^{(p)} :\ t = -\tau_p, \dots, T_p - \tau_p,\ p = 1, 2, \dots\} \text{ is bounded.}$$

By (A1), (5.5) and (5.7), extracting subsequences and using the diagonalization process we obtain that there exist a strictly increasing sequence of natural numbers $\{p_i\}_{i=1}^{\infty}$ such that for each integer t there exists

$$y_t = \lim_{i\to\infty} y_t^{(p_i)}. \tag{5.9}$$

Since the set $\mathcal{A}$ is closed and (5.2), (5.7) are true for each integer t we have

$$\phi(y_t) = \phi(y_{t+1}),\ y_{t+1} \in a(y_t).$$

Property (1) implies that

$$y_t \in \Omega(a) \text{ for each integer } t.$$

On the other hand it follows from (5.8) and (5.9) that

$$\rho(y_0, \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves (2) and Theorem 2.5.

6. An auxiliary result for Theorems 2.6 and 2.7

Lemma 6.1. *Assume that $\epsilon, M > 0$. Then there exist $\delta > 0$, a natural number L and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each integer $T \geq 2L$, each function $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^{T} \subset X$ such that*

$$\rho(x_0, \widehat{\theta}) \leq M,$$

for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t),$$
$$\rho_1((x_t, x_{t+1}), \mathcal{A}) \leq \delta$$

and that

$$\psi(x_0) \leq \psi(x_T) + \delta$$

the inequality

$$\rho(x_t, \Omega(a)) \leq \epsilon$$

is valid for each $t \in \{L, \dots, T-L\}$.

Proof. In view of (A1), there exists $M_0 > 1$ such that

$$\phi(B(\widehat{\theta}, M)) \subset [-M_0+1, M_0+1]. \tag{6.1}$$

There exists $M_1 > M + M_0$ such that

$$\{x \in X :\ \widehat{\phi}(x) \leq M_0\} \subset B(\widehat{\theta}, M_1). \tag{6.2}$$

Assume that the lemma does not hold. Then for each integer $k \geq 1$ there exist an integer

$$T_k \geq 2k,\ \psi_k \in \mathcal{L} \tag{6.3}$$

such that

$$(\phi, \psi_k) \in \mathcal{E}(M_1, k^{-1}) \tag{6.4}$$

and a sequence $\{x_t^{(k)}\}_{t=0}^{T_k} \subset X$ such that

$$\rho(x_0^{(k)}, \widehat{\theta}) \leq M, \tag{6.5}$$

for each $t \in \{0, \dots, T_k - 1\}$,

$$\psi_k(x_{t+1}^{(k)}) \le \psi_k(x_t^{(k)}),\ \rho_1((x_t^{(k)}, x_{t+1}^{(k)}), \mathcal{A}) \le k^{-1}, \tag{6.6}$$

$$\psi_k(x_0^{(k)}) \le \psi_k(x_T^{(k)}) + 1/k \tag{6.7}$$

and

$$\max\{\rho(x_t^{(k)}, \Omega(a)) :\ t = k, \dots, T_k - k\} > \epsilon. \tag{6.8}$$

In view of (6.1) and (6.5), for each integer $k \ge 1$,

$$\phi(x_0^{(k)}) \le M_0 - 1. \tag{6.9}$$

It follows from (6.4), (6.5) and (6.9) that for each integer $k \ge 1$,

$$|\psi_k(x_0^{(k)}) - \phi(x_0^{(k)})| \le 1/k,$$

$$\psi_k(x_0^{(k)}) \le \phi(x_0^{(k)}) + 1/k \le M_0. \tag{6.10}$$

By (2.7), (6.2), (6.3), (6.7) and (6.10), for each integer $k \ge 1$ and each $t \in \{0, \dots, T_k\}$,

$$\widehat{\phi}(x_t^{(k)}) \le \psi_k(x_t^{(k)}) \le \psi_k(x_0^{(k)}) \le M_0,$$

$$x_t^{(k)} \in B(\widehat{\theta}, M_1). \tag{6.11}$$

Equations (6.4) and (6.11) imply that for each integer $k \ge 1$ and each $t \in \{0, \dots, T_k\}$,

$$|\phi(x_t^{(k)}) - \psi_k(x_t^{(k)})| \le k^{-1}. \tag{6.12}$$

It follows from (6.6), (6.7) and (6.12) that for each integer $k \ge 1$ and each $t \in \{0, \dots, T_k - 1\}$ we have

$$\phi(x_t^{(k)}) - \phi(x_{t+1}^{(k)}) \le \psi_k(x_t^{(k)}) - \psi_k(x_{t+1}^{(k)}) + 2/k \le 3/k \tag{6.13}$$

and in view of (6.6) and (6.12),

$$\phi(x_t^{(k)}) \ge \psi_k(x_t^{(k)}) - 1/k \ge \psi_k(x_{t+1}^{(k)}) - 1/k \ge \phi(x_{t+1}^{(k)}) - 2/k. \tag{6.14}$$

By (6.7) and (6.12),

$$|\phi(x_0^{(k)}) - \phi(x_{T_k}^{(k)})| \le |\psi_k(x_0^{(k)}) - \psi_k(x_{T_k}^{(k)})| + 2/k \le 3/k. \tag{6.15}$$

Equation (6.8) implies that for each integer $k \ge 1$ there exists

$$\tau_k \in \{k, \dots, T_k - k\} \tag{6.16}$$

such that

$$\rho(x_{\tau_k}^{(k)}, \Omega(a)) > \epsilon. \tag{6.17}$$

Let $k \ge 1$ be an integer. Define a finite sequence

$$y_t^{(k)} = x_{t+\tau_k}^{(k)},\ t = -\tau_k, \dots, T_k - \tau_k. \tag{6.18}$$

In view of (6.17) and (6.18), for each integer $k \ge 1$ and each $t \in \{-\tau_k, \dots, T_k - \tau_k\}$,

$$\rho(y_t^{(k)}, \widehat{\theta}) \le M_1.$$

Extracting a subsequence and using a diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{k_i\}_{i=1}^{\infty}$ such that for each integer t there exists

$$y_t = \lim_{i\to\infty} y_t^{(k_i)}. \tag{6.19}$$

Let t be an integer. By (6.6), (6.18) and (6.19),

$$\rho_1((y_t, y_{t+1}), \mathcal{A}) = \lim_{i\to\infty} \rho_1((y_t^{(k_i)}, y_{t+1}^{(k_i)}), \mathcal{A}) = \lim_{i\to\infty} \rho_1((x_{t+\tau_{k_i}}^{(k_i)}, x_{t+\tau_{k_i}+1}^{(k_i)}), \mathcal{A}) = 0$$

and

$$y_{t+1} \in a(y_t).$$

It follows from (6.13), (6.14), (6.18) and (6.19) that

$$|\phi(y_t) - \phi(y_{t+1})| = \lim_{i\to\infty} |\phi(y_t^{(k_i)}) - \phi(y_{t+1}^{(k_i)})| = \lim_{i\to\infty} |\phi(x_{t+\tau_{k_i}}^{(k_i)}) - \phi(x_{t+\tau_{k_i}}^{(k_i)})| = 0.$$

In view of property (1),

$$y_t \in \Omega(a)$$

for every integer t and in particular,

$$y_0 \in \Omega(a).$$

On the other hand it follows from (6.17)-(6.19) that for every integer $k \geq 1$,

$$\rho(y_0^{(k)}, \Omega(a)) = \rho(x_{\tau_k}^{(k)}, \Omega(a)) > \epsilon.$$

and

$$\rho(y_0, \Omega(a)) \geq \epsilon.$$

The contradiction we have reached completes the proof of Lemma 6.1. □

Lemma 6.2. *Assume that (P1) holds and that $M, \epsilon > 0$. Then there exists $\delta > 0$ such that for each*

$$z_1, z_2 \in \Omega(a) \cap B(\widehat{\theta}, M)$$

satisfying

$$|\phi(z_1) - \phi(z_2)| \leq \delta$$

the inequality $\rho(z_1, z_2) \leq \epsilon$ holds.

Proof. Assume that the lemma does not hold. Then for each integer $k \geq 1$ there exist

$$z_{k,1},\ z_{k,2} \in \Omega(a) \cap B(\widehat{\theta}, M)$$

such that

$$|\phi(z_{k,1}) - \phi(z_{k,2})| \leq k^{-1},\ \rho(z_{k,1}, z_{k,2}) > \epsilon.$$

By (A1), $\{z_{k,1}\}_{k=1}^{\infty}$ has a convergent subsequence. We may assume without loss of generality that $\{z_{k,1}\}_{k=1}^{\infty}$ converges. Analogously we may assume that the sequence $\{z_{k,2}\}_{k=1}^{\infty}$ converges too. Set

$$z_i = \lim_{k\to\infty} z_{k,i},\ i = 1, 2.$$

By the equations above,

$$z_1, z_2 \in \Omega(a),$$
$$\rho(z_1, z_2) \geq \epsilon,\ \phi(z_1) = \phi(z_2).$$

This contradicts (P1). The contradiction we have reached completes the proof of Lemma 6.2. □

Lemma 6.3. *Assume that property (P1) holds, $\epsilon \in (0,1)$, $M > 0$. Then there exist $\delta > 0$, a natural number L and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each integer $T \geq 2L$, each function $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ such that*

$$\rho(x_0, \widehat{\theta}) \leq M, \tag{6.20}$$

for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t),\ \rho_1((x_t, x_{t+1}), \mathcal{A}) \leq \delta \tag{6.21}$$

and that

$$\psi(x_0) \leq \psi(x_T) + \delta \tag{6.22}$$

there exists $z \in \Omega(a)$ such that the inequality

$$\rho(x_t, z) \leq \epsilon$$

holds for each $t \in \{L, \dots, T-L\}$,

Proof. There exists $M_0 > 1$ such that

$$\phi(B(\widehat{\theta}, M)) \subset [-M_0 + 1, M_0 - 1]. \tag{6.22}$$

In view of (2.3), there exists $M_1 > M + M_0$ such that

$$\{x \in X : \ \widehat{\phi}(x) \leq M_0\} \subset B(\widehat{\theta}, M_1). \tag{6.23}$$

Lemma 6.2 implies that there exists $\epsilon_1 \in (0, \epsilon/4)$ such that the following property holds:

(i) for each

$$z_1, z_2 \in \Omega(a) \cap B(\widehat{\theta}, M_1 + 1)$$

satisfying

$$|\phi(z_1) - \phi(z_2)| \leq \epsilon_1$$

the inequality $\rho(z_1, z_2) \leq \epsilon/4$ holds.

In view of the continuity of ϕ, there exists $\epsilon_0 \in (0, \epsilon_1)$ such that the following property holds:

(ii) for each

$$z_1, z_2 \in B(\widehat{\theta}, M_1 + 4)$$

satisfying $\rho(z_1, z_2) \leq \epsilon_0$ we have

$$|\phi(z_1) - \phi(z_2)| \leq \epsilon_1/8.$$

Lemma 6.1 implies that there exist

$$\delta \in (0, \epsilon_0/8),$$

a natural number L and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that

$$\mathcal{U} \subset \{\psi \in \mathcal{L} : \ (\phi, \psi) \in \mathcal{E}(M_1 + 2, \epsilon_0/8)\}$$

and that the following property holds:

(iii) for each integer $T \geq 2L$, each function $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ such that

$$\rho(x_0, \widehat{\theta}) \leq M$$

and that for all integers $t = 0, \ldots, T-1$, equations (6.21) and (6.22) are valid we have

$$\rho(x_t, \Omega(a)) \leq \epsilon_0/4,\ t \in \{L, \ldots, T-L\}. \tag{6.24}$$

Assume that an integer

$$T \geq 2L,\ \psi \in \mathcal{U}, \tag{6.25}$$

a sequence $\{x_t\}_{t=0}^T \subset X$ satisfies (6.20) and that for all integers $t = 0, \ldots, T-1$, equations (6.21) and (6.22) are true. In view of property (iii), (6.24) is valid. Equations (6.20) and (6.22) imply that

$$\phi(x_0) \leq M_0 - 1. \tag{6.26}$$

By the choice of $\mathcal{U}$, (6.20), (6.25) and (6.26),

$$\psi(x_0) \leq \phi(x_0) + 4^{-1} \leq M_0. \tag{6.27}$$

Equations (2.7), (6.21), (6.23) and (6.27) imply that for all $t = 0, \ldots, T$,

$$\psi(x_t) \leq M_0,\ \rho(x_t, \widehat{\theta}) \leq M_1. \tag{6.28}$$

Let

$$t \in \{L, \ldots, T-L\}. \tag{6.29}$$

In view of (6.24) and (6.29), there exists

$$z_t \in \Omega(a) \tag{6.30}$$

such that

$$\rho(x_t, z_t) \leq \epsilon_0/4. \tag{6.31}$$

It follows from (6.28) and (6.31) that

$$\rho(z_t, \widehat{\theta}) \leq M_1 + 2^{-1}. \tag{6.32}$$

By (6.21) and (6.22), for each $t, s \in \{0, \ldots, T\}$,

$$|\psi(x_t) - \psi(x_s)| \leq \delta. \tag{6.33}$$

By the choice of $\mathcal{U}$ and equations (6.25), (6.28) and (6.33), for each $s, t \in \{0, \ldots, T\}$,

$$\begin{aligned} &|\phi(x_t) - \phi(x_s)| \\ &\leq |\phi(x_t) - \psi(x_t)| + |\psi(x_t) - \psi(x_s)| + |\psi(x_s) - \phi(x_s)| \\ &\leq \epsilon_0/8 + \delta + \epsilon_0 < \epsilon_0/2. \end{aligned} \tag{6.34}$$

Property (ii) and equations (6.28), (6.31) and (6.32) imply that for each $t \in \{L, \ldots, T-L\}$,

$$|\phi(x_t) - \phi(z_t)| \leq \epsilon_1/8.$$

In view of the equation above and (6.34), for each $t \in \{L, \ldots, T-L\}$,

$$\begin{aligned} &|\phi(z_L) - \phi(z_t)| \\ &\leq |\phi(z_L) - \phi(x_L)| + |\phi(x_L) - \phi(x_t)| + |\phi(x_t) - \phi(z_t)| \\ &\leq \epsilon_1/8 + \epsilon_0/2 + \epsilon_1/8 \leq 3\epsilon_1/4. \end{aligned}$$

Property (i) and equations (6.30)-(6.32) imply that for each $t \in \{L, \dots, T-L\}$,

$$\rho(z_t, z_L) \le \epsilon/4,$$

$$\rho(x_t, z_L) \le \rho(x_t, z_t) + \rho(z_t, z_L) \le \epsilon_0/4 + \epsilon/4 < \epsilon.$$

Lemma 6.3 is proved. □

7. Proofs of Theorems 2.6 and 2.7

We prove Theorems 2.6 and 2.7 simultaneously. We may assume that $\epsilon < 1 < M$. There exists $M_0 > M + 1$ such that

$$\phi(B(\widehat{\theta}, M)) \subset [-M_0 + 1, M_0 - 1]. \tag{7.1}$$

In view of (2.3), there exist $M_1 > M + M_0$ such that

$$\{x \in X : \widehat{\phi}(x) \le M_0\} \subset B(\widehat{\theta}, M_1) \tag{7.2}$$

and $M_2 > M_1 + 1$ such that

$$\phi(B(\widehat{\theta}, M_1)) \subset [-M_2, M_2]. \tag{7.3}$$

Lemmas 6.1 and 6.3 imply that there exist

$$\delta \in (0, \epsilon),$$

a natural number L_0 and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that

$$\mathcal{U} \subset \{\psi \in \mathcal{L} : (\phi, \psi) \in \mathcal{E}(M_1 + 2, 1/8)\} \tag{7.4}$$

and that the following property holds:

(i) for each integer $T \ge 2L_0$, each function $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ such that

$$\rho(x_0, \widehat{\theta}) \le M_1$$

and that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \le \psi(x_t),\ \rho_1((x_t, x_{t+1}), \mathcal{A}) \le \delta \tag{7.5}$$

and that

$$\psi(x_0) \le \psi(x_T) + \delta \tag{7.6}$$

we have

$$\rho(x_t, \Omega(a)) \le \epsilon,\ t \in \{L_0, \dots, T - L_0\}$$

and if property (P1) holds, then there exists $z \in \Omega(a)$ such that

$$\rho(x_t, z) \le \epsilon,\ t = L_0, \dots, T - L_0.$$

Choose integers

$$Q > 2 + (2M_2 + 4)\delta^{-1} \tag{7.7}$$

and

$$L > Q(4L_0 + 8). \tag{7.8}$$

Suppose that an integer

$$T > L,\ \psi \in \mathcal{U} \tag{7.9}$$

and that a sequence $\{x_t\}_{t=0}^T \subset X$ is such that

$$\rho(x_0, \widehat{\theta}) \le M \tag{7.10}$$

and that for each $t \in \{0, \dots, T\}$ (7.5) is true. It follows from (7.1) and (7.10) that

$$\phi(x_0) \le M_0 - 1. \tag{7.11}$$

In view (7.4) and (7.9)-(7.11),

$$\psi(x_0) \le M_0.$$

Together with (2.7), (7.2) and (7.5) this implies that

$$\psi(x_t) \le M_0,\ \rho(x_t, \widehat{\theta}) \le M_1,\ t = 0, \dots, T. \tag{7.12}$$

By induction we define a strictly increasing finite sequence $t_i \in \{0, \dots, T\}$, $i = 0, \dots, q$. Set

$$t_0 = L_0.$$

If

$$\psi(x_T) \ge \phi(x_{t_0}) - \delta,$$

then set $t_1 = T$ and complete to construct the sequence.

Assume that

$$\psi(x_T) < \psi(x_{t_0}) - \delta.$$

Evidently, there is an integer $t_1 \in (t_0, T]$ satisfying

$$\psi(x_{t_1}) < \psi(x_0) - \delta \tag{7.13}$$

and that if an integer S satisfies $t_0 < S < t_1$, then

$$\psi(x_S) \ge \psi(x_{t_0}) - \delta. \tag{7.14}$$

If $t_1 = T$, then we complete to construct the sequence.

Assume that $k \in \{1, 2, \dots\}$ and that we defined a strictly increasing sequence $t_0, \dots, t_k \in \{0, 1, \dots, T\}$ such that

$$t_0 = L_0,\ t_k \le T$$

and that for each $i \in \{0, \dots, k-1\}$,

$$\psi(x_{t_{i+1}}) < \psi(x_{t_i}) - \delta$$

and if an integer S satisfies $t_i < S < t_{i+1}$, then

$$\psi(x_S) \ge \psi(x_{t_i}) - \delta.$$

(It is not difficult to see that the assumption is true with $k = 1$).

If $t_k = T$, then we complete to construct the sequence. Assume that $t_k < T$. If

$$\psi(x_T) \ge \psi(x_{t_k}) - \delta,$$

then we set $t_{k+1} = T$ and complete to construct the sequenced.

Assume that

$$\psi(x_T) < \psi(x_{t_k}) - \delta.$$

Evidently, there is a natural number $t_{k+1} \in (t_k, T]$ for which

$$\psi(x_{t_{k+1}}) < \psi(x_{t_k}) - \delta$$

and that if an integer S satisfies $t_k < S < t_{k+1}$, then

$$\psi(x_S) \geq \psi(x_{t_k}) - \delta.$$

Evidently, the assumption made for k is true for $k+1$ too. Therefore by induction, we constructed the strictly increasing finite sequence of integers $t_i \in [0, T]$, $i = 0, \dots, q$ such that

$$t_0 = L_0,\ t_q = T \tag{7.15}$$

and that for every i satisfying $0 \leq i < q - 1$,

$$\psi(x_{t_{i+1}}) < \psi(x_{t_i}) - \delta \tag{7.16}$$

and for each $i \in \{0, \dots, q-1\}$ and each integer S satisfies $t_i < S < t_{i+1}$, we have

$$\psi(x_S) \geq \psi(x_{t_i}) - \delta. \tag{7.17}$$

By (7.3), (7.4), (7.9) and (7.12),

$$\psi(x_{t_0}) - \psi(x_{t_{q-1}}) \leq \phi(x_{t_0}) - \phi(x_{t_{q-1}}) + 1 \leq 2M_2 + 1. \tag{7.18}$$

It follows from (7.5), (7.7), (7.16) and (7.18) that

$$2M_2 + 1 \geq \psi(x_{t_0}) - \psi(x_{t_{q-1}})$$

$$\sum\{\psi(x_{t_i}) - \psi(x_{t_{i+1}}) :\ i \text{ is an integer},\ 0 \leq i \leq q-2\} \geq \delta(q-1)$$

and

$$q \leq 1 + \delta^{-1}(2M_2 + 1) < Q. \tag{7.19}$$

Set

$$E = \{i \in \{0, \dots, q-1\} :\ t_{i+1} - t_i \geq 2L_0 + 4\}. \tag{7.20}$$

Let

$$i \in E. \tag{7.21}$$

By (7.4), (7.5), (7.20) and (7.21),

$$t_{i+1} - 1 - t_i \geq 2L_0 + 3. \tag{7.22}$$

Equations (7.17), (7.21) and (7.22) imply that

$$\psi(x_{t_{i+1}-1}) \geq \psi(x_{t_i}) - \delta.$$

Equation above, (7.5), (7.9), (7.12), (7.17), (7.21), (7.22) and property (i) applied to the program $\{x_t\}_{t=t_i}^{t_{i+1}-1}$ imply that

$$\rho(x_t, \Omega(a)) \leq \epsilon,\ t = t_i + L_0, \dots, t_{i+1} - 1 - L_0$$

and if property (P1) holds, then there exists

$$z_i \in \Omega(a)$$

such that

$$\rho(x_t, z_i) \leq \epsilon,\ t = t_i + L_0, \dots, t_{i+1} - 1 - L_0.$$

Set

$$a_i = t_i + L_0,\ b_i = t_{i+1} - L_0 - 1,\ i \in E.$$

By (7.8), (7.15), (7.19) and (7.20),

$$\mathrm{Card}(\{0,\dots,T\}\setminus \cup_{i\in E}\{a_i,\dots,b_i\})$$

$$\le \mathrm{Card}(\cup\{\{t_i,\dots,t_{i+1}\}:\ i\in\{0,\dots,q-1\}\setminus E\})$$

$$+\mathrm{Card}(\cup\{\{t_i,\dots,t_i+L_0-1\}\cup\{t_{i+1}-L_0,\dots,t_{i+1}\}:\ i\in E\})$$

$$\le q(2L_0+5)+(2L_0+2)q+L_0\le q(4L_0+7)+L_0$$

$$\le (4L_0+7)(Q-1)+L_0\le (4L_0+7)Q<L.$$

Theorems 2.6 and 2.7 are proved.

8. An extension of the weak turnpike result

Lemma 8.1. *Let $M>0$. Then there exist $M_1>0$ and a neighborhood V of ϕ in $\mathcal{L}$ such that for each integer $T\ge 1$, each function*

$$\psi\in V \tag{8.1}$$

and each sequence $\{x_t\}_{t=0}^T\subset X$ such that

$$\rho(x_0,\widehat{\theta})\le M,\ \psi(x_{t+1})\le\psi(x_t), t=0,\dots,T-1 \tag{8.2}$$

the inequality $\rho(x_t,\widehat{\theta})\le M_1$ holds for all $t=0,\dots,T$.

Proof. There exists $M_0>0$ such that

$$\phi(B(\widehat{\theta},M))\subset[-M_0,M_0].$$

In view of (2.3), there exist $M_1>M+M_0+1$ such that

$$\{x\in X:\ \widehat{\phi}(x)\le M+M_0+1\}\subset B(\widehat{\theta},M_1).$$

Set

$$V=\{\psi\in\mathcal{L}:\ (\phi,\psi)\in\mathcal{E}(M_1,1/4)\}.$$

Assume that $T\ge 1$ is an integer, (8.1) holds and $\{x_t\}_{t=0}^T\subset X$ satisfies (8.2). By the relations above, (8.1) and (8.2),

$$\phi(x_0)\le M_0,\ \psi(x_0)<M_0+1/4,\ \psi(x_t)<M_0+1/4,\ t=0,\dots,T,$$

$$\rho(x_t,\widehat{\theta})\le M_1,\ t=0,\dots,T.$$

Lemma 8.1 is proved. □

For each $\epsilon,M>0$ denote by $\mathcal{V}(M,\epsilon)$ the set of all nonempty sets $\mathcal{B}\subset X\times X$ such that for each $(\xi_1,\xi_2)\in(B(\widehat{\theta},M)\times B(\widehat{\theta},M))\cap\mathcal{B}$,

$$\rho_1((\xi_1,\xi_2),\mathcal{A})\le\epsilon.$$

Theorems 2.6 and 2.7 and Lemma 8.1 imply the following result.

Theorem 8.2. *Let $\epsilon, M > 0$. Then there exist natural numbers L, Q, numbers $\delta > 0, M_0 > M$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each integer $T > L$, each $\mathcal{B} \in \mathcal{V}(M_0, \delta)$, each $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T$ such that*

$$\rho(x_0, \widehat{\theta}) \le M$$

and that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \le \psi(x_t)$$

and

$$(x_t, x_{t+1}) \in \mathcal{B}$$

there exist nonnegative integers $a_i < b_i \le T$, $i = 1, \dots, q$, where $q \in \{1, \dots, Q\}$ is an integer such that

$$a_{i+1} > b_i,\ i \in \{1, \dots, q\} \setminus \{q\},$$

for each $i \in \{1, \dots, q\}$,

$$\rho(x_t, \Omega(a)) \le \epsilon,\ t = a_i, \dots, b_i$$

and that

$$Card(\{0, \dots, T\} \setminus \cup_{i=1}^q \{a_i, \dots, b_i\}) \le L$$

and if (P1) holds then there exist $z_i \in \Omega(a)$, $i = 1, \dots, q$ such that

$$\rho(x_t, z_i) \le \epsilon,\ t \in \{a_i, \dots, b_i\},\ i = 1, \dots, q.$$

9. Stability of the turnpike phenomenon

In this section we state the main result of the paper (Theorem 9.2) which shows that the turnpike phenomenon is stable under perturbations of the set-valued mapping and the objective function.

Denote by $\mathfrak{M}$ the set of all pairs $(\mathcal{B}, \psi)$ such that $\psi \in \mathcal{L}$ and $\mathcal{B}$ is a nonempty subset of $X \times X$ such that

$$\psi(z) \le \psi(y) \text{ for each } (y, z) \in \mathcal{B}. \tag{9.1}$$

We use the following assumption.

(C1) For each $M, \epsilon > 0$ there exist $\delta > 0$, $M_0 > M$, $\epsilon_0 \in [0, \epsilon]$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each $z \in B(\widehat{\theta}, M)$ satisfying $\rho(x, \Omega(a)) \le \delta$ there exist $\xi_1, \xi_2 \in \Omega(a)$ such that

$$\rho(z, \xi_i) \le \epsilon,\ i = 1, 2$$

and such that for each integer $T \ge 1$ and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ such that $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ the following inequality holds:

$$\begin{aligned}
&\sup\{\psi(z_T) :\ \{z_t\}_{t=0}^T \subset X,\ z_0 = \xi_1,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T-1\} - \epsilon_0 \\
&\le \sup\{\psi(z_T) :\ \{z_t\}_{t=0}^T \subset X,\ z_0 = z,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T-1\} \\
&\le \sup\{\psi(z_T) :\ \{z_t\}_{t=0}^T \subset X,\ z_0 = \xi_2,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T-1\} + \epsilon_0. \quad (9.1)
\end{aligned}$$

Note that as special cases we can have $\epsilon_0 = \epsilon$ and $\epsilon_0 = 0$. Clearly, (9.1) holds with $\epsilon_0 = 0$ if

$$\{\xi \in X :\ (\xi_1, \xi) \in \mathcal{B}\} \subset \{\xi \in X :\ (z, \xi) \in \mathcal{B}\} \subset \{\xi \in X :\ (\xi_2, \xi) \in \mathcal{B}\}.$$

We also use the following assumption.

(C2) For each $M, \epsilon > 0$ there exist $\delta > 0$, $M_0 > M$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each $\xi_1, \xi_2 \in B(\widehat{\theta}, M) \cap \Omega(a)$ satisfying $\phi(\xi_1) \le \phi(\xi_2) - \epsilon$, each pair of natural numbers T_1, T_2 and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ such that $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ the following inequality holds:

$$\psi(\xi_1) + \delta,$$

$$\sup\{\psi(z_{T_2}) : \ \{z_t\}_{t=0}^{T_2} \subset X, \ z_0 = \xi_1, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, T_2 - 1\} + \delta$$

$$\le \sup\{\psi(z_{T_1}) : \ \{z_t\}_{t=0}^{T_1} \subset X, \ z_0 = \xi_2, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, T_1 - 1\}.$$

The next assumption is a strong version of (C1).

(C1') For each $M, \epsilon > 0$ there exist $\delta > 0$, $M_0 > M$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each $z \in B(\widehat{\theta}, M)$ satisfying $\rho(z, \Omega(a)) \le \delta$ there exist $\xi_1, \xi_2 \in \Omega(a)$ such that

$$\rho(z, \xi_i) \le \epsilon, \ i = 1, 2$$

and such that for each $(\mathcal{B}, \psi) \in \mathfrak{M}$ satisfying $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ the following inclusion holds:

$$\{\xi \in X : \ (\xi_1, \xi) \in \mathcal{B}\} \subset \{\xi \in X : \ (z, \xi) \in \mathcal{B}\} \subset \{\xi \in X : \ (\xi_2, \xi) \in \mathcal{B}\}.$$

Clearly, (C1)' implies (C1). The next assumption is a strong version of (C2).

(C2') For each $M, \epsilon > 0$ there exist $\delta > 0$, $M_0 > M$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each $\xi \in B(\widehat{\theta}, M) \cap \Omega(a)$ and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ satisfying $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ there exist $\eta_1, \eta_2 \in X$ such that

$$\rho(\eta_i, \xi) \le \epsilon, \ i = 1, 2, \ (\eta_i, \eta_i) \in \mathcal{B}, \ i = 1, 2, \ (\eta_1, \xi), (\xi, \eta_2) \in \mathcal{B}.$$

Note that (C1') and (C2') hold for the von Neumann-Gale model [14,18,19,25].

Proposition 9.1. *(C2') implies (C2).*

Proof. Assume that (C2') holds. Let $\epsilon \in (0, 1)$, $M > 0$. In view of the continuity of ϕ, there exists $\epsilon_0 \in (0, \epsilon/4)$ such that for each $\eta_1, \eta_2 \in B(\widehat{\theta}, M + 2)$ satisfying $\rho(\eta_1, \eta_2) \le \epsilon_0$ we have

$$|\phi(\eta_1) - \phi(\eta_2)| \le \epsilon/8. \tag{9.2}$$

Assumption (C2') implies that there exist $\delta \in (0, \epsilon_0)$, $M_0 > M + 1$ and a neighborhood $\mathcal{U}_0$ of ϕ in $\mathcal{L}$ such that the following property holds:

(i) for each $\xi \in B(\widehat{\theta}, M) \cap \Omega(a)$ and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ satisfying $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ there exist $\eta_1, \eta_2 \in X$ such that

$$\rho(\eta_i, \xi) \le \epsilon_0, \ i = 1, 2, \ (\eta_i, \eta_i) \in \mathcal{B}, \ i = 1, 2, \ (\eta_1, \xi), (\xi, \eta_2) \in \mathcal{B}.$$

Define

$$\mathcal{U} = \mathcal{U}_0 \cap \{\psi \in \mathcal{L} : \ (\phi, \psi) \in \mathcal{E}(M_0, \epsilon/8)\}. \tag{9.3}$$

Assume that

$$\xi_1, \xi_2 \in B(\widehat{\theta}, M) \cap \Omega(a) \tag{9.4}$$

satisfy

$$\phi(\xi_1) \le \phi(\xi_2) - \epsilon, \tag{9.5}$$

T_1, T_2 are natural numbers and

$$(\mathcal{B}, \psi) \in \mathfrak{M},\ \psi \in \mathcal{U},\ \mathcal{B} \in \mathcal{V}(M_0, \delta). \tag{9.6}$$

Property (i) and (9.3), (9.4) and (9.6) imply that there exist $\eta_1, \eta_2 \in X$ such that

$$(\eta_i, \eta_i) \in \mathcal{B},\ i = 1, 2, \tag{9.7}$$

$$\rho(\eta_1, \xi_1),\ \rho(\xi_2, \eta_2) \in \mathcal{B},\ \rho(\xi_i, \eta_i) \leq \epsilon_0,\ i = 1, 2. \tag{9.8}$$

By (9.1) and (9.8),

$$\begin{aligned}
&\psi(\xi_1),\ \sup\{\psi(z_{T_2}) :\ \{z_t\}_{t=0}^{T_2} \subset X,\ z_0 = \xi_1,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T_2 - 1\} \\
&\leq \phi(\eta_1), \qquad (9.9)
\end{aligned}$$

$$\begin{aligned}
&\sup\{\psi(z_{T_1}) :\ \{z_t\}_{t=0}^{T_1} \subset X,\ z_0 = \xi_2,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T_1 - 1\} \\
&\geq \sup\{\psi(z_{T_1 - 1}) :\ \{z_t\}_{t=0}^{T_1 - 1} \subset X,\ z_0 = \eta_2, \\
&(z_t, z_{t+1}) \in \mathcal{B},\ t \in \{0, \dots, T_1 - 1\} \setminus \{T_1 - 1\}\} \geq \phi(\eta_2). \qquad (9.10)
\end{aligned}$$

In view of (9.4) and (9.8), for $i = 1, 2$,

$$\rho(\widehat{\theta}, \eta_i) \leq M + 1 \leq M_0. \tag{9.11}$$

Equations (9.3), (9.6) and (9.11) imply that

$$|\phi(\eta_i) - \psi(\eta_i)| \leq \epsilon/8,\ i = 1, 2. \tag{9.12}$$

It follows from (9.2), (9.4), (9.5), (9.11) and (9.12) that

$$\begin{aligned}
\psi(\eta_2) &\geq \phi(\eta_2) - \epsilon/8 \geq \phi(\xi_2) - \epsilon/8 - \epsilon/8 \\
&\geq \phi(\xi_1) + \epsilon - \epsilon/8 - \epsilon/8 \\
&\geq \phi(\eta_1) + \epsilon - 3\epsilon/8 \geq \psi(\eta_1) + \epsilon/2. \qquad (9.13)
\end{aligned}$$

By (9.6), (9.10) and (9.13),

$$\begin{aligned}
&\psi(\xi_1),\ \sup\{\psi(z_{T_2}) :\ \{z_t\}_{t=0}^{T_2} \subset X,\ z_0 = \xi_1,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T_2 - 1\} \\
&\leq \psi(\eta_1) \leq \psi(\eta_2) - \epsilon/2 \\
&\leq \sup\{\psi(z_{T_1}) :\ \{z_t\}_{t=0}^{T_1} \subset X,\ z_0 = \xi_2,\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, T_1 - 1\} - \epsilon/2.
\end{aligned}$$

Thus (C2) holds and Lemma 9.1 is proved. □

We prove the following result.

Theorem 9.2. *Assume that assumptions (C1) and (C2) hold. Let $\epsilon, M > 0$. Then there exist a natural number L, $M_0 > M$, $\delta \in (0, \epsilon)$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for every integer $T \geq 2L$, each $(\mathcal{B}, \psi) \in \mathfrak{M}$ satisfying $\psi \in \mathcal{U}$ and $\mathcal{B} \in \mathcal{V}(M_0, \delta)$ and every sequence $\{x_t\}_{t=0}^{T} \subset X$ which satisfies*

$$\rho(x_0, \widehat{\theta}) \leq M,$$

for all integers $t = 0, \dots, T - 1$,

$$\psi(x_{t+1}) \leq \psi(x_t)$$

and

$$(x_t, x_{t+1}) \in \mathcal{B},$$

$$\psi(x_T) \geq \sup\{\psi(z_T) : \{z_t\}_{t=0}^T \subset X, \ z_0 = x_0 \textit{ for all } t = 0, \dots, T-1, \\ \psi(z_{t+1}) \leq \psi(z_t), \ (z_t, z_{t+1}) \in \mathcal{B}\} - \delta$$

there exist integer $\tau_1 \in \{0, \dots, L\}$, $\tau_2 \in \{T - L, \dots, T\}$ *such that*

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = \tau_1, \dots, \tau_2$$

and if (P1) holds then there exist $w \in \Omega(a)$ *such that*

$$\rho(x_t, w) \leq \epsilon, \ t \in \tau_1, \dots, \tau_2.$$

Moreover, if $\rho(x_0, \Omega) \leq \delta$*, then* $\tau_1 = 0$ *and if* $\rho(x_T, \Omega) \leq \delta$*, then* $\tau_2 = T$*.*

10. Auxiliary result for Theorem 9.2

Proposition 10.1. *Let* $x \in \Omega(a)$*. Then there exists a bounded sequence* $\{y_i\}_{i=-\infty}^{\infty} \subset \Omega(a)$ *such that* $y_0 = x$ *and for all integers* t*,*

$$(y_t, y_{t+1}) \in \mathcal{A}, \ \phi(y_t) = \phi(y_0), \ y_t \in \Omega(a).$$

Proof. Let $k \geq 1$ be an integer. There exists $\delta_k \in (0, 1/k)$ such that for each $z \in B(x, \delta_k)$,

$$|\phi(x) - \phi(z)| \leq 1/k. \tag{10.1}$$

There exists a trajectory $\{x_t^{(k)}\}_{t=0}^{\infty} \in Y(0, \infty, a)$ such that

$$\liminf_{t \to \infty} \rho(x_t^{(k)}, x) < \delta_k. \tag{10.2}$$

In view of (10.1) and (10.2), we may assume without loss of generality that

$$\rho(x_0^{(k)}, x) < \ \delta_k, \ |\phi(x_0^{(k)}) - \phi(x)| \leq 1/k. \tag{10.3}$$

Clearly, the sequence $\{\phi(x_t^{(k)})\}_{t=0}^{\infty}$ is decreasing. Together with (10.1)-(10.3) this implies that

$$|\phi(x_t^{(k)}) - \phi(x)| \leq 1/k \text{ for all integers } t \geq 0. \tag{10.4}$$

By (10.2), there exists an integer

$$t_k \geq 4k \tag{10.5}$$

such that

$$\rho(x_{t_k}^{(k)}, x) < \delta_k. \tag{10.6}$$

For each integer $t \geq -t_k$ define

$$y_t^{(k)} = x_{t+t_k}^{(k)}. \tag{10.7}$$

It follows from (10.3) and (10.7) that

$$\rho(y_{-t_k}^{(k)}, x) < \delta_k, \ \rho(y_0^{(k)}, x) < \delta_k \tag{10.8}$$

and for all integers $t \geq -t_k$,

$$|\phi(y_t^{(k)}) - \phi(x)| = |\phi(x_{t+t_k}^{(k)}) - \phi(x)| \leq 1/k. \tag{10.9}$$

In view of (2.3), (2.4) and (10.9) the set

$$\{y_t^{(k)} : \ t \geq -t_k \text{ is an integer}, \ k = 1, 2, \dots\}$$

is bounded. Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and a sequence $\{y_t\}_{t=-\infty}^{\infty} \subset X$ such that for each integer t,

$$\lim_{j\to\infty} y_t^{(k_j)} = y_t. \tag{10.10}$$

By (10.8) and (10.10), $y_0 = x$, the sequence $\{y_i\}_{i=-\infty}^{\infty}$ is bounded and $\phi(y_t) = \phi(x)$ for all integers t. Since the set $\mathcal{A}$ is closed we have $(y_t, y_{t+1}) \in \mathcal{A}$ for all integers t. Property (1) of Theorem 2.5 and the equation above imply that $y_t \in \Omega(a)$ for all integers t. Proposition 10.1 is proved. □

Lemma 10.2. *Assume that* $\{x_t\}_{-\infty}^{t=0} \subset X$ *satisfies*

$$x_{t+1} \in a(x_t),\ \phi(x_t) = \phi(x_0),\ t = -1, -2, \dots,$$

$$x_0 \in \Omega(a).$$

Then $x_t \in \Omega(a)$ *for every integer* $t \le 0$.

Proof. Proposition 10.1 implies that there exists a sequence $\{y_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$ such that

$$y_0 = x_0$$

and for every integer t,

$$y_{t+1} \in a(y_t),\ \phi(y_t) = \phi(y_0).$$

For every integer $t > 0$ set

$$x_t = y_t.$$

Property (1) of Theorem 2.5 implies that $\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$. Lemma 10.2 is proved. □

Analogously to Lemma 10.2 we can prove the next result.

Lemma 10.3. *Assume that* $\{x_t\}_{t=0}^{\infty} \subset X$ *satisfies*

$$x_{t+1} \in a(x_t),\ \phi(x_t) = \phi(x_0),\ t = 1, 2, \dots,$$

$$x_0 \in \Omega(a).$$

Then $x_t \in \Omega(a)$ *for every integer* $t \ge 0$.

Lemma 10.4. *Let* $\epsilon \in (0, 1)$, $M > 1$. *Then there exist* $\delta > 0$ *and a neighborhood* $\mathcal{U}$ *of* ϕ *in* $\mathcal{L}$ *such that for each natural number* $T \ge 2$, *each* $\psi \in \mathcal{U}$ *and each sequence* $\{x_t\}_{t=0}^{T} \subset X$ *satisfying* $\rho(x_0, \widehat{\theta}) \le M$ *and for each* $t \in \{0, \dots, T-1\}$,

$$\psi(x_{t+1}) \le \psi(x_t),\ \rho_1((x_t, x_{t+1}), \mathcal{A}) \le \delta,$$

$$\psi(x_0) \le \psi(x_T) + \delta,$$

$$\rho(x_j, \Omega(a)) \le \delta,\ j = 0, T$$

the inequality $\rho(x_t, \Omega(a)) \le \epsilon$ *holds for all* $t = 0, \dots, T$.

Proof. Lemma 8.1 implies that there exists $M_0 > M$ and a neighborhood V_0 of ϕ in $\mathcal{L}$ such that the following property holds:

(a) for each integer $T \geq 1$, each function $\psi \in V_0$ and each sequence $\{x_t\}_{t=0}^T \subset X$ such that

$$\rho(x_0, \widehat{\theta}) \leq M, \ \psi(x_{t+1}) \leq \psi(x_t), t = 0, \dots, T-1$$

the inequality $\rho(x_t, \widehat{\theta}) \leq M_0$ holds for all $t = 0, \dots, T$.

Assume that the lemma does not hold. Then for each integer $k \geq 1$ there exists an integer

$$T_k \geq 2, \ \psi_k \in \mathcal{L}$$

such that

$$(\phi, \psi_k) \in \mathcal{E}(M_0, 2^{-k}\epsilon), \ \psi_k \in V_0 \tag{10.11}$$

and a sequence $\{x_t^{(k)}\}_{t=0}^{T_k} \subset X$ such that

$$\rho(x_0^{(k)}, \widehat{\theta}) \leq M, \tag{10.12}$$

for each $t \in \{0, \dots, T_k - 1\}$,

$$\psi_k(x_0^{(k)}) \leq \psi(x_{T_k}^{(k)}) + 2^{-k}\epsilon, \tag{10.13}$$

$$\psi_k(x_{t+1}^{(k)}) \leq \psi_k(x_t^{(k)}), \ \rho_1((x_t^{(k)}, x_{t+1}^{(k)}), \mathcal{A}) \leq 2^{-k}\epsilon, \tag{10.14}$$

$$\rho(x_i^{(k)}, \Omega(a)) \leq 2^{-k}\epsilon, \ i = 0, \dots, T_k \tag{10.15}$$

and

$$\max\{\rho(x_t^{(k)}, \Omega(a)) : \ t = 0, \dots, T_k\} > \epsilon. \tag{10.16}$$

In view of (10.11)-(10.14), for each integer $k \geq 1$,

$$\rho(x_t^{(k)}, \widehat{\theta}) \leq M_0, \ t = 0, \dots, T_k, \tag{10.17}$$

$$\psi_k(x_0^{(k)}) - 2^{-k}\epsilon \leq \psi_k(x_t^{(k)}) \leq \psi_k(x_0^{(k)}), \ t = 0, \dots, T_k. \tag{10.18}$$

Equations (10.15) and (10.16) imply that for each integer $k \geq 1$ there exists

$$\tau_k \in \{1, \dots, T_k - 1\}$$

such that

$$\rho(x_{\tau_k}^{(k)}, \Omega(a)) > \epsilon. \tag{10.19}$$

Extracting a subsequence, re-indexing and using a diagonalization process we obtain that at least one of the following cases holds:

(a) $\lim_{k\to\infty} \tau_k = \infty$, $\lim_{k\to\infty}(T_k - \tau_k) = \infty$;

(b) $\limsup_{k\to\infty} \tau_k < \infty$, $\lim_{k\to\infty}(T_k - \tau_k) = \infty$;

(C) $\lim_{k\to\infty} \tau_k = \infty$, $\limsup_{k\to\infty}(T_k - \tau_k) < \infty$;

(d) $\limsup_{k\to\infty} \tau_k < \infty$, $\limsup_{k\to\infty}(T_k - \tau_k) < \infty$.

Assume that the case (a) holds. For each integer $k \geq 1$ set

$$y_t^{(k)} = x_{t+\tau_k}^{(k)}, \ t = -\tau_k, \dots, T_k - \tau_k. \tag{10.20}$$

Extracting a subsequence, using a diagonalization process and re-indexing we may assume without loss of generality that for each integer t there exists

$$z_t = \lim_{i\to\infty} y_t^{(k_i)}. \tag{10.21}$$

It follows from (10.11), (10.17), (10.18), (10.20) and (10.21) that for each integer t,

$$|\phi(z_t) - \phi(z_{t+1})| = \lim_{k\to\infty} |\phi(y_t^{(k)}) - \phi(y_{t+1}^{(k)})| = \lim_{k\to\infty} |\phi(x_{t+\tau_k}^{(k)}) - \phi(x_{t+\tau_k+1}^{(k_i)})|$$

$$= \lim_{k\to\infty} |\psi_k(x_{t+\tau_k}^{(k)}) - \psi_k(x_{t+\tau_k+1}^{(k_i)})| = 0. \tag{10.22}$$

By (10.14) and (10.20)-(10.22), for each integer t,
(10.23)

$$\rho_1((z_t, z_{t+1}), \mathcal{A}) \le \lim_{k\to\infty} \rho_1((y_t^{(k)}, y_{t+1}^{(k)}), \mathcal{A}) = \lim_{k\to\infty} \rho_1((x_{t+\tau_k}^{(k)}, x_{t+\tau_k+1}^{(k)}), \mathcal{A}) = 0.$$

It follows from property (1) of Theorem 2.5 and (10.22) and (10.23) that

$$z_t \in \Omega(a)$$

for all integers t. On the other hand it follows from (10.11) and (10.20) that for every integer $k \ge 1$,

$$\rho(y_0^{(k)}, \Omega(a)) = \rho(x_{\tau_k}^{(k)}, \Omega(a)) \ge \epsilon.$$

and

$$\rho(y_0, \Omega(a)) \ge \epsilon.$$

The contradiction we have reached proves that the case (a) does not hold.

Assume that the case (b) holds. In view (10.12), extracting a subsequence, using a diagonalization process and re-indexing we may assume without loss of generality that

$$\tau_k = \tau_1,\ k = 1, 2, \dots \tag{10.24}$$

and that for each integer $t \ge 0$ there exists

$$z_t = \lim_{k\to\infty} x_t^{(k)}. \tag{10.25}$$

It follows from (10.11), (10.12), (10.18) and (10.25) that for each integer $t \ge 0$,

$$|\phi(z_t)-\phi(z_{t+1})| = \lim_{k\to\infty} |\phi(x_t^{(k)})-\phi(x_{t+1}^{(k)})| = \lim_{k\to\infty} |\psi_k(x_t^{(k)})-\psi_k(x_{t+1}^{(k)})| = 0. \tag{10.26}$$

By (10.14) and (10.25), for every integer $t \ge 0$,

$$\rho_1((z_t, z_{t+1}), \mathcal{A}) \le \lim_{k\to\infty} \rho_1((x_t^{(k)}, x_{t+1}^{(k)}), \mathcal{A}) = 0. \tag{10.27}$$

It follows from (10.15) and (10.25) that

$$z_0 \in \Omega(a).$$

Lemma 10.3, the equation above, (10.26) and (10.27) imply that

$$z_t \in \Omega(a)$$

for all integers $t \ge 0$. On the other hand it follows from (10.11), (10.24) and (10.25) that

$$\rho(z_{\tau_1}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{\tau_1}^{(k)}, \Omega(a)) \ge \epsilon.$$

The contradiction we have reached proves that the case (b) does not hold.

Assume that the case (c) holds. We may assume without loss of generality that

$$T_k - \tau_k = T_1 - \tau_1, \ k = 1, 2, \dots \tag{10.28}$$

For each integer $k \geq 1$ set

$$y_t^{(k)} = x_{t+\tau_k}^{(k)}, \ t = 0, -1, -2, \dots . \tag{10.29}$$

In view of (10.17), extracting a subsequence, using a diagonalization process and re-indexing we may assume without loss of generality that for each integer $t \leq 0$ there exists

$$z_t = \lim_{k\to\infty} y_t^{(k)}. \tag{10.30}$$

By (10.14), (10.29) and (10.30), for every integer $t \leq 0$,

$$\rho_1((z_t, z_{t+1}), \mathcal{A}) = \lim_{k\to\infty} \rho_1((y_t^{(k)}, y_{t+1}^{(k)}), \mathcal{A}) = \lim_{k\to\infty} \rho_1((x_{t+\tau_k}^{(k)}, x_{t+\tau_k+1}^{(k)}), \mathcal{A}) = 0. \tag{10.31}$$

It follows from (10.11), (10.17), (10.18), (10.29) and (10.30) that for each integer $t \leq 0$,

$$|\phi(z_t) - \phi(z_{t+1})| = \lim_{k\to\infty} |\phi(y_t^{(k)}) - \phi(y_{t+1}^{(k)})| = \lim_{k\to\infty} |\phi(x_{t+\tau_k}^{(k)}) - \phi(x_{t+\tau_k+1}^{(k)})|$$

$$= \lim_{k\to\infty} |\psi_k(x_{t+\tau_k}^{(k)}) - \psi_k(x_{t+\tau_k+1}^{(k_i)})| = 0. \tag{10.32}$$

By (10.15), (10.29) and (10.30),

$$\rho(z_0, \Omega(a)) = \lim_{k\to\infty} \rho(y_0, \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_k}^{(k)}, \Omega(a)) = 0. \tag{10.33}$$

It follows from Lemma 10.2 and (10.31)-(10.33) that

$$z_t \in \Omega(a)$$

for all integers $t \leq 0$. On the other hand it follows from (10.28)-(10.30) that

$$\rho(z_{\tau_1 - T_1}, \Omega(a)) = \lim_{k\to\infty} \rho(y_{\tau_1 - T_1}^{(k)}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{\tau_k}^{(k)}, \Omega(a)) \geq \epsilon.$$

The contradiction we have reached proves that the case (c) does not hold.

Therefore case (d) holds. Extracting a subsequence, using a diagonalization process and re-indexing we may assume without loss of generality that

$$T_k = T_1, \ \tau_k = \tau_1, \ k = 1, 2, \dots \tag{10.34}$$

and that for each integer $t \in \{0, \dots, T_1\}$ there exists

$$y_t = \lim_{k\to\infty} x_t^{(k)}. \tag{10.35}$$

By (10.15), (10.34) and (10.35),

$$\rho(y_0, \Omega(a)) = \lim_{k\to\infty} \rho(x_0^{(k)}, \Omega(a)) = 0, \ \rho(y_{T_1}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_k}^{(k)}, \Omega(a)) = 0. \tag{10.36}$$

By (10.14) and (10.35), for every integer $t = 0, \dots, T_1 - 1$,

$$\rho_1((y_t, y_{t+1}), \mathcal{A}) = \lim_{k\to\infty} \rho_1((x_t^{(k)}, x_{t+1}^{(k)}), \mathcal{A}) = 0. \tag{10.37}$$

It follows from (10.11), (10.17), (10.18) and (10.35) that for each integer $t \in \{0, \dots, T_1 - 1\}$,

$$(10.38)\quad |\phi(y_t) - \phi(y_{t+1})| = \lim_{k\to\infty} |\phi(x_t^{(k)}) - \phi(x_{t+1}^{(k)})| = \lim_{k\to\infty} |\psi_k(x_t^{(k)}) - \psi_k(x_{t+1}^{(k)})| = 0.$$

It follows from (10.19), (10.34), (10.35) and (10.38) that

$$\rho(y_{T_1}, \Omega(a)) = \lim_{k\to\infty} \rho(x_{T_1}^{(k)}, \Omega(a)) \geq \epsilon.$$

Property (1) of Theorem 2.5, Proposition 10.1 (10.37) and (10.38) imply that

$$y_t \in \Omega(a)$$

for all integers $t = 0, \dots, T_1$. The contradiction we have reached proves Lemma 10.4. □

Lemma 10.5. *Assume that property (P1) holds and that $\epsilon \in (0, 1)$, $M > 1$. Then there exist $\delta > 0$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that for each natural number $T \geq 2$, each $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ satisfying*

$$(10.39)\quad \rho(x_0, \widehat{\theta}) \leq M$$

and for each $t \in \{0, \dots, T - 1\}$,

$$(10.40)\quad \psi(x_{t+1}) \leq \psi(x_t),\ \rho_1((x_t, x_{t+1}), \mathcal{A}) \leq \delta,$$

$$(10.41)\quad \psi(x_0) \leq \psi(x_T) + \delta,$$

$$(10.42)\quad \rho(x_j, \Omega(a)) \leq \delta,\ j = 0, T$$

there exists $\eta \in \Omega(a)$ such that the inequality $\rho(x_t, \eta) \leq \epsilon$ holds for all $t = 0, \dots, T$.

Proof. Lemma 8.1 implies that there exists $M_0 > M$ and a neighborhood $\mathcal{U}_0$ of ϕ in $\mathcal{L}$ such that the following property holds:

(a) for each integer $S \geq 1$, each function $\psi \in \mathcal{U}_0$ and each sequence $\{x_t\}_{t=0}^S \subset X$ such that

$$\rho(x_0, \widehat{\theta}) \leq M,\ \psi(x_{t+1}) \leq \psi(x_t), t = 0, \dots, S - 1$$

the inequality $\rho(x_t, \widehat{\theta}) \leq M_1$ holds for all $t = 0, \dots, S$.

Lemma 6.2 implies that there exists $\epsilon_0 \in (0, \epsilon/4)$ such that the following property holds:

(b) for each $\eta_1, \eta_2 \in \Omega(a) \cap B(\widehat{\theta}, M_1 + 2)$ which satisfy $|\phi(\eta_1) - \phi(\eta_2)| \leq 2\epsilon_0$ we have $\rho(\eta_1, \eta_2) \leq \epsilon/8$.

It follows from the uniform continuity of the function ϕ on compact sets that there exists $\epsilon_1 \in (0, \epsilon_0/4)$ such that the following property holds:

(c) for each $\eta_1, \eta_2 \in B(\widehat{\theta}, M_1 + 4)$ satisfying $\rho(\eta_1, \eta_2) \leq \epsilon_1$ we have

$$|\phi(\eta_1) - \phi(\eta_2)| \leq \epsilon_0/8.$$

Lemma 10.4 implies that there exist $\delta \in (0, \epsilon_1/4)$ and a neighborhood $\mathcal{U}$ of ϕ in $\mathcal{L}$ such that

$$\mathcal{U} \subset \mathcal{U}_0 \cap \{\psi \in \mathcal{L} :\ (\phi, \psi) \in \mathcal{E}(M_1 + 4, \epsilon_0/16)\}$$

and such that the following property holds:

(d) for each integer $T \geq 2$, each $\psi \in \mathcal{U}$ and each sequence $\{x_t\}_{t=0}^T \subset X$ satisfying (10.39) and for each $t \in \{0, \dots, T-1\}$, satisfying (10,40)-(10.42) we have

$$\rho(x_t, \Omega(a)) \leq \epsilon_1, \ t = 0, \dots, T. \tag{10.43}$$

Assume that $T \geq 2$ is an integer ,

$$\psi \in \mathcal{U}, \tag{10.44}$$

$\{x_t\}_{t=0}^T \subset X$, (10.39) holds and for each $t \in \{0, \dots, T-1\}$, (10.40)-(10.42) hold. Property (d) and equations (10.39)-(10.42), (10.44) imply that (10.43) holds for all $t = 0, \dots, T$. In view of (10.43), for each $t \in \{0, \dots, T\}$, there exists

$$\eta_t \in \Omega(a) \tag{10.45}$$

such that

$$\rho(x_t, \eta_t) \leq \epsilon_1. \tag{10.46}$$

Property (a), (10.40) and (10.44) imply that

$$\rho(x_t, \widehat{\theta}) \leq M_1, \ \rho(\eta_t, \widehat{\theta}) \leq M_1 + 1, \ t = 0, \dots, T. \tag{10.47}$$

It follows from the choice of $\mathcal{U}$, (10.44) and (10.47) that for all $t = 0, \dots, T$,

$$|\phi(x_t) - \psi(x_t)| \leq \epsilon_0/16. \tag{10.48}$$

In view of (10.40) and (10.41), for each $t \in \{0, \dots, T\}$,

$$\psi(x_0) - \delta \leq \psi(x_t) \leq \psi(x_0). \tag{10.49}$$

Let $t \in \{0, \dots, T\}$. Equations (10.48) and (10.49) imply that

$$|\phi(x_t) - \phi(x_0)| \leq \epsilon_0/8 + |\psi(x_t) - \psi(x_0)| \leq \delta + \epsilon_0/8. \tag{10.50}$$

Property (c), (10.46) and (10.47) imply that

$$|\phi(x_t) - \phi(\eta_t)| \leq \epsilon_0/8. \tag{10.51}$$

By (10.50) and (10.51),

$$|\phi(\eta_0) - \phi(\eta_t| \leq |\phi(\eta_0) - \phi(x_0)| + |\phi(x_0) - \phi(x_t)| + |\phi(x_t) - \phi(\eta_t|$$

$$\leq \epsilon_0/4 + \delta + \epsilon_0/8 < \epsilon_0. \tag{10.52}$$

Property (b), (10.47) and (10.52) imply that

$$\rho(\eta_0, \eta_t) \leq \epsilon/8.$$

Together with (10.46) this implies that

$$\rho(x_t, \eta_0) \leq \epsilon_1 + \epsilon/8 < \epsilon.$$

Lemma 10.5 is proved. □

11. Proof of Theorem 9.2

Lemma 8.1 implies that there exist $M_1 > M+1$ and a neighborhood $\mathcal{U}_0$ of ϕ in $\mathcal{L}$ such that the following property holds:

(a) for each integer $S \geq 1$, each function $\psi \in \mathcal{U}_0$ and each sequence $\{x_t\}_{t=0}^S \subset X$ such that

$$\rho(x_0, \widehat{\theta}) \leq M, \ \psi(x_{t+1}) \leq \psi(x_t), t = 0, \dots, S-1$$

the inequality $\rho(x_t, \widehat{\theta}) \leq M_1$ holds for all $t = 0, \dots, S$.

Lemmas 10.4 and 10.5 imply that there exist $\delta_0 \in (0, \epsilon)$ and a neighborhood $\mathcal{U}_1$ of ϕ in $\mathcal{L}$ such that the following property holds:

(b) for each natural number $S \geq 2$, each $\psi \in \mathcal{U}_1$ and each sequence $\{x_t\}_{t=0}^S \subset X$ satisfying $\rho(x_0, \widehat{\theta}) \leq M_1 + 1$ and for each $t \in \{0, \dots, S-1\}$,

$$\psi(x_{t+1}) \leq \psi(x_t), \ \rho_1((x_t, x_{t+1}), \mathcal{A}) \leq \delta_0,$$

$$\psi(x_0) \leq \psi(x_S) + \delta_0,$$

$$\rho(x_j, \Omega(a)) \leq \delta_0, \ j = 0, S$$

the inequality $\rho(x_t, \Omega(a)) \leq \epsilon$ holds for all $t = 0, \dots, S$ and if (P1) holds, then $\rho(x_t, \eta) \leq \epsilon$, $t = 0, \dots, S$, where $\eta \in \Omega(a)$.

It follows from the continuity of the function ϕ that there exists $\delta_1 \in (0, \delta_0)$ such that the following property holds:

(c) for each $\eta_1, \eta_2 \in B(\widehat{\theta}, M_1 + 1)$ satisfying $\rho(\eta_1, \eta_2) \leq 2\delta_1$ we have

$$|\phi(\eta_1) - \phi(\eta_2)| \leq \delta_0/16.$$

Assumption (C2) implies there exist $\delta_2 \in (0, \delta_1)$, $M_2 > M_1+2$ and a neighborhood $\mathcal{U}_2$ of ϕ in $\mathcal{L}$ such that the following property holds:

(d) for each $\xi_1, \xi_2 \in B(\widehat{\theta}, M_1+2) \cap \Omega(a)$ satisfying $\phi(\xi_1) \leq \phi(\xi_2) - \delta_1/8$, each pair of natural numbers S_1, S_2 and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ such that $\psi \in \mathcal{U}_2$ and $\mathcal{B} \in \mathcal{V}(M_2, \delta_2)$ we have

$$\psi(\xi_1) + \delta_2 \leq \psi(\xi_2),$$

$$\psi(\xi_2) + \delta_2,$$

$$\sup\{\psi(z_{S_2}) : \ \{z_t\}_{t=0}^{S_2} \subset X, \ z_0 = \xi_1, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, S_2 - 1\} + \delta_2$$
$$\leq \sup\{\psi(z_{S_1}) : \ \{z_t\}_{t=0}^{S_1} \subset X, \ z_0 = \xi_2, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, S_1 - 1\}.$$

Assumption (C1) implies that there exist $\delta_3 \in (0, \delta_2)$, $M_3 > M_2$ and a neighborhood $\mathcal{U}_3$ of ϕ in $\mathcal{L}$ such that the following property holds:

(e) for each $z \in B(\widehat{\theta}, M_2)$ satisfying $\rho(z, \Omega(a)) \leq \delta_3$ there exist $\xi_1, \xi_2 \in \Omega(a)$ such that

$$\rho(z, \xi_i) \leq \delta_2, \ i = 1, 2$$

and such that for each integer $S \geq 1$ and each $(\mathcal{B}, \psi) \in \mathfrak{M}$ satisfying $\psi \in \mathcal{U}_3$ and $\mathcal{B} \in \mathcal{V}(M_2, \delta_3)$ we have

$$\psi(\xi_1) - \delta_2/16 \leq \psi(z) \leq \psi(\xi_2) + \delta_2/16,$$

$$\sup\{\psi(z_S) : \ \{z_t\}_{t=0}^S \subset X, \ z_0 = \xi_1, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, S-1\} - \delta_2/16$$
$$\leq \sup\{\psi(z_S) : \ \{z_t\}_{t=0}^S \subset X, \ z_0 = z, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, S-1\}$$
$$\leq \sup\{\psi(z_S) : \ \{z_t\}_{t=0}^S \subset X, \ z_0 = \xi_2, \ (z_t, z_{t+1}) \in \mathcal{B}, \ t = 0, \dots, S-1\} + \delta_2/16.$$

Theorem 8.2 implies that there exist a natural number L, $\delta_4 > 0$, $M_0 > M_3 + 1$ and a neighborhood $\mathcal{U}_4$ of ϕ in $\mathcal{L}$ such that the following property holds:

(f) for each integer $T > L$, each $\mathcal{B} \in \mathcal{V}(M_0, \delta_4)$, each $\psi \in \mathcal{U}_4$ and each sequence $\{x_t\}_{t=0}^T$ such that

$$\rho(x_0, \widehat{\theta}) \leq M$$

and that for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t)$$

and

$$(x_t, x_{t+1}) \in \mathcal{B}$$

we have

$$\mathrm{Card}(\{0, \dots, T\} :\ \rho(x_t, \Omega(a)) \geq \delta_3\} \leq L.$$

Set

$$\delta = \min\{\delta_i :\ i = 0, 1, 2, 3, 4\}/4, \tag{11.1}$$

$$\mathcal{U} = \cap_{i=0}^4 \mathcal{U}_i \cap \{\psi \in \mathcal{L} :\ (\phi, \psi) \in \mathcal{E}(M_0 + 1, \delta)\}. \tag{11.2}$$

Assume that an integer

$$T \geq 2L,\ (\mathcal{B}, \psi) \in \mathfrak{M},\ \psi \in \mathcal{U},\ \mathcal{B} \in \mathcal{V}(M_0, \delta), \tag{11.3}$$

$\{x_t\}_{t=0}^T \subset X$,

$$\rho(x_0, \widehat{\theta}) \leq M, \tag{11.4}$$

for all integers $t = 0, \dots, T-1$,

$$\psi(x_{t+1}) \leq \psi(x_t),\ (x_t, x_{t+1}) \in \mathcal{B}, \tag{11.5}$$

$$\psi(x_T) \geq \sup\{\psi(z_T) :\ \{z_t\}_{t=0}^T \subset X,\ z_0 = x_0 \text{ for all } t = 0, \dots, T-1,$$

$$\psi(z_{t+1}) \leq \psi(z_t),\ \ (z_t, z_{t+1}) \in \mathcal{B}\} - \delta. \tag{11.6}$$

Property (f) and (11.1)-(11.5) imply that

$$\mathrm{Card}(\{0, \dots, T\} :\ \rho(x_t, \Omega(a)) \geq \delta_3\} \leq L. \tag{11.7}$$

In view of (11.7), there exist

$$\tau_1 \in \{0, \dots, L\},\ \tau_2 \in \{T - L, \dots, T\} \tag{11.8}$$

such that

$$\rho(x_{\tau_i}, \Omega(a)) < \delta_3,\ i = 1, 2. \tag{11.9}$$

If $\rho(x_0, \Omega(a)) \leq \delta$, then we set $\tau_1 = 0$ and if $\rho(x_T, \Omega(a)) \leq \delta$, then we set $\tau_2 = T$. Property (a) and equations (11.2)-(11.5) imply that

$$\rho(x_t, \widehat{\theta}) \leq M_1,\ t = 0, \dots, T. \tag{11.10}$$

In view of (11.5) and (11.8),

$$\psi(x_{\tau_2}) \leq \psi(x_{\tau_1}). \tag{11.11}$$

We show that

$$\psi(x_{\tau_1}) \leq \psi(x_{\tau_2}) + \delta_0.$$

Assume the contrary. Then

$$\psi(x_{\tau_1}) > \psi(x_{\tau_2}) + \delta_0. \tag{11.12}$$

Property (e) applied to $z = x_{\tau_i}$, $i = 1, 2$, (11.9) and (11.10) imply that for $i = 1, 2$ there exist

$$\xi_{i,1}, \xi_{i,2} \in \Omega(a) \tag{11.13}$$

such that

$$\rho(x_{\tau_i}, \xi_{i,j}) \le \delta_2,\ j = 1, 2 \tag{11.14}$$

and that for $i = 1, 2$ and each integer $S \ge 1$ we have

$$\psi(\xi_{i,1}) - \delta_2/16 \le \psi(x_{\tau_i}) \le \psi(\xi_{i,2}) + \delta_2/16, \tag{11.15}$$

$$\begin{aligned} &\sup\{\psi(z_S) :\ \{z_t\}_{t=0}^S \subset X,\ z_0 = \xi_{i,1},\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, S-1\} - \delta_2/16 \\ &\le \sup\{\psi(z_S) :\ \{z_t\}_{t=0}^S \subset X,\ z_0 = x_{\tau_i},\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, S-1\} \\ &\le \sup\{\psi(z_S) :\ \{z_t\}_{t=0}^S \subset X,\ z_0 = \xi_{i,2},\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, S-1\} + \delta_2/16. \end{aligned} \tag{11.16}$$

By (11.1)-(11.3), (11.10) and (11.12),

$$\begin{aligned} \phi(x_{\tau_1}) - \phi(x_{\tau_2}) &\ge \psi(x_{\tau_1}) - \psi(x_{\tau_2}) - |\phi(x_{\tau_1}) - \psi(x_{\tau_1})| \\ &- |\phi(x_{\tau_2}) - \psi(x_{\tau_2})| > \delta_0 - 2\delta > \delta_0/2. \end{aligned} \tag{11.17}$$

Property (c) and equations (11.10), (11.14) and (11.17) imply that for $j, p \in \{1, 2\}$,

$$\begin{aligned} \phi(\xi_{1,p}) - \phi(\xi_{2,j}) &\ge \phi(x_{\tau_1}) - \phi(x_{\tau_2}) - |\phi(x_{\tau_1}) - \phi(\xi_{1,p})| \\ &- |\phi(\xi_{2,j}) - \phi(x_{\tau_2})| > \delta_0/2 - \delta_0/8 = 3\delta_0/8. \end{aligned} \tag{11.18}$$

Property (d) and (11.1)-(11.3), (11.10), (11.13), (11.14) and (11.18) imply that for each pair of integers $S_1, S_2 \ge 1$ and each $j, p \in \{1, 2\}$,

$$\begin{aligned} &\sup\{\psi(z_{S_2}) :\ \{z_t\}_{t=0}^{S_2} \subset X,\ z_0 = \xi_{2,j},\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, S_2 - 1\} + \delta_2 \\ &\le \psi(\xi_{1,p}) \\ &\le \sup\{\psi(z_{S_1}) :\ \{z_t\}_{t=0}^{S_1} \subset X,\ z_0 = \xi_{1,p},\ (z_t, z_{t+1}) \in \mathcal{B},\ t = 0, \dots, S_1 - 1\}, \end{aligned} \tag{11.19}$$

$$\psi(\xi_{2,j}) + \delta_2 \le \psi(\xi_{1,p}). \tag{11.20}$$

By (11.6), (11.8), (11.15), (11.16), (11.19) and (11.20),

$$\begin{aligned} &\psi(x_T) \\ &\le \sup\{\psi(z_{T-\tau_2}) :\ \{z_t\}_{t=0}^{T-\tau_2} \subset X, \\ &z_0 = x_{\tau_2},\ (z_t, z_{t+1}) \in \mathcal{B},\ t \in \{0, \dots, T - \tau_2\} \setminus \{T - \tau_2\}\} \\ &\le \sup\{\psi(z_{T-\tau_2}) :\ \{z_t\}_{t=0}^{T-\tau_2} \subset X, \\ &z_0 = \xi_{2,2},\ (z_t, z_{t+1}) \in \mathcal{B},\ t \in \{0, \dots, T - \tau_2\} \setminus \{T - \tau_2\}\} + \delta_2/16 \\ &\le \sup\{\psi(z_{T-\tau_1}) :\ \{z_t\}_{t=0}^{T-\tau_1} \subset X, \\ &z_0 = \xi_{1,1},\ (z_t, z_{t+1}) \in \mathcal{B},\ t \in \{0, \dots, T - \tau_1 - 1\}\} + \delta_2/16 - \delta_2 \\ &\le \sup\{\psi(z_{T-\tau_1}) :\ \{z_t\}_{t=0}^{T-\tau_1} \subset X, \\ &z_0 = x_{\tau_1},\ (z_t, z_{t+1}) \in \mathcal{B},\ t \in \{0, \dots, T - \tau_1 - 1\}\} + \delta_2/16 - \delta_2 + \delta_2/16 \end{aligned}$$

$$\leq \sup\{\psi(z_T) : \{z_t\}_{t=0}^T \subset X,$$
$$z_0 = x_0, (z_t, z_{t+1}) \in \mathcal{B}, t \in \{0, \ldots, T-1\}\} - \delta_2/2$$
$$\leq \psi(x_T) + \delta - \delta_2/2.$$

This contradicts (11.1). The contradiction we have reached proves that

$$\psi(x_{\tau_1}) \leq \psi(x_{\tau_2}) + \delta_0. \tag{11.21}$$

Property (b) equations (11.1)-(11.3), (11.5), (11.10), (11.14) and (11.21) imply that

$$\rho(x_t, \Omega(a)) \leq \epsilon, \ t = \tau_1, \ldots, \tau_2$$

and that if (P1) holds then there exists $\eta \in \Omega(a)$ such that

$$\rho(x_t, \eta) \leq \epsilon, \ t = \tau_1, \ldots, \tau_2.$$

Theorem 9.2 is proved.

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