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SUBDIFFERENTIAL AND OPTIMALITY CONDITIONS FOR CONVEX SET FUNCTIONS

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Abstract. In this paper, we study a subdifferential and optimality conditions for convex set functions. We define a subdifferential for set functions. We investigate the subdifferential precisely and show optimality conditions in terms of convex analysis on an embedding space. We study a robust approach for an uncertain problem as an application.

1. Introduction

In optimization theory, subdifferentials play a central role. There are so many useful subdifferentials, the subdifferential in convex analysis, Clarke subdifferential, Greenberg-Pierskalla subdifferential, and so on. Especially, optimality conditions and duality results in terms of subdifferentials have been investigated extensively, see [1, 3, 7, 21, 24, 26, 28–38, 42]. Usually, objective functions of these problems are real-valued functions. For multi-objective optimization, vector-valued functions and set-valued functions have been studied by various researchers. On the other hand, optimization problems whose objective function is a set function have been investigated extensively, for Morris functions [5, 6, 8, 9, 17–20, 23, 25, 43], for more simple set functions [40, 41], and so on. Especially, in [40], we study Fenchel duality for convex set functions. We introduce Fenchel conjugate for set functions, and investigate Fenchel duality in terms of convex analysis on an embedding space. However, a subdifferential for set functions has not been investigated yet. It is expected to study such a subdifferential since subdifferentials play an important role in optimization problems.

In this paper, we study a subdifferential and optimality conditions for convex set functions. We define a subdifferential for set functions. We investigate the subdifferential precisely and show optimality conditions in terms of convex analysis on an embedding space. We study a robust approach for an uncertain problem as an application.

The remainder of the paper is organized as follows. In Section2, we introduce some preliminaries. In Section 3, we study a subdifferential and optimality conditions. We define a subdifferential and show optimality conditions for convex set functions. Especially, we show a Karush-Kuhn-Tucker type necessary and sufficient

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optimality condition for convex set functions. In Section 4, we discuss about our results and study applications to uncertain problems.

2. Preliminaries

Let $\langle v, x\rangle$ denote the inner product of two vectors v and x in the n-dimensional Euclidean space $\mathbb{R}^n$. Given nonempty sets A, $B \subset \mathbb{R}^n$, and $\Gamma \subset \mathbb{R}$, we define $A+B$ and ΓA as follows:

$$\begin{aligned} A+B &= \{x+y \in \mathbb{R}^n \mid x \in A, y \in B\}, \\ \Gamma A &= \{\gamma x \in \mathbb{R}^n \mid \gamma \in \Gamma, x \in A\}. \end{aligned}$$

We define $A+\emptyset = \Gamma\emptyset = \emptyset A = \emptyset$. A set A is said to be convex if for each x, $y \in A$, and $\alpha \in [0,1]$, $(1-\alpha)x + \alpha y \in A$. Let $\mathcal{A}_0$ be the following family of nonempty convex sets:

$$\mathcal{A}_0 = \{A \subset \mathbb{R}^n \mid A : \text{nonempty convex}\}.$$

Clearly, $\mathcal{A}_0$ is closed under addition and multiplication by positive scalars. A subfamily $\mathcal{A} \subset \mathcal{A}_0$ is said to be convex if for each A, $B \in \mathcal{A}$, and $\alpha \in [0,1]$, $(1-\alpha)A + \alpha B \in \mathcal{A}$. Let $\mathcal{C} \subset \mathcal{A}_0$ be the family of all nonempty compact convex subsets of $\mathbb{R}^n$, that is,

$$\mathcal{C} = \{A \subset \mathbb{R}^n \mid A : \text{nonempty compact convex}\}.$$

Let A, $B \in \mathcal{C}$. We define their Hausdorff distance $d_H(A,B)$ by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}.$$

Let $\equiv$ be a binary relation on $\mathcal{C}^2$ defined by

$$(A,B) \equiv (C,D) \text{ if and only if } A+D = B+C,$$

then $\equiv$ is an equivalence relation on $\mathcal{C}^2$. Denote the equivalence class of $(A,B) \in \mathcal{C}^2$ as $[A,B] = \{(C,D) \in \mathcal{C}^2 \mid (A,B) \equiv (C,D)\}$, and the quotient space of $\mathcal{C}^2$ by $\equiv$ as $(\mathcal{C}^2/\equiv) = \{[A,B] \mid (A,B) \in \mathcal{C}^2\}$. On the quotient space, we define addition, scalar multiplication, and norm as follows:

$$[A,B] + [C,D] = [A+C, B+D],$$

$$\lambda \cdot [A,B] = \begin{cases} [\lambda A, \lambda B] & \text{if } \lambda \geq 0, \\ [(-\lambda)B, (-\lambda)A] & \text{if } \lambda < 0. \end{cases}$$

$$\|[A,B]\| = d_H(A,B),$$

Then, $\mathcal{C}^2/\equiv$ is a normed space. Additionally, by the following function $\psi : \mathcal{C} \to \mathcal{C}^2/\equiv$;

$$\psi(A) = [A, \{0\}],$$

$\mathcal{C}$ can be regarded as a subset of the embedding space $\mathcal{C}^2/\equiv$. For more detail, see [10–12, 27, 40].

Let F be a set function from $\mathcal{A}_0$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. We denote the domain of F by dom$F$, that is, dom$F = \{A \in \mathcal{A}_0 \mid F(A) < +\infty\}$. F is said to be proper if for all $A \in \mathcal{A}_0$, $F(A) > -\infty$ and dom$F$ is nonempty. A proper set function F on $\mathcal{A}_0$ is said to be convex if for each A, $B \in \mathrm{dom}F$, and $\alpha \in [0, 1]$,

$$F((1-\alpha)A + \alpha B) \leq (1-\alpha)F(A) + \alpha F(B).$$

F is said to be concave if $-F$ is a convex set function. The epigraph of F is defined as epi$F = \{(A, \alpha) \in \mathcal{A}_0 \times \mathbb{R} \mid F(A) \leq \alpha\}$. In [40], we show that a proper set function F is a convex set function if and only if epi$F$ is convex. F is said to be affine if F is a convex and concave set function. F is said to be linear if F is an affine set function and $F(\{0\}) = 0$. We can check that F is linear if and only if for each A, $B \in \mathrm{dom}F$, and $\lambda \geq 0$,

$$F(A+B) = F(A) + F(B), F(\lambda A) = \lambda F(A), \tag{2.1}$$

see Theorem 2.5 in [40]. A set function F is said to be lower semicontinuous (lsc) on $\mathcal{C}$ in terms of the Hausdorff distance if for each $\{B_k\} \subset \mathcal{C}$ and $B \in \mathcal{C}$ with $d_H(B_k, B)$ converges to 0,

$$\liminf_{k\to\infty} F(B_k) \geq F(B).$$

F is said to be continuous on $\mathcal{C}$ in terms of the Hausdorff distance if F and $-F$ are lsc in terms of the Hausdorff distance.

We show important examples of convex and linear set functions.

Example 2.1 ([40])**.** Let f be a real-valued convex function on $\mathbb{R}^n$. Let

$$F_0(A) = \sup_{x\in A} f(x),$$

then F_0 is a convex set function.

Let $v \in \mathbb{R}^n$, then the following set function V is linear: for each $A \in \mathcal{A}_0$,

$$V(A) = \sup_{x\in A} \langle v, x\rangle .$$

Hence,

$$\{V : \mathcal{A}_0 \to \overline{\mathbb{R}} \mid v \in \mathbb{R}^n, V(A) = \sup_{x\in A} \langle v, x\rangle\} \subsetneq \{V : \mathcal{A}_0 \to \overline{\mathbb{R}} \mid V : \text{ linear }\}.$$

The converse inclusion does not hold. Actually, let V_0 be the following function: for each closed convex set $A \subset \mathbb{R}$,

$$V_0(A) = \begin{cases} b-a & A = [a,b], a, b \in \mathbb{R}, \\ \infty & \text{otherwise.} \end{cases}$$

Then V_0 is a linear set function. However, there does not exist $v \in \mathbb{R}$ such that V_0 is defined by v.

We define the following set $\mathcal{F}_L$ as follows:

$$\mathcal{F}_L = \{V : \mathcal{A}_0 \to \mathbb{R} \cup \{+\infty\}, \text{ linear}\}.$$

Let F be a proper set function on $\mathcal{A}_0$. Then, we define the Fenchel conjugate of F as follows: $F^* : \mathcal{F}_L \to \overline{\mathbb{R}}$,

$$F^*(V) = \sup_{A\in \mathrm{dom}F} \{V(A) - F(A)\}.$$

We define the Fenchel biconjugate as follows: $F^{**} : \mathcal{A}_0 \to \overline{\mathbb{R}}$,

$$F^{**}(A) = \sup_{V \in \mathrm{dom}F^*} \{V(A) - F^*(V)\}.$$

In [40], we show the following Fenchel duality theorem for convex set functions.

Theorem 2.2 ([40])**.** *Let F and G be proper convex set functions from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$. Assume that $\mathrm{dom}F \cup \mathrm{dom}G \subset \mathcal{C}$, $\mathrm{dom}F \cap \mathrm{dom}G$ is nonempty, and F is continuous on $\mathcal{C}$. Then*

$$\inf_{A \in \mathcal{A}_0} \{F(A) + G(A)\} = \max_{V \in \mathcal{F}_L} \{-F^*(V) - G^*(-V)\}.$$

In convex analysis, the following theorems for the subdifferential play an important role.

Theorem 2.3 ([4])**.** *Let X be a Hausdorff locally convex space, X^* the dual space of X, f a convex function on X, $x_0 \in X$, and $\partial f(x_0) = \{v \in X^* \mid \forall x \in X, f(x) \geq f(x_0) + \langle v, x - x_0 \rangle\}$. If f is finite and continuous at x_0, then $\partial f(x_0)$ is nonempty.*

Theorem 2.4 ([3])**.** *Let X be a normed space, f and g proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$. Assume that there exists $x_0 \in \mathrm{dom}f \cap \mathrm{dom}g$ such that f is continuous at x_0. Then*

$$\partial(f + g)(x_0) = \partial f(x_0) + \partial g(x_0).$$

3. Subdifferential and optimality conditions

In this section, we study a subdifferential and optimality conditions. We define a subdifferential for set functions. We show optimality conditions in terms of convex analysis on the embedding normed space $\mathcal{C}^2/\equiv$.

We define the subdifferential for a set function F at $A_0 \in \mathcal{A}_0$ as follows:

$$\partial F(A_0) = \{V \in \mathcal{F}_L \mid \forall A \in \mathcal{A}_0, F(A) \geq F(A_0) + V(A) - V(A_0), V(A_0) \in \mathbb{R}\}.$$

In the following theorem, we show some properties of $\partial F(A_0)$. The proofs are easy and will be omitted.

Theorem 3.1. *Let F be a proper set function on $\mathcal{A}_0$, $A_0 \in \mathrm{dom}F$, and $\alpha > 0$. Then the following statements hold:*

(i) *$\partial F(A_0)$ is convex,*
(ii) *$\partial(\alpha F)(A_0) = \alpha \partial F(A_0)$,*
(iii) *A_0 is a global minimizer of F over $\mathcal{A}_0$ if and only if $0 \in \partial F(A_0)$.*

Next, we investigate the nonemptyness of the subdifferential.

Theorem 3.2. *Let F be a proper set function on $\mathcal{A}_0$ and $A_0 \in \mathrm{dom}F$. If F is continuous convex on $\mathcal{C}$, and $\mathrm{dom}F \subset \mathcal{C}$, then $\partial F(A_0)$ is nonempty.*

Proof. Assume that F is continuous convex on $\mathcal{C}$, and $\mathrm{dom}F \subset \mathcal{C}$. We define the following function $\bar{F}$ on $(\mathcal{C}^2/\equiv)$ as follows: for each $[A, \{0\}] \in \{[S, \{0\}] \in (\mathcal{C}^2/\equiv) \mid S \in \mathrm{dom}F\}$,

$$\bar{F}([A, \{0\}]) = F(A),$$

and for each $[A, B] \notin \{[S, \{0\}] \in (\mathcal{C}^2/\equiv) \mid S \in \mathrm{dom}F\}$, $\bar{F}([A, B]) = \infty$. Since $A_0 \in \mathrm{dom}F$, $\bar{F}([A_0, \{0\}])$ is finite. By the similar way in the proof of Theorem 3.2

in [40], we can show that epi$\bar{F}$ is closed convex and epi$(-\bar{F})$ is closed. Hence $\bar{F}$ is convex, and continuous at $[A_0, \{0\}]$. By Theorem 2.3, $\partial\bar{F}([A_0, \{0\}])$ is nonempty. Let $v \in \partial\bar{F}([A_0, \{0\}])$, and V a set function as follows: for each $A \in \mathcal{C}$,

$$V(A) = v([A, \{0\}]),$$

and for each $A \in \mathcal{A}_0 \setminus \mathcal{C}$, $V(A) = \infty$. For each $A \in \mathrm{dom}F$,

$$\begin{aligned} F(A) &= \bar{F}([A, \{0\}]) \\ &\geq \bar{F}([A_0, \{0\}]) + v([A, \{0\}]) - v([A_0, \{0\}]) \\ &= F(A_0) + V(A) - V(A_0). \end{aligned}$$

This shows that $V \in \partial F(A_0)$. □

In the following theorem, we show the subdifferential sum formula for convex set functions in terms of convex analysis on the embedding normed space $\mathcal{C}^2/\equiv$.

Theorem 3.3. *Let F and G be proper convex set functions from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$, and $A_0 \in \mathrm{dom}F \cap \mathrm{dom}G$. Assume that $\mathrm{dom}F \cup \mathrm{dom}G \subset \mathcal{C}$, and F is continuous on $\mathcal{C}$. Then,*

$$\partial(F + G)(A_0) \supset \partial F(A_0) + \partial G(A_0).$$

Additionally, assume that $V \in \partial(F + G)(A_0)$ satisfies $\mathrm{dom}V = \mathcal{C}$, and for each $\{A_k\}$, $\{B_k\} \subset \mathcal{C}$, A, $B \in \mathcal{C}$,

$$d_H(A_k + B, B_k + A) \to 0 \Longrightarrow V(A_k) - V(B_k) \to V(A) - V(B). \tag{3.1}$$

Then $V \in \partial F(A_0) + \partial G(A_0)$.

Proof. Let $V_1 \in \partial F(A_0)$ and $V_2 \in \partial G(A_0)$. By the definition of the subdifferential, for each $A \in \mathcal{A}_0$,

$$F(A) \geq F(A_0) + V_1(A) - V_1(A_0),$$

$$G(A) \geq G(A_0) + V_2(A) - V_2(A_0).$$

We can easily show that $V_1 + V_2 \in \partial(F + G)(A_0)$.

Conversely, let $V \in \partial(F + G)(A_0)$. We define $\bar{F}$ and $\bar{G}$ by the similar way in the proof of Theorem 3.2. Then, $\bar{F}$ and $\bar{G}$ are proper convex functions, $[A_0, \{0\}] \in \mathrm{dom}\bar{F} \cap \mathrm{dom}\bar{G}$, and $\bar{F}$ is continuous at $[A_0, \{0\}]$. Let v be the following function on $\mathcal{C}^2/\equiv$: for each $[A, B] \in \mathcal{C}^2/\equiv$,

$$v([A, B]) = V(A) - V(B).$$

Then, v is well-defined, real-valued and linear. Actually, if $(A, B) \equiv (C, D)$, then $A + D = B + C$. By Equation 2.1,

$$V(A) + V(D) = V(A + D) = V(B + C) = V(B) + V(C).$$

This shows that $V(A) - V(B) = V(C) - V(D)$. Let $[A, B]$, $[C, D] \in \mathcal{C}^2/\equiv$, and $\lambda > 0$. Then,

$$\begin{aligned} v([A, B] + [C, D]) &= v([A + C, B + D]) = V(A + C) - V(B + D) \\ &= V(A) + V(C) - V(B) - V(D) \\ &= v([A, B]) + V([C, D]), \end{aligned}$$

$$\begin{aligned} v(\lambda[A,B]) &= v([\lambda A, \lambda B]) \\ &= V(\lambda A) - V(\lambda B) \\ &= \lambda V(A) - \lambda V(B) \\ &= \lambda v(\lambda[A,B]), \end{aligned}$$

$$\begin{aligned} v(-\lambda[A,B]) &= v([-\lambda B, -\lambda A]) \\ &= V(-\lambda B) - V(-\lambda A) \\ &= -\lambda V(B) + \lambda V(A) \\ &= -\lambda v([A,B]). \end{aligned}$$

This shows that v is linear. Next, we show that v is continuous on $\mathcal{C}^2/\equiv$. Actually, if $\{[A_k, B_k]\} \subset \mathcal{C}^2/\equiv$ converges to $[A,B]$, then

$$d_H(A_k + B, B_k + A) = \|[A_k + B, B_k + A]\| = \|[A_k, B_k] - [A,B]\| \to 0.$$

By the assumption,

$$v([A_k, B_k]) = V(A_k) - V(B_k) \to V(A) - V(B) = v([A,B]).$$

This shows that v is continuous. Additionally, $v \in \partial(\bar{F} + \bar{G})([A_0, \{0\}])$. Actually, let $[A,B] \in \mathcal{C}^2/\equiv$. If there exists $B_0 \in \mathcal{C}$ such that $[A,B] = [B_0, \{0\}]$, then

$$\begin{aligned} (\bar{F} + \bar{G})(A,B) &= (\bar{F} + \bar{G})(B_0, \{0\}) \\ &= F(B_0) + G(B_0) \\ &\geq F(A_0) + G(A_0) + V(B_0) - V(A_0) \\ &= \bar{F}([A_0, \{0\}]) + \bar{G}([A_0, \{0\}]) + v([B_0, \{0\}]) - v([A_0, \{0\}]) \\ &= (\bar{F} + \bar{G})([A_0, \{0\}]) + v([A,B]) - v([A_0, \{0\}]). \end{aligned}$$

On the other hand if there does not exists $B_0 \in \mathcal{C}$ such that $[A,B] = [B_0, \{0\}]$, then $(\bar{F} + \bar{G})(A,B) = \infty$. This shows that $v \in \partial(\bar{F} + \bar{G})([A_0, \{0\}])$. Hence, by Theorem 2.4, there exist $v_1 \in \partial\bar{F}([A_0, \{0\}])$ and $v_2 \in \partial\bar{G}([A_0, \{0\}])$ such that $v = v_1 + v_2$. By the similar way in the proof of Theorem 3.2, let V_1 and V_2 be set functions as follows: for each $A \in \mathcal{C}$,

$$V_1(A) = v_1([A, \{0\}]), V_2(A) = v_2([A, \{0\}]).$$

Then, $V_1 \in \partial F(A_0)$, $V_2 \in \partial G(A_0)$, and $V = V_1 + V_2$. This completes the proof. □

Next, we show a relation between the subdifferential and Fenchel conjugate.

Theorem 3.4. *Let F be a set function and $V_0 \in \mathcal{F}_L$. Then, the following statements are equivalent.*

(i) $V_0 \in \partial F(A_0)$,
(ii) $F(A_0) + F^*(V_0) = V_0(A_0)$,
(iii) $F(A_0) + F^*(V_0) \leq V_0(A_0)$.

Proof. Let $V_0 \in \partial F(A_0)$, then by the definition of the subdifferential, $F(A) \geq F(A_0) + V_0(A) - V_0(A_0)$ for each $A \in \mathrm{dom}F$. This shows that

$$V_0(A_0) - F(A_0) \leq F^*(V_0) = \sup_{A \in \mathrm{dom}F} \{V_0(A) - F(A)\} \leq V_0(A_0) - F(A_0).$$

This shows that (i)→(ii)→(iii). The proof of (iii)→(i) is similar and will be omitted. □

We define the indicator function of $\mathcal{A}$ as follows:

$$\delta_{\mathcal{A}}(A) = \begin{cases} 0 & A \in \mathcal{A}, \\ \infty & \textit{otherwise}. \end{cases}$$

For a convex subfamily $\mathcal{A}$ of $\mathcal{A}_0$, the normal cone of $\mathcal{A}$ at $A_0 \in \mathcal{A}$ is defined by

$$N_{\mathcal{A}}(A_0) = \{V \in \mathcal{F}_L \mid \forall A \in \mathcal{A}, V(A) - V(A_0) \leq 0\}.$$

We can easily see that $N_{\mathcal{A}}(A_0)$ is a convex cone, and

$$N_{\mathcal{A}}(A_0) = \partial \delta_{\mathcal{A}}(A_0).$$

Next, we show characterizations of a global minimizer of F in terms of the subdifferential. In Theorem 3.1, we show that A_0 is a global minimizer of F over $\mathcal{A}_0$ if and only if $0 \in \partial F(A_0)$. In the following theorem, we show a necessary and sufficient optimality condition for constrained optimization in terms of the subdifferential.

Theorem 3.5. *Let F be a proper convex set function from $\mathcal{A}_0$ to $\mathbb{R} \cup \{+\infty\}$, and $\mathcal{A} \subset \mathcal{C}$ is convex. Assume that* $\mathrm{dom}F \subset \mathcal{C}$*, $A_0 \in$* $\mathrm{dom}F \cap \mathcal{A}$*, and F is continuous on $\mathcal{C}$. Then, A_0 is a global minimizer of F over $\mathcal{A}$ if and only if*

$$0 \in \partial F(A_0) + N_{\mathcal{A}}(A_0).$$

Proof. Assume that A_0 is a global minimizer of F over $\mathcal{A}$. By Theorem 2.2,

$$\inf_{A \in \mathcal{A}_0} \{F(A) + \delta_{\mathcal{A}}(A)\} = \max_{V \in \mathcal{F}_L} \{-F^*(V) - \delta^*_{\mathcal{A}}(-V)\}.$$

Hence, there exists $V_0 \in \mathcal{F}_L$ such that

$$\begin{aligned} F(A_0) &= \inf_{A \in \mathcal{A}_0} \{F(A) + \delta_{\mathcal{A}}(A)\} \\ &= \max_{V \in \mathcal{F}_L} \{-F^*(V) - \delta^*_{\mathcal{A}}(-V)\} \\ &= -F^*(V_0) - \delta^*_{\mathcal{A}}(-V_0). \end{aligned}$$

By the definition of the Fenchel conjugate,

$$\begin{aligned} F(A_0) + F^*(V_0) &= -\delta^*_{\mathcal{A}}(-V_0) \\ &= \inf_{A \in \mathrm{dom}\delta_{\mathcal{A}}} \{V_0(A) + \delta_{\mathcal{A}}(A)\} \\ &\leq V_0(A_0) + \delta_{\mathcal{A}}(A_0) \\ &= V_0(A_0). \end{aligned}$$

By Theorem 3.4, $V_0 \in \partial F(A_0)$. Similarly, we can show that $-V_0 \in N_{\mathcal{A}}(A_0)$. Hence, $0 \in \partial F(A_0) + N_{\mathcal{A}}(A_0)$ holds.

Conversely, if $0 \in \partial F(A_0) + N_{\mathcal{A}}(A_0)$, then there exists $V \in \partial F(A_0)$ such that $-V \in N_{\mathcal{A}}(A_0)$. Since $-V \in N_{\mathcal{A}}(A_0)$, $-V(A) - (-V(A_0)) \leq 0$ for each $A \in \mathcal{A}$. By the definition of the subdifferential,

$$F(A) \geq F(A_0) + V(A) - V(A_0) \geq F(A_0),$$

that is, A_0 is a global minimizer of F over $\mathcal{A}$. This completes the proof. □

Next, we consider the following optimization problem:

$$\begin{array}{ll} \text{Minimize} & F(A), \\ \text{subject to} & A \in \mathcal{A} = \{A \in \mathcal{C} \mid \forall i \in I, G_i(A) \leq 0\}, \end{array}$$

where F is a convex set function on $\mathcal{C}$, I is an index set, and G_i is a convex set function on $\mathcal{C}$ for each $i \in I$. In the following theorem, we investigate Karush-Kuhn-Tucker (KKT) type necessary and sufficient optimality condition for the problem.

Theorem 3.6. *Let F be a real-valued continuous convex set function on $\mathcal{C}$, I an index set, G_i a real-valued convex set function on $\mathcal{C}$ for each $i \in I$, $A_0 \in \mathcal{A} = \{A \in \mathcal{C} \mid \forall i \in I, G_i(A) \leq 0\}$, and $I(A_0) = \{i \in I \mid G_i(A_0) = 0\}$. Assume that the following equation holds:*

$$N_{\mathcal{A}}(A_0) = \bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0), \tag{3.2}$$

where $\mathbb{R}_+^{(I)} = \{\lambda \in \mathbb{R}^I \mid \forall i \in I, \lambda_i \geq 0, \{i \in I \mid \lambda_i \neq 0\}$: finite$\}$. Then, A_0 is a global minimizer of F over $\mathcal{A}$ if and only if there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that

$$0 \in \partial F(A_0) + \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0).$$

Proof. Let A_0 is a global minimizer of F over $\mathcal{A}$. By Theorem 3.5, $0 \in \partial F(A_0) + N_{\mathcal{A}}(A_0)$. By the assumption, there exist $V \in \partial F(A_0)$ and $\lambda \in \mathbb{R}_+^{(I)}$ such that $-V \in \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0)$. This shows that $0 \in \partial F(A_0) + \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0)$.

Conversely, assume that there exists $\bar{\lambda} \in \mathbb{R}_+^{(I)}$ such that $0 \in \partial F(A_0) + \sum_{i \in I(A_0)} \bar{\lambda}_i \partial G_i(A_0)$. Then,

$$\begin{aligned} 0 &\in \partial F(A_0) + \sum_{i \in I(A_0)} \bar{\lambda}_i \partial G_i(A_0) \\ &\subset \partial F(A_0) + \bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0) \\ &= \partial F(A_0) + N_{\mathcal{A}}(A_0). \end{aligned}$$

By Theorem 3.5, A_0 is a global minimizer of F over $\mathcal{A}$. This completes the proof. □

4. Discussion and applications

In this section, we discuss about our results and study applications to uncertain problems with motion uncertainty. We regard a decision variable set as an error caused by a motion, and introduce robust approach for the uncertain problem.

4.1. **Subdifferential sum formula.** The following equation is known as a subdifferential sum formula in convex analysis:

$$\partial(f+g)(x_0) = \partial f(x_0) + \partial g(x_0).$$

In Theorem 3.3, we show the following similar statement:

$$\partial(F+G)(A_0) \supset \partial F(A_0) + \partial G(A_0).$$

Additionally, if $V \in \partial(F+G)(A_0)$ satisfies $\mathrm{dom} V = \mathcal{C}$ and the continuity condition (3.1), then $V \in \partial F(A_0) + \partial G(A_0)$. By the condition (3.1), we can show that v in Theorem 3.3 is continuous on $\mathcal{C}^2/\equiv$. Unfortunately, we can not show that v is continuous even if we assume that V is continuous on $\mathcal{C}$. Actually, if $\{[A_k, B_k]\} \subset \mathcal{C}^2/\equiv$ converges to $\{[A, B]\}$, then $d_H(A_k + B, B_k + A)$ converges to 0. However, we can not prove that A_k and B_k converge some sets in $\mathcal{C}$. Hence, we can not apply the continuity of V. See the following example.

Example 4.1. For a nonempty compact convex subset $A \subset \mathbb{R}^2$, we define a set function V as follows:

$$V(A) = \sup\{x_1 \mid (x_1, x_2) \in A\}.$$

Then V is real-valued linear and continuous on $\mathcal{C}$. Let $A = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, $B = \{x \in \mathbb{R}^2 \mid \|x - (2, 0)\| \leq 1\}$, $A_k = \{(\frac{1}{k}, k)\}$, and $B_k = \{(2 + \frac{1}{k}, k)\}$. Then, $d_H(A_k + B, B_k + A)$ converges to 0 and A_k and B_k do not converge. Hence, we can not apply the continuity of V.

On the other hand, the continuity condition (3.1) holds. Therefore, we can apply Theorem 3.3 to the function V.

4.2. **Necessary and sufficient constraint qualification.** (3.2) in Theorem 3.6 is a constraint qualification for KKT optimality condition. We show the following theorem for the normal cone and constraint functions without proof.

Theorem 4.2. *Let I be an index set, G_i a real-valued convex set function on $\mathcal{C}$ for each $i \in I$, $A_0 \in \mathcal{A} = \{A \in \mathcal{C} \mid \forall i \in I, G_i(A) \leq 0\}$, and $I(A_0) = \{i \in I \mid G_i(A_0) = 0\}$. Then,*

$$N_{\mathcal{A}}(A_0) \supset \bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0).$$

By Theorem 4.2, the constraint qualification (3.2) and the following inclusion are equivalent:

$$N_{\mathcal{A}}(A_0) \subset \bigcup_{\lambda \in \mathbb{R}_+^{(I)}} \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0).$$

Similar constraint qualifications have been studied for convex and quasiconvex optimization, see [21, 30–33, 38]. These constraint qualifications are known as necessary and sufficient constraint qualifications. However, we show that (3.2) is only sufficient. Actually, let $V \in N_{\mathcal{A}}(A_0)$, then A_0 is a global minimizer of V over $\mathcal{A}$. By KKT condition, we can show that

$$0 \in \partial V(A_0) + \sum_{i \in I(A_0)} \lambda_i \partial G_i(A_0).$$

Although we can show that $\partial v(x_0) = \{v\}$ for $v \in \mathbb{R}^n$, we can not show that $\partial V(x_0) = \{V\}$ for a linear set function V. Hence, we can not clarify whether (3.2) is a necessary constraint qualification or not by the usual way. The difficulty causes from "$-[A, B] = [B, A]$" and "$\{0\} \neq A + (-A)$".

4.3. **Application to motion uncertainty.** Optimization problems with data uncertainty have been studied extensively, see [2, 13–16, 22, 39, 42]. On the other hand, in [40, 41], we study uncertain problems with motion uncertainty. We regard a decision variable set as an error caused by a motion, and investigate a robust approach for the problem with motion uncertainty.

Let I be an index set, f a real-valued convex function on $\mathbb{R}^n$, g_i a real-valued convex function on $\mathbb{R}^n$ for each $i \in I$. We study the following convex programming problem (P):

$$(P)\begin{cases}\text{minimize } f(x),\\ \text{subject to } g_i(x) \leq 0, \forall i \in I.\end{cases}$$

For such a problem, we may not be able to choose an exact vector because of an error by a motion. In [40, 41], we introduce the following worst case approach with motion uncertainty. Let F be the following function on $\mathcal{A}_0$: for each $A \in \mathcal{A}_0$,

$$F(A) = \sup_{x \in A} f(x).$$

We define G_i similarly, and consider the following robust problem (RP) with motion uncertainty:

$$(RP)\begin{cases}\text{minimize } F(A),\\ \text{subject to } G_i(A) \leq 0, \forall i \in I.\end{cases}$$

We can solve the problem (RP) by using our results, for example Theorem 3.5.

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References

[1] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, *Generalized Concavity*, Math. Concepts Methods Sci. Engrg. Plenum Press, New York, 1988.

[2] A. Beck and A. Ben-Tal, *Duality in robust optimization: Primal worst equals dual best*, Oper. Res. Lett. **37** (2009), 1–6.

[3] R. I. Boţ, *Conjugate Duality in Convex Optimization*, Springer-Verlag, Berlin. 2010.

[4] R. I. Boţ, S. M. Grad and G. Wanka, *Duality in Vector Optimization*, Springer-Verlag, Berlin. 2009.

[5] J. H. Chou, W. S Hsia and T. Y. Lee, *Epigraphs of convex set functions*, J. Math. Anal. Appl. **118** (1986), 247–254.

[6] J. H. Chou, W. S Hsia and T. Y. Lee, *Convex programming with set functions*, Rocky Mountain J. Math. **17** (1987), 535–543.

[7] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech. Opér **15** (1973), 437–448.

[8] W. S. Hsia, J. H. Lee and T. Y. Lee, *Convolution of set functions*, Rocky Mountain J. Math. **21** (1991), 1317–1325.

[9] W. S. Hsia and T. Y. Lee, *Some minimax theorems on set functions*, Bull. Inst. Math. Acad. Sinica **25** (1997), 29–33.

[10] D. Kuroiwa, *On derivatives of set-valued maps and optimality conditions for set optimization*, J. Nonlinear Convex Anal. **10** (2009), 41–50.

[11] D. Kuroiwa, *Generalized minimality in set optimization*, in: Set Optimization and Applications - The State of the Art, Springer Proceedings in Mathematics & Statistics, vol. 151, Springer, 2015, pp. 293–311.

[12] D. Kuroiwa and T. Nuriya, *A generalized embedding vector space in set optimization*, in: Nonlinear Analysis and Convex Analysis, Yokohama Publisher, Yokohama, 2007, pp. 297–303.

[13] V. Jeyakumar and G. Y. Li, *Strong duality in robust convex programming: complete characterizations*, SIAM J. Optim. **20** (2010), 3384–3407.

[14] V. Jeyakumar and G. Y. Li, *Characterizing robust set containments and solutions of uncertain linear programs without qualifications*, Oper. Res. Lett. **38** (2010), 188–194.

[15] V. Jeyakumar and G. Y. Li, *Robust Farkas' lemma for uncertain linear systems with applications*, Positivity **15** (2011), 331–342.

[16] V. Jeyakumar and G. Y. Li, *Strong duality in robust semi-definite linear programming under data uncertainty*, Optimization **63** (2014), 713–733.

[17] D. Kuroiwa, G. M. Lee and S. Suzuki, *Surrogate duality for optimization problems involving set functions*, Linear Nonlinear Anal. **5** (2019), 269–277.

[18] H. C. Lai and L. J. Lin, *Moreau-Rockafellar type theorem for convex set functions*, J. Math. Anal. Appl. **132** (1988), 558–571.

[19] H. C. Lai and L. J. Lin, *The Fenchel-Moreau theorem for set functions*, Proc. Amer. Math. Soc. **103** (1988), 85–90.

[20] T. Y. Lee, *Generalized convex set functions*, J. Math. Anal. Appl. **141** (1989), 278–290.

[21] C. Li, K. F. Ng and T. K. Pong, *Constraint qualifications for convex inequality systems with applications in constrained optimization*, SIAM J. Optim. **19** (2008), 163–187.

[22] G. Y. Li, V. Jeyakumar and G. M. Lee, *Robust conjugate duality for convex optimization under uncertainty with application to data classification*, Nonlinear Anal. **74** (2011), 2327–2341.

[23] L. J. Lin, *On the optimality of differentiable nonconvex n-set functions*, J. Math. Anal. Appl. **168** (1992), 351–366.

[24] J. J. Moreau, *Inf-convolution, sous-additivité, convexité des fonctions numériques*, J. Math. Pures Appl. **49** (1970), 109–154.

[25] R. J. T. Morris, *Optimal constrained selection of a measurable subset*, J. Math. Anal. Appl. **70** (1979), 546–562.

[26] J. P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res. **15** (1990), 597–625.

[27] H. Rådström, *An embedding theorem for spaces of convex sets*, Proc. Amer. Math. Soc. **3** (1952), 165–169.

[28] R. T. Rockafellar, *Convex Analysis*, Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J. 1970.

[29] S. Suzuki and D. Kuroiwa, *On set containment characterization and constraint qualification for quasiconvex programming*, J. Optim. Theory Appl. **149** (2011), 554–563.

[30] S. Suzuki and D. Kuroiwa, *Optimality conditions and the basic constraint qualification for quasiconvex programming*, Nonlinear Anal. **74** (2011), 1279–1285.

[31] S. Suzuki and D. Kuroiwa, *Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming*, Nonlinear Anal. **75** (2012), 2851–2858.

[32] S. Suzuki and D. Kuroiwa, *Necessary and sufficient constraint qualification for surrogate duality*, J. Optim. Theory Appl. **152** (2012), 366–377.

[33] S. Suzuki and D. Kuroiwa, *Some constraint qualifications for quasiconvex vector-valued systems*, J. Global Optim. **55** (2013), 539–548.

[34] S. Suzuki and D. Kuroiwa, *Characterizations of the solution set for quasiconvex programming in terms of Greenberg-Pierskalla subdifferential*, J. Global Optim. **62** (2015), 431–441.

[35] S. Suzuki and D. Kuroiwa, *A constraint qualification characterizing surrogate duality for quasiconvex programming*, Pac. J. Optim. **12** (2016), 87–100.

[36] S. Suzuki and D. Kuroiwa, *Nonlinear Error Bounds for Quasiconvex Inequality Systems*, Optim. Lett. **11** (2017), 107–120.

[37] S. Suzuki and D. Kuroiwa, *Duality Theorems for Separable Convex Programming without Qualifications*, J. Optim. Theory Appl. **172** (2017), 669-683.

[38] S. Suzuki and D. Kuroiwa, *Generators and constraint qualifications for quasiconvex inequality systems*, J. Nonlinear Convex Anal. **18** (2017), 2101–2121.

[39] S. Suzuki and D. Kuroiwa, *Surrogate duality for robust quasiconvex vector optimization*, Appl. Anal. Optim. **2** (2018), 27–39.

[40] S. Suzuki and D. Kuroiwa, *Fenchel duality for convex set functions*, Pure Appl. Funct. Anal. **3** (2018), 505–517.

[41] S. Suzuki and D. Kuroiwa, *Duality theorems for convex and quasiconvex set functions*, SN Operations Research Forum, **1** (2020), 4 (13pages).

[42] S. Suzuki, D. Kuroiwa and G. M. Lee, *Surrogate duality for robust optimization*, European J. Oper. Res. **231** (2013), 257–262.

[43] C. Zălinescu, *On several results about convex set functions*, J. Math. Anal. Appl. **328** (2007), 1451–1470.

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