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LINKING AND THE LERAY-SCHAUDER INDEX

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ABSTRACT. We show how infinite dimensional linking can be used to solve problems that could not be solved before. We study applications to n-dimensional nonlinear partial differential equations.

1. Linking pairs

Many problems arising in science and engineering call for the solving of the Euler equations of functionals, i.e., equations of the form

$$G'(u) = 0, \tag{1.1}$$

where $G(u)$ is a C^1 functional (usually representing the energy) arising from the given data. As an illustration, the equation

$$-\Delta u(x) = f(x, u(x))$$

is the Euler equation of the functional

$$G(u) = \frac{1}{2}\|\nabla u\|^2 - \int F(x, u(x))\, dx$$

on an appropriate space, where

$$F(x, t) = \int_0^t f(x, s)\, ds, \tag{1.2}$$

and the norm is that of L^2. The solving of the Euler equations is tantamount to finding critical points of the corresponding functional. The history of this approach goes back to the calculus of variations. Then the desire was to find extrema of certain expressions G (functionals). Following the approach of calculus, one tried to find all critical points of G, substitute them back in G and see which one gives the required extremum. This worked fairly well in one dimension where $G'(u) = 0$ is an ordinary differential equation. However, in higher dimensions, it turned out that it was easier to find the extrema of G than solve $G'(u) = 0$. This led to the approach of solving equations of the form $G'(u) = 0$ by finding extrema of G.

The classical approach was to look for maxima or minima. If the functional is bounded from below and one is looking for a minimum, one can obtain a minimizing

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sequence satisfying

$$G(u_k) \to a = \inf G. \tag{1.3}$$

If such a sequence converges or has a convergent subsequence, then we indeed obtain a minimum. However, in dealing with such sequences it is difficult, in general, to establish the convergence of a subsequence because there is very little with which to work.

Luckily, there is some help. In such a case, one can show that there is a sequence, called a Palais-Smale PS sequence, satisfying

$$G(u_k) \to a, \quad G'(u_k) \to 0 \tag{1.4}$$

where $a = \inf G$. It is much easier to establish the existence of a convergent subsequence of a PS sequence than of a minimizing sequence. In fact, a minimizing sequence may not have a convergent subsequence while a PS sequence for the same functional does.

Actually, one can do better. If the functional $G(u)$ is bounded from below, then there exists a sequence (called a Cerami sequence) satisfying

$$G(u_k) \to a, \quad (1 + \|u_k\|)G'(u_k) \to 0 \tag{1.5}$$

for $a = \inf G$. As in the case of a PS sequence, if a Cerami sequence has a convergent subsequence, it will produce a minimum. The advantage of obtaining such a sequence is that the additional structure allows one to prove the convergence of a subsequence in cases where a corresponding PS sequence need not have a converging subsequence.

However, when the functional is not semi-bounded, the methods for producing critical points become more complicated. It appears that no one procedure works in all cases. The same is true even for semibounded functionals if one wishes to obtain critical points which are not extrema. We present an approach which produces sequences similar to (1.5) when one is searching for critical points whether or not they are extrema.

This method of detecting critical points is called **linking,** initiated by Ambrosetti and Rabinowitz ([1, 13]. It was discovered that there are pairs of sets A, B such that whenever they separate a functional G, i.e., satisfy

$$a_0 := \sup_A G \le b_0 := \inf_B G,$$

one obtains a Cerami sequence of the form

$$G(u_k) \to a < \infty, \ (1 + \|u_k\|)\|G'(u_k)\| \to 0, \tag{1.6}$$

provided the functional is bounded on bounded sets. If this sequence has a convergent subsequence, we obtain a critical point. The main question is to identify such pairs of sets. We now describe a method of obtaining them.

2. A general linking theorem

A basic question is how to find linking subsets. Once we have them, we will have a very useful theorem:

Theorem 2.1. *Let G be a C^1-functional on E, and let A, B be subsets of E such that A links B and*

$$a_0 := \sup_A G \leq b_0 := \inf_B G \leq b_1 := \sup_{c(A)} G < \infty,$$

where c(A) is the convex hull of A. Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a,\ b_0 \leq a \leq b_1,\ (1 + \|u_k\|)\|G'(u_k)\| \to 0. \tag{2.1}$$

If $\{u_k\}$ has a convergent subsequence, then there is a solution $u \in E$ of

$$G(u) = a, \quad G'(u) = 0.$$

All the theorem requires is that A links B and

$$a_0 \leq b_0 \leq b_1 < \infty.$$

Finding sets A and B which separate the functional G is quite easy, but determining whether or not the set A links the set B is quite another story. There are many criteria which are used to determine whether or not the set A links the set B, but the most general one is the following (cf. [14, 22]):

Let E be a Banach space. Let Φ be the set of all mappings $\Gamma(t) \in C(E \times [0,1], E)$ having the following properties:

a): for each $t \in [0,1), \Gamma(t)$ is a homeomorphism of E onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0,1)$

b): $\Gamma(0) = I$

c): for each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$ and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of E.

d): For each $t_0 \in [0,1)$ and each bounded set $A \subset E$ we have

$$\sup_{\substack{0 \leq t \leq t_0 \\ u \in A}} \{\|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\|\} < \infty.$$

We have

Definition 2.2. A subset A of E links a subset B of E if $A \cap B = \phi$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \phi$, i.e.,

$$\bigcup_{t \in (0,1]} \Gamma(t)A \cap B \neq \phi.$$

This says that if $\Gamma(t)$ takes A into u_0, it must intersect B.

Now that we have the general definition of linking, it appears that the only way we can check to see if two sets link, is to require that one of them is contained in a finite-dimensional subspace. The reason is that in order to verify the definition, we need to invoke the Brouwer fixed point theorem. This is not easy to do, and the following result is what is used in most cases (cf. [14, 22]).

Theorem 2.3. *Let N be a finite dimensional subspace of a Banach space E, and let Ω be a bounded open subset of N containing a point p. Let F be a continuous map of E onto N such that F is bijective on $\overline{\Omega}$. Then $\partial\Omega$ links $F^{-1}(p)$.*

Proof. Assume that $\partial\Omega$ does not link $F^{-1}(p)$. Then there is a $\Gamma \in \Phi$ such that

$$\Gamma(t)\partial\Omega \cap F^{-1}(p) = \phi, \quad 0 \leq t \leq 1,$$

or, equivalently,

$$F(\Gamma(t)\partial\Omega) \cap \{p\} = \phi, \quad 0 \leq t \leq 1. \tag{2.2}$$

Let

$$\gamma(t) = F \circ \Gamma(t).$$

Then $\gamma(t) \in C(\overline{\Omega}, N)$ for each $t \in [0,1]$ and

$$\gamma(t)x \neq p, \quad x \in \partial\Omega, \ t \in [0,1].$$

Also

$$\gamma(0)x = F(x), \quad x \in \overline{\Omega}. \tag{2.3}$$

If $\Gamma(1)E = \{u_0\}$, then

$$\gamma(1)x = F(u_0) \neq p, \quad x \in \overline{\Omega}, \tag{2.4}$$

since

$$F(\Gamma(1)\partial\Omega) \cap \{p\} = \phi$$

by (2.2).

In view of (2.2) and (2.3), the Brouwer degree satisfies

$$i(\gamma(t), \Omega, p) = i(\gamma(0), \Omega, p) = 1$$

for all $t \in [0,1]$. But this contradicts (2.4). Hence $\partial\Omega$ links $F^{-1}(p)$. □

3. Linking sets

In order to find sets that link in the sense of Definition 2, we apply Theorem 3. We give a partial list below.

Example 1. Let B be an open set in E, and let A consist of two points e_1, e_2 with $e_1 \in B$ and $e_2 \notin \bar{B}$. Then A links ∂B. ∂B links A as well if ∂B is bounded.

Example 2. Let M, N be closed subspaces such that $\dim N < \infty$ and $E = M \oplus N$. Let

$$\mathcal{B}_R = \{u \in E : \|u\| < R\} \tag{3.1}$$

and take $A = \partial\mathcal{B}_R \cap N, \ B = M$. Then A links B.

Example 3. Take M, N as before and let $v_0 \neq 0$ be an element of N. We write $N = \{v_0\} \oplus N'$. We take

$$\begin{aligned} A &= \{v' \in N' : \|v'\| \leq R\} \cup \{sv_0 + v' : v' \in N', s \geq 0, \|sv_0 + v'\| = R\}, \\ B &= \{w \in M : \|w\| \geq \delta\} \cup \{sv_0 + w : w \in M, s \geq 0, \|sv_0 + w\| = \delta\}, \end{aligned}$$

where $0 < \delta < R$. Then A links B.

Example 4. Let M, N be as in Example 2. Take $A = \partial\mathcal{B}_\delta \cap N$, and let v_0 be any element in $\partial\mathcal{B}_1 \cap N$. Take B to be the set of all u of the form

$$u = w + sv_0,\ w \in M,$$

satisfying any of the following:

(a): $\|w\| \le R,\ s = 0$
(b): $\|w\| \le R,\ s = 2R_0$
(c): $\|w\| = R,\ 0 \le s \le 2R_0$,

where $0 < \delta < \min(R, R_0)$. Then A and B link each other.

Example 5. Let M, N be closed subspaces of E such that

$$E = M \oplus N,$$

with one of them being finite-dimensional. Let w_0 be an element of $M \setminus \{0\}$, and let $0 < \delta < r < R$. Take

$$\begin{aligned} A &= \{v \in N : \delta \le \|v\| \le R\} \cup \{sw_0 + v : v \in N, s \ge 0, \|sw_0 + v\| = \delta\} \\ &\cup \{sw_0 + v : v \in N, s \ge 0, \|sw_0 + v\| = R\}, \\ \\ B &= \partial\mathcal{B}_r \cap M,\ 0 \le \delta < r < R. \end{aligned}$$

Then A and B link each other.

Example 6. Let M, N be closed subspaces of E such that

$$E = M \oplus N,$$

with one of them being finite-dimensional. Let w_0 be an element of $M \setminus \{0\}$, and let $0 \le r < R$,

$$\begin{aligned} A &= \{w \in M : \|w\| = R\}, \\ B &= \{v \in N : \|v\| \ge r\} \cup \{u = v + sw_0 : v \in N, s \ge 0, \|u\| = r\}. \end{aligned}$$

Then A links B.

Example 7. Let M, N be as in Example 2. Take $A = \partial\mathcal{B}_\delta \cap N$, and let v_0 be any element in $\partial\mathcal{B}_1 \cap N$. Take B to be the set of all u of the form

$$u = w + sv_0, \quad w \in M,$$

satisfying any of the following:

(a): $s = 0$
(b): $s = 2R_0$

where $0 < \delta < R_0$. Then A links B.

Example 8. Let N be a finite dimensional subspace of a Hilbert space E with orthogonal complement $M \oplus Y$, where Y is a finite dimensional subspace of E

orthogonal to both M and N, and let $\delta < R$ be positive numbers. Let $y_1, \cdots, y_n$ be an orthogonal basis for Y and let

$$Y_+ = \{y \in Y : (y, y_k) \geq 0, \quad 1 \leq k \leq n\},$$
$$\Omega = \{v + y : v \in N,\ y \in Y_+,\ \|v + y\| < R\},$$

and

$$F(v + w + y) = v + \frac{\|w + y\|}{\|y\|} \sum_1^n |(y, y_k)| y_k, \quad v \in N,\ y \in Y,\ w \in M. \tag{3.2}$$

Then $A_R = \partial\Omega$ links $B = F^{-1}(Y_+ \cap \partial\mathcal{B}_\delta) = \{w + y \in M \oplus Y : \|w + y\| = \delta\}$.

Note that all of these examples follow from Theorem 2.3. Related research can be found in [2–4, 6, 10, 11, 31, 32] and their references.

4. The limitation

The only reason these examples work is because we are able to use the Brouwer fixed point theorem in finite-dimensional spaces. However, there are many applications for which we would like to obtain critical points if both sets are infinite-dimensional. It is not obvious how to proceed. It is not clear that we can obtain similar results in such cases. We now describe one method that works in the infinite-dimensional case. It was initiated by Kryszewski and Szulkin [9]. It involves adjusting the topology of the underlying space. Our aim is to find a counterpart of Theorem 3 that holds true when N is infinite dimensional. We adjust our definitions of the functional G and the mapping F to accommodate infinite dimensions. These definitions reduced to the usual when N is finite dimensional. We can then prove the counterpart of Theorem 3 when N is infinite dimensional. In order to do so, we make adjustments to the topology of the space and introduce infinite dimensional splitting. This allows us to use a form of compactness on the subspace N. We lose the Brouwer index, but we are able to replace it with the Leray-Schauder index. We carry out the details in Sections 5 and 6. In Sections 7 - 16 we solve several equations which require infinite dimensional splitting. In Sections 7 - 10 we study general semilinear partial differential equations, and in Sections 11, 12 we consider the wave equation. In Sections 13 - 15 we study the n-dimensional radially symmetric wave equation and in Section 16 the non-periodic Schrodinger equation. In all cases we obtain results stronger than those previously known.

5. Flows

Let $\mathcal{Q}$ be a set of positive functions $\rho(t)$ on $[0, \infty)$, which are
(a) locally Lipschitz continuous,
(b) nondecreasing
(c) satisfy

$$\int_0^\infty \frac{dt}{\rho(t)} = \infty. \tag{5.1}$$

Moreover, $\mathcal{Q}$ is to satisfy

$$\rho_1, \rho_2 \in \mathcal{Q} \Longrightarrow \max(\rho_1, \rho_2) \in \mathcal{Q},$$

and contain functions of the form

$$(1 + |t|)^\beta, \quad \beta \le 1.$$

Let $Q \ne \phi$ be a subset of a Banach space E, and let Σ_Q be the set of all continuous maps $\sigma = \sigma(t)$ from $E \times [0,1]$ to E such that

(1) $\sigma(0)$ is the identity map,
(2) for each $t \in [0,1]$, $\sigma(t)$ is a homeomorphism of E onto E,
(3) $\sigma'(t) = d\sigma(t)/dt$ is piecewise continuous and satisfies

$$\|\sigma'(t)u\| \le C\rho(d(\sigma(t)u, Q)), \quad u \in E, \tag{5.2}$$

for some $\rho \in \mathcal{Q}$. If $Q = \{0\}$, we write $\Sigma = \Sigma_Q$. The mappings in Σ_Q are called *flows*. We note the following.

Remark 5.1. If $\sigma_1,\ \sigma_2$ are in Σ_Q, define $\sigma_3 = \sigma_1 \circ \sigma_2$ by

$$\sigma_3(s) = \begin{cases} \sigma_1(2s), & 0 \le s \le \frac{1}{2}, \\ \sigma_2(2s-1)\sigma_1(1), & \frac{1}{2} < s \le 1. \end{cases}$$

Then $\sigma_3 \in \Sigma_Q$, and $\sigma_3(1) = \sigma_2(1)\sigma_1(1)$.

Proof. The first two properties are obvious. To check the third, note that

$$\sigma_3'(s) = \begin{cases} 2\sigma_1'(2s), & 0 \le s \le (\frac{1}{2})_-, \\ 2\sigma_2'(2s-1)\sigma_1(1), & (\frac{1}{2})_+ \le s \le 1. \end{cases}$$

Thus, if

$$\|\sigma_i'(t)u\| \le C_i\rho_i(d(\sigma_i(t)u, Q)), \quad u \in E,\ i = 1, 2, \tag{5.3}$$

then

$$\|\sigma_3'(s)u\| \le \begin{cases} 2\|\sigma_1'(2s)u\|, & 0 \le s \le (\frac{1}{2})_-, \\ 2\|\sigma_2'(2s-1)\sigma_1(1)u\|, & (\frac{1}{2})_+ \le s \le 1, \end{cases}$$

or

$$\|\sigma_3'(s)u\| \le \begin{cases} 2C_1\rho(d(\sigma_3(s)u, Q)), & 0 \le s \le (\frac{1}{2})_-, \\ 2C_2\rho(d(\sigma_3(s)u, Q)), & (\frac{1}{2})_+ \le s \le 1, \end{cases}$$

where $\rho = \max(\rho_1, \rho_2)$. We can now take $C_3 = 2\max(C_1, C_2)$. □

6. Infinite dimensional splitting

The idea of splitting the topologies of subspaces originated in [9]. Let N be a closed, separable subspace of a Hilbert space E. We can define a new norm $|v|_w$ satisfying $|v|_w \le \|v\|\ \forall v \in N$ and such that the topology induced by this norm is equivalent to the weak topology of N on bounded subsets of N. This can be done as follows: Let $\{e_k\}$ be an orthonormal basis for N. Define

$$(u, v)_w = \sum_{k=1}^{\infty} \frac{(u, e_k)(v, e_k)}{2^k}, \quad u, v \in N.$$

This is a scalar product. The corresponding norm squared is

$$|v|_w^2 = \sum_{k=1}^{\infty} \frac{|(v, e_k)|^2}{2^k}, \quad v \in N.$$

Then $|v|_w$ satisfies $|v|_w \leq \|v\|,\ v \in N$. If $v_j \to v$ weakly in N, then there is a $C > 0$ such that

$$\|v_j\|, \|v\| \leq C, \quad \forall j > 0.$$

For any $\varepsilon > 0$, there exist $K > 0, M > 0$, such that $1/2^K < \varepsilon^2/(8C^2)$ and $|(v_j - v, e_k)| < \varepsilon/2$ for $1 \leq k \leq K, j > M$. Therefore,

$$\begin{aligned} |v_j - v|_w^2 &= \sum_{k=1}^{\infty} \frac{|(v_j - v, e_k)|^2}{2^k} \\ &\leq \sum_{k=1}^{K} \frac{\varepsilon^2/4}{2^k} + \sum_{k=K+1}^{\infty} \frac{4C^2}{2^k} \\ &\leq \frac{\varepsilon^2}{4} \sum_{k=1}^{\infty} \frac{1}{2^k} + \frac{4C^2}{2^K} \sum_{k=1}^{\infty} \frac{1}{2^k} \\ &\leq \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}. \end{aligned}$$

Therefore, $v_j \to v$ weakly in N implies $|v_j - v|_w \to 0$.

Conversely, let $\|v_j\|, \|v\| \leq C$ for all $j > 0$ and $|v_j - v|_w \to 0$. Let $\varepsilon > 0$ be given. If $h = \sum_{k=1}^{\infty} \alpha_k e_k \in N$, take K so large that $\|h_K\| < \varepsilon/(4C)$, where $h_K = \sum_{k=K+1}^{\infty} \alpha_k e_k$. Take M so large that $|v_j - v|_w^2 < \varepsilon^2/(4\sum_{k=1}^{K} 2^k|\alpha_k|^2)$ for all $j > M$. Then

$$\begin{aligned} &|(v_j - v, h - h_K)|^2 = |\sum_{k=1}^{K} \alpha_k (v_j - v, e_k)|^2 \\ &\leq \sum_{k=1}^{K} 2^k |\alpha_k|^2 \sum_{k=1}^{\infty} \frac{|(v_j - v, e_k)|^2}{2^k} \\ &< \varepsilon^2/4 \end{aligned}$$

for $j > M$. Also, $|(v_j - v, h_K)| \leq 2C\|h_K\| < \varepsilon/2$. Therefore,

$$|(v_j - v, h)| < \varepsilon, \quad \forall j > M,$$

that is, $v_j \to v$ weakly in N.

For $u = v + h,\ u_1 = v_1 + h_1 \in E = N \oplus N^\perp$ with $v, v_1 \in N, h, h_1 \in N^\perp$, we define the scalar product $(u, u_1)_w = (v, v_1)_w + (h, h_1)$. Thus, the corresponding norm satisfies $|u|_w \leq \|u\|\ \forall u \in E$.

We denote E equipped with this scalar product and norm by E_w. It is a scalar product space with the same elements as E. In particular, if $(u_n = v_n + w_n)$ is $\|\cdot\|$-bounded and $u_n \overset{|\cdot|_w}{\to} u$, then $v_n \rightharpoonup v$ weakly in N, $w_n \to w$ strongly in $N^\perp$, $u_n \rightharpoonup v + w$ weakly in E.

For $u \in E$ and $Q \subset E$, we define

$$d_w(u, Q) = \inf_{v \in Q} |u - v|_w.$$

Let L be a bounded, convex, closed subset of N. Then L is $|\cdot|_w$-compact. In fact, since L is bounded with respect to both norms $|\cdot|_w$ and $\|\cdot\|$, for any $v_n \in L$, there is a renamed subsequence such that $v_n \rightharpoonup v_0$ weakly in E. Then $v_0 \in L$ since L is convex, and on the bounded set L the $|\cdot|_w$-topology is equivalent to the weak topology. Thus, $v_n \overset{|\cdot|_w}{\to} v_0$ and L is $|\cdot|_w$-compact.

Let L be a compact subset of E_w. We define $\Sigma_w(L)$ to be the set of all $\sigma(t) \in \Sigma : [0,1] \times E \mapsto E$ such that

(1) $\sigma(t)$ is $|\cdot|_w$-continuous.
(2) There is a finite dimensional subspace E_f of E such that $\dim E_f > 0$ and $\sigma(t)u - u \in E_f$, $(t, u) \in I \times L$. (E_f does not depend on t.)

Here we use E_f to denote various finite-dimensional subspaces of E when exact dimensions are irrelevant. Note that $\Sigma_w(L)$ is not empty since $\sigma(t) \equiv 1$ is a member.

We let Σ_{wQ} denote the set of those $\sigma \in \Sigma_w$ which satisfy

$$|\sigma'(t)u|_w \le C\rho(d_w(\sigma(t)u, Q)), \quad u \in E, \tag{6.1}$$

where $Q \subset E$.

We have

Lemma 6.1. *If L is compact in E_w and $\sigma \in \Sigma_w(L)$, then*

$$\tilde{L} = \{\sigma(t)L : t \in I\}$$

is compact in E_w.

Proof. Supose $\{t_k\} \subset I$, $\{u_k\} \subset L$ are sequences. Then there are renamed subsequences such that

$$t_k \to t_0, \quad |u_k - u_0|_w \to 0.$$

Thus $I \times L$ is a compact subset of $I \times E_w$. By definition, there is a finite dimensional subspace E_f containing the set $\{\sigma(t)u - u,\ t \in I,\ u \in L\}$. Since this set is bounded, every sequence has a convergent subsequence. Since every sequence in L has a convergent subsequence, the same must be true of $\tilde{L}$. □

Lemma 6.2. *If $\sigma_1,\ \sigma_2 \in \Sigma_w(L)$, then $\sigma_3 = \sigma_1 \circ \sigma_2 \in \Sigma_w(L)$.*

Proof. By the definition of $\Sigma_w(L)$, for any $(s_0, u_0) \in I \times L$, there is a $|\cdot|_w$-neighborhood $U_{(s_0,u_0)}$ such that $\{u - \sigma_1(t)u : (t, u) \in U_{(s_0,u_0)} \cap (I \times L)\} \subset E_f$.

Note that, $I \times L \subset \bigcup_{(s,u)\in I\times L} U_{(s,u)}$. Since L is $|\cdot|_w$-compact, $I \times L \subset \bigcup_{i=1}^{j_0} U_{(s_i,u_i)}$ where $(s_i, u_i) \in (I \times L)$. Consequently, $\{u - \sigma_1(t)u : (t,u) \in (I \times L)\} \subset E_f$. The same is true of σ_2. Since

$$\sigma_3(s) = \begin{cases} \sigma_1(2s), & 0 \le s \le \frac{1}{2}, \\ \sigma_2(2s-1)\sigma_1(1), & \frac{1}{2} < s \le 1, \end{cases}$$

$u - \sigma_3(t)u \in E_f$ as well. □

Concerning the mapping F we define

Definition 6.3. Let N be a closed separable subspace of a Hilbert space E. We shall call a map F of E onto N an N-weakly continuous mapping if F is a $|\cdot|_w$-continuous map from E onto N satisfying

- $F_N = I$ and it maps bounded sets into bounded sets;
- There exists a fixed finite-dimensional subspace E_0 of E such that $F(u-v) - (F(u) - F(v)) \in E_0$, $\forall\, u,\ v \in E$;
- F maps finite-dimensional subspaces of E to finite-dimensional subspaces of E;

Note that every continuous map F of E onto N satisfying $F_N = I$ is N-weakly continuous when N is finite dimensional.

Our counterpart of Theorem 2.3 for infinite dimensional subspaces is:

Theorem 6.4. *Let N be a closed, separable subspace of a Banach space E, and let Ω be a bounded, convex, open subset of N containing a point p. Let F be an N-weakly continuous mapping. Assume*

$$\sigma(t)\partial\Omega \cap F^{-1}(p) = \phi, \quad 0 \le t \le 1,$$

for some $\sigma \in \Sigma_w(\overline{\Omega})$. Then

$$\sigma(t)\Omega \cap F^{-1}(p) \ne \phi, \quad 0 \le t \le 1.$$

Proof. Assume that there is a $\sigma \in \Sigma_w(\overline{\Omega})$ such that

$$\sigma(t)\partial\Omega \cap F^{-1}(p) = \phi, \quad 0 \le t \le 1, \tag{6.2}$$

and

$$\sigma(t)\Omega \cap F^{-1}(p) = \phi, \quad 0 \le t \le 1,$$

or, equivalently,

$$F(\sigma(t)\Omega) \cap \{p\} = \phi, \quad 0 < t \le 1. \tag{6.3}$$

Let

$$\gamma(t)x = F(\sigma(t)x), \quad (t,x) \in I \times \overline{\Omega}.$$

Then $\gamma(t) \in C(I \times \overline{\Omega}, E_w \cap N)$ and

$$\gamma(t)x \ne p, \quad x \in \partial\Omega,\ t \in [0,1]. \tag{6.4}$$

Also

$$\gamma(0)x = F(x) = x, \quad x \in \overline{\Omega}. \tag{6.5}$$

By hypothesis, there exists a fixed finite-dimensional subspace E_0 of E such that $F(u-v) - (F(u) - F(v)) \in E_0,\ \forall\ u,\ v \in E$. Take $u = \sigma(t)x,\ \ v = x$. Since $\overline{\Omega}$ is compact in E_w and $\sigma \in \Sigma_w(\overline{\Omega})$, there is a finite dimensional subspace E_1 of E such that $\dim E_1 > 0$ and $\sigma(t)u - u \in E_1,\ (t,u) \in I \times \overline{\Omega}$. Hence

$$\begin{aligned}\gamma(t)x &= P_0(F\sigma(t)x - F(x) - F[\sigma(t)x - x]) \\ &\quad + FP_1[\sigma(t)x - x] + x \\ &= x - \varphi(t)x, \quad (t,x) \in I \times \overline{\Omega},\end{aligned}$$

where $\varphi(t)x = -P_0(F\sigma(t)x - Fx - F[\sigma(t)x-x]) - FP_1[\sigma(t)x-x]$, and the $P_0,\ P_1$ are projections onto the finite dimensional subspaces $E_0,\ E_1$. Thus, $\varphi(t)$ is a compact map from $I \times \overline{\Omega}$ to $I \times E_f$. In view of (6.2), the Leray-Schauder degree i satisfies

$$i(\gamma(t), \Omega, p) = i(\gamma(0), \Omega, p) = 1$$

for all $t \in [0,1]$. But this contradicts (6.3). Hence

$$\sigma(t)\Omega \cap F^{-1}(p) \neq \phi, \quad 0 \leq t \leq 1.$$

□

Definition 6.5. Let N be a closed separable subspace of a Hilbert space E. A C' functional $G(u)$ on E will be called an N-weak-to-weak continuously differentiable functional on E if

$$|v_n - v|_w \to 0 \tag{6.6}$$

implies

$$|G'(v_n) - G'(v)|_w \to 0. \tag{6.7}$$

Thus,

$$v_n = Pu_n \to v \text{ weakly in } E, \quad w_n = (I-P)u_n \to w \text{ strongly in } E \tag{6.8}$$

implies

$$G'(v_n + w_n) \to G'(v+w) \text{ weakly in } E, \tag{6.9}$$

where P is the projection of E onto N.

Note that every C' functional is N-weak-to-weak continuously differentiable when $\dim N < \infty$.

Our counterpart to Theorem 2.1 for infinite dimensional subspaces is:

Theorem 6.6. *Let N be a closed separable subspace of a Hilbert space E, and let Ω be a bounded, convex, open subset of N containing a point p. Let G be an N-weak-to-weak continuously differentiable functional on E. Let F be an N-weakly continuous mapping. Assume $d = d(A,B) > 0$, and*

$$a_0 := \sup_A G \leq b_0 := \inf_B G \leq b_1 := \sup_{\overline{\Omega}} G < \infty,$$

where $A = \partial\Omega$ *and* $B = F^{-1}(p)$. *Then for each* $\rho \in \mathcal{Q}$ *and* $\beta > 0$ *satisfying*

$$\beta \int_0^d \frac{dt}{\rho(t)} > b_1 - b_0, \tag{6.10}$$

there is a sequence $\{u_k\} \subset E$ *such that*

$$G(u_k) \to c, \ b_0 \le c \le b_1, \quad \rho(d_w(u_k, B))\|G'(u_k)\| \le \beta. \tag{6.11}$$

Proof. If the theorem were false, then there would be a $\delta > 0$ such that

$$\rho(d_w(u, B))\|G'(u)\| > \beta \tag{6.12}$$

when

$$u \in U = \{u \in E : b_0 - 3\delta \le G(u) \le b_1 + 3\delta\}. \tag{6.13}$$

For $u \in \hat{E} = \{u \in E : G'(u) \neq 0\}$, let $h(u) = G'(u)/\|G'(u)\|$. Then by (6.12)

$$(G'(u), h(u)) > \beta/\rho(d_w(u, B)), \quad u \in U. \tag{6.14}$$

For each $u \in U$ there is an $\tilde{E} = E_w$ neighborhood $\mathcal{W}(u)$ of u such that

$$(G'(v), h(u)) > \beta/\rho(d_w(v, B)), \quad v \in \mathcal{W}(u) \cap U. \tag{6.15}$$

For otherwise there would be a sequence $\{v_k\} \subset U$ such that

$$|v_k - u|_w \to 0 \text{ and } (G'(v_k), h(u)) \le \beta/\rho(d_w(v_k, B)). \tag{6.16}$$

$$(G'(v_k), h(u)) \to (G'(u), h(u)) \le \beta/\rho(d_w(u, B)), \tag{6.17}$$

by (6.7) in view of (6.16). This contradicts (6.14). Thus (6.15) holds.

Let $\tilde{U}$ be the set U with the inherited topology of $\tilde{E}$. It is a metric space, and $\mathcal{W}(u) \cap \tilde{U}$ is an open set in this space. Thus, $\{\mathcal{W}(u) \cap \tilde{U}\}, u \in \tilde{U}$, is an open covering of the paracompact space $\tilde{U}$ (cf., e.g., [8]). Consequently, there is a locally finite refinement $\{\mathcal{W}_\tau\}$ of this cover. For each τ there is an element u_τ such that $\mathcal{W}_\tau \subset \mathcal{W}(u_\tau)$. Let $\{\psi_\tau\}$ be a partition of unity subordinate to this covering. Each ψ_τ is locally Lipschitz continuous with respect to the norm $|u|_w$ and consequently with respect to the norm of E. Let

$$Y(u) = \sum \psi_\tau(u) h(u_\tau), \quad u \in \tilde{U}. \tag{6.18}$$

Then $Y(u)$ is locally Lipschitz continuous with respect to both norms. Moreover,

$$\|Y(u)\| \le \sum \psi_\tau(u)\|h(u_\tau)\| \le 1 \tag{6.19}$$

and

$$(G'(u), Y(u)) = \sum \psi_\tau(u)(G'(u), h(u_\tau)) \ge \beta/\rho(d_w(u, B)), \quad u \in \tilde{U}. \tag{6.20}$$

Reduce δ to satisfy

$$\beta \int_\delta^d \frac{dt}{\rho(t)} \ge b_1 - b_0 + \delta.$$

Let

$$\begin{aligned} Q_0 &= \{u \in E : b_0 - 2\delta \le G(u) \le b_1 + 2\delta\}, \\ Q_1 &= \{u \in E : b_0 - \delta \le G(u) \le b_1 + \delta\}, \\ Q_2 &= E \setminus Q_0, \\ \eta(u) &= d_w(u, Q_2)/[d_w(u, Q_1) + d_w(u, Q_2)]. \end{aligned}$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous (with respect to the E_w norm) on E and satisfies

$$\begin{cases} \eta(u) = 1, & u \in Q_1, \\ \eta(u) = 0, & u \in \bar{Q}_2, \\ \eta(u) \in (0,1), & \text{otherwise}. \end{cases} \tag{6.21}$$

Let

$$\tilde{W}(u) = -\eta(u) Y(u) \rho(d_w(u, B)).$$

Then

$$\|\tilde{W}(u)\| \le \rho(d_w(u, B)) \le \rho(d(u, B)), \quad u \in \tilde{U}.$$

Then, for each $v \in U$ there is a unique solution $\sigma(t)v$ of

$$\sigma'(t) = \tilde{W}(\sigma(t)),\ t \in \mathbb{R}^+, \quad \sigma(0) = v. \tag{6.22}$$

Take

$$T = \int_\delta^d \frac{dt}{\rho(t)} \ge (b_1 - b_0 + \delta)/\beta. \tag{6.23}$$

Let

$$K = \{(u,t) : u = \sigma(t)v,\ v \in \bar{\Omega},\ t \in [0,T]\}.$$

Then K is a compact subset of $\tilde{E} \times \mathbb{R}$. To see this, let (u_k, t_k) be any sequence in K. Then $u_k = \sigma(t_k)v_k$, where $v_k \in \bar{\Omega}$. Since Ω is bounded, there is a subsequence such that $v_k \to v_0$ weakly in E and $t_k \to t_0$ in $[0,T]$. Since $\bar{\Omega}$ is convex and bounded, v_0 is in $\bar{\Omega}$ and $|v_k - v_0|_w \to 0$. Since $\sigma(t)$ is continuous in $\tilde{E} \times \mathbb{R}$, we have

$$u_k = \sigma(t_k)v_k \rightharpoonup \sigma(t_0)v_0 \in K.$$

Each $u_0 \in U$ has a neighborhood $\mathcal{W}(u_0)$ in $\tilde{E}$ and a finite dimensional subspace $S(u_0)$ such that $Y(u) \subset S(u_0)$ for $u \in \mathcal{W}(u_0) \cap U$. Since $\sigma(t)u$ is continuous in $\tilde{E} \times \mathbb{R}$, for each $(u_0, t_0) \in K$ there is a neighborhood $\mathcal{W}(u_0, t_0) \subset \tilde{E} \times \mathbb{R}$ and a finite dimensional subspace $S(u_0, t_0) \subset E$ such that $z_t(u) \subset S(u_0, t_0)$ for $(u,t) \in \mathcal{W}(u_0, t_0)$, where

$$z_t(u) := u - \sigma(t)u = \begin{cases} \int_0^t Y(\sigma(s)u)\rho(d_w(\sigma(s), B))ds, & u \in U, \\ 0, & u \notin U. \end{cases} \tag{6.24}$$

Since K is compact, there is a finite number of points $(u_j, t_j) \subset K$ such that $K \subset \mathcal{W} = \cup \mathcal{W}(u_j, t_j)$. Let S be a finite dimensional subspace of E containing p and all the $S(u_j, t_j)$ and such that $FS \ne \{0\}$. Then for $v \in \bar{\Omega}$ and $t \in [0,T]$ we have $z_t(v) \in S$. Thus $\sigma \in \Sigma_w(\overline{\Omega})$.

We also have

$$\begin{aligned} dG(\sigma(t)v)/dt &= -\eta(\sigma(t)v)(G'(\sigma(t)v), Y(\sigma(t)v))\rho(d_w(\sigma(t)v, B)) \\ &\leq -\beta\eta(\sigma). \end{aligned}$$

Let $v \in A$. If there is a $t_1 \leq T$ such that $\sigma(t_1)v \notin Q_1$, then

$$G(\sigma(T)v) \leq G(\sigma(t_1)v) \leq b_0 - \delta. \tag{6.25}$$

On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0, T]$, then we have by (6.25)

$$G(\sigma(T)v) \leq b_1 - \beta T \leq b_0 - \delta.$$

Hence

$$G(\sigma(T)v) \leq b_0 - \delta, \quad v \in A. \tag{6.26}$$

Let $u(t)$ be the solution of

$$u'(t) = -\rho(u(t)),\ t \in [0, T], \quad u(0) = d = d(A, B).$$

Then,

$$d(\sigma(t)v, B) \geq u(t),\ t \in [0, T],\ v \in A.$$

But

$$\int_{u(t)}^{d} \frac{d\tau}{\rho(\tau)} = t, \quad t \in [0, T].$$

Consequently,

$$u(t) \geq u(T) \geq \delta, \quad t \in [0, T],$$

since

$$T = \int_{\delta}^{d} \frac{dt}{\rho(t)} \geq (b_0 - a_0 + \delta)/\beta.$$

Thus,

$$d(\sigma(t)v, B) \geq \delta, \quad t \in [0, T],\ v \in A.$$

Consequently, $\sigma(t)v \cap B = \phi,\ t \in (0, T]$. This means that

$$\sigma(t)v \cap B = \phi, \quad v \in A,\ t \in (0, T].$$

Hence,

$$\sigma(t)A \cap B = \phi, \quad t \in (0, T],$$

and

$$\sup_{\sigma(T)A} G \leq b_0 - \delta.$$

But $\sigma \in \Sigma_w(\overline{\Omega})$. By Theorem 6.4, this implies

$$\sigma(t)\Omega \cap B \neq \phi, \quad 0 < t \leq T.$$

Thus, there is a $u \in \Omega$ such that $\sigma(T)u \in B$. But that would mean that $G(\sigma(T)u) \geq b_0$, contradicting (6.26). This completes the proof. □

Theorem 6.7. *Let N be a closed, separable subspace of a Hilbert space E with orthogonal complement $M \oplus Y$, where Y is a finite dimensional subspace of E orthogonal to both M and N, and let $\delta < R_0$ be positive numbers. Let $y_1, \cdots, y_n$ be an orthogonal basis for Y and let*

$$Y_+ = \{y \in Y : (y, y_k) \geq 0, \quad 1 \leq k \leq n\},$$

$$\Omega_R = \{v + y : v \in N,\ y \in Y_+,\ \|v + y\| < R\}, \quad R > R_0,$$

and

$$F(v + w + y) = v + \frac{\|w + y\|}{\|y\|} \sum_1^n |(y, y_k)| y_k, \quad v \in N,\ y \in Y,\ w \in M. \tag{6.27}$$

Let G be a an N-weak-to-weak continuously differentiable functional on E and assume

$$-\infty < \sup_{A_R} G \leq b_0 = \inf_B G \leq \sup_{\overline{\Omega}_R} G \leq b_1 < \infty, \quad R > R_0, \tag{6.28}$$

holds with $A_R = \partial\Omega$ and $B = F^{-1}(Y_+ \cap \partial\mathcal{B}_\delta) = \{w + y \in M \oplus Y : \|w + y\| = \delta\}$. Then for each sequence $\nu_k \to \infty$ there is a $\beta > 0$ and a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c,\ b_0 \leq c \leq b_1, \quad (\nu_k + |u_k|_w)\|G'(u_k)\| \leq \beta. \tag{6.29}$$

Proof. If $y \in Y \backslash \{0\}$, let

$$\tilde{y} = \frac{1}{\|y\|} \sum_1^n |(y, y_k)| y_k.$$

Then $\|\tilde{y}\| = 1$, $\tilde{y} \in Y_+ \cap \partial\mathcal{B}_1$, and

$$F(v + w + y) = v + \|w + y\| \cdot \tilde{y}.$$

Consequently,

$$F^{-1}(\delta\tilde{y}) = \{w + y : w \in M,\ y \in Y, \|w + y\| = \delta\}.$$

Thus, if $z \in Y_+ \cap \partial\mathcal{B}_1$, then

$$F^{-1}(\delta z) = \{w \in M,\ y \in Y : \|w + y\| = \delta,\ \tilde{y} = z\}$$

and

$$\begin{aligned} F^{-1}(Y_+ \cap \partial\mathcal{B}_\delta) &= \{w \in M,\ y \in Y : \|w + y\| = \delta,\ \tilde{y} \in Y_+ \cap \partial\mathcal{B}_1\} \\ &= \{w \in M,\ y \in Y : \|w + y\| = \delta\}. \end{aligned}$$

Apply Theorem 6.6.

□

Definition 6.8. We shall say that a set $A \subset E$ links a set $B \subset E$ **weakly** if $d = d(A, B) > 0$ and whenever

$$a_0 := \sup_A G \leq b_0 := \inf_B G \leq b_1 := \sup_{\overline{\Omega}} G < \infty$$

holds for some N-weak-to-weak continuously differentiable functional G on E, then for each $\rho \in \mathcal{Q}$ and $\beta > 0$ satisfying

$$\beta \int_0^d \frac{dt}{\rho(t)} > b_1 - b_0, \tag{6.30}$$

there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c,\ b_0 \le c \le b_1, \quad \rho(d_w(u_k, B))\|G'(u_k)\| \le \beta. \tag{6.31}$$

Corollary 6.9. *Let N be a closed, separable subspace of a Hilbert space E with orthogonal complement $M \oplus Y$, where Y is a finite dimensional subspace of E orthogonal to both M and N, and let $\delta < R$ be positive numbers. Let $y_1, \cdots, y_n$ be an orthogonal basis for Y and let*

$$Y_+ = \{y \in Y : (y, y_k) \ge 0, \quad 1 \le k \le n\},$$

$$\Omega = \{v + y : v \in N,\ y \in Y_+,\ \|v + y\| < R\},$$

and

$$F(v + w + y) = v + \frac{\|w + y\|}{\|y\|} \sum_1^n |(y, y_k)| y_k, \quad v \in N,\ y \in Y,\ w \in M. \tag{6.32}$$

Then F is N-weakly continuous and $A_R = \partial\Omega$ links $B = F^{-1}(Y_+ \cap \partial\mathcal{B}_\delta) = \{w + y \in M \oplus Y : \|w + y\| = \delta\}$ weakly.

Remark 6.10. It follows from Theorem 6.7 that Examples 2 - 8 produce weakly linking sets when the subspace N is separable but not finite dimensional.

7. Applications

We consider semilinear partial differential equations of the form

$$\mathcal{A}u = f(x, u), u \in D \tag{7.1}$$

in unbounded domains. Included is the case of the Schrödinger operator $\mathcal{A} = -\Delta + \mathcal{V}(x)$ on $D = H^1(\mathbb{R}^n)$, where $\mathcal{V}(x)$ is a given potential. One wishes to find nontrivial solutions and, in particular, the so called "minimizing solutions." These are solutions that minimize the corresponding energy functional. If they are not trivial, they are called "ground state solutions."

The existence of solutions depends both on the linear operator $\mathcal{A}$ and the nonlinear term $f(x, u)$. We shall study the problem for the case when $\mathcal{A}$ is selfadjoint, having a nonempty resolvent set, and $f(x, u)$ is superlinear. The results are stated in the next section and proved in Sections 9 and 10.

8. Superlinear problems

Let $\Omega \subset \mathbb{R}^n$ be an open set and $\mathcal{A}$ a selfadjoint operator on $L^2(\Omega)$. We assume that $\sigma_e(\mathcal{A})$ is not the whole of $\mathbb{R}$. (The essential spectrum $\sigma_e(\mathcal{A})$ of a selfadjoint operator $\mathcal{A}$ consists of those points of the spectrum that are not isolated eigenvalues of finite multiplicity.) For convenience, we assume there is an interval $[0, b]$ satisfying $[0, b] \cap \sigma_e(\mathcal{A}) = \phi$, but $[a, b] \cap \sigma(\mathcal{A}) \neq \phi$, where $0 < a < b$. We let $D = D(|\mathcal{A}|^{(1/2)})$. With the scalar product $(u, v)_D = (|\mathcal{A}|^{(1/2)}u, |\mathcal{A}|^{(1/2)}v)$, it becomes a Hilbert space. We let

$$N = E(-\infty, 0),\ M = E(b, \infty),\ Y = E[a, b]$$

be orthogonal invariant subspaces of $\mathcal{A}$ with $D = N \oplus Y \oplus M$. Hence,

$$(\mathcal{A}v, v) \leq 0, \quad v \in N,$$

$$(\mathcal{A}w, w) \geq b\|w\|^2, \quad w \in M,$$

and

$$a\|y\|^2 \leq (\mathcal{A}y, y) \leq b\|y\|^2, \quad y \in Y.$$

We assume that $C_0^\infty(\Omega) \subset D \subset H^{m,2}(\Omega)$ for some $m > 0$. In particular,

$$\|u\|_{m,2} \leq C\|u\|_D, \quad u \in D. \tag{8.1}$$

Let q be a number satisfying

$$\begin{array}{lllll} 2 & < & q & < 2^* := 2n/(n-2m), & 2m < n \\ 2 & < & q & < \infty, & n \leq 2m. \end{array}$$

We assume that D is compact in $L^q_{loc}(\Omega)$ and

$$\|u\|_q \leq C\|u\|_D, \quad u \in D, \tag{8.2}$$

where $\|\cdot\|_q$ is the norm of $L^q(\Omega)$. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|f(x, t)| \leq V(x)^2(|t| + 1), \quad x \in \Omega,\ |t| \geq \delta, \tag{8.3}$$

and

$$|f(x, t)| \leq \sigma|t|, \quad |t| < \delta,\ x \in \Omega,\ t \in \mathbb{R}, \tag{8.4}$$

for some $\sigma < a,\ \delta > 0$, where $V(x) > 0$ is a function in $L^2(\Omega)$ such that

$$\|Vu\| \leq C\|u\|_D, \quad u \in D$$

and multiplication by $V(x)$ is a compact operator from D to $L^2(\Omega)$. Assume that

$$F(x, t) := \int_0^t f(x, s)\, ds$$

satisfies

$$F(x, t) \geq 0, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{8.5}$$

and

$$F(x, t)/t^2 \to \infty \text{ as } t^2 \to \infty. \tag{8.6}$$

We shall prove:

Theorem 8.1. *Under the above hypotheses there is a nontrivial solution of*

$$\mathcal{A}u = f(x,u), u \in D. \tag{8.7}$$

Theorem 8.2. *Assume, in addition, that*

$$H(x,t) := tf(x,t) - 2F(x,t) \geq -W(x) \in L^1(\Omega). \tag{8.8}$$

Let $\mathcal{M}$ be the collection of solutions of (8.7). Then there is a nontrivial solution that minimizes the energy functional

$$G(u) = (\mathcal{A}u, u) - 2\int_\Omega F(x,u), \quad u \in D \tag{8.9}$$

over $\mathcal{M}\backslash\{0\}$.

Remark 8.3. A nontrivial solution that minimizes the energy functional is called a **ground state solution.**

9. Some lemmas

Before proving our main theorems (Theorem 8.1 and Theorem 8.2), we shall prove a few lemmas. We define

$$G(u) = (\mathcal{A}u, u) - 2\int_\Omega F(x,u), \quad u \in D, \tag{9.1}$$

where we write $u = v + y + w,\ v \in N,\ y \in Y,\ w \in M$.

Lemma 9.1. *Let $r > 0$ and $q \in [2, 2^*)$, where $2^* = 2n/(n-2)$. If $\{u_k\}$ is a bounded sequence in $E := H^1(\mathbb{R}^n)$, and*

$$\sup_{y\in\mathbb{R}^n}\int_{B(y,r)} |u_k|^q dx \to 0, \quad k \to \infty, \tag{9.2}$$

where $B(y,r) := \{u \in E : \|u - y\| \leq r\}$, then $u_k \to 0$ in $L^p(\mathbb{R}^n)$ for $q < p < 2^*$.

Proof. We consider $n \geq 3$ and make use of the fact that

$$\int_{B(y,r)} |u(x)|^q dx \leq C\Big(\int_{B(y,r)} (u^2 + |\nabla u|^2)dx\Big)^{q/2}, \quad 2 \leq q \leq 2^*,\ u \in H^1(\mathbb{R}^n).$$

Choose

$$p_1 = q\frac{2^*-p}{2^*-q},\ p_2 = 2^*\frac{p-q}{2^*-q},\ t = \frac{2^*-q}{2^*-p} > 1,\ t' = \frac{t}{t-1} > 1.$$

Then $p_1 t = q,\ p_2 t' = 2^*,\ 1/t + 1/t' = 1,\ p_1 + p_2 = p$. By Hölder's Inequality, we have

$$\begin{aligned}
&\int_{B(y,r)} |u_k|^p dx \\
&\leq \Big(\int_{B(y,r)} |u_k|^{p_1 t} dx\Big)^{1/t}\Big(\int_{B(y,r)} |u_k|^{p_2 t'} dx\Big)^{1/t'} \\
&\leq c\Big(\int_{B(y,r)} |u_k|^q dx\Big)^{1/t}\Big(\int_{B(y,r)} |u_k|^{2^*} dx\Big)^{1/t'} \\
&\leq c\Big(\int_{B(y,r)} |u_k|^q dx\Big)^{1/t}\Big(\int_{B(y,r)} (u_k^2 + |\nabla u_k|^2)dx\Big)^{p_2/2}.
\end{aligned}$$

Covering $\mathbb{R}^n$ by balls with radius r in such a way that each point of $\mathbb{R}^n$ is contained in at most $n+1$ balls, we have

$$\int_{\mathbb{R}^n} |u_k|^p dx \leq (n+1)c \sup_{y\in\mathbb{R}^n} \Big(\int_{B(y,r)} |u_k|^q dx \Big)^{1/t},$$

which implies the conclusion of the lemma. □

Lemma 9.2. *Assume that* $\rho_k = \|u_k\|_D \to \infty$ *and* $\tilde{u}_k = u_k/\rho_k \to \tilde{u}$ *a.e. If* $\tilde{u} \not\equiv 0$, *then*

$$\int_\Omega F(x,u_k)/\rho_k^2 \to \infty. \tag{9.3}$$

Proof. Let Ω_0 be the subset of Ω where $\tilde{u} \neq 0$. If the measure of Ω_0 is positive, then

$$\int_\Omega F(x,u_k)/\rho_k^2 \geq \int_{\Omega_0} \frac{F(x,u_k)}{u_k^2} \tilde{u}_k^2 \to \infty,$$

since the integrand is bounded below and $u_k^2 \to \infty$ on Ω_0. □

Lemma 9.3.

$$v_k = Pu_k \to v \text{ weakly in } D, \quad g_k = (I-P)u_k \to g \text{ strongly in } D \tag{9.4}$$

implies

$$G'(v_k + g_k) \to G'(v+g) \text{ weakly in } D, \tag{9.5}$$

where P *is the projection of* D *onto* N.

Proof. Since the u_k are bounded in D, there is a renamed subsequence converging to a limit u weakly in D, $Vu_k \to Vu$ in $L^2(\Omega)$ and a.e. in Ω. Let $\varepsilon > 0$ and $h \in D$ be given. Then $f(x,u_k)h(x)$ converges to $f(x,u)h(x)$ a.e. and is dominated by $(|Vu_k| + V)|Vh|$ which converges to $(|Vu| + V)|Vh|$ in $L^1(\Omega)$, we have

$$\int_\Omega f(x,u_k)h(x)dx \to \int_\Omega f(x,u)h(x)dx \text{ as } k \to \infty.$$

Thus,

$$\begin{aligned}(G'(u_k),h)/2 &= (\mathcal{A}u_k,h) - \int_\Omega f(x,u_k(x))h(x) \\ &\to (\mathcal{A}u,h) - \int_\Omega f(x,u(x))h(x) \\ &= (G'(u),h)/2.\end{aligned}$$

This gives (9.5). □

Lemma 9.4. *For each* $\rho > 0$ *sufficiently small there is an* $\varepsilon > 0$ *such that*

$$G(h) \geq \varepsilon, \quad h = y + w \in Y \oplus M,\ \|h\|_D = \rho. \tag{9.6}$$

Proof. By (8.3) and (16.13),

$$2\int_{|h|<\delta} |F(x,h)| \leq \sigma \int_{|h|<\delta} h^2$$

and

$$\int_{|h|>\delta} |F(x,h)| \le C\int_{|h|>\delta}(|h|^2+|h|)$$
$$\le C\int_{|h|>\delta}(|h|^q/\delta^{q-2}+|h|^q/\delta^{q-1})$$
$$\le C\int_{|h|>\delta}|h|^q$$
$$\le C\|h\|_q^q.$$

Consequently,

$$G(h) \ge \|h\|_D^2 - \sigma\int_{|h|<\delta} h^2 - C\int_{|h|>\delta}(|h|^q+|h|) \ge (1-a^{-1}\sigma - C'\|h\|_q^{q-2})\|h\|_D^2.$$

We take $\|h\|_D^2$ sufficiently small. □

Lemma 9.5. *Let*

(9.7) $$Q_R = \{v+y : v\in N,\ y\in Y_+ : \|v+y\|_D \le R\}.$$

Then there is an $R>0$ such that

(9.8) $$G(u)\le 0, \quad u\in \partial Q_R.$$

Proof. If not, $\exists R_k\to\infty,\ u_k = v_k+y_k \in \partial Q_{R_k}$, such that $G(u_k)>0$. If $y_k=0$, then

$$G(v_k) = -\|v_k\|_D^2 - 2\int_\Omega F(x,v_k) \le -\|v_k\|_D^2 \le 0.$$

Hence, $y_k\ne 0$ and

$$\|v_k\|_D^2 + \|y_k\|_D^2 = R_k^2.$$

Let $\tilde u_k = u_k/R_k = \tilde v_k + \tilde y_k$. Then

$$\|\tilde v_k\|_D^2 + \|\tilde y_k\|_D^2 = 1.$$

Since $\dim Y<\infty$, there are renamed subsequences such that $\tilde y_k\to\tilde y$ in D and $\tilde u_k = u_k/R_k = \tilde v_k+\tilde y_k\to\tilde u$ *a.e.* Since,

$$0 < G(u_k)/R_k^2 \le \|\tilde y_k\|_D^2 - \|\tilde v_k\|_D^2 - 2\int_\Omega F(x,u_k)/R_k^2,$$

we have by hypothesis

$$\|\tilde y_k\|_D^2 - (1-\|\tilde y_k\|_D^2)\ge 0,$$

or

$$\|\tilde y_k\|_D^2 \ge \frac{1}{2}.$$

Thus, $\tilde u\not\equiv 0$. Lemma 9.2 implies

(9.9) $$\int_\Omega F(x,u_k)/R_k^2 \to\infty.$$

Since,

$$0 < G(u_k)/R_k^2 = \|\tilde y_k\|_D^2 - \|\tilde v_k\|_D^2 - 2\int_\Omega F(x,u_k)/R_k^2 \to -\infty,$$

this produces a contradiction, and the lemma follows. □

Lemma 9.6. *For any $R > 0$*

$$b_1 = \sup_{Q_R} G < \infty. \tag{9.10}$$

Proof. If not, there is a sequence $u_k = v_k + y_k \in Q_R$, such that $G(u_k) \to \infty$. Consequently,

$$\|y_k\|_D^2 - \|v_k\|_D^2 - 2\int_\Omega F(x, u_k) \geq G(u_k) \to \infty.$$

Thus $\|y_k\|_D^2 \to \infty$. Since the sequence u_k is bounded, there are subsequences such that $y_k \to y$ in Y and $u_k = v_k + y_k \to u$ *a.e.* This provides a contradiction. □

Lemma 9.7. *If u_k, g_k are bounded sequences in D, then there are renamed subsequences such that $u_k \to u, g_k \to g$ a.e. and*

$$\int_\Omega f(x, u_k)g_k \to \int_\Omega f(x, u)g \tag{9.11}$$

and

$$\int_\Omega F(x, u_k) \to \int_\Omega F(x, u). \tag{9.12}$$

Proof. Since u_k, g_k are bounded in D, there are renamed subsequences for which they converge weakly and a.e. to limits u, g. On renamed subsequences $Vu_k \to Vu$ and $Vg_k \to Vg$ in $L^2(\Omega)$. Since $f(x, u_k)g_k \to f(x, u)g$ a.e., and it is dominated by $|(|Vu_k| + V)|Vg_k|$ which converges to $|(|Vu| + V)|Vg|$ in $L^1(\Omega)$, we see that (9.11) holds. The same argument applies to (9.12). □

10. Proofs of the theorems

Proof of Theorem 8.1. Let $0 < \rho < \delta < R$ be such that Lemmas 9.4 and 9.5 hold. Then

$$\sup_A G < \inf_B G,$$

where $A = \partial Q_R$ and $B = \{w \in M : \|w\|_D = \rho.\}$ We let Ω be the interior of Q_R. By Theorem 6.7, there is a sequence $\{u_k\} \subset D$ satisfying (6.29) with $c \geq b_0 > 0$. Let $\rho_k = \|u_k\|_D$, and assume that $\rho_k \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in D, and $V\tilde{u}_k \to V\tilde{u}$ in $L^2(\Omega)$ and a.e. This implies

$$\begin{aligned} 1 = &\|\tilde{u}_k\|_D^2 \leq [|(G'(u_k), v_k)|/2 + |(G'(u_k), w_k)|/2 + |(G'(u_k), y_k)|/2]/\rho_k^2 \\ &+ \int_\Omega |f(u_k)| \cdot (|w_k| + |y_k| + |v_k|)/\rho_k^2. \end{aligned}$$

Since $|f(u_k)| \cdot (|w_k| + |y_k| + |v_k|)/\rho_k^2$ is dominated by $|(|V\tilde{u}_k| + V\rho_k^{-1})(|V\tilde{w}_k| + |V\tilde{y}_k| + |V\tilde{v}_k|)$ which converges in $L^1(\Omega)$, we have in the limit

$$1 \leq \int_\Omega |V\tilde{u}|(|V\tilde{w}| + |V\tilde{y}| + |V\tilde{v}|).$$

This shows that $\tilde{u} \not\equiv 0$. Then by Lemma 9.2

$$G(u_k)/\rho_k^2 = \|\tilde{w}_k\|_D^2 + \|\tilde{y}_k\|_D^2 - \|\tilde{v}_k\|_D^2 - 2\int_\Omega F(x, u_k)/\rho_k^2 \to -\infty. \tag{10.1}$$

But this contradicts (6.29). Hence, the ρ_k are bounded.

Consequently, there is a renamed subsequence converging to a limit u weakly in D and a.e. in Ω. For any $\varphi \in C_0^\infty(\Omega)$, we have

$$(G'(u_k), \varphi)/2 = (\mathcal{A}u_k, \varphi) - \int_\Omega f(x, u_k(x))\varphi(x) \to 0.$$

Hence,

$$(G'(u), \varphi)/2 = (\mathcal{A}u, \varphi) - \int_\Omega f(x, u)\varphi(x) = 0,$$

showing that $G'(u) = 0$.

To show that $u \neq 0$, note that

$$G(u_k) = (\mathcal{A}u_k, u_k) - 2\int_\Omega F(x, u_k). \tag{10.2}$$

By Lemma 9.7,

$$(\mathcal{A}u_k, u_k) = (G'(u_k), u_k)/2 - \int_\Omega f(x, u_k)u_k(x) \to -\int_\Omega f(x, u)u(x) = (\mathcal{A}u, u).$$

Since

$$\int_\Omega F(x, u_k) \to \int_\Omega F(x, u), \tag{10.3}$$

we see that $G(u_k) \to G(u)$. But $G(u_k) \to c \geq b_0 > 0$. Hence, $G(u) > 0$. Since $G(0) = 0$, we see that $u \neq 0$. □

Proof of Theorem 8.2. By Theorem 8.1, $\mathcal{M}\backslash\{0\} \neq \phi$. Let

$$\gamma = \inf_{\mathcal{M}\backslash\{0\}} G.$$

We must show that $\gamma \neq -\infty$. Let $\{u_k\}$ be a sequence in $\mathcal{M}\backslash\{0\}$ such that

$$G(u_k) \to \gamma.$$

Thus

$$G(u_k) = \|w_k\|_D^2 + \|y_k\|_D^2 - \|v_k\|_D^2 - 2\int_\Omega F(x, u_k) \to \gamma. \tag{10.4}$$

Note that

$$\int_\Omega H(x, u_k(x)) = G(u_k) - (G'(u_k), u_k)/2 \to \gamma,$$

where

$$H(x, t) := tf(x, t) - 2F(x, t).$$

Also, $H(x, u_k(x)) \geq -W(x)$ *a.e.* by (8.8). Hence,

$$\gamma \geq -\int_\Omega W(x). \tag{10.5}$$

Since $u_k \in \mathcal{M}\backslash\{0\}$, we have

$$(G'(u_k), w_k)/2 = \|w_k\|_D^2 - \int_\Omega f(x, u_k(x))w_k(x) = 0,$$

$$(G'(u_k), y_k)/2 = \|y_k\|_D^2 - \int_\Omega f(x, u_k(x))y_k(x) = 0$$

and

$$(G'(u_k), v_k)/2 = -\|v_k\|_D^2 - \int_\Omega f(x, u_k(x))v_k(x) = 0.$$

Let $\rho_k = \|u_k\|_D$. Assume that $\rho_k \to \infty$. Let $\tilde{u}_k = u_k/\rho_k$. Then $\|\tilde{u}_k\|_D = 1$. Hence, there is a renamed subsequence such that $\tilde{u}_k \rightharpoonup \tilde{u}$ in D, and $V\tilde{u}_k \to V\tilde{u}$ in $L^2(\Omega)$ and a.e. This implies

$$1 = \|\tilde{u}_k\|_D^2 \le \int_\Omega |f(u_k)| \cdot (|w_k| + |y_k| + |v_k|)/\rho_k^2. \tag{10.6}$$

Since $|f(u_k)| \cdot (|w_k| + |y_k| + |v_k|)/\rho_k^2$ is dominated by $|(|V\tilde{u}_k| + V\rho_k^{-1})(|V\tilde{w}_k| + |V\tilde{y}_k| + |V\tilde{v}_k|)$, in the limit we have,

$$1 \le C\|V\tilde{u}\|^2.$$

This shows that $\tilde{u} \not\equiv 0$.

Then by Lemma 9.2

$$G(u_k)/\rho_k^2 = \|\tilde{w}_k\|_D^2 + \|\tilde{v}_k\|_D^2 - \|\tilde{v}_k\|_D^2 - 2\int_\Omega F(x, u_k)/\rho_k^2 \to -\infty. \tag{10.7}$$

But this contradicts (10.5). Hence, the ρ_k are bounded.

Consequently, there is a renamed subsequence converging to a limit u weakly in D and a.e. in Ω. For any $\varphi \in C_0^\infty(\Omega)$, we have

$$(G'(u_k), \varphi)/2 = (w_k, \varphi)_D + (y_k, \varphi)_D - (v_k, \varphi)_D - \int_\Omega f(x, u_k(x))\varphi(x) \to 0.$$

Hence,

$$(G'(u), \varphi)/2 = (w, \varphi)_D + (y, \varphi)_D - (v, \varphi)_D - \int_\Omega f(x, u)\varphi(x) = 0,$$

showing that $G'(u) = 0$. Thus $u \in \mathcal{M}$.

I claim that $u \neq 0$. To see this, note that

$$(G'(u_k), \hat{u}_k)/2 = (w_k, \hat{u}_k)_D + (y_k, \hat{u}_k)_D - (v_k, \hat{u}_k)_D - \int_\Omega f(x, u_k(x))\hat{u}_k(x) = 0,$$

where $\hat{u}_k = w_k - v_k$. Thus,

$$\begin{aligned}\|w_k\|_D^2 + \|y_k\|_D^2 + \|v_k\|_D^2 &= \int_\Omega f(x, u_k)\hat{u}_k = \int_{|u_k|<\delta} + \int_{|u_k|>\delta} \\ &\le \sigma\|u_k\| \cdot \|\hat{u}_k\| + C\|u_k\|_q^{q-1}\|\hat{u}_k\|_q.\end{aligned}$$

Hence,

$$\varepsilon\|u_k\|_D^2 \le \|u_k\|_D^2 - \sigma\|u_k\|^2 \le C'\|u_k\|_q^{q-1}\|\hat{u}_k\|_D$$

for some $\varepsilon > 0$. Since $u_k \neq 0$, this shows that $\|u_k\|_q \ge c > 0$. By Corollary 9.1, (9.2) cannot hold. Hence, there is a $B(z, r)$ such that

$$\int_{B(z,r)} |u_k|^q dx \ge \alpha > 0,$$

showing that $u \neq 0$ in $B(z,r)$. From this we imply that $u \not\equiv 0$.

To show that $G(u) = \gamma$, note that

$$G(u_k) = \|w_k\|_D^2 + \|y_k\|_D^2 - \|v_k\|_D^2 - 2\int_\Omega F(x,u_k). \tag{10.8}$$

$$\|w_k\|_D^2 = (G'(u_k), w_k)/2 - \int_\Omega f(x,u_k)w_k(x) \to -\int_\Omega f(x,u)w(x) = \|w\|_D^2,$$

$$\|y_k\|_D^2 = (G'(u_k), y_k)/2 - \int_\Omega f(x,u_k)y_k(x) \to -\int_\Omega f(x,u)y(x) = \|y\|_D^2,$$

and

$$\|v_k\|_D^2 = -(G'(u_k), v_k)/2 + \int_\Omega f(x,u_k)v_k(x) \to \int_\Omega f(x,u)v(x) = \|v\|_D^2.$$

Since

$$\int_\Omega F(x,u_k) \to \int_\Omega F(x,u), \tag{10.9}$$

we see that $G(u_k) \to G(u)$. But $G(u_k) \to \gamma$. Hence, $G(u) = \gamma$. Thus, u is a ground state solution. □

11. The semilinear wave equation

In this section we study periodic solutions of the Dirichlet problem for the semilinear wave equation:

$$\Box u := u_{tt} - u_{rr} = p(t,r,u), \quad t \in \mathbb{R}, \quad 0 < r < R, \tag{11.1}$$

$$u(t,R) = u(t,0) = 0, \quad t \in \mathbb{R}, \tag{11.2}$$

$$u(t+T,r) = u(t,r), \quad t \in \mathbb{R}, \quad 0 \le r \le R. \tag{11.3}$$

Our basic assumption is that the ratio R/T is rational. Thus, we can write

$$2R/T = a/b, \tag{11.4}$$

where a, b are relatively prime positive integers. We also assume

$$|p(t,r,s)| \le C(|s|+1), \quad |s| > \delta \tag{11.5}$$

and

$$|p(r,t,s)| \le \sigma|s|, \quad |s| < \delta, \tag{11.6}$$

for some $\sigma < \alpha = \frac{\pi^2}{R^2 b^2}$ and $\delta > 0$. We have

Theorem 11.1. *Under assumptions (11.4) - (11.6), the operator □ has a selfadjoint extension L having discrete spectrum except for the point* 0. *Assume that*

$$P(t,r,s) \ge 0, \tag{11.7}$$

where

$$P(t,r,s) = \int_0^s p(t,r,\sigma)\, d\sigma, \tag{11.8}$$

and

$$P(t,r,s)/s^2 \to \infty, \quad |s| \to \infty. \tag{11.9}$$

Then (11.1) - (11.3) has at least one nontrivial solution.

An important aspect of this theorem is that all rational values of R/T are allowed.

12. The spectrum of the linear operator

In considering problem (11.1)–(11.3), we shall need to calculate the spectrum of the linear operator $\Box$.

Theorem 12.1. *Consider the operator*

$$\Box u = u_{tt} - u_{rr} \tag{12.1}$$

applied to functions $u(t,r)$ *in* $C^\infty(\bar{Q})$ *satisfying*

$$u(t+T,r) = u(t,r), \quad t \in \mathbb{R}, \quad 0 \le r \le R \tag{12.2}$$

$$u(t,R) = u(t,0) = 0, \quad t \in \mathbb{R}, \tag{12.3}$$

where $Q = [0,T]\times[0,R]$. *Then* $\Box$ *is symmetric on* $L^2(Q)$. *Assume that* $2R/T = a/b$, *where* a, b *are relatively prime integers (i.e.,* $(a,b) = 1$*). Then* $\Box$ *has a selfadjoint extension having no essential spectrum other than* $\{0\}$.

Proof. If

$$\psi_{jk}(t,r) = \sin(j\pi r/R)e^{2\pi ikt/T}, \tag{12.4}$$

then

$$\Box\psi_{jk} = [(j\pi/R)^2 - (2\pi k/T)^2]\psi_{jk}. \tag{12.5}$$

Thus $\psi_{jk}(t,r)$ is an eigenfunction of $\Box$ with eigenvalue

$$\lambda_{jk} = (j\pi/R)^2 - (2\pi k/T)^2. \tag{12.6}$$

It is easily checked that the functions ψ_{jk}, when normalized, form a complete orthonormal sequence in $L^2(Q)$. We shall show that the corresponding eigenvalues (12.6) are not dense in $\mathbb{R}$. It will then follow that $\Box$ has a selfadjoint extension L with spectrum equal to the closure of the set $\{\lambda_{jk}\}$ (cf., e.g., [24]). Now

$$\lambda_{jk} = \frac{\pi^2}{R^2b^2}(bj - a|k|)(bj + a|k|). \tag{12.7}$$

Hence

$$|\lambda_{jk}| \ge \frac{\pi^2}{R^2b^2}|(bj + a|k|)| \tag{12.8}$$

when $bj \ne ak$, and

$$\lambda_{jk} = 0, \quad bj = ak. \tag{12.9}$$

Thus

$$\lim_{\substack{j,|k|\to\infty \\ bj\ne ak}} |\lambda_{jk}| = \infty \tag{12.10}$$

and

$$\lim_{\substack{j,|k|\to\infty \\ bj=ak}} |\lambda_{jk}| = 0. \tag{12.11}$$

Hence, the point 0 is the only limit point of eigenvalues. Consequently, it is in $\sigma_e(L)$. This completes the proof. □

Proof of Theorem 11.1. We apply Theorem 8.1. Let β be a number greater than α. Then $L = \Box$ has no essential spectrum in the interval $(0, \beta)$, but it has spectrum in that interval. By Theorem 8.1 there is a nontrivial solution of

$$Lu(t, r) = p(t, r, u).$$

This is precisely what we want.

Theorem 12.2. *Assume, in addition, that*

$$H(r, t, s) = sp(r, t, s) - 2P(r, t, s) \geq -W(r, t), \quad (r, t) \in Q,\ s \in \mathbb{R}, \tag{12.12}$$

where $W(r, t) \in L^1(Q)$. Let $\mathcal{M}$ be the collection of solutions of (11.1). Then there is a nontrivial solution that minimizes the energy functional

$$G(u) = (Lu, u) - 2\int_Q P(r, t, u), \quad u \in D \tag{12.13}$$

over $\mathcal{M}\backslash\{0\}$.

Such solutions are called *ground state solutions.*

Proof. This follows from Theorem 8.2. □

13. Radially symmetric wave equations

In this section we study radially symmetric periodic solutions of the Dirichlet problem for the semilinear wave equation

$$\Box u := u_{tt} - \Delta u = f(t, x, u), \quad t \in \mathbb{R}, \quad x \in \mathcal{B}_R, \tag{13.1}$$

$$u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial\mathcal{B}_R, \tag{13.2}$$

$$u(t + T, x) = u(t, x), \quad t \in \mathbb{R}, \quad x \in \mathcal{B}_R, \tag{13.3}$$

where

$$\mathcal{B}_R = \{x \in \mathbb{R}^n : |x| < R\}. \tag{13.4}$$

In this case we have

$$f(t, x, u) = f(t, |x|, u), \quad x \in \mathcal{B}_R.$$

Our basic assumption is that the ratio R/T is rational. Thus, we can write

$$8R/T = a/b, \tag{13.5}$$

where a, b are relatively prime positive integers. We show that

$$n \not\equiv 3 \pmod{(4, a)} \tag{13.6}$$

implies that the linear problem corresponding to (13.1) – (13.3) has no essential spectrum. If

$$n \equiv 3 \qquad (\mathrm{mod}\ (4, a)) \tag{13.7}$$

then the essential spectrum of the linear operator consists of precisely one point λ_0, where

$$\lambda_0 = -(n-3)(n-1)/4R^2. \tag{13.8}$$

This shows that the spectrum has at most one limit point.

Let q be any number satisfying

$$\begin{aligned} 2 &< q \le 2^* = 2n/(n-2), \quad n > 2 \\ 2 &< q < \infty, \qquad\qquad\quad n \le 2 \end{aligned}$$

and let $u(r, t)$ be a Carathéodory function on $\mathbb{R} \times \mathbb{R}$. By the Sobolev inequality,

$$\|u\|_q \le C\|u\|_H, \quad u \in H,$$

where

$$\|u\|_q := (\int_{\mathbb{R}^n} |u(|x|)|^q dx)^{1/q} = (c_n \int_{\mathbb{R}} |u(r)|^q r^{n-1} dr)^{1/q}, \ \|u\| = \|u\|_2.$$

We consider the nonlinear case for $f(t, r, s)$ satisfying

$$|f(t, r, s)| \le C(|s|^{q-1} + 1), \quad |s| > \delta, \, r = |x|, \tag{13.9}$$

and

$$|f(r, t, s)| \le \sigma|s|, \quad |s| < \delta, \tag{13.10}$$

for some $\sigma < \alpha =$ smallest positive eigenvalue and $\delta > 0$. We have

Theorem 13.1. *Under assumptions (11.2) - (11.4), the operator* $\Box$ *has a selfadjoint extension* L *having discrete spectrum except for the point* λ_0, *where*

$$\lambda_0 = -(n-3)(n-1)/4R^2$$

when $n \equiv 3$*(mod* $(4, a)$*). Assume that*

$$F(t, r, s) \ge 0, \tag{13.11}$$

where

$$F(t, r, s) = \int_0^s f(t, r, \sigma)\, d\sigma, \tag{13.12}$$

and

$$F(t, r, s)/s^2 \to \infty, \quad |s| \to \infty. \tag{13.13}$$

Then the problem (13.1) – (13.3) has at least one nontrivial solution.

An important aspect of this theorem is that all rational values of R/T are allowed.

Theorem 13.2. *Assume, in addition, that*

$$H(r,t,s) = sp(r,t,s) - 2P(r,t,s) \geq -W(r,t), \quad (r,t) \in Q,\ s \in \mathbb{R}, \tag{13.14}$$

where $W(r,t) \in L^1(Q)$. Let $\mathcal{M}$ be the collection of solutions of (13.1) – (13.3) . Then there is a nontrivial solution that minimizes the energy functional

$$G(u) = (Lu,u) - 2\int_Q P(r,t,u), \quad u \in D \tag{13.15}$$

over $\mathcal{M}\backslash\{0\}$.

Such solutions are called *ground state solutions.*

Before proving Theorems 13.1 and 13.2, we shall need to determine the spectrum of the linear term.

14. The spectrum of the linear operator

In proving Theorem 13.1 we shall need to calculate the spectrum of the linear operator $\Box$ applied to periodic rotationally symmetric functions. Specifically, we shall need

Theorem 14.1. *Let L_0 be the operator*

$$L_0 u = u_{tt} - u_{rr} - r^{-1}(n-1)u_r \tag{14.1}$$

applied to functions $u(t,r)$ in $C^\infty(\bar{Q})$ satisfying

$$u(T,r) = u(0,r),\ u_t(T,r) = u_t(0,r), \quad 0 \leq r \leq R \tag{14.2}$$

$$u(t,R) = u_r(t,0) = 0, \quad t \in \mathbb{R} \tag{14.3}$$

where $Q = [0,T] \times [0,R]$. Then L_0 is symmetric on $L^2(Q,\rho)$, where $\rho = r^{n-1}$. Assume that $8R/T = a/b$, where a, b are relatively prime integers (i.e., $(a,b) = 1$). Then L_0 has a selfadjoint extension L having no essential spectrum other than the point $\lambda_0 = -(n-3)(n-1)/4R^2$. If $n \not\equiv 3\ (mod(4,a))$, then L has no essential spectrum. If $n \equiv 3\ (mod(4,a))$, then the essential spectrum of L is precisely the point λ_0.

Proof. Let $\nu = (n-2)/2$ and let γ be a positive root of $J_\nu(x) = 0$, where J_ν is the Bessel function of the first kind. Set

$$\varphi(r) = J_\nu(\gamma r/R)/r^\nu. \tag{14.4}$$

Then

$$\varphi'' + (n-1)\varphi'/r = (x^2 J_\nu'' + xJ_\nu' - \nu^2 J_\nu)/r^{\nu+2} = -\gamma^2 J_\nu/R^2 r^\nu, \tag{14.5}$$

where $x = \gamma r/R$. If

$$\psi(t,r) = \varphi(r)e^{2\pi ikt/T}, \tag{14.6}$$

then

$$L_0\psi = [(\gamma/R)^2 - (2\pi k/T)^2]\psi. \tag{14.7}$$

Let γ_j be the j-th positive root of $J_\nu(x) = 0$, and set

$$\psi_{jk}(t,r) = r^{-\nu} J_\nu(\gamma_j r/R) e^{2\pi ikt/T}. \tag{14.8}$$

Then $\psi_{jk}(t,r)$ is an eigenfunction of L_0 with eigenvalue

$$\lambda_{jk} = (\gamma_j/R)^2 - (2\pi k/T)^2. \tag{14.9}$$

It is easily checked that the functions ψ_{jk}, when normalized, form a complete orthonormal sequence in $L^2(Q,\rho)$. We shall show that the corresponding eigenvalues (14.9) are not dense in $\mathbb{R}$. It will then follow that L_0 has a selfadjoint extension L with spectrum equal to the closure of the set $\{\lambda_{jk}\}$ (cf., e.g., [24]). Now

$$\gamma_j = \beta_j - (\mu - 1)/8\beta_j + O(\beta_j^{-3}) \text{ as } \beta_j \to \infty \tag{14.10}$$

where

$$\beta_j = \pi(j + \frac{1}{2}\,\nu - \frac{1}{4}), \quad \mu = 4\nu^2 \tag{14.11}$$

(cf., e.g., [33]). Thus

$$\begin{aligned} \lambda_{jk}R^2 &= [\beta_j - \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &\cdot\ [\beta_j + \tau_k - (\mu-1)/8\beta_j + O(\beta_j^{-3})] \\ &= \beta_j^2 - \tau_k^2 - (\mu-1)/4 + O(\beta_j^{-2}) \end{aligned}$$

where $\tau_k = 2k\pi R/T$. (We may assume $k \geq 0$.) Now

$$\beta_j - \tau_k = \pi(j + \frac{1}{2}\,\nu - \frac{1}{4} - ak/4b) = \pi[(4j + n - 3)b - ak]/4b. \tag{14.12}$$

Since the expression in the brackets is an integer, we see that either $\beta_j = \tau_k$ or

$$|\beta_j - \tau_k| \geq \pi/4b. \tag{14.13}$$

Thus

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j = \tau_k}} \lambda_{jk} = -(\mu - 1)/4R^2 = \lambda_0 \tag{14.14}$$

and

$$\lim_{\substack{j,|k|\to\infty \\ \beta_j \neq \tau_k}} |\lambda_{jk}| = \infty. \tag{14.15}$$

If $n - 3$ is not a multiple of $(4, a)$, then

$$\beta_j - \tau_k = \pi[(4j + n - 3) - ak/b]/4 \tag{14.16}$$

can never vanish. To see this, note that if $(b, k) \neq b$, then ak/b is not an integer. Hence $\beta_j \neq \tau_k$. If $b = (b, k)$, then

$$(n - 3) \neq ak' - 4j \quad \forall j,\ k' = k/b. \tag{14.17}$$

Thus in this case we always have $\beta_j \neq \tau_k$ and $|\lambda_{jk}| \to \infty$ as $j, k \to \infty$. On the other hand, if $n \equiv 3\ (mod(4, a))$, then there is an infinite number of positive integers j, k' such that

$$n - 3 = ak' - 4j. \tag{14.18}$$

Hence, the point λ_0 is a limit point of eigenvalues. Consequently, it is in $\sigma_e(L)$. This completes the proof. □

15. Proof of Theorems 13.1 and 13.2.

Proof of Theorem 13.1. In all cases L has no essential spectrum in an interval $[\alpha, \beta)$ with $0 < \alpha < \beta$. We let M be the subspace of $E = D(|L|^{1/2})$ on which $L \geq \beta$, N the subspace on which $L \leq 0$, and Y the subspace on which $\alpha \leq L \leq \beta$.

By Theorem 8.1 there is a nontrivial solution of

$$Lu(t,r) = f(t,r,u).$$

This is precisely what we want. □

Proof of Theorem 13.2. This follows from Theorem 8.2. □

16. Schrödinger operators

In [28, 30] we proved

Theorem 16.1. *Let $\mathcal{A} = -\Delta + \mathcal{V}(x)$ on $H^1(\mathbb{R}^n)$. Assume*

(1) $\mathcal{V}$ is continuous, 1-periodic in $x_1, \cdots, x_k$ and $(a,b) \subset \rho(\mathcal{A})$, $a < 0 < b$,

(2) $f(x,t)$ is continuous, 1-periodic in $x_1, \cdots, x_k$ and

$$|f(x,t)| \leq C(|t|^{p-1} + 1)$$

for some $p \in (2, 2^*)$, $2^* := 2n/(n-2)$, $n > 2$, $2^* := \infty$, $n \leq 2$.

(3)

$$|f(x,t)| \leq \sigma |t|, \quad |t| < \delta,\ x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

for some $\sigma < min[-a, b]$, $\delta > 0$.

(4)

$$F(x,t) \geq 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

(5)

$$F(x,t)/t^2 \to \infty \text{ as } t^2 \to \infty$$

uniformly in x.

(6)

$$2F(x,t+s) - 2F(x,t) - (2rs - (r-1)^2 t) f(x,t) \geq -W(x), \quad x \in \mathbb{R}^n,\ s,t \in \mathbb{R},\ r \in [0,1],$$

where $W(x) \in L^1(\mathbb{R}^n)$.

Then

$$\mathcal{A}u = f(x,u), \quad u \in D. \tag{16.1}$$

has a nontrivial ground state solution.

Theorems 8.1 and 8.2 allow us to dispense with periodicity.

In proving a non-periodic counterpart of Theorem 16.1 we shall use special norms. Let $\alpha \geq 0, \delta > 0, r, t \geq 1$ be parameters with t allowed to be ∞. For $\alpha > 0$ we define

$$\begin{aligned} M_{\alpha,r,t,\delta}(\mathcal{V}) &= \left(\int\left(\int_{|x-y|<\delta}|\mathcal{V}(x)|^r\omega_\alpha(x-y)\mathrm{d}x\right)^{t/r}\mathrm{d}y\right)^{1/t}, \\ &\quad 1 \leq t < \infty \\ &= \sup_y\left(\int_{|x-y|<\delta}|\mathcal{V}(x)|^r\omega_\alpha(x-y)\mathrm{d}x\right)^{1/r}, \\ &\quad t = \infty \end{aligned}$$

where $\omega_\alpha(x)$ is given by

$$\begin{aligned} \omega_s(x) &= |x|^{s-n}, && 0 < s < n, \\ &= 1 - \log|x|^2, && s = n, \\ &= 1, && s > n. \end{aligned}$$

For $\alpha = 0$ we put

$$M_{0,r,t,\delta}(\mathcal{V}) = \|\mathcal{V}\|_t. \tag{16.2}$$

If we define

$$\begin{aligned} \mathcal{V}_{\alpha,r,\delta}(y) &= \left(\int_{|x-y|<\delta}|\mathcal{V}(x)|^r\omega_\alpha(x-y)\mathrm{d}x\right)^{1/r}, \quad \alpha > 0, \\ &= |\mathcal{V}(y)|, \quad \alpha = 0, \end{aligned} \tag{16.3}$$

then we have

$$M_{\alpha,r,t,\delta}(\mathcal{V}) = \|\mathcal{V}_{\alpha,r,\delta}\|_t. \tag{16.4}$$

We also put

$$M_{\alpha,r,t}(\mathcal{V}) = M_{\alpha,r,t,1}(\mathcal{V}), \quad \mathcal{V}_{\alpha,r}(y) = \mathcal{V}_{\alpha,r,1}(y). \tag{16.5}$$

If we define

$$M_{\alpha,p}(\mathcal{V}) = \sup_y \int_{|x|<1}|\mathcal{V}(x-y)|^p|x|^{\alpha-n}\mathrm{d}x, \tag{16.6}$$

then

$$M_{\alpha,r}(\mathcal{V}) = M_{\alpha,r,\infty}(\mathcal{V}), \quad 0 < \alpha < n. \tag{16.7}$$

The following was proved in [15, 29].

Theorem 16.2. *Let $P(\mathrm{D})$ be an elliptic operator of order m, and let $\mathcal{V}(x)$ be a function in $M_{\alpha,r,t}$, where $1 < p < \infty, 1 \leq r < \infty, 1 \leq t \leq \infty$, $\alpha \geq 0$ ($\alpha \neq 0$ if $t < r$) and*

$$\alpha/nr \leq m/n - 1/t. \tag{16.8}$$

Assume that one of the following holds:

(a) $r \neq 1$,
(b) *inequality (16.8) is strict,*
(c) $p = 2, r = 1, t = \infty$.

If $t = \infty$, *assume in addition that*

$$M_{\alpha,r,t,\delta}(\mathcal{V}) \to 0 \text{ as } \delta \to 0 \tag{16.9}$$

and

$$\mathcal{V}_{\alpha,r}(y) \to 0 \text{ as } |y| \to \infty \tag{16.10}$$

hold. If $\overline{\{P(\xi), \xi \in E^n\}} \neq \mathbb{R}$, *then* $P(\mathrm{D}) + \mathcal{V}$ *has an s-extension* B *such that*

$$\sigma_{\mathrm{e}}(B) = \sigma(P_0) = \overline{\{P(\xi), \xi \in E^n\}} \tag{16.11}$$

holds.

Using these norms we can prove:

Theorem 16.3. *Let* $\mathcal{A}$ *be a selfadjoint extension of* $-\Delta + \mathcal{V}(x)$ *on* $H^1(\mathbb{R}^n)$ *satisfying the hypotheses of Theorem 16.2 for m=2. Assume*

(1) There is an interval $[0, b]$ *satisfying* $[0, b] \cap \sigma_{\mathrm{e}}(\mathcal{A}) = \phi$, *but* $[a, b] \cap \sigma(\mathcal{A}) \neq \phi$, *where* $0 < a < b$.

(2) $f(x, t)$ *is a Caratheódory function on* $\mathbb{R}^n \times \mathbb{R}$ *satisfying*

$$|f(x, t)| \leq V(x)^2(|t| + 1), \quad x \in \mathbb{R}^n, \ |t| \geq \delta, \tag{16.12}$$

and

$$|f(x, t)| \leq \sigma|t|, \quad |t| < \delta, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{16.13}$$

for some $\sigma < a$, $\delta > 0$, *where* $V(x) > 0$ *is a function in* $L^2(\mathbb{R}^n)$ *such that*

$$\|Vu\| \leq C\|u\|_D, \quad u \in D$$

and multiplication by $V(x)$ *is a compact operator from* D *to* $L^2(\mathbb{R}^n)$.

(3)

$$F(x, t) \geq 0, \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}.$$

(4)

$$F(x, t)/t^2 \to \infty \text{ as } t^2 \to \infty$$

uniformly in x.

(5)

$$H(x, t) := tf(x, t) - 2F(x, t) \geq -W(x), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R},$$

where $W(x) \in L^1(\mathbb{R}^n)$.

Then

$$\mathcal{A}u = f(x, u), \quad u \in D. \tag{16.14}$$

has a nontrivial ground state solution.

Proof. If $\mathcal{V}$ satisfies the hypotheses of Theorem 16.2 with $m = 2$, then $-\Delta + \mathcal{V}(x)$ will satisfy the hypotheses of Theorem 16.3. Apply Theorems 8.1 and 8.2. □

Remark 16.4. Note that

$$2F(x,t+s) - 2F(x,t) - (2rs - (r-1)^2 t)f(x,t) \geq -W(x), \quad x \in \mathbb{R}^n,\ s,t \in \mathbb{R},\ r \in [0,1],$$

implies

$$H(x,t) := tf(x,t) - 2F(x,t) \geq -W(x), \quad x \in \Omega,\ t \in \mathbb{R} \tag{16.15}$$

(just take $s = -t$ and $r = 0$).

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