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SURJECTIVITY, ZEROS AND FIXED POINTS OF SOME SEMILINEAR MAPPINGS IN NORMED SPACES

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Abstract. We present various sufficient conditions for the surjectivity of weakly coercive mappings and the existence of zeros of some mappings of the form $L+g$ between normed spaces, where L is a linear Fredholm mapping of index zero and g is L-compact.

1. Introduction

The topological degree has been one the favorite tools of Louis Nirenberg in dealing with nonlinear problems in partial differential equations and geometry. He also delivered a superb series of lectures on nonlinear functional analysis [13] at the Courant Institute, during the academic year 1973-74. Besides a clear presentation of Brouwer and Leray-Schauder topological degrees and of several of its extensions like essential maps and stable homotopy groups, those lecture notes describe striking applications to bifurcation theory, monotone operators and min-max theorems.

A welcome addition to [13] is the survey paper [14] based on the Hermann Weyl lectures given by Nirenberg in Princeton in March 1980. The part devoted to topological methods deals with recent extensions of Leray-Schauder degree theory, in particular to Fredholm maps with positive or zero index, and to degree theories associated to closed orbits of ordinary differential equations.

In this paper are stated and proved a number of conditions for the surjectivity and the existence of zeros of some nonlinear perturbations of linear Fredholm operators L of index zero between normed and prehilbertian spaces. Some fixed point theorems are obtained when L is the identity. Some of those results may be new even in the finite-dimensional case, as shown by an example at the end of the paper.

The used tool in the whole paper is the coincidence degree theory of mappings of the form $f = L - g$ when g satisfies a suitable compactness condition, which reduces to the usual one when L is the identity [11,12]. The requested concepts and results of coincidence degree theory are briefly recalled in Section 2 for the reader's convenience.

In Section 3, which deals with weakly coercive mappings, namely such that $\|f(x)\| \to \infty$ when $\|x\| \to \infty$, we show in Theorem 3.1 that a weakly coercive

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mapping $f = L - g$, with range in a prehilbertian space H with inner product $\langle \cdot, \cdot \rangle$, is surjective when L is invertible and the asymptotic condition

$$\liminf_{\|x\| \to \infty, x \in D(L)} \frac{\langle Lx, f(x) \rangle}{\|Lx\|} > -\infty$$

is satisfied. The special cases of L-coercive mappings, such that

$$\frac{\langle Lx, f(x) \rangle}{\|Lx\|} \to \infty \text{ when } \|x\| \to \infty \text{ in } D(L),$$

and of L-monotone mappings, such that

$$\langle L(x-y), f(x) - f(y) \rangle \geq 0 \text{ for all } x, y \in D(L),$$

are considered as well.

A second class of surjectivity results for weakly coercive mappings is based upon the Theorem 4.2, showing that such a mapping $f = L - g$ between normed spaces is surjective when the coincidence degree $d_L[f, B_R, z]$ is different from zero for some $z \in Y$ and open balls B_R of center 0 and sufficiently large radius R. This theorem extends to the frame of coincidence degree some interesting results contained in Section 5.4 of Berger's monograph [1] and attributed to Plastock [15] (although not contained in this reference). Special cases deal with odd mappings, mappings with linear growth and locally one-to-one mappings. See also [3] and [4] for the finite-dimensional case.

In Section 5, we introduce and name Poincaré-Bohl coincidence theorem an existence result for a zero of $f = L - g$ defined in the closure of some open bounded neighborhood Ω of the origin, and taking values in a prehilbertian space H, when L is invertible and, for some (not necessarily continuous) mapping $h : X \to H$, one has

$$\langle h(x), f(x) \rangle \geq 0 \text{ and } \langle h(x), L(x) \rangle > 0 \textit{ on } \partial\Omega.$$

Special cases of this Theorem 5.1 are a fixed point theorem of Krasnosel'skii (see [9] for a special case and [10], Thm. 21.4) when $L = I$ and

$$\langle x, g(x) \rangle \leq \|x\|^2 \text{ on } \partial\Omega,$$

whose particular case for $\Omega = B_R$ is named here Hadamard fixed point theorem, because of the first use by Hadamard of this assumption in his proof of the Brouwer fixed point theorem given in [7].

When Ω is the interior of a closed convex neighborhood C of the origin, some arguments of convex analysis are used to prove in Theorem 5.6 a new variant of the fixed point theorem of Krasnosel'skii, namely the existence of a fixed point of g when

$$\langle \nu_C(x), g(x) \rangle \leq \langle \nu_C(x), x \rangle \text{ on } \partial C,$$

for some outer normal field $\nu_C : \partial C \to \partial B_1$ to ∂C.

The special case of Theorem 5.6 where $H = \mathbb{R}^n$ contains and unifies the well known existence theorems of Hadamard and of Poincaré-Miranda, as well as new existence conditions.

2. The spaces and the mappings

In the whole paper, X and Y always denote real normed spaces, and H a real prehilbertian (or Hilbert space if explicitly mentioned) with inner product $\langle\cdot,\cdot\rangle$. To avoid heavy notations, we use the same symbol $\|\cdot\|$ for the norms in X and Y.

2.1. **L-compact and L-completely continuous mappings in normed spaces.** We consider semilinear mappings $f = L - g$, where $L : D(L) \subseteq X \to Y$ is a linear Fredholm mapping of index zero, and $g : \overline{\Omega} \subseteq X \to Y$ is L-compact on $\overline{\Omega}$ for some open bounded set $\Omega \subseteq X$.

Recall that $L : D(L) \subseteq X \to Y$ is **Fredholm of index zero** if L has a closed range $R(L)$, and if the dimensions of its kernel $N(L)$ and the codimension of its range are finite and equal. On the other hand, g is **L-compact** on $\overline{\Omega}$ if there exists a linear mapping $A : X \to Y$ with finite rank such that $L + A$ is invertible and $(L + A)^{-1}g : \overline{\Omega} \to X$ is compact, i.e. continuous and such that $(L + A)^{-1}g(\overline{\Omega})$ is relatively compact. We denote by $\mathcal{F}(L)$ the set of such linear mappings A. For example, if P is a continuous projector in X such that $R(P) = N(L)$, Q a continuous projector in Y such that $N(Q) = R(L)$, and $J : N(L) \to R(Q)$ an isomorphism, then $L + JP$ is invertible.

We denote by $\mathcal{C}_L(\overline{\Omega}, Y)$ the set of mappings $f = L - g$ with $L : D(L) \subseteq X \to Y$ Fredholm of index 0 and $g : \overline{\Omega} \to Y$ L-compact on $\overline{\Omega}$. In the case where $X = Y$ and $L = I$, we can take $A = 0$ and the I-compactness of g on $\overline{\Omega}$ reduces to the usual compactness of g on $\overline{\Omega}$. So $\mathcal{C}_I(\overline{\Omega}, X)$ is the set of compact perturbations of identity on $\overline{\Omega}$. If X and Y have the same finite dimension, we can always take $L = 0$ and define $\mathcal{C}_0(\overline{\Omega}, Y)$ as the set of continuous mappings $f : \overline{\Omega} \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

When $f = L - g$ with $g : X \to Y$ L-compact on $\overline{B}$ for each open bounded $B \subseteq X$, g is said to be **L-completely continuous** on X and we write $f \in \mathcal{K}_L(X, Y)$. For $X = Y$ and $L = I$, the elements of $\mathcal{K}_I(X, X)$ are the completely continuous perturbations of identity in X.

See [11, 12] for more details.

2.2. **Coincidence degree of L-compact perturbations of linear Fredholm mappings of index zero.** Given $y \notin f(D(L) \cap \partial\Omega)$, one can associate to $f = L - g \in \mathcal{C}_L(\overline{\Omega}, Y)$ an integer $d_L[f, \Omega, y]$, the **coincidence degree** of f (or of L and g) in Ω at y, which has the following fundamental properties:

(1) **Normalization** : *if* $N(L) = \{0\}$, $d_L[L, \Omega, y] = 1$ *if* $y \in L(\Omega)$ *and* $d_L[L, \Omega, y] = 0$ *if* $y \notin L(\overline{\Omega})$.
(2) **Additivity** : *if* $\Omega_1 \subseteq \Omega$ *and* $\Omega_2 \subseteq \Omega$ *are open and disjoints and* $y \notin f[D(L) \cap (\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)]$, *then* $d_L[f, \Omega, y] = d_L[f, \Omega_1, y] + d_L[f, \Omega_2, y]$.
(3) **Homotopy invariance** : *if* $F = L - G \in \mathcal{C}(\overline{\Omega} \times [0, 1], Y)$ *is such that* $y \notin F[(D(L) \cap \partial\Omega) \times [0, 1]]$, *then* $d_L[F(\cdot, \lambda), \Omega, y]$ *is constant in* $[0, 1]$.

By definition,

$$d_L[f, \Omega, y] := d_{LS}[I - (L + A)^{-1}(g + A), \Omega, (L + A)^{-1}y], \tag{2.1}$$

where $A \in \mathcal{F}(L)$, and d_{LS} denotes the Leray-Schauder degree [3, 13]. The right-hand member of (2.1) does not depend upon the choice of A when A remains in the same equivalence class in $\mathcal{F}(L)$ defined by the relation $B \sim A$ if the Leray-Schauder index $i_{LS}[I + (L + B)^{-1}(A - B), 0] = 1$. Clearly, when $X = Y$ and $f \in \mathcal{C}_I(\overline{\Omega}, X)$, we can take $A = 0$ and

$$d_I[f, \Omega, y] = d_{LS}[I - g, \Omega, y].$$

When $f \in \mathcal{C}_0(\overline{\Omega}, \mathbb{R}^n)$, we can take $d_0[f, \Omega, y) = d_B[f, \Omega, y]$, where d_B denotes the Brouwer degree. See [11, 12] for more details.

Useful consequences of the fundamental properties of the coincidence degree are the following ones :

4. **Existence** : *If* $d_L[f, \Omega, y] \neq 0$*, then* $y \in f(D(L) \cap \Omega)$.
5. **Excision** : *If* $\Omega_1 \subseteq \Omega$ *is open and* $y \notin f[D(L) \cap (\overline{\Omega} \setminus \Omega_1)]$*, then* $d_L[f, \Omega, y] = d_L[f, \Omega_1, y]$.
6. **Generalized Borsuk theorem** : *If* Ω *is a symmetric open bounded neighborhood of* 0 *and* f *is odd, then* $d_L[f, \Omega, 0] = 1$ *(mod. 2)*.

2.3. **The case where $(L+A)^{-1}$ is continuous for some A in $\mathcal{F}(L)$.** The coincidence degree has further properties when the following supplementary assumption holds :

$$(L + A)^{-1} \quad \textit{is continuous for some} \quad A \in \mathcal{F}(L). \tag{2.2}$$

If condition (2.2) holds for one $A \in \mathcal{F}(L)$, it holds for all $A \in \mathcal{F}(L)$, because of the easily verified formula for $B \in \mathcal{F}(L)$ (apply $L + B$ to both members)

$$(L + B)^{-1} = [I - (L + B)^{-1}(B - A)](L + A)^{-1}.$$

Assumption (2.2) holds in particular when X and Y are Banach spaces and L is closed, namely when $(x_n) \to x$ and $Lx_n \to y$ imply that $x \in D(L)$ and $y = Lx$.

Lemma 2.1. *If* $\Omega \subseteq X$ *is open and bounded,* $f \in C_L(\overline{\Omega}, Y)$*,* L *satisfies condition (2.2), and* $y \notin f(D(L) \cap \partial\Omega)$*, there exists* $\delta > 0$ *such that*

$$\|f(x) - y\| \geq \delta \quad \textit{for all} \quad x \in D(L) \cap \partial\Omega. \tag{2.3}$$

Proof. If (2.3) does not hold, there is a sequence $(x_k)_{k \in \mathbb{N}}$ in $D(L) \cap \partial\Omega$ such that

$$Lx_k - g(x_k) - y \to 0 \quad \text{as} \quad k \to \infty,$$

and hence, using condition (2.2), such that

$$x_k - (L + A)^{-1}[g(x_k) + Ax_k + y] \to 0 \quad \text{as} \quad k \to \infty.$$

By the L-compactness of g on $\overline{\Omega}$, $(L+A)^{-1}(g+A)(\overline{\Omega})$ is relatively compact, and there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ such that the sequence $((L + A)^{-1}[g(x_{k_n}) + Ax_{k_n}])_{n \in \mathbb{N}}$ converges to some $v \in X$ and hence the sequence $(x_{k_n})_{n \in \mathbb{N}}$ converges to $-v - (L + A)^{-1}y := x \in \partial\Omega$ when $n \to \infty$. Therefore, by continuity of $(L + A)^{-1}(g + A)$,

$$x - (L + A)^{-1}[g(x) + Ax + y] = 0,$$

so that $x \in D(L) \cap \partial\Omega$, a contradiction to the assumption $y \notin f(D(L) \cap \partial\Omega)$. □

A consequence of Lemma 2.1 is the invariance of the coincidence degree for small perturbations of y and an openness property of f.

Lemma 2.2. *If $f \in \mathcal{C}_L(\overline{\Omega}, Y)$ for some open bounded set $\Omega \subseteq X$, $y \notin f(D(L) \cap \partial\Omega)$, L satisfies condition (2.2), and*

$$d_L[f, \Omega, y] \neq 0,$$

then there exists $\delta > 0$ such that

$$d_L[f, \Omega, z] = d_L[f, \Omega, y] \quad \text{when} \quad \|z - y\| < \delta.$$

Furthermore, $f(\Omega)$ is a neighborhood of y.

Proof. By Lemma 2.1, there exists $\delta > 0$ such that condition (2.3) holds, and the homotopy $F : \overline{\Omega} \times [0, 1] \to Y$ defined by

$$F(x, \lambda) := Lx - g(x) - (1 - \lambda)y - \lambda z$$

is such that, for all $x \in D(L) \cap \partial\Omega$ and $\lambda \in [0, 1]$,

$$\|F(x, \lambda)\| \quad \geq \quad \|f(x) - y\| - \lambda\|z - y\| \geq \delta - \|z - y\| > 0$$

if $\|z - y\| < \delta$. If it is the case, $0 \notin F((D(L) \cap \partial\Omega) \times [0, 1])$ and

$$d_L[f, \Omega, z] = d_L[F(\cdot, 1), 0] = d_L[F(\cdot, 0), 0] = d_L[f, \Omega, y] \neq 0.$$

Consequently $z \in f(D(L) \cap \Omega)$ for all $z \in y + B_\delta$. □

3. Weakly coercive mappings : asymptotic conditions for surjectivity

3.1. **Weakly coercive mappings.** We say that $f : D(f) \subseteq X \to Y$ is **weakly coercive** if

$$\lim_{\|x\| \to \infty, x \in D(f)} \|f(x)\| = +\infty. \tag{3.1}$$

The following examples of continuous real functions of one variable show that surjectivity and weak coercivity are independent concepts :

(a) $f(x) = x^3$ is onto and weakly coercive
(b) $f(x) = x^2$ is not onto and weakly coercive
(c) $f(x) = x^2 \sin x$ is onto and not weakly coercive.

3.2. **An asymptotic condition for surjectivity of some weakly coercive mappings.** In this subsection, we introduce a sufficient condition for a weakly coercive mapping $f \in \mathcal{K}_L(X, H)$ to be onto.

Theorem 3.1. *Any weakly coercive mapping $f = L - g \in \mathcal{K}_L(X, H)$, such that*

$$N(L) = \{0\} \quad \text{and} \quad \liminf_{\|x\| \to \infty, x \in D(L)} \frac{\langle Lx, f(x) \rangle}{\|Lx\|} > -\infty. \tag{3.2}$$

is onto.

Proof. By assumption (3.2), there exists $c \in \mathbb{R}$ and $r > 0$ such that

$$\langle Lx, f(x) \rangle \geq c\|Lx\| \quad \text{when} \quad \|x\| \geq r. \tag{3.3}$$

Let $y \in H$ and let us define the homotopy $F : D(L) \times [0, 1] \to H$ by

$$F(x, \lambda) = (1 - \lambda)Lx + \lambda[f(x) - y] = Lx - \lambda[g(x) + y].$$

Clearly, $F(x,0) = Lx = 0$ if and only if $x = 0$. If $\lambda \in (0,1]$ and $F(x,\lambda) = 0$, then

$$\|f(x) - y\| = \frac{1-\lambda}{\lambda}\|Lx\|, \tag{3.4}$$

and, using (3.3), if furthermore $\|x\| \geq r$,

$$(1-\lambda)\|Lx\|^2 = -\lambda\langle Lx, f(x)\rangle + \lambda\langle Lx, y\rangle \leq -\lambda c\|Lx\| + \|y\|\|Lx\|.$$

As $N(L) = \{0\}$, this implies that, for $\|x\| \geq r$ and $\lambda \in (0,1]$,

$$\frac{1-\lambda}{\lambda}\|Lx\| \leq |c| + \|y\|,$$

which, combined to (3.4), gives

$$\|f(x) - y\| \leq |c| + \|y\| \;\text{ if }\; F(x,\lambda) = 0,\ \|x\| \geq r,\ \lambda \in (0,1]. \tag{3.5}$$

Now, the weak coerciveness of f (and hence of $f(\cdot) - y$) implies the existence of $R > r$ such that

$$\|x\| \leq R \text{ when } \|f(x) - y\| \leq |c| + \|y\|. \tag{3.6}$$

It follows from (3.5), (3.6) and $N(L) = \{0\}$ that $F(x,\lambda) \neq 0$ when $(x,\lambda) \in (D(L) \cap \partial B_R) \times [0,1]$, and the homotopy invariance of the coincidence degree implies that

$$\begin{aligned} d_L[f, B_R, y] &= d_L[F(\cdot,1), B_R, y] = d_L[F(\cdot,0), B_R, y] \\ &= d_L[L, B_R, 0] = 1. \end{aligned}$$

By the existence property of the coincidence degree, $y \in f(D(L) \cap B_R)$. □

The special case where $X = H$ and $L = I$ is of interest.

Corollary 3.2. *Any weakly coercive mapping $f \in \mathcal{K}_I(H,H)$ such that*

$$\liminf_{\|x\|\to\infty} \frac{\langle x, f(x)\rangle}{\|x\|} > -\infty. \tag{3.7}$$

is onto.

When $X = H = \mathbb{R}^n$ with the usual inner product, this surjectivity result goes as follows.

Corollary 3.3. *Any weakly coercive mapping $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ satisfying condition (3.7) is onto.*

Remark 3.4. In the example (a) of Subsection 3.1, condition (3.7) is satisfied, although in example (b), $\liminf_{|x|\to\infty} \frac{x^3}{|x|} = -\infty$. In example (c), neither the weak coercivity nor the condition (3.7) hold.

3.3. **L-coercive mappings.** We can reinforce the coerciveness condition to obtain surjectivity without assumption (3.2). A mapping $f \in \mathcal{K}_L(X,H)$ is said to be **L-coercive** if

$$N(L) = \{0\} \text{ and } \frac{\langle Lx, f(x)\rangle}{\|Lx\|} \to \infty \text{ if } \|x\| \to \infty \text{ in } D(L). \tag{3.8}$$

When $X = H$ and $L = I$, we recover the classical concept of **coerciveness** of f, namely,

$$\frac{\langle x, f(x)\rangle}{\|x\|} \to \infty \quad \text{if} \quad \|x\| \to \infty.$$

As, for $x \neq 0$,

$$\frac{\langle Lx, f(x)\rangle}{\|Lx\|} \leq \left| \frac{\langle Lx, f(x)\rangle}{\|Lx\|} \right| \leq \|f(x)\|,$$

we see that any L-coercive mapping is weakly coercive. On the other hand, condition (3.8) trivially implies the assumption (3.2) of Theorem 3.1, so that the following results hold.

Corollary 3.5. *Any L-coercive mapping $f \in \mathcal{K}_L(X,H)$ is onto.*

Corollary 3.6. *Any coercive mapping $f \in \mathcal{K}_I(H,H)$ is onto.*

Corollary 3.7. *Any coercive mapping $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is onto.*

3.4. **L-monotone mappings.** A mapping $f = L - g : D(L) \subseteq X \to H$ is called **L-monotone** if it satisfies the condition

$$N(L) = \{0\} \text{ and } \langle L(x-y), f(x) - f(y)\rangle \geq 0 \text{ for all } x, y \in D(L). \tag{3.9}$$

When $X = H$ and $L = I$, the condition (3.9) reduces to the classical definition of **monotonicity**

$$\langle x - y, f(x) - f(y)\rangle \geq 0 \text{ for all } x, y \in H.$$

If we observe that the condition (3.9) for $y = 0$ implies that

$$\langle Lx, f(x)\rangle \geq \langle Lx, f(0)\rangle \geq -\|f(0)\|\|Lx\| \text{ for all } x \in D(L),$$

we see that assumption (3.2) of Theorem 3.1 is satisfied for all L-monotone mappings, so that the following results immediately follow.

Corollary 3.8. *Any L-monotone weakly coercive mapping $f \in \mathcal{K}_L(X,H)$ is onto.*

Corollary 3.9. *Any monotone weakly coercive mapping $f \in \mathcal{K}_I(H,H)$ is onto.*

Corollary 3.10. *Any monotone weakly coercive mapping $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is onto.*

Remark 3.11. When H is a Hilbert space, the statements of Corollaries 3.6 and 3.9 remain true for monotone mappings $f : H \to H$ which are continuous on finite-dimensional subspaces of H. See [4, 13] for details and proofs.

4. Weakly coercive mappings : degree conditions for surjectivity

4.1. **A degree condition for the surjectivity of weakly coercive mappings.** Another way for obtaining the surjectivity of weakly coercive mappings $f \in \mathcal{K}_L(X,Y)$ is based upon the following considerations.

Lemma 4.1. *If the mapping $f \in \mathcal{K}_L(X,Y)$ is weakly coercive, then, for each $y \in Y$, there exists $R_y > 0$ such that, for any $R \geq R_y$, $d_L[f, B_R, y]$ is defined and constant.*

Proof. It follows from the definition of weak coerciveness that, given $y \in Y$, there exists $R_y > 0$ such that

$$\|f(x)\| > \|y\| \ \text{ for all } \ x \in D(L) \ \text{ verifying } \ \|x\| \geq R_y.$$

Consequently, $y \notin f(D(L) \cap \partial B_R)$ for all $R \geq R_y$, so that $d_L[f, B_R, y]$ is well defined, and $y \notin f(\overline{B}_R \setminus B_{R_y}, y]$. The excision property of the coincidence degree implies that

$$d_L[f, B_R, y] = d_L[f, B_{R_y}, y] \ \text{ for all } \ R \geq R_y.$$

□

We have now the following sufficient condition for surjectivity.

Theorem 4.2. *Any mapping weakly coercive $f \in \mathcal{K}_L(X,Y)$ such that $d_L[f, B_{R_z}, z] \neq 0$ for some $z \in Y$ is onto.*

Proof. Let $y \in Y$ and let us consider the homotopy $F : D(L) \times [0,1] \to Y$ defined by

$$F(x, \lambda) = f(x) - (1-\lambda)z - \lambda y = Lx - g(x) - (1-\lambda)z - \lambda y.$$

Using the weak coerciveness of f, there exists $R_y > 0$ such that

$$\|f(x)\| > \|y\| \ \text{ when } \ x \in D(L) \ \text{ and } \ \|x\| \geq R_y$$

and there exists $R_z > 0$ such that

$$\|f(x)\| > \|z\| \ \text{ when } \ x \in D(L) \ \text{ and } \ \|x\| \geq R_z.$$

Thus, for $x \in D(L)$ such that $\|x\| \geq R_* := \max\{R_y, R_z\}$ and $\lambda \in [0,1]$,

$$\begin{aligned} \|f(x)\| &= (1-\lambda)\|f(x)\| + \lambda\|f(x)\| > (1-\lambda)\|z\| + \lambda\|y\| \\ &\geq \|(1-\lambda)z + \lambda y\| \end{aligned}$$

and hence

$$\|F(x,\lambda)\| \geq \|f(x)\| - \|(1-\lambda)z + \lambda y\| > 0$$

for all $x \in (D(L) \cap \partial B_{R_*}) \times [0,1]$. The homotopy invariance and the excision property of the coincidence degree imply that

$$\begin{aligned} d_L[f, B_{R_y}, y] &= d_L[f, B_{R_*}, y] = d_L[F(\cdot,1), B_{R_*}, 0] \\ &= d_L[F(\cdot,0), B_{R_*}, 0] = d_L[f, B_{R_*}, z] \\ &= d_L[f, B_{R_z}, z] \neq 0. \end{aligned}$$

The existence of $x \in B_{R_y}$ such that $f(x) = y$ follows from the existence property of the coincidence degree. □

Corollary 4.3. *Any weakly coercive mapping* $f \in \mathcal{K}_I(X,X)$ *such that* $d_{LS}[f, B_{R_z}, z] \neq 0$ *for some* $z \in X$ *is onto.*

Corollary 4.4. *Any weakly coercive mapping* $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ *such that* $d_B[f, B_{R_z}, z] \neq 0$ *for some* z *is onto.*

4.2. **Odd mappings.** A first application of Theorem 4.2 involve the odd mappings.

Corollary 4.5. *Any weakly coercive odd mapping* $f \in \mathcal{K}_L(X,Y)$ *is onto.*

Proof. As f is odd, $f(0) = 0$ and, by the generalized Borsuk theorem, $d_L[f, B_{R_0}, 0] = 1$ (mod. 2). The result follows from Theorem 4.2. □

Corollary 4.6. *Any weakly coercive odd mapping* $f \in \mathcal{K}_I(X,X)$ *is onto.*

Corollary 4.7. *Any weakly coercive odd mapping* $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ *is onto.*

4.3. **Mappings with linear growth.** Let $L : D(L) \subseteq X \to Y$ be linear, Fredholm of index zero, such that $N(L) = \{0\}$ and $L^{-1} : Y \to X$ is continuous.

From the identity $x = L^{-1}Lx$ for all $x \in D(L)$, we deduce that

$$\|Lx\| \geq \|L^{-1}\|^{-1}\|x\| \text{ for all } x \in D(L) \tag{4.1}$$

and hence

$$\|Lx\| \to \infty \text{ if } x \in D(L) \text{ and } \|x\| \to \infty. \tag{4.2}$$

We say that the mapping $h : X \to Y$ is **L-quasibounded** if

$$|h|_L := \limsup_{\|x\| \to \infty} \frac{\|h(x)\|}{\|Lx\|} < +\infty. \tag{4.3}$$

$|h|_L$ is called the **L-quasinorm** of h. For $X = Y$ and $L = I$ the above concepts reduce to the classical ones of **quasiboundedness** and **quasinorm** introduced by Granas [6].

The mapping $h : X \to Y$ is called **asymptotically linear** if there exists a linear continuous mapping $C : X \to Y$ such that

$$\lim_{\|x\| \to \infty} \frac{h(x) - Cx}{\|x\|} = 0. \tag{4.4}$$

It is easy to see that at most one C satisfies (4.4). If it exists, it is denoted by h'_∞ and named the **derivative at infinity** of h. The concept was introduced by Krasnosel'skii [8].

If h is asymptotically linear with derivative at infinity h'_∞, and if L^{-1} exists and is bounded, then, for $x \in D(L) \setminus \{0\}$, we obtain, using (4.1),

$$\begin{aligned} \frac{\|h(x)\|}{\|Lx\|} &\leq \frac{\|h(x) - h'_\infty x\|}{\|Lx\|} + \frac{\|h'_\infty x\|}{\|Lx\|} \\ &\leq \|L^{-1}\| \frac{\|h(x) - h'_\infty x\|}{\|x\|} + \|L^{-1}\| \frac{\|h'_\infty x\|}{\|x\|}, \end{aligned}$$

and hence

$$\begin{aligned}\limsup_{\|x\|\to\infty}\frac{\|h(x)\|}{\|Lx\|} &\le \|L^{-1}\|\lim_{\|x\|\to\infty}\frac{\|h(x)-h'_\infty x\|}{\|x\|}+\|L^{-1}\|\sup_{\|u\|=1}\|h'_\infty u\| \\ &= \|L^{-1}\|\|h'_\infty\|,\end{aligned}$$

which shows that h is L-quasibounded and $|h|_L \le \|L^{-1}\|\|h'_\infty\|$.

We have a surjectivity theorem for some quasibounded and L-completely continuous perturbations of L.

Theorem 4.8. *Any mapping $f = L - g \in \mathcal{K}_L(X,Y)$ such that $N(L) = \{0\}$, $L^{-1} : Y \to X$ is bounded, $g : X \to Y$ is L-quasibounded and $|g|_L < 1$ is weakly coercive and onto.*

Proof. Define the homotopy $H : D(L)\times[0,1] \to Y$ by

$$H(x,\lambda) = Lx - \lambda g(x).$$

Let $\varepsilon > 0$ be such that

$$|g|_L + \varepsilon < 1.$$

By the quasiboundedness of g, there exists $R_0 > 0$ such that

$$\|g(x)\| \le (|g|_L+\varepsilon)\|Lx\| \ \text{ when } \ \|x\| \ge R_0.$$

Consequently, for all $x \in D(L)$ such that $\|x\| \ge R_0$ and $\lambda \in [0,1]$,

$$\|H(x,\lambda)\| = \|Lx - \lambda g(x)\| \ge [1-(|g|_L+\varepsilon)]\|Lx\| > 0, \tag{4.5}$$

so that, using (4.2), $\|H(x,1)\| = \|f(x)\| \to \infty$ when $x \in D(L)$ and $\|x\| \to \infty$, and f is weakly coercive. From (4.5) and the homotopy invariance property of the coincidence degree, we obtain

$$\begin{aligned}d_L[f,B_{R_0},0] &= d_L[H(\cdot,1),B_{R_0},0] = d_L[H(\cdot,0),B_{R_0},0] \\ &= d_L[L,B_{R_0},0] = 1.\end{aligned}$$

The result follows from Theorem 4.2. □

We also have a surjectivity theorem for asymptotically linear mappings.

Theorem 4.9. *Any mapping $f = L - g \in \mathcal{K}_L(X,Y)$ such that $g : X \to X$ is asymptotically linear and $L - g'_\infty : D(L) \subseteq X \to Y$ has a continuous inverse $(L - g'_L)^{-1} : Y \to X$ is weakly coercive and onto.*

Proof. Let us introduce the homotopy $F : D(L)\times[0,1] \to Y$ by

$$F(x,\lambda) = Lx - \lambda g(x) - (1-\lambda)g'_\infty x.$$

Let $\varepsilon > 0$ be such that $\varepsilon\|(L-g'_\infty)^{-1}\| < 1$. By assumption, there exists $R_0 > 0$ such that one has

$$\|g(x) - g'_\infty x\| \le \varepsilon\|x\| \ \text{ when } \ \|x\| \ge R_0.$$

Consequently, for all $\lambda \in [0,1]$ and $x \in D(L)$ such that $\|x\| \geq R_0$, we have, using (4.1) applied to $L - g'_\infty$,

$$\begin{aligned}\|F(x,\lambda)\| &= \|Lx - g'_\infty x - \lambda(g(x) - g'_\infty x)\| \\ &\geq \|Lx - g'_\infty x\| - \|g(x) - g'_\infty x)\| \\ &\geq (\|L - g'_\infty\|^{-1})^{-1}\|x\| - \varepsilon\|x\| \\ &= (\|L - g'_\infty\|^{-1})^{-1}\left[1 - \varepsilon\|(L - g'_\infty)^{-1}\|\right]\|x\|.\end{aligned}$$

By the choice of ε, $F(x,\lambda) \neq 0$ for all $(x,\lambda) \in (D(L) \cap \partial B_{R_0}) \times [0,1]$, and, taking $\lambda = 1$, we obtain the weak coercivity of f. From the homotopy invariance of the coincidence degree, we get

$$\begin{aligned}d_L[f, B_{R_0}, 0] &= d_L[F(\cdot,1), B_{R_0}, 0] = d_L[F(\cdot,0), B_{R_0}, 0] \\ &= d_L[L - g'_\infty, B_{R_0}, 0] = \pm 1.\end{aligned}$$

The result follows from Theorem 4.2. □

4.4. **Locally one-to-one mappings.** In this whole subsection, we assume that L satisfies Assumption (2.2). We start with a property of mappings which are one-to-one on a closed ball centered at the origin.

Lemma 4.10. *Let $U \subseteq X$ be an open neighborhood of 0 and $f = L - g : U \to Y$ be such that $f(0) = 0$. If $f \in \mathcal{C}_L(\overline{B}_R, Y)$ is one-to-one on some closed ball $\overline{B}_R \subseteq U$, then $f(D(L) \cap U)$ is a neighborhood of $f(0) = 0$.*

Proof. The injectivity of f on $\overline{B}_R$ implies that $d_L[f, B_R, 0]$ is well defined because $f(x) \neq 0 = f(0)$ for each $x \in \partial B_R$. To show that $d_L[f, B_R, 0] \neq 0$, define the homotopy $H : D(L) \cap \overline{B}_R \times [0,1] \to Y$ by

$$\begin{aligned}H(x,\lambda) &= f\left(\frac{1}{1+\lambda}x\right) - f\left(\frac{-\lambda}{1+\lambda}x\right) \\ &= Lx - g\left(\frac{1}{1+\lambda}x\right) + g\left(\frac{-\lambda}{1+\lambda}x\right).\end{aligned}$$

Notice that $H(x,0) = f(x)$ and $H(x,1) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right)$ is odd. Now, $H(x,\lambda) \neq 0$ for any $(x,\lambda) \in \partial B_R \times [0,1]$, because, if $H(x,\lambda) = 0$, then

$$f\left(\frac{1}{1+\lambda}x\right) = f\left(\frac{-\lambda}{1+\lambda}x\right),$$

and, f being one-to-one on $\overline{B}_R$, this gives

$$\frac{1}{1+\lambda}x = \frac{-\lambda}{1+\lambda}x$$

that is $x = 0$. Using the homotopy invariance property and the generalized Borsuk theorem [11], we get

$$d_L[f, B_R, 0] = d_L[H(\cdot,0), B_R, 0] = d_L[H(\cdot,1), B_R, 0] = 1 \text{ (mod 2)}.$$

Thus, by Lemma 2.2, $f(D(L) \cap U) \supset f(B_R)$ is a neighborhood of 0.

We say that $f : U \to Y$ is **locally one-to-one** or **locally injective** if, for each $x \in U$ there is a neighborhood $V \subseteq U$ of x such that f is one-to-one on V. Any one-to-one mapping on U is of course locally one-to-one on U, but the converse is not true, as shown by the classical example of $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (e^x \cos y, e^x \sin y)$.

We extend to the frame of coincidence degree the **theorem of invariance of domain**.

Theorem 4.11. *Let $U \subseteq X$ be open, L satisfies condition (2.2), and let $f = L - g : U \to Y$ be locally one-to-one and such that $f \in \mathcal{C}_L(\overline{\Omega}, Y)$ for each open bounded set $\Omega \subseteq U$. Then f is an open mapping. In particular, $f(D(L) \cap U)$ is open in Y.*

Proof. Let $V \subseteq U$ be open. We show that $f(D(L) \cap V)$ is a neighborhood of each of its points. Let $y_0 \in f(D(L) \cap V)$ and $x_0 \in D(L) \cap V$ be such that $f(x_0) = y_0$. Since f is locally one-to-one, there exists a closed ball $x_0 + \overline{B}_R \subseteq V$ on which f is one-to-one. Define $\widetilde{V} = V - x_0 = \{\widetilde{x} = x - x_0 : x \in V\}$ and $\widetilde{f} : D(L) \cap \widetilde{V} \to Y$ by

$$\widetilde{f}(\widetilde{x}) = f(x) - f(x_0) = f(x_0 + \widetilde{x}) - f(x_0).$$

Clearly $\overline{B}_R \subseteq \widetilde{V}$, $\widetilde{f} \in \mathcal{C}_L(\overline{B}_R, Y)$ is one-to-one on $\overline{B}_R$ and $\widetilde{f}(0) = 0$. According to Lemma 4.10,

$$\widetilde{f}(D(L) \cap \widetilde{V}) = f(D(L) \cap V) - f(x_0) = f(D(L) \cap V) - y_0$$

is a neighborhood of 0, thus $f(D(L) \cap V)$ is a neighborhood of y_0. □

Now we can state and prove another surjectivity theorem for f.

Theorem 4.12. *If L satisfies condition (2.2), any weakly coercive, locally one-to-one mapping $f \in \mathcal{K}_L(X, Y)$ is onto.*

Proof. Theorem 4.11 implies that $f(D(L))$ is open. To show that $f(D(L))$ is closed, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $f(D(L))$ which converges to y, and let us show that $y = f(x)$ for some $x \in D(L)$. We have $y_n = f(x_n)$ $(n \in \mathbb{N})$ for some sequence $(x_n)_{n\in\mathbb{N}}$ in $D(L)$, and $(x_n)_{n\in\mathbb{N}}$ is bounded because, if it is not the case, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ tends to infinity and, by weak coercivity, $\|f(x_{n_k})\| \to \infty$, a contradiction with $\|f(x_{n_k})\| = \|y_{n_k}\| \to \|y\|$ when $k \to \infty$. On the other hand, as $(L + A)^{-1}$ is continuous,

$$(L + A)^{-1} f(x_n) = x_n - (L + A)^{-1}[g(x_n) + Ax_n] \to (L + A)^{-1} y \ \text{ as } \ n \to \infty,$$

and, as $(x_n)_{n\in\mathbb{N}}$ is bounded, a subsequence $((L+A)^{-1}[g(x_{n_k})+Ax_{n_k}])_{k\in\mathbb{N}}$ converges to some z, so that $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x := z - (L + A)^{-1} y$ as $k \to \infty$. By continuity of $(L + A)^{-1}(g + A)$,

$$x - (L + A)^{-1}[g(x) + Ax] = (L + A)^{-1} y,$$

that is $x \in D(L)$ and $f(x) = y$. So, $f(D(L))$, open and closed, is equal to Y. □

5. Boundary conditions for the existence of a zero

5.1. **Some coincidence and fixed point theorems.** The main result of this subsection is an existence condition that we name the **Poincaré-Bohl coincidence theorem**.

Theorem 5.1. *If $\Omega \subseteq X$ is an open, bounded neighborhood of 0, any mapping $f \in \mathcal{C}_L(\overline{\Omega}, H)$ such that*

$$\langle h(x), f(x)\rangle \geq 0 \ \ and \ \ \langle h(x), Lx\rangle > 0 \ \ on \ \ D(L) \cap \partial\Omega \tag{5.1}$$

for some (not necessarily continuous) mapping $h : \partial\Omega \to H$, has a zero in $\overline{\Omega}$.

Proof. If f has a zero in $D(L) \cap \partial\Omega$, the result is proved. If not, let us define the homotopy $H : (D(L) \cap \overline{\Omega}) \times [0, 1] \to H$ by

$$H(x, \lambda) = (1 - \lambda)Lx + \lambda f(x) = Lx - \lambda g(x).$$

$H(\cdot, 1)$ has no zero on $D(L) \cap \partial\Omega$, and, for $(x, \lambda) \in (D(L) \cap \partial\Omega) \times [0, 1)$, we have

$$\langle h(x), H(x, \lambda)\rangle = (1 - \lambda)\langle h(x), Lx\rangle + \lambda\langle h(x), f(x)\rangle > 0.$$

Therefore, $H(x, \lambda) \neq 0$ for all $(x, \lambda) \in (D(L) \cap \partial\Omega) \times [0, 1]$. On the other hand, $N(L) = \{0\}$. Indeed, assume by contradiction that a certain nonzero $\nu \in N(L)$ exists. Since $0 \in \Omega$ and Ω is open, there is an open ball $B_\rho \subset \Omega$ and, for $0 < \lambda < \rho\|\nu\|^{-1}$, $u = \lambda\nu \in B_\rho \cap N(L) \subset \Omega$ and $u \neq 0$. Since Ω is bounded, $ru \in \Omega$ for $r > 1$ sufficiently large. By connexity, the segment $[u, ru]$ intersects $\partial\Omega$, that is, the exists $r^* \in (1, r)$ such that $r^*u \in \partial\Omega$. In that case, we would have $\langle h(r^*u), L(r^*u)\rangle = 0$, a contradiction with the second condition in (5.1). Then, by the homotopy invariance and the normalization property of the coincidence degree,

$$d_L[f, \Omega, 0] = d_L[H(\cdot, 1), \Omega, 0] = d_L[H(\cdot, 0), \Omega, 0] = d_L[L, \Omega, 0] = 1.$$

The result follows from the existence property of the coincidence degree. □

The special case of Theorem 5.1 where $X = H$ and $L = I$ is of interest.

Corollary 5.2. *If $\Omega \subseteq H$ is an open, bounded neighborhood of 0, any mapping $f \in \mathcal{C}_I(\overline{\Omega}, H)$ such that*

$$\langle h(x), f(x)\rangle \geq 0 \ \ and \ \ \langle h(x), x\rangle > 0 \ \ when \ \ x \in D(L) \cap \partial\Omega. \tag{5.2}$$

for some (not necessarily continuous) mapping $h : \partial\Omega \to H$, has a zero in $\overline{\Omega}$.

When $h = I$, Corollary 5.2 gives the **Krasnosel'skii fixed point theorem** [9,10] in a prehilbertian space.

Corollary 5.3. *If $\Omega \subseteq H$ is an open, bounded neighborhood of 0, any mapping $f = I - g \in \mathcal{C}_I(\overline{\Omega}, H)$ such that*

$$\langle x, g(x)\rangle \leq \|x\|^2 \ \ when \ \ x \in \partial\Omega, \tag{5.3}$$

has a fixed point in $\overline{\Omega}$.

Proof. The first condition in (5.2) with $L = h = I$ is equivalent to (5.3), and the second one is trivially satisfied. □

Corollary 5.3 with $\Omega = B_R$ gives the **Hadamard fixed point theorem** [7] in a prehilbertian space. If g satisfies the **Rothe condition** $g(\partial B_R) \subseteq \overline{B}_R$, then

$$\langle x, g(x)\rangle \le \|x\|\|g(x)\| \le \|x\|^2 \text{ for all } x \in \partial B_R,$$

and the existence of a fixed point for g follows from the Hadamard theorem. This is a fortiori the case if g satisfies the **Schauder condition** $g(\overline{B}_R) \subseteq \overline{B}_R$.

5.2. **The case of a convex neighborhood of the origin in a Hilbert space.** The following result is an easy consequence of the projection theorem on a closed convex set C in a Hilbert space H (see e.g. [2, 5]).

Proposition 5.4. *Let H be a Hilbert space and $C \subseteq H$ be a non empty closed convex neighborhood of the origin. Then, for each $x \in \partial C$, there exists $\nu_C(x) \in \partial B_1$ such that*

$$\langle \nu_C(x), y - x\rangle \le 0 \text{ for all } y \in C. \tag{5.4}$$

The mapping ν_C is called an **outer normal field** to ∂C. Notice that ν_C needs not to be continuous and is the usual outer normal field to ∂C when ∂C is smooth. The condition (5.4) can be written

$$C \subseteq \{y \in H : \langle \nu_C(x), y - x\rangle \le 0\}$$

which means that C is contained in one of the half-spaces bounded by the hyperplane containing x

$$H_x = \{u \in H : \langle \nu_C(x), u - x\rangle = 0\},$$

a **supporting hyperplane** to C at x.

A useful property of the outer normal field is the following one.

Proposition 5.5. *Let H be a Hilbert space and $C \subseteq H$ be a non empty closed convex neighborhood of the origin. Then, for each $x \in \partial C$, one has*

$$\langle \nu_C(x), x\rangle \ge \mathit{dist}(0, \partial C) > 0. \tag{5.5}$$

Proof. As $0 \in \text{int } C$, there exists $u \in \partial C$ such that

$$\text{dist}(0, \partial C) = \|u\| = \min_{v \in \partial C} \|v\|,$$

so that $\|v\| \ge \|u\|$ for all $v \in \partial C$. Then $\overline{B}_{\|u\|} \subseteq C$ because if there is $z \in \overline{B}_{\|u\|}$ such that $z \in H \setminus C$ (open), then there is some $w \in \partial C$ such that $\|w\| < \|z\| \le \|u\|$, a contradiction. Consequently, $\|u\|\nu_C(x) \in C$ and we obtain (5.5) by taking $y = \|u\|\nu_C(x)$ in (5.4). □

We can use Theorem 5.1 to obtain the following fixed point result for compact mappings in a Hilbert space.

Theorem 5.6. *If H is a Hilbert space and $C \subseteq H$ a closed, bounded convex neighborhood of the origin, any mapping $f = I - g \in \mathcal{C}_I(C, H)$ such that*

$$\langle \nu_C(x), g(x)\rangle \le \langle \nu_C(x), x\rangle \text{ when } x \in \partial C \tag{5.6}$$

for some outer normal vector field ν_C to ∂C, has a fixed point in C.

Proof. It follows from Proposition 5.5 that $h = \nu_C$ verifies the assumptions of Theorem 5.1 with $X = H$ and $L = I$. □

Remark 5.7. Except for the case where $C = \overline{B}_R$, condition (5.6) is distinct from Krasnosel'skii's condition (5.3), because, for $\overline{\Omega}$ a bounded closed convex neighborhood of the origin, the mapping $\frac{x}{\|x\|}$ needs not to be an outer normal field to $\partial\Omega$.

5.3. **The case of a convex neighborhood of the origin in $\mathbb{R}^n$.** If $H = \mathbb{R}^n$, Theorem 5.6 takes the following form, that we name the **Poincaré-Hadamard existence theorem**.

Corollary 5.8. *If $C \subseteq \mathbb{R}^n$ is a compact convex neighborhood of 0, any mapping $f \in C(C, \mathbb{R}^n)$ such that*

$$\langle \nu_C(u), f(u)\rangle \geq 0 \text{ when } u \in \partial C$$

for some outer normal field $\nu_C : \partial C \to \partial B(1)$, has a zero in C.

For example, for $p > 1$, let us take $C = B^p_R$, the closed ball centered at the origin and of radius R in the norm $|x|_p = (\sum_{j=1}^n |x_j|^p)^{1/p}$. Its boundary $\partial B^p_R = \{x \in \mathbb{R}^n : \sum_{j=1}^n |x_j|^p = R^p\}$ is smooth enough so that the outer normal field to ∂B^p_R is uniquely defined by

$$\nu_{B^p_R}(x) = \frac{1}{(\sum_{j=1}^n |x_j|^{2p-2})^{1/2}}(|x_1|^{p-2}x_1, \ldots, |x_n|^{p-2}x_n) \quad (x \in \partial B^p_R).$$

The application of Corollary 5.8 provides the following result.

Corollary 5.9. *For $p > 1$, any mapping $f \in C(B^p_R, \mathbb{R}^n)$ such that*

$$\sum_{j=1}^n |x_j|^{p-2} x_j f_j(x) \geq 0 \text{ for all } u \in \partial B^p_R$$

has a zero in B^p_R.

In particular, for $p = 2$, we recover the **Hadamard existence theorem**.

Corollary 5.10. *Any mapping $f \in C(B^2_R, \mathbb{R}^n)$ such that*

$$\langle u, f(u)\rangle \geq 0 \text{ for all } u \in \partial B^2_R,$$

has a zero in B^2_R.

The case where $C = [-R, R]^n$ is the closed ball centered at the origin in norm $|x|_\infty = \max_{1\leq j\leq n}\{|x_j|\}$ provides the n-dimensional version of the Bolzano intermediate value theorem known as the **Poincaré-Miranda existence theorem** on an hypercube.

Corollary 5.11. *Any mapping $f \in C([-R, R]^n, \mathbb{R}^n)$ such that*

$$(sgn\ u_i) f_i(u) \geq 0 \text{ if } u \in [-R, R]^n,\ |u_i| = R \quad (i = 1, \ldots, n) \tag{5.7}$$

has a zero in $[-R, R]^n$.

The condition (5.7) tells that, for each $j = 1, \dots, n$, the component f_j takes opposite signs on the opposite j^{th} hyperfaces of the hypercube $[-R, R]^n$. It comes from the fact that, for each $j = 1, \dots, n$, the unit vector e^j is a constant outer normal field to $\partial[-R, R]^n$ on the face $\{x \in [-R, R]^n : x_j = R\}$ and $-e^j$ a constant outer normal field to $\partial[-R, R]^n$ on the hyperface $\{x \in [-R, R]^n : x_j = -R\}$. Historical and bibliographical informations about the Poincaré-Miranda theorem and its extensions can be found in [4].

The special case of Corollary 5.8 when $C = B_R^1$, the closed ball of radius R centered at 0 in the norm $|x|_1 = \sum_{j=1}^n |x_j|$, does not seem to be known. As,

$$\partial B_R^1 = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n |x_j| = R \right\} = \left\{ x \in \mathbb{R}^n : \sum_{j=1}^n (\text{sgn } x_j) x_j = R \right\},$$

if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$ $(j = 1, \dots, n)$, and if we denote by E_n the set of those 2^n ε, then

$$\partial B_R^1 = \bigcup_{\varepsilon \in E_n} [\{x \in \mathbb{R}^n : \langle \varepsilon, x \rangle = R\} \cap \partial B_R^1] = \bigcup_{\varepsilon \in E_n} F_R^\varepsilon,$$

where the

$$F_R^\varepsilon := \{x \in \mathbb{R}^n : \langle \varepsilon, x \rangle = R\} \cap \partial B_R^1 \quad (\varepsilon \in E_n)$$

are the 2^n hyperfaces of B_R^1. Notice that B_R^1 and ∂B_R^1 are symmetrical with respect to the origin and that

$$\begin{aligned} -F_R^\varepsilon &= \{-x \in \mathbb{R}^n : \langle \varepsilon, x \rangle = R\} \cap \partial B_R^1 \\ &= \{x \in \mathbb{R}^n : \langle -\varepsilon, x \rangle = R\} \cap \partial B_R^1 = F_R^{-\varepsilon} \end{aligned}$$

so that F_R^ε and $F_R^{-\varepsilon}$ are opposite hyperfaces of ∂B_R^1. The constant field $\frac{\varepsilon}{\sqrt{n}}$ can be taken as outer normal field to ∂B_R^1 on the face F_R^ε, and Corollary 5.8 provides the following result.

Corollary 5.12. *Any mapping $f \in C(B_R^1, \mathbb{R}^n)$ such that*

$$\langle \varepsilon, f(u) \geq 0 \quad \text{if} \quad u \in \partial F_R^\varepsilon \quad (\varepsilon \in E_n) \tag{5.8}$$

has a zero in B_R^1.

The condition (5.8) tells that, for each couple of opposite hyperfaces F_R^ε and $F_R^{-\varepsilon}$ of the hyperoctaedron B_R^1, the corresponding linear combinations $\langle \varepsilon, f(u) \rangle$ takes opposite signs.

As an illustration, we explicit the conditions (5.8) when $n = 2$ (losange).

$$\begin{aligned} f_1(u) + f_2(u) \geq 0 \quad &\text{if} \quad u_1 \geq 0,\ u_2 \geq 0,\ u_1 + u_2 = R, \\ f_1(u) + f_2(u) \leq 0 \quad &\text{if} \quad u_1 \leq 0,\ u_2 \leq 0,\ u_1 + u_2 = -R, \\ f_2(u) - f_1(u) \geq 0 \quad &\text{if} \quad u_1 \leq 0,\ u_2 \geq 0,\ u_2 - u_1 = R, \\ f_2(u) - f_1(u) \leq 0 \quad &\text{if} \quad u_1 \geq 0,\ u_2 \leq 0,\ u_1 - u_2 = -R. \end{aligned}$$

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