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NONLINEAR ROBIN PROBLEMS WITH LOCALLY DEFINED REACTION

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Abstract. We consider a nonlinear Robin problem driven by a $p-$ Laplacian. The reaction consistes of two terms. The first one is parametric and only locally defined, while the second one is $(p-1)$- superlinear. Using cutt-off techniques together with critical point theory and critical groups, we show that for big values of the parameter $\lambda > 0$, the problem has at least three nontrivial solutions, all with sign information (positive, negative and nodal). In the semilinear case ($p = 2$), we produce a second nodal solution, for a total of four nontrivial solutions, all with sign information.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper we study the following parametric nonlinear Robin problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2} u(z) = \lambda f(z, u(z)) + g(z, u(z)) \\ \hfill \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2} u = 0 \text{ on } \partial\Omega, \end{cases}$$

with $\lambda > 0$, $1 < p < \infty$. By $\triangle_p$ we denote the $p-$Laplace differential operator defined by

$$\Delta_p u = div\left(|Du|^{p-2} Du\right), \text{ for all } u \in W^{1,p}(\Omega),$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^N$. The potential function ξ satisfies $\xi \in L^\infty(\Omega)$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The reaction of the problem (right-hand side) consists of two terms. One is the parametric term $\lambda f(z, x)$ with $\lambda > 0$ being the parameter. The other one is a perturbation $g(z, x)$. Both functions f and g are Carathéodory functions (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ and $z \to g(z, x)$ are measurable functions, while for a.a. $z \in \Omega$, $x \to f(z, x)$ and $x \to g(z, x)$ are continuous). The interesting feature of our work here, is that the parametric term $\lambda f(z, \cdot)$ is only locally defined, namely the conditions imposed on $f(z, \cdot)$ concern only its behavior near zero. There are no hypotheses on $f(z, \cdot)$ for large values of $x \in \mathbb{R}$.

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In the boundary condition, $\frac{\partial u}{\partial n_p}$ denotes the conormal derivative of u corresponding to the $p-$Laplacian and is interpreted using the nonlinear Green's identity (see Papageorgiou-Radulescu-Repovs [13], Corollary 1.5.17, p.35). Specifically, for $u \in C^1\left(\overline{\Omega}\right)$, we have

$$\frac{\partial u}{\partial n_p} = |Du|^{p-2}\frac{\partial u}{\partial n},$$

where $n\left(.\right)$ is the outward unit normal on $\partial\Omega$. Using cut-off techniques together with variational tools based on the critical point theory and Morse theory (critical groups), we show that for all $\lambda > 0$ big, problem (P_λ) has at least three nontrivial smooth solutions, all with sign information. More precisely, we prove that there exist two solutions with fixed sign (one positive and the other negative) and a third solution which is nodal (that is, sign changing). In the semilinear case (that is, $p = 2$), by strengthening the regularity of the functions $f\left(z, \cdot\right)$ and $g\left(z, \cdot\right)$ (we assume that both are C^1 functions), we produce a second nodal solution, for a total of four nontrivial solutions, all with sign information. Finally, for both the nonlinear and the semilinear problems, we show that the solutions produced converge to zero in $C^1\left(\overline{\Omega}\right)$ as $\lambda \to \infty$.

The first paper dealing with equations which have reaction terms that are only locally defined is the work of Wang [14]. In that paper, the author deals with a semilinear Dirichlet equation driven by the Laplacian and with a reaction of the form $x \to \lambda\left|x\right|^{q-2}x + g\left(z, x\right)$, where $1 < q < 2$. So, in the reaction we encounter a parametric concave term and a perturbation $g \in C\left(\Omega \times \mathbb{R}, \mathbb{R}\right)$, which is odd in $x \in \mathbb{R}$ for $|x|$ small, and $\lim_{x\to 0}\frac{g(z,x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$. No other conditions are imposed on g. In particular, there are no conditions on $g\left(z, \cdot\right)$ for $|x|$ big. The symmetry of the reaction near zero permits the use of a symmetric mountain pass theorem, and so the author shows that for all $\lambda > 0$, the problem has a sequence $\{u_n\}_{n\geq 1} \subseteq H_0^1\left(\Omega\right)$ of weak solutions such that $\left\|u_n\right\|_\infty \to 0$ as $n \to \infty$. No sign information is given for the solutions produced. Later, Li-Wang [7] extended the result to Schrödinger equations, and in addition proved that the solutions are nodal.

More recently, Papageorgiou-Radulescu [9] and Papageorgiou-Radulescu-Repovs [12] extended the aforementioned works to nonlinear, nonhomogeneous Robin problems, while very recently Aizicovici-Papageorgiou-Staicu [1] obtained similar results for anisotropic (p, q)-equations. All these papers impose a local symmetry condition on the reaction, which permits the use of some version of the symmetric mountain pass theorem. No such symmetry condition is employed here.

2. Mathematical Background - Hypotheses

In the analysis of problem (P_λ) we will use the the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, and the Banach space $C^1(\overline{\Omega})$. By $\|.\|$ we will denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p\right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\|.\|_p$ stands for the L^p-norm. The space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \left\{u \in C^1\left(\overline{\Omega}\right) : u\left(z\right) \geq 0 \text{ for all } z \in \Omega\right\}.$$

This cone has a nonempty interior given by|

$$int\, C_+ = \left\{u \in C_+ : u\left(z\right) > 0 \text{ for all } z \in \overline{\Omega}\right\},$$

If $u,\ v \in W^{1,p}(\Omega$ and $u\left(z\right) \leq v\left(z\right)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \left\{y \in W^{1,p}(\Omega) : u\left(z\right) \leq y\left(z\right) \leq v\left(z\right) \text{ for a.a. } z \in \Omega\right\}.$$

Also by $int_{C^1(\overline{\Omega})}[u, v]$ with denote the interior in $C^1(\overline{\Omega})$ of $[u, v] \bigcap C^1(\overline{\Omega})$.

On $\partial\Omega$ we consider the $(N-1)$ $-$dimensional Hausdorff (surface) measure $\sigma\left(\cdot\right)$. Having this measure, we can define in the usual way the boundary Lebesgue spaces $L^s\left(\partial\Omega\right)$ $(1 \leq s \leq \infty)$. We recall that there exists a unique continuous linear linear map $\gamma_0 : W^{1,p}(\Omega \to L^p\left(\partial\Omega\right)$ known as the "trace map", such that

$$\gamma_0\left(u\right) = u\mid_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega)\bigcap C(\overline{\Omega}).$$

So, the trace map extends to all Sobolev functions the notion of boundary value. We know that γ_0 is compact from $W^{1,p}(\Omega)$ into $L^p\left(\partial\Omega\right)$, $\operatorname{Im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$ and $ker\gamma_0 = W_0^{1,p}(\Omega)$

In the sequel for the sake of notational simplicity, we drop the use of the trace map γ_0. All restrictions of Sobolev functions to $\partial\Omega$ are understood in the sense of traces.

If $x \in \mathbb{R}$, then we set

$$x^{\pm} = \max\left\{\pm x, 0\right\}.$$

For $u \in W^{1,p}\left(\Omega\right)$, we define $u^{\pm}\left(z\right) = u\left(z\right)^{\pm}$ for a.a. $z \in \Omega$. We know that

$$u^{\pm} \in W^{1,p}\left(\Omega\right),\ u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

Given a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$, we say that it satisfies the Ambrosetti-Rabinowitz condition (the AR-condition for short), if there exist $M > 0$ and $q > p$ such that:

$$0 < qF_0\left(z, x\right) \leq f_0\left(z, x\right)x \text{ for a.a. } z \in \Omega,\ \text{all } |x| \geq M,$$

where $F_0\left(z, x\right) = \int\limits_0^x f_0\left(z, s\right)ds$, and

$$0 < \operatorname*{essinf}_{\Omega} F_0\left(\cdot, \pm M\right).$$

This condition is very convenient for the verification of the Palais-Smale condition (the PS-condition for short).

Recall that if X is a Banach space and $\varphi \in C^1\left(X, \mathbb{R}\right)$, then we say that φ satisfies the PS-condition, if every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi\left(u_n\right)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'\left(u_n\right) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence.

By $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ we denote the nonlinear operator defined by

$$\langle A(u), h\rangle = \int_\Omega |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u,\ h \in W^{1,p}(\Omega).$$

This operator has the following properties (see Gasinski-Papageorgiou [3], Problem 2.192, p.279):

- it is bounded (that is, it maps bounded sets to bounded sets);
- it is continuous and monotone (hence maximal monotone too);
- it is of type $(S)_+$, that is, *for every sequence* $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ *such that* $u_n \xrightarrow{w} u$ *in* $W^{1,p}(\Omega)$ *and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0,$$

one has

$$u_n \to u \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty.$$

Here $\xrightarrow{w}$ designates the weak convergence in $W^{1,p}(\Omega)$ and $\langle\cdot,\cdot\rangle$ denotes the duality brackets for the pair $\left(W^{1,p}(\Omega)^*, W^{1,p}(\Omega)\right)$.

Let $\mathcal{S} \subseteq W^{1,p}(\Omega)$. We say that S is downward directed (resp. upward directed), if for all $u_1,\ u_2 \in \mathcal{S}$ we can find $\widehat{u} \in \mathcal{S}$ such that $\widehat{u} \leq u_1$ and $\widehat{u} \leq u_2$ (resp. for all $v_1,\ v_2 \in \mathcal{S}$, we can find $\widehat{v} \in \mathcal{S}$ such that $v_1 \leq \widehat{v}$ and $v_2 \leq \widehat{v}$).

Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\} \text{ (the critical set of } \varphi),$$

and

$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \text{ (the sublevel of } \varphi \text{ at } c).$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subset Y_1 \subset X$. For every $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}- relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Recall that for $k \in -\mathbb{N}$ we have $H_k(Y_1, Y_2)$. Suppose $u \in K_\varphi$ is isolated and let $c = \varphi(u)$. Then the *critical groups of* φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Now suppose that $\varphi \in C^1(X, \mathbb{R})$ satisfies the $PS-$condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the *critical groups of* φ *at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \in \mathbb{N}_0.$$

By the second deformation theorem (see Papageorgiou-Radulescu-Repovs [13], Theorem 5.3.12, p.386), this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Indeed if $c' < c < \inf \varphi(K_\varphi)$, then $\varphi^{c'}$ is a strong deformation retract of φ^c (see [13], p.386) and so,

$$H_k(X, \varphi^c) = H_k\left(X, \varphi^{c'}\right) \text{ for all } k \in \mathbb{N}_0$$

(see [13], Corollary 6.1.24, p.468).

Suppose that K_φ is finite. We introduce the following quantities:

$$M(t,u)=\sum_{k\in\mathbb{N}_0} rank\ C_k(\varphi,u)\,t^k \text{ for all } t\in\mathbb{R},\ \text{all } u\in K_\varphi,$$

$$P(t,\infty)=\sum_{k\in\mathbb{N}_0} rank\ C_k(\varphi,\infty)\,t^k \text{ for all } t\in\mathbb{R}.$$

Then the "Morse relation" says that

$$\sum_{u\in K_\varphi} M(t,u)=P(t,\infty)+(1+t)\,Q(t)\,, \tag{2.1}$$

where

$$Q(t)=\sum_{k\in\mathbb{N}_0}\beta_k t^k$$

is a formal series in $t\in\mathbb{R}$ with nonnegative integer coefficients.

Now we introduce the hypotheses on the data of problem (P_λ).

$\mathbf{H}(\xi)$: $\xi\in L^\infty(\Omega)$, $\xi(z)\geq 0$ for a.a. $z\in\Omega$;

$\mathbf{H}(\beta)$: $\beta\in C^{0,\alpha}(\Omega)$ with $\alpha\in(0,1)$, $\beta(z)\geq 0$ for all $z\in\Omega$;

$\mathbf{H}_0$: $\xi\not\equiv 0$ or $\beta\not\equiv 0$.

Remark: If $\beta\equiv 0$, then we recover the Neumann problem.

$\mathbf{H}(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$ and

(i) there exists $r\in(p,p^*)$ such that

$$\lim_{x\to 0}\frac{f(z,x)}{|x|^{r-2}x}=0 \text{ uniformly for a.a. } z\in\Omega,$$

where

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if } p<N\\ +\infty & \text{if } N\leq p;\end{cases}$$

(ii) if $F(z,x)=\int_0^x f(z,s)\,ds$, then there exists $\tau\in(r,p^*)$ such that

$$\lim_{x\to\infty}\frac{F(z,x)}{x^\tau}=+\infty \text{ uniformly for a.a. } z\in\Omega.$$

Remarks: We emphasize that this reaction term is only locally defined. No conditions are imposed on $f(z,x)$ for $|x|$ big. We also point out that no sign condition is imposed on $f(z,\cdot)$.

$\mathbf{H}(g)$: $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $g(z,0)=0$ for a.a. $z\in\Omega$ and

(i) there exist $a\in L^\infty(\Omega)$ and $1<p<d<p^*$ such that

$$|g(z,x)|\leq a(z)\left[1+|x|^{d-1}\right] \text{ for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R};$$

(ii) If $G(z,x)=\int_0^x g(z,s)\,ds$, then there exists $q\in(p,r)$ (see hypothesis $\mathbf{H}(f)(i)$) and $M>0$ such that

$$0<qG(z,x)\le g(z,x)\,x \text{ for a.a. } z\in\Omega,\ \text{all } |x|\ge M,$$

and

$$0\le \operatorname*{essinf}_{\Omega} G(\cdot,\pm M);$$

(iii) there exists $c_0>0$ such that

$$0\le g(z,x)\,x\le c_0|x|^r \text{ for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R}.$$

Remarks: We see that for a.a. $z\in\Omega$, $g(z,\cdot)$ satisfies the AR-condition (see $\mathbf{H}(g)(ii)$). Moreover, $g(z,\cdot)$ satisfies a global sign condition (see $\mathbf{H}(g)(iii)$).

In what follows by $\gamma:W^{1,p}(\Omega)\to\mathbb{R}$ we denote the C^1-functional defined by

$$\gamma(u)=\|Du\|_p^p+\int_\Omega \xi(z)|u|^p\,dz+\int_{\partial\Omega}\beta(z)|u|^p\,d\sigma \text{ for all } u\in W^{1,p}(\Omega).$$

Hypotheses $\mathbf{H}(\xi),\mathbf{H}(\beta),\mathbf{H}_0$ together with Lemma 4.11 of Mugnai-Papageorgiou [8] and Proposition 2.3 of Gasinski-Papageorgiou [4] imply that

$$C_1\|u\|^p\le\gamma(u) \text{ for some } C_1>0,\ \text{all } u\in W^{1,p}(\Omega). \tag{2.2}$$

On account of hypotheses $\mathbf{H}(f)(i)$, (ii), we can find $\delta_0>0$ such that

$$|f(z,x)|\le|x|^{r-1},\ \ |F(z,x)|\le\frac{1}{r}|x|^r,\ F(z,x)\ge|x|^\tau \quad \text{for a.a. } z\in\Omega,\ \text{all } |x|\le\delta_0. \tag{2.3}$$

Let $\theta\in(0,\delta_0)$ and consider the cut-off function $\eta\in C_c^1(\mathbb{R})$ such that

$$\operatorname{supp}\eta\subseteq[-\theta,\theta],\ 0\le\eta\le1,\ \eta|_{[-\frac{\theta}{2},\frac{\theta}{2}]}\equiv1. \tag{2.4}$$

Using this cut-off function, we introduce the following modification of the parametric, locally defined reaction term

$$\widehat{f}_\lambda(z,x)=\eta(x)\lambda f(z,x)+[1-\eta(x)]|x|^{r-2}x. \tag{2.5}$$

This is a Carathéodory function. We consider the positive and negative truncations of $\widehat{f}_\lambda(z,\cdot)$, namely the Carathéodory functions

$$\widehat{f}_\lambda^\pm(z,x)=\widehat{f}_\lambda\left(z,\pm x^\pm\right).$$

We set

$$\widehat{F}_\lambda^\pm(z,x)=\int_0^x\widehat{f}_\lambda^\pm(z,s)\,ds.$$

Also, we introduce the positive and negative truncations of $g(z,\cdot)$, namely the Carathéodory functions

$$g_\pm(z,x)=g\left(z,\pm x^\pm\right).$$

We set

$$G_{\pm}(z,x)=\int_0^x g_{\pm}(z,x)\,ds.$$

Finally we define

$$\widehat{\zeta}_\lambda^{\pm}(z,x)=\widehat{f}_\lambda^{\pm}(z,x)+g_{\pm}(z,x)\ \text{ for }\ (z,x)\in\Omega\times\mathbb{R}.$$

These are Carathéodory functions.

Proposition 2.1. *If hypotheses* $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then for every* $\lambda>0$, *the functions* $\widehat{\zeta}_\lambda^{\pm}(z,\cdot)$ *satisfy the AR condition.*

Proof. On account of hypothesis $\mathbf{H}(g)(ii)$, it suffices to show that $\widehat{f}_\lambda^{+}(z,\cdot)$ satisfies the AR condition. First we note that (2.3), (2.4) and (2.5) imply

$$\left|\widehat{f}_\lambda(z,x)\right|\le C_2|x|^{r-1}\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }\ x\in\mathbb{R}, \tag{2.6}$$

with $C_2=C_2(\lambda)>0$, hence

$$\left|\widehat{F}_\lambda(z,x)\right|\le\frac{C_2}{r}|x|^r\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }\ x\in\mathbb{R}. \tag{2.7}$$

Let $x>\theta$. We have

$$\begin{aligned}\widehat{F}_\lambda^{+}(z,x)&=\int_0^x\widehat{f}_\lambda^{+}(z,s)\,ds=\int_0^x\widehat{f}_\lambda(z,s)\,ds\\&=\int_0^x\left[\eta(s)\lambda f(z,s)+[1-\eta(s)]s^{r-1}\right]ds\ (\text{see }(2.5))\\&=\int_0^\theta\left[\eta(s)\lambda f(z,s)+[1-\eta(s)]s^{r-1}\right]ds+\int_\theta^x s^{r-1}ds\ (\text{see }(2.4))\\&\le C_3\lambda\theta^r+\tfrac{1}{r}x^r\ \text{ for some }C_3>0.\end{aligned} \tag{2.8}$$

Since $x>\theta$, from (2.4) and (2.5) it follows that

$$\widehat{f}_\lambda^{+}(z,x)=x^{r-1}. \tag{2.9}$$

Then with $q\in(p,r)$ as in hypothesis $\mathbf{H}(g)(ii)$, we have

$$\widehat{f}_\lambda^{+}(z,x)x-q\widehat{F}_\lambda^{+}(z,x)\ge\left[1-\frac{q}{r}\right]x^r-qC_3\lambda\theta^r\ (\text{see }(2.8),(2.9)). \tag{2.10}$$

Choose $M_+>\max\{M,\theta\}$ (see $\mathbf{H}(g)(ii)$) big such that

$$\left[1-\frac{q}{r}\right]M_+^r>qC_2\lambda\theta^r\ (\text{recall }q<r).$$

So, from (2.10) we have

$$\widehat{f}_\lambda^{+}(z,x)x\ge q\widehat{F}_\lambda^{+}(z,x)\ \text{ for a.a. }\ z\in\Omega,\ \text{ all }\ x\ge M_+.$$

Also note that for $x \geq M_+$, we have

$$\begin{aligned}\widehat{F}_\lambda^+(z,x) &= \int_0^\theta \widehat{f}_\lambda^+(z,s)\,ds + \int_\theta^x \widehat{f}_\lambda^+(z,s)\,ds \\ &\geq -C_2\int_0^\theta s^{r-1}ds + \frac{1}{r}\left[x^r - \theta^r\right] \text{ (see (2.6) and (2.9))} \\ &= \frac{1}{r}x^r - \frac{C_4}{r}\theta^r \text{ for some } C_4 > 0.\end{aligned}$$

Choosing M_+ even bigger if necessary, we may assume that

$$M_+^r > C_4\theta^r.$$

Therefore we have

$$\operatorname*{essinf}_{\Omega} \widehat{F}_\lambda^+(\cdot, M_+) > 0 \text{ and } \widehat{F}_\lambda^+(z,x) > 0 \text{ for a.a. } z \in \Omega, \text{ all } x \geq M_+.$$

This proves that $\widehat{\zeta}_\lambda^+(z,\cdot)$ satisfies the AR condition. Similarly we show that $\widehat{\zeta}_\lambda^-(z,\cdot)$ satisfies the AR condition. ☐

3. Nonlinear problems

Let by $\widehat{\varphi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functionals defined by

$$\widehat{\varphi}_\lambda^\pm(u) = \frac{1}{p}\gamma(u) - \int_\Omega \left[\widehat{F}_\lambda^\pm(z,x) + G^\pm(z,u)\right] dz \text{ for all } u \in W^{1,p}(\Omega).$$

Proposition 3.1. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold and* $\lambda \geq 1$, *then we can find* $\rho_\lambda > 0$ *and* $\widehat{m}_\lambda > 0$ *such that*

$$\widehat{\varphi}_\lambda^\pm(u) \geq \widehat{m}_\lambda > 0 \textit{ for all } u \in W^{1,p}(\Omega) \textit{ with } \|u\| = \rho_\lambda.$$

Proof. Using (2.2), (2.7), hypothesis $\mathbf{H}(g)(ii)$ and the fact that $\lambda \geq 1$, we obtain

$$\widehat{\varphi}_\lambda^\pm(u) \geq C_1\|u\|^p - \lambda C_5\|u\|^r \text{ for some } C_5 > 0, \text{ all } u \in W^{1,p}(\Omega),$$

hence

$$\widehat{\varphi}_\lambda^\pm(u) \geq \left[C_1 - \lambda C_5\|u\|^{r-p}\right]\|u\|^p.$$

Therefore if $\rho_\lambda \in \left(0, \left(\frac{C_1}{\lambda C_5}\right)^{\frac{1}{r-p}}\right)$, then

$$\widehat{\varphi}_\lambda^\pm(u) \geq \widehat{m}_\lambda := \rho_\lambda^p\left[C_1 - \lambda C_5^{r-p}\rho_\lambda^{r-p}\right] > 0$$
$$\text{for all } u \in W^{1,p}(\Omega) \text{ with } \|u\| = \rho_\lambda.$$

☐

Proposition 3.2. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then there exist* $\widetilde{u} \in W^{1,p}(\Omega)$, $\widetilde{u} \geq 0$ *and* $\widetilde{\lambda}_1 \geq 1$ *such that for all* $\lambda \geq \widetilde{\lambda}_1$ *we have*

$$\widehat{\varphi}_\lambda^\pm(\pm\widetilde{u}) < 0 \textit{ and } \|\widetilde{u}\| > \rho_\lambda.$$

Proof. Let $\widetilde{u} = \frac{\theta}{2} \in W^{1,p}(\Omega)$. Then from (2.3), (2.5) and hypothesis $\mathbf{H}(g)(iii)$, we have

$$\begin{aligned} \widehat{\varphi}_{\lambda}^{\pm}(\widetilde{u}) &\leq \frac{\widetilde{u}^p}{p}\left[\|\xi\|_{\infty}|\Omega|_N + \|\beta\|_{L^{\infty}(\partial\Omega)}\sigma(\partial\Omega)\right] - \int_{\Omega}\lambda F(z,\widetilde{u})\,dz \\ &\leq C_6\widetilde{u}^p - \lambda\widetilde{u}^{\tau} \text{ for some } C_6 > 0 \text{ (see (2.3)).} \end{aligned}$$

Here by $|\cdot|_N$ we denote the Lebesgue measure in $\mathbb{R}^N$.

We choose $\widetilde{\lambda}_0 \geq 1$ such that

$$\widehat{\varphi}_{\lambda}^{\pm}(\widetilde{u}) < 0 \text{ for all } \lambda \geq \widetilde{\lambda}_0. \tag{3.1}$$

From the proof of Proposition 3.1, we know that

$$\rho_{\lambda} \to 0+ \text{ as } \lambda \to \infty.$$

So, we can find $\widetilde{\lambda}_1 \geq \widetilde{\lambda}_0 \geq 1$ such that

$$\|\widetilde{u}\| > \rho_{\lambda} \text{ for all } \lambda \geq \widetilde{\lambda}_1.$$

We conclude that for $\widetilde{u} = \frac{\theta}{2} \in int\ C_+$ and for $\lambda \geq \widetilde{\lambda}_1$ we have

$$\widehat{\varphi}_{\lambda}^{\pm}(\pm\widetilde{u}) < 0 \text{ and } \|\widetilde{u}\| > \rho_{\lambda}.$$

□

From Proposition 2.1, we know that the integrands $\widehat{\zeta}_{\lambda}^{\pm}(\cdot,\cdot)$ satisfy the AR-condition. So, we have the following result (see Ambrosetti-Rabinowitz [2]):

Proposition 3.3. *If hypotheses $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ hold, then for every $\lambda > 0$, the functionals $\widehat{\varphi}_{\lambda}^{\pm}$ satisfy the PS-condition.*

We consider the following nonlinear parametric Robin problem

$$(Q_{\lambda}) \qquad \begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{f}_{\lambda}(z,u(z)) + g(z,u(z)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega,\ \lambda > 0,\ 1 < p < \infty. \end{cases}$$

Using variational tools, we can show the existence of constant sign solutions of (Q_{λ}) when $\lambda \geq 1$ is big.

Proposition 3.4. *If hypotheses $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ hold, and $\lambda \geq \widetilde{\lambda}_1$ (see Proposition 3.2), then problem (Q_{λ}) has at least two constant sign solutions $u_{\lambda} \in int\ C_+$ and $v_{\lambda} \in -int\ C_+$.*

Proof. Propositions 3.1, 3.2 and 3.3 permit the use of the mountain pass theorem [2]. So, we can find $u_{\lambda} \in W^{1,p}(\Omega)$ such that

$$u_{\lambda} \in K_{\widehat{\varphi}_{\lambda}^{+}} \text{ and } \widehat{\varphi}_{\lambda}^{+}(0) = 0 < \widehat{m}_{\lambda} \leq C_{\lambda} = \widehat{\varphi}_{\lambda}^{+}(u_{\lambda}). \tag{3.2}$$

From (3.2) we have that $u_{\lambda} \neq 0$ and

$$\left(\widehat{\varphi}_{\lambda}^{+}\right)'(u_{\lambda}) = 0.$$

Hence

$$\begin{aligned}\langle A(u_\lambda), h\rangle &+ \int_\Omega \xi(z)|u_\lambda(z)|^{p-2}u_\lambda(z)hdz \\ &+\int_{\partial\Omega}\beta(z)|u_\lambda(z)|^{p-2}u_\lambda(z)hd\sigma \\ &= \int_\Omega \left[\widehat{f}_\lambda^+(z,u_\lambda)+g_+(z,u_\lambda)\right]hdz \text{ for all } h\in W^{1,p}(\Omega).\end{aligned} \tag{3.3}$$

In (3.3) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. We obtain

$$C_1\left\|u_\lambda^-\right\|^p \le 0 \text{ (see (3.2))},$$

therefore

$$u_\lambda \ge 0,\ u_\lambda \ne 0.$$

Then from (3.2) we have

$$\begin{cases} -\Delta_p u_\lambda(z) + \xi(z)u_\lambda(z)^{p-1} = \widehat{f}_\lambda(z,u_\lambda(z)) + g(z,u_\lambda(z)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for a.a. } z\in\Omega, \\ \frac{\partial u_\lambda}{\partial n_p} + \beta(z)u_\lambda^{p-1} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

From (3.4) and Proposition 2.10 of Papageorgiou-Radulescu [10], we infer that $u_\lambda \in L^\infty(\Omega)$. Then we apply Theorem 2 of Lieberman [6] and obtain that

$$u_\lambda \in C_+\backslash\{0\}.$$

From (3.4) it follows

$$\Delta_p u_\lambda(z) \le \left[\|\xi\|_\infty + 2\|u_\lambda\|_\infty^{r-p}\right]u_\lambda(z)^{p-1} \text{ for a.a. } z\in\Omega$$

(see (2.3), (2.5) and hypothesis $\mathbf{H}(g)(iii)$) and by the nonlinear maximum principle we get

$$u_\lambda \in int\ C_+.$$

Similarly, working this time with $\widehat{\varphi}_\lambda^-$, we produce a negative solution

$$v_\lambda \in -int\ C_+.$$

□

Next we determine the behavior of u_λ and v_λ as $\lambda\to\infty$.

Proposition 3.5. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta)$, $\mathbf{H}_0$, $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then*

$$u_\lambda \to 0 \text{ and } v_\lambda\to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } \lambda\to+\infty.$$

Proof. Let $\lambda_n\to+\infty$ and consider $u_n = u_{\lambda_n}\in int\ C_+$ be positive solutions of problem (Q_{λ_n}), $n\in\mathbb{N}$. From the proof of Proposition 3.4, we know that

$$\widehat{m}_{\lambda_n} \le C_{\lambda_n} = \widehat{\varphi}_{\lambda_n}^+(u_n) = \inf_{\gamma\in\Gamma}\max_{0\le s\le 1}\widehat{\varphi}_{\lambda_n}^+(\widetilde{\gamma}(s)), \tag{3.5}$$

where

$$\Gamma = \left\{\widetilde{\gamma}\in C\left([0,1], W^{1,p}(\Omega)\right) : \widetilde{\gamma}(0)=0, \widetilde{\gamma}(1)=\widetilde{u}\right\}$$

From (3.5) we have

$$\widehat{\varphi}_{\lambda_n}^+(u_n) \le \max_{0\le s\le 1}\widehat{\varphi}_{\lambda_n}^+(s\widetilde{u}). \tag{3.6}$$

Also (2.3), (2.4), (2.5) and hypothesis $\mathbf{H}(g)(iii)$ imply that

$$\widehat{\varphi}_{\lambda_n}(s\widetilde{u}) \leq C_7 s^p - \lambda_n C_8 s^\tau \text{ for some } C_7 > 0,\ C_8 > 0.$$

We consider the function

$$\mu_{\lambda_n}(s) = C_7 s^p - C_8 s^\tau \text{ for all } s \geq 0, \text{ with } n \in \mathbb{N}.$$

Evidently since $p < \tau$, we can find $s_0 > 0$ such that

$$0 < \mu_{\lambda_n}(s_0) = \max_{s \geq 0} \mu_{\lambda_n}(s),$$

hence

$$\mu'_{\lambda_n}(s_0) = 0,$$

therefore

$$s_0 = s_0(\lambda_n) = \left[\frac{pC_7}{\lambda_n \tau C_8}\right]^{\frac{1}{\tau - p}}. \tag{3.7}$$

Using (3.7) we obtain

$$\mu_{\lambda_n}(s_0) \leq C_7 \left[\frac{pC_7}{\lambda_n \tau C_8}\right]^{\frac{p}{\tau - p}} = C_9 \lambda^{-\frac{p}{\tau - p}} \text{ for some } C_9 > 0, \text{ all } n \in \mathbb{N}. \tag{3.8}$$

From (3.6) we have

$$\widehat{\varphi}^+_{\lambda_n}(u_n) \leq \mu_{\lambda_n}(s_0) \leq C_9 \lambda^{-\frac{p}{\tau - p}} \text{ for all } n \in \mathbb{N} \text{ (see (3.8))},$$

hence

$$q\widehat{\varphi}^+_{\lambda_n}(u_n) + \left\langle \left(\widehat{\varphi}^+_{\lambda_n}\right)'(u_n), u_n \right\rangle \leq qC_9 \lambda^{-\frac{p}{\tau - p}} \text{ for all } n \in \mathbb{N},$$

therefore

$$\begin{aligned} &\left[\frac{q}{p} - 1\right] \gamma(u_n) \\ &+ \int_\Omega \left[\left(\widehat{f}^+_{\lambda_n}(z, u_n) + g_+(z, u_n)\right) u_n - q\widehat{F}^+_{\lambda_n}(z, u_n) + G_+(z, u_n)\right] dz \\ &\leq qC_9 \lambda^{-\frac{p}{\tau - p}}, \end{aligned}$$

and in view of Proposition 2.1 and hypothesis $\mathbf{H}(g)(ii)$ we conclude that

$$\|u_n\|^p \leq C_{10} \text{ for some } C_{10} > 0, \text{all } n \in \mathbb{N}.$$

Therefore $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Then Proposition 2.10 of Papageorgiou-Radulescu [10] implies that we can find $C_{11} > 0$ such that

$$\|u_n\|_\infty \leq C_{11} \text{ for all } n \in \mathbb{N}$$

Invoking Theorem 2 of Lieberman [6], we can find $\alpha \in (0, 1)$ and $C_{12} > 0$ such that

$$u_n \in C^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_{12} \text{ for all } n \in \mathbb{N}.$$

We know that $C^{1,\alpha}\left(\overline{\Omega}\right)$ is compactly embedded in $C^1\left(\overline{\Omega}\right)$, so for at least a subsequence we have

$$u_n \to \overline{u} \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

By (3.5) and (3.8) we infer

$$\widehat{\varphi}^+_{\lambda_n}(u_n) \to 0^+ \text{ as } n \to \infty. \tag{3.9}$$

Moreover, we have

$$\left\langle \left(\widehat{\varphi}_{\lambda_n}^{+}\right)'(u_n), h\right\rangle = 0 \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{3.10}$$

Since $\lambda_n \to +\infty$, from (3.9) and (3.10) it follows that $\overline{u} = 0$. Therefore we conclude that

$$u_n \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

Similarly, working this time with $\widehat{\varphi}_{\lambda_n}^{-}(\cdot)$ we show that

$$v_{\lambda_n} \to 0 \text{ in } C^1\left(\overline{\Omega}\right) \text{ as } n \to \infty.$$

□

Now we will produce extremal constant sign solutions for problem (Q_λ), that is, we will show that for $\lambda > 0$ big, problem (Q_λ) has a smallest positive solution and a biggest negative solution

So, we consider the following two solution sets

$$\widehat{\mathcal{S}}_\lambda^{+} = \{u : u \text{ is a positive solution of } (Q_\lambda)\},$$

$$\widehat{\mathcal{S}}_\lambda^{-} = \{u : u \text{ is a negative solution of } (Q_\lambda)\}.$$

From Proposition 3.4 it follows that for $\lambda \geq \widetilde{\lambda}_1$

$$\varnothing \neq \widehat{\mathcal{S}}_\lambda^{+} \subseteq int\ C_+ \text{ and } \varnothing \neq \widehat{\mathcal{S}}_\lambda^{-} \subseteq -int\ C_+ .$$

Moreover, from Papageorgiou-Radulescu-Repovs [11] (see the proof of Proposition 7), we know that

$$\widehat{\mathcal{S}}_\lambda^{+} \text{ is downward directed}$$

and

$$\widehat{\mathcal{S}}_\lambda^{-} \text{ is upward directed.}$$

Proposition 3.6. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g),$ *hold and* $\lambda \geq \widetilde{\lambda}_1$, *then problem* (Q_λ) *has a smallest positive solution* $u_\lambda^* \in int\ C_+$ *and a biggest negative solution* $v_\lambda^* \in -int\ C_+$.

Proof. By Lemma 3.10, p.178 of Hu-Papageorgiou [5], we can find a decreasing sequence $\{u_n\}_{n\geq 1} \subseteq \widehat{\mathcal{S}}_\lambda^{+}$ such that

$$\inf_{n\geq 1} u_n = \inf \widehat{\mathcal{S}}_\lambda^{+}.$$

We have

$$\begin{aligned} \langle A(u_n), h\rangle + \int_\Omega \xi(z) u_n(z)^{p-1} h dz + \int_{\partial\Omega} \beta(z) u_n(z)^{p-1} h d\sigma \\ = \int_\Omega \left[\widehat{f}_\lambda(z, u_n) + g_+(z, u_n)\right] h dz \\ \text{for all } n \in \mathbb{N}, \text{ all } h \in W^{1,p}(\Omega), \end{aligned} \tag{3.11}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{3.12}$$

In (3.11) we chose $h = u_n \in W^{1,p}(\Omega)$ and using (3.12) and (2.2), we infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.13}$$

In (3.11) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.13). We obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda^* \rangle = 0,$$

hence

$$u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \tag{3.14}$$

(see Section 2). We pass to the limit as $n \to \infty$ in (3.11) and use (3.14). Then

$$\begin{aligned}&\langle A(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1} h dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1} h d\sigma \\ &= \int_\Omega \left[\widehat{f}_\lambda(z, u_\lambda^*) + g(z, u_\lambda^*)\right] h dz \text{ for all } h \in W^{1,p}(\Omega),\end{aligned}$$

hence $u_\lambda^* \in \widehat{\mathcal{S}}_\lambda^+ \cup \{0\}$. If we show that $u_\lambda^* \neq \{0\}$, then u_λ^* is the desired minimal positive solution 0f (Q_λ).

We argue indirectly. So, suppose that $u_\lambda^* = 0$. Then $u_n \to 0$ in $W^{1,p}(\Omega)$ (see (3.14)). We set

$$y_n = \frac{u_n}{\|u_n\|},\ n \in \mathbb{N}.$$

Then

$$\|y_n\| = 1, y_n > 0 \text{ for all } n \in \mathbb{N}.$$

We may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and } L^p(\partial\Omega). \tag{3.15}$$

From (3.11) we have

$$\begin{aligned}&\langle A(y_n), h\rangle + \int_\Omega \xi(z) {y_n}^{p-1} h dz + \int_{\partial\Omega} \beta(z) {y_n}^{p-1} h d\sigma \\ &= \int_\Omega \left[\tfrac{\widehat{f}_\lambda(z,u_n)}{\|u_n\|^{p-1}} + \tfrac{g(z,u_n)}{\|u_n\|^{p-1}}\right] h dz \text{ for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{3.16}$$

By (2.3) and (2.5) we see that

$$\left\{\frac{\widehat{f}_\lambda(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{r'}(\Omega) \text{ is bounded, where } \frac{1}{r} + \frac{1}{r'} = 1. \tag{3.17}$$

Similarly from hypothesis $\mathbf{H}(g)(i)$ it follows that

$$\left\{\frac{g(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{r'}(\Omega) \text{ is bounded.} \tag{3.18}$$

If in (3.16) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.15), (3.17) and (3.18), we obtain

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,$$

hence

$$y_n \to y \text{ in } W^{1,p}(\Omega) \text{ (see Section 2), with } \|y\| = 1. \tag{3.19}$$

On account of (3.17), (3.18), (2.3), (2.5) and hypothesis $\mathbf{H}(g)(iii)$, we have

$$\frac{\widehat{f}_\lambda(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} 0 \text{ and } \frac{g(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \xrightarrow{w} 0 \text{ in } L^{r'}(\Omega). \tag{3.20}$$

So, if in (3.16)we pass to the limit as $n \to \infty$ and use (3.19) and (3.20), then

$$\langle A(y), h\rangle + \int_\Omega \xi(z) y^{p-1} h dz + \int_{\partial\Omega} \beta(z) y^{p-1} h d\sigma = 0 \text{ for all } h \in W^{1,p}(\Omega).$$

Let $h = y \in W^{1,p}(\Omega)$. Then

$$C_1 \|y\|^p \le 0 \text{ (see (2.2))},$$

hence $y = 0$, which contradicts (3.19). Therefore $u_\lambda^* \neq 0$ and so

$$u_\lambda^* \in \widehat{\mathcal{S}}_\lambda^+ \text{ and } u_\lambda^* = \inf \widehat{\mathcal{S}}_\lambda^+ .$$

Similarly, working with $\widehat{\mathcal{S}}_\lambda^-$, we produce $v_\lambda^* \in \widehat{\mathcal{S}}_\lambda^-$ with $v_\lambda^* = \sup \widehat{\mathcal{S}}_\lambda^-$. In this case, since $\widehat{\mathcal{S}}_\lambda^-$ is upward directed, we can find $\{v_n\}_{n\ge1} \subseteq \widehat{\mathcal{S}}_\lambda^-$ increasing, such that

$$\sup_{n\ge1} v_n = \sup \widehat{\mathcal{S}}_\lambda^-.$$

□

We will use these two extremal constant sign solutions in order to produce a nodal solution for problem (Q_λ) when λ is big enough.

Proposition 3.7. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta)$, $\mathbf{H}_0$, $\mathbf{H}(f)$, $\mathbf{H}(g)$ *hold, then there exists* $\widetilde{\lambda}_2 \ge \widetilde{\lambda}_1$ *such that for all* $\lambda \ge \widetilde{\lambda}_2$, *problem* (Q_λ) *has a nodal solution* $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$.

Proof. Let $u_\lambda^* \in int\ C_+$ and $v_\lambda^* \in -int\ C_+$ be the two extremal constant sign solutions of problem (Q_λ) produced in Proposition 3.6. We introduce the following Carathéodory function

$$\widehat{k}_\lambda(z,x) = \begin{cases} \widehat{f}_\lambda(z, v_\lambda^*(z)) + g(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z) \\ \widehat{f}_\lambda(z,x) + g(z,x) & \text{if } v_\lambda^*(z) \le x \le u_\lambda^*(z) \\ \widehat{f}_\lambda(z, u_\lambda^*(z)) + g(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{3.21}$$

We consider the positive and negative truncations of $\widehat{k}_\lambda(z,\cdot)$, namely the Carathéodory functions

$$\widehat{k}_\lambda^\pm(z,x) = \widehat{k}_\lambda\left(z, \pm x^\pm\right). \tag{3.22}$$

We set

$$\widehat{K}_\lambda(z,x) = \int_0^x \widehat{k}_\lambda(z,s)\,ds \text{ and } \widehat{K}_\lambda^\pm(z,x) = \int_0^x \widehat{k}_\lambda^\pm(z,s)\,ds$$

and introduce the C^1-functionals $\widehat{\psi}_\lambda$, $\widehat{\psi}_\lambda^{\pm}: W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{K}_\lambda(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega)$$

and

$$\widehat{\psi}_\lambda^{\pm}(u) = \frac{1}{p}\gamma(u) - \int_\Omega \widehat{K}_\lambda^{\pm}(z,u)\,dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (3.21), (3.22) and the nonlinear regularity theory, we show easily that

$$K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}),\ K_{\widehat{\psi}_\lambda^+} \subseteq [0, u_\lambda^*] \cap C_+, K_{\widehat{\psi}_\lambda^-} \subseteq [v_\lambda^*, 0] \cap (-C_+).$$

The extremality of u_λ^*, v_λ^* implies that

$$K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}),\ K_{\widehat{\psi}_\lambda^+} = \{0, u_\lambda^*\},\ K_{\widehat{\psi}_\lambda^-} = \{0, v_\lambda^*\}. \tag{3.23}$$

Note that $\widehat{\psi}_\lambda^+$ is coercive (see (3.21), (3.22)). Also it is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$\widehat{\psi}_\lambda^+(\widetilde{u}_\lambda^*) = \inf\left\{\widehat{\psi}_\lambda^+(u) : u \in W^{1,p}(\Omega)\right\}. \tag{3.24}$$

Let

$$u_* = \min\left\{\frac{\theta}{2}, \min_{\overline{\Omega}} u_\lambda^*\right\} > 0$$

(recall that $u_\lambda^* \in int\ C_+$). Then

$$\widehat{\psi}_\lambda^+(u_*) \le C_{13}u_*^p - \lambda C_{14}u_*^\tau \text{ for some } C_{13},\ C_{14} > 0$$

(see (2.3), (2.5) and hypothesis $\mathbf{H}(g)(iii)$). So, we can find $\widetilde{\lambda}_2^+ \ge \widetilde{\lambda}_1$ such that

$$\widehat{\psi}_\lambda^+(u_*) < 0 \text{ for all } \lambda \ge \widetilde{\lambda}_2^+,$$

hence

$$\widehat{\psi}_\lambda^+(u_\lambda^*) < 0 = \widehat{\psi}_\lambda^+(0) \text{ for all } \lambda \ge \widetilde{\lambda}_2^+ \text{ (see (3.24))},$$

therefore

$$\widetilde{u}_\lambda^* \ne 0 \text{ for all } \lambda \ge \widetilde{\lambda}_2^+. \tag{3.25}$$

From (3.24) we have

$$\widetilde{u}_\lambda^* \in K_{\widehat{\psi}_\lambda^+},$$

hence

$$\widetilde{u}_\lambda^* = u_\lambda^* \in int\ C_+ \text{ (see (3.24), (3.25))}.$$

It is clear from (3.22) that

$$\widehat{\psi}_\lambda^+ |_{C_+} = \widehat{\psi}_\lambda |_{C_+},$$

hence u_λ^* is a local $C^1(\overline{\Omega})-$minimizer of $\widehat{\psi}_\lambda$, therefore

$$u_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}_\lambda \text{ for all } \lambda \ge \widetilde{\lambda}_2^+ \tag{3.26}$$

(see Papageorgiou-Radulescu [10], Proposition 2.12).

Similarly, working this time with $\widehat{\psi}_\lambda^-$, we produce $\widetilde{\lambda}_2^- \ge \widetilde{\lambda}_1$ such that

$$v_\lambda^* \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}_\lambda \text{ for all } \lambda \ge \widetilde{\lambda}_2^-. \tag{3.27}$$

Let

$$\widetilde{\lambda}_2 = \max\left\{\widetilde{\lambda}_2^+, \widetilde{\lambda}_2^-\right\}$$

and let $\lambda \geq \widetilde{\lambda}_2$. We may assume that

$$\widehat{\psi}_\lambda(v^*) \leq \widehat{\psi}_\lambda(u^*).$$

The reasoning is similar if the opposite inequality holds, using (3.27) instead of (3.26). Also, we may assume that

$$K_{\widehat{\psi}_\lambda} \text{ is finite.} \tag{3.28}$$

Otherwise, we already have an infinity of smooth nodal solutions.

Using (3.26), (3.28) and Theorem 5.7.6, p. 448, of Papageorgiou-Radulescu-Repovs [13], we can find $\rho \in (0,1)$ small, such that

$$\begin{aligned} &\widehat{\psi}_\lambda(v_\lambda^*) \leq \widehat{\psi}_\lambda(u_\lambda^*) < \inf\left\{\widehat{\psi}_\lambda(u) : \|u - u_\lambda^*\| = \rho\right\} =: \widehat{m}_\lambda, \\ &\|u_\lambda^* - v_\lambda^*\| > \rho. \end{aligned} \tag{3.29}$$

Evidently, $\widehat{\psi}_\lambda(\cdot)$ is coercive (see (3.21)). Therefore

$$\widehat{\psi}_\lambda \text{ satisfies the PS-condition} \tag{3.30}$$

(see Papageorgiou-Radulescu-Repovs [13], Proposition 5.1.15, p.369).

Then (3.29), (3.30) permit the use of the mountain pass theorem. So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{\widehat{\psi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1\left(\overline{\Omega}\right), \ \widehat{m}_\lambda \leq \widehat{\psi}_\lambda(y_\lambda) \tag{3.31}$$

(see (3.23) and (3.29)). From (3.29) and (3.31) it follows that

$$y_\lambda \notin \{u_\lambda^*, v_\lambda^*\}. \tag{3.32}$$

Since y_λ is a critical point of $\widehat{\psi}_\lambda(\cdot)$ of mountain pass type, we have

$$C_1\left(\widehat{\psi}_\lambda, y_\lambda\right) \neq 0. \tag{3.33}$$

(see Papageorgiou-Radulescu-Repovs [13], Theorem 6.5.8, p.527).

On the other hand, if $u \in C^1\left(\overline{\Omega}\right)$ and

$$\|u\|_{C^1(\overline{\Omega})} \leq \rho_0 \leq \min\left\{\frac{\theta}{2}, \min\left\{\min_{\overline{\Omega}} u_\lambda^*, \min_{\overline{\Omega}}(-v_\lambda^*)\right\}\right\}$$

(recall that $u_\lambda^* \in int\ C_+$, $v_\lambda^* \in -int\ C_+$, see Proposition 3.6), then

$$\begin{aligned} \widehat{\psi}_\lambda(u) &= \frac{1}{p}\gamma(u) - \int_\Omega [\lambda F(z,u) + G(z,u)]\,dz \ \text{(see (2.3), (2.5), (3.21))} \\ &\geq \frac{1}{p}\gamma(u) - \frac{1}{r}[\lambda + C_0]\|u\|_r^r \ \text{(see (2.3), and } \mathbf{H}(g)(iii) \\ &\geq \frac{C_1}{p}\|u\|^p - \frac{1}{r}[\lambda + C_0]\|u\|^r \ \text{(see (2.2))}. \end{aligned}$$

Since $r > p$, for $\rho_0 \in (0,1)$ small, we have

$$\widehat{\psi}_\lambda(u) > 0 \text{ for all } 0 < \|u\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

hence $u = 0$ is a local $C^1(\overline{\Omega})$ −minimizer of $\widehat{\psi}_\lambda(\cdot)$, therefore $u = 0$ is a local $W^{1,p}(\Omega)$-minimizer of $\widehat{\psi}_\lambda(\cdot)$ (see [10]), and we conclude that

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \tag{3.34}$$

(where $\delta_{k,l}$ denotes the Kronecker symbol defined by $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$ if $k \neq l$). Comparing (3.33) and (3.34), we infer that $y_\lambda \neq 0$ and so, $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$ is a nodal solution of the problem (Q_λ), for $\lambda \geq \widetilde{\lambda}_2$. □

In view of Proposition 3.5, we arrive at:

Proposition 3.8. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then*

$$u_\lambda^*,\ v_\lambda^*,\ y_\lambda \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to +\infty.$$

Then Proposition 3.8 and (2.5) lead to the following multiplicity theorem for (P_λ).

Theorem 3.9. *If hypotheses* $\mathbf{H}(\xi), \mathbf{H}(\beta), \mathbf{H}_0, \mathbf{H}(f), \mathbf{H}(g)$ *hold, then there exists* $\widetilde{\lambda}_3 \geq \widetilde{\lambda}_2$ *such that for* $\lambda \geq \widetilde{\lambda}_3$, *problem* (P_λ) *has at least three nontrivial solutions*

$$u_\lambda \in int\ C_+,\ v_\lambda \in -int\ C_+ \text{ and } y_\lambda \in [v_\lambda, u_\lambda] \cap C^1(\overline{\Omega}),\ \text{nodal.}$$

Moreover,

$$u_\lambda,\ v_\lambda,\ y_\lambda \to 0 \text{ in } C^1(\overline{\Omega}) \text{ as } \lambda \to +\infty.$$

4. Semilinear problems

In the semilinear case ($p = 2$), under stronger regularity hypotheses on $f(z, \cdot)$ and $g(z, \cdot)$, we can improve Theorem 3.9 by producing a second nodal solution of (P_λ) for a total of four nontrivial solutions, all with sign information.

So, now the problem under consideration is the following

$$(SP_\lambda) \qquad \begin{cases} -\Delta u(z) + \xi(z) u(z) = \lambda f(z, u(z)) + g(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n_p} + \beta(z) u = 0 \text{ on } \partial\Omega,\ \lambda > 0. \end{cases}$$

The conditions on the two nonlinearities $f(z, x)$ and $g(z, x)$ are the following.

$\mathbf{H}(f)'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and

(*i*) there exists $r \in (2, 2^*)$ such that

$$\lim_{x \to 0} \frac{f(z, x)}{|x|^{r-2} x} = 0 \text{ uniformly for a.a. } z \in \Omega;$$

(*ii*) if $F(z, x) = \int_0^x f(z, s)\, ds$, then there exists $\tau \in (r, 2^*)$ such that

$$\lim_{x \to \infty} \frac{F(z, x)}{x^\tau} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

Remark: Hypothesis $\mathbf{H}(f)'(i)$ implies that

$$0 = f_x'(z, 0) = \lim_{x \to 0} \frac{f(z, x)}{x} \text{ uniformly for a.a. } z \in \Omega.$$

H (g): $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that $g(z,0)=0$ for a.a. $z\in\Omega$, $g(z,\cdot)\in C^1(\mathbb{R})$ and

(i) there exist $a\in L^\infty(\Omega)$ and $2<d<2^*$ such that

$$\left|g_x'(z,x)\right|\le a(z)\left[1+|x|^{d-2}\right] \text{ for a.a. } z\in\Omega,\text{ all } x\in\mathbb{R};$$

(ii) If $G(z,x)=\int_0^x g(z,s)\,ds$, then there exist $q\in(2,r)$ and $M>0$ such that

$$0<qG(z,x)\le g(z,x)\,x \text{ for a.a. } z\in\Omega,\text{ all } |x|\ge M,$$

and

$$0\le \operatorname*{essinf}_{\Omega} G(\cdot,\pm M);$$

(iii) there exists $c_0>0$ such that

$$0\le g(z,x)\,x\le c_0|x|^r \text{ for a.a. } z\in\Omega,\text{ all } x\in\mathbb{R}.$$

Remark: Hypothesis $\mathbf{H}(g)'(iii)$ implies that

$$0=g'(z,x)=\lim_{x\to0}\frac{g(z,x)}{x} \text{ uniformly for a.a. } z\in\Omega.$$

$\mathbf{H}_1$: For every $\lambda>0$ and $\rho>0$, there exists $\xi_\rho^\lambda>0$ such that for a.a. $z\in\Omega$, the function $x\to\lambda f(z,x)+g(z,x)+\xi_\rho^\lambda x$ is nondecreasing on $[-\rho,\rho]$.

Remark: This is a lower Lipschitz condition. It is satisfied if for every $\lambda>0$ and $\rho>0$, there exists $\widehat{\xi}_\rho^\lambda>0$ such that

$$\lambda f_x'(z,x)+g_x'(z,x)\ge-\widehat{\xi}_\rho^\lambda \text{ for a.a. } z\in\Omega.,\text{ all } |x|\le\rho.$$

In what follows we set

$$\zeta_\lambda(z,x)=\widehat{f}_\lambda(z,x)+g(z,x),\ \widehat{F}_\lambda(z,x)=\int_0^x\widehat{f}_\lambda(z,s)\,ds$$

and we consider the C^1-functional $\widehat{\varphi}_\lambda:W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u)=\frac{1}{p}\gamma(u)-\int_\Omega\left[\widehat{F}_\lambda(z,x)+G(z,u)\right]dz \text{ for all } u\in W^{1,p}(\Omega).$$

Theorem 4.1. *If hypotheses* $\mathbf{H}(\xi),\mathbf{H}(\beta),\mathbf{H}_0,\mathbf{H}(f)',\mathbf{H}(g)',\mathbf{H}_1$ *hold, then there exists* $\widetilde{\lambda}_3\ge1$ *such that for all* $\lambda\ge\widetilde{\lambda}_3$, *problem* (P_λ) *has at least four nontrivial solutions*

$$u_\lambda\in int\ C_+,\ v_\lambda\in-int\ C_+,\ \textit{and } y_\lambda,\ \widehat{y}_\lambda\in int_{C^1(\overline{\Omega})}[v_\lambda,u_\lambda],\ \textit{nodal.}$$

Proof. From Theorem 3.9, we know that there exists $\widetilde{\lambda}_3\ge1$ such that for all $\lambda\ge\widetilde{\lambda}_3$ problem (P_λ).has at least three nontrivial solutions

$$u_\lambda\in int\ C_+,\ v_\lambda\in-int\ C_+ \text{ and } y_\lambda\in[v_\lambda,u_\lambda]\cap C^1\left(\overline{\Omega}\right) \text{ nodal.} \tag{4.1}$$

Let $\rho = \max\{\|u_\lambda\|_\infty, \|v_\lambda\|_\infty\}$ and let $\widehat{\xi}_\rho^\lambda > 0$ be as postulated by hypothesis $\mathbf{H}_1$. We have

$$\begin{aligned}
&-\Delta y_\lambda + \left[\xi(z) + \widehat{\xi}_\rho^\lambda\right] y_\lambda = \lambda f(z, y_\lambda) + g(z, y_\lambda) + \widehat{\xi}_\rho^\lambda y_\lambda \\
&\leq \lambda f(z, u_\lambda) + g(z, u_\lambda) + \widehat{\xi}_\rho^\lambda u_\lambda \text{ (see (4.1) and } \mathbf{H}_1) \\
&= -\Delta u_\lambda + \left[\xi(z) + \widehat{\xi}_\rho^\lambda\right] u_\lambda
\end{aligned}$$

hence

$$\Delta(u_\lambda - y_\lambda) \leq \left[\|\xi\|_\infty + \widehat{\xi}_\rho^\lambda\right](u_\lambda - y_\lambda),$$

therefore $u_\lambda - y_\lambda \in int\ C_+$ (by the Hopf boundary point theorem). Similarly we show that

$$y_\lambda - v_\lambda \in int\ C_+.$$

It follows that

$$y_\lambda \in int_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda]. \tag{4.2}$$

Consider the homotopy

$$h_t(u) = h(t, u) = (1-t)\widehat{\psi}_\lambda(u) + t\widehat{\varphi}_\lambda(u) \text{ for all } (t, u) \in [0,1] \times H^1(\Omega).$$

Suppose that we could find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{y_n\}_{n\geq 1} \subseteq H^1(\Omega)$ such that

$$t_n \to t \text{ in } [0,1],\ y_n \to y \text{ in } H^1(\Omega), h_t'(y_n) = 0 \text{ for all } n \in \mathbb{N}.$$

We have

$$\begin{aligned}
&\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n h dz + \int_{\partial\Omega} \beta(z) y_n h d\sigma \\
&= (1-t_n)\int_\Omega k_\lambda(z, y_n) h dz + t_n \int_\Omega \zeta_\lambda(z, y_n) h dz \text{ for all } h \in H^1(\Omega).
\end{aligned} \tag{4.3}$$

By (4.3), using standard regularity theory, we show that in fact we have

$$y_n \to y \text{ in } C^1\left(\overline{\Omega}\right)$$

hence

$$y_n \in [v_\lambda, u_\lambda] \text{ for all } n \geq n_0 \text{ (see (4.2))}.$$

This contradicts (3.28). Then, the homotopy invariance property of critical groups (see Papageorgiou-Radulescu-Repovs [13], Theorem 6.3.8, p.505) implies that

$$C_k\left(\widehat{\psi}_\lambda, y_\lambda\right) = C_k(\widehat{\varphi}_\lambda, y_\lambda) \text{ for all } k \in \mathbb{N}_0, \tag{4.4}$$

hence

$$C_1(\widehat{\varphi}_\lambda, y_\lambda) \neq 0 \text{ (see (3.33))}. \tag{4.5}$$

But $\widehat{\varphi}_\lambda \in C^2\left(H^1(\Omega), \mathbb{R}\right)$. So, by (4.5) and Theorem 6.5.11, p.530 of Papageorgiou-Radulescu-Repovs [13], we have

$$C_k(\widehat{\varphi}_\lambda, y_\lambda) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0,$$

hence

$$C_k\left(\widehat{\psi}_\lambda, y_\lambda\right) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0, \text{ (see (4.4))}. \tag{4.6}$$

Recall that u_λ, v_λ are local minimizers of $\widehat{\psi}_\lambda(\cdot)$ (see the proof of Proposition 3.7). Hence

$$C_k\left(\widehat{\psi}_\lambda, u_\lambda\right) = C_k\left(\widehat{\psi}_\lambda, v_\lambda\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.7}$$

Also from (3.34) we have

$$C_k\left(\widehat{\psi}_\lambda, 0\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.8}$$

The functional $\widehat{\psi}_\lambda(\cdot)$ is coercive (see (3.21)). Hence we obtain

$$C_k\left(\widehat{\psi}_\lambda, \infty\right) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.9}$$

Suppose that $K_{\widehat{\psi}_\lambda} = \{0, u_\lambda, v_\lambda, y_\lambda\}$. Then from (4.6), (4.7), (4.8), (4.9) and the Morse relation with $t = -1$ (see (2.1)) it follows

$$3(-1)^0 + (-1)^1 = (-1)^0,$$

therefore $(-1)^0 = 0$, a contradiction.

So, there exists $\widehat{y}_\lambda \in K_{\widehat{\psi}_\lambda}$, $\widehat{y}_\lambda \notin \{0, u_\lambda, v_\lambda, y_\lambda\}$, and since $\lambda \geq \widetilde{\lambda}_3$, this is the second nodal solution for problem (P_λ). Finally, using the Hopf boundary point theorem, we conclude that

$$\widehat{y}_\lambda \in int_{C^1(\overline{\Omega})}[v_\lambda, u_\lambda].$$

$\square$

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