Pure and Applied Functional Analysis
Volume 8, Number 1, 2023, 261–277

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

EXISTENCE OF RENORMALIZED SOLUTIONS TO NONLINEAR ELLIPTIC ANISOTROPIC EQUATIONS WITH VARIABLE EXPONENTS AND L^{∞}-DATA

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Abstract. In this paper, we use truncation techniques and the generalized monotonicity method to prove the existence of a renormalized solution to non-linear elliptic anisotropic problem with variable exponent and L^{∞}-data. The functional setting involves Sobolev spaces with variable exponents $W^{1,\overrightarrow{p}(.)}(\Omega)$. The main contribution of our work is to prove the existence of renormalized solutions when the second term f belongs to L^{∞} and the exponents are able to vary.

1. Introduction

In this paper our aim is to study the existence of at least one renormalized solution to the following class of non-linear elliptic inclusion problem

$$(E,f)\begin{cases} \beta(u)-\displaystyle\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,\frac{\partial u}{\partial x_i})+\text{div F(u)}\ni f & \text{in}\ \ \Omega \\ \\ u=\ 0 & \text{on}\ \ \partial\Omega, \end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N\geq 3)$ with Lipschitz boundary $\partial\Omega$, the right-hand side $f\in L^{\infty}(\Omega)$, $F:\mathbb{R}\longrightarrow\mathbb{R}^{\mathbb{N}}$ locally Lipschitz continuous and $\beta:\mathbb{R}\longrightarrow 2^{\mathbb{R}}$ a set-valued, maximal monotone mapping such that $0\in\beta(0)$.
In recent years, the theory of Lebesgue and Sobolev spaces with variable exponents has experienced a revival of interest through the study of various mathematical problems with variable exponent. The interest in working on such problems is linked to a large scale of applications that involve some nonhomogeneous materials (blood for example). They appear in models of electrorheological fluids, stationary thermo-rheological viscous flows of non-Newtonian fluids (see [16, 26] for more details), models of propagation of epidemic disease (see [4]), image restoration (see [12]).
We recall that the concept of renormalized solution was first introduced by P.L Lions and R.J Diperna [13], and used later by P.L Lions and Murat to tacle elliptic equation with low summability data i.e when the data is L^1 or a measure. A similar concept that of entropy solution was introduced by P. Bénilan, L. Boccardo,

2020 *Mathematics Subject Classification.* 35J60, 35J67.
Key words and phrases. Generalized Lebesgue-Sobolev spaces, anisotropic Sobolev spaces, weak solution, renormalized solution, Marcinkiewicz spaces.

T. Gallouët and collaborators, see [6] and references therein. Moreover, equations with non-standard growth and L^1 data in Lebesgue and Sobolev space with variable exponent, are also investigated by several authors. For more details, one can refer to [28] and references therein.
Problems of kind (E, f) have been studied in generalized Orlicz spaces by Wittbold et al. in [17] and in the weighted Sobolev spaces by Akdim and Allalou in [2]. Papers [3,20,28] are the closest to the results presented here. Namely, in [3], Akdim et al. proved the existence of at least one renormalized solutions of problem (E, f) in anisotropic space with constant exponents $\overrightarrow{p} = (p_1, ..., p_n)$ where p_i is constant for any $i = 1, ..., N$ and $f \in L^\infty(\Omega)$. In [20], Konaté and Ouaro studied the problem (E, f) when $divF = 0$ and f a measure data. Wittbold et al. in [28], considered the problem (E, f) with an isotropic operator and inclusion equation when $f \in L^1(\Omega)$. More precisely, they proved the existence and uniqueness of renormalized solutions with variable exponent.
Our aim is to extend the main result of authors in [3] to the context of anisotropic Sobolev space with variable exponent $\overrightarrow{p}(.) = \big(p_1(.), ..., p_N(.)\big)$. More precisely, we prove the existence of renormalized solutions to the general elliptic problem (E, f). The novelty in our work is that we are dealing with vector $\overrightarrow{p}(.) = \big(p_1(.), ..., p_N(.)\big)$ where for any $i = 1, ..., N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous fonction. One of our main arguments is the Poincaré inequality, but we cannot use it directly in anisotropic Sobolev space with variable exponent. To overcome this difficulty, we choose an appropiate constant exponent in order to be able to use Poincaré inequality.

Before presenting our main results, we first give the following assumptions.
We denote by

$$p_M(x) := \max(p_1(x), ..., p_N(x)) \text{ and } p_m(x) := \min\left(p_1(x), ..., p_N(x)\right).$$

The vector $\overrightarrow{p}(.) = \big(p_1(.), ..., p_N(.)\big)$ is such that for any $i = 1, ..., N,\ p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x\in\Omega} p_i(x) \leq p_i^+ := \sup_{x\in\Omega} p_i(x) < \infty. \tag{1.1}$$

For any $i = 1, ..., N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Caratheodory function satisfaying the following.
• there exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1\bigg(j_i(x) + |\xi|^{p_i(x)-1}\bigg), \tag{1.2}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;
• for $\xi,\ \eta\ \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_i(x, \xi) - a_i(x, \eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } |\xi - \eta| \geq 1 \\ C_2|\xi - \eta|^{p_i^-} & \text{if } |\xi - \eta| < 1 \end{cases} \tag{1.3}$$

and,
• there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{1.4}$$

for $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.
The hypotheses on a_i are classical in the study of nonlinear problems (see [3]).
Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p}-N}{\overline{p}(N-1)} \tag{1.5}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.6}$$

where $\dfrac{N}{\overline{p}} = \displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-}$.
Now, we give the followings.

Theorem 1.1. *Under assumptions* (1.1)-(1.6) *and* $f \in L^\infty(\Omega)$ *there exists at least one renormalized solution* (u,b) *to problem* (E,f) *in the sense that:*

(i): $u : \Omega \longrightarrow \mathbb{R}$ *is measurable,* $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$, $b \in L^\infty(\Omega)$, $u(x) \in dom(\beta(x))$, $b(x) \in \beta(u(x))$ *for a.e. in* Ω,

(ii): *For all* $h \in C_c^1(\mathbb{R})$ *and* $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\varphi]dx + \int_\Omega bh(u)\varphi dx$$
$$- \int_\Omega F(u).\nabla[h(u)\varphi]dx = \int_\Omega fh(u)\varphi dx, \tag{1.7}$$

(iii): $\displaystyle\int_{\{l<|u|<l+1\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}dx \longrightarrow 0$ *as* $l \longrightarrow +\infty$.

The rest of the paper is organized as follows. In Section 2, we introduce some fundamental preliminary results which are useful in this work. Then, we prove existence results in section 3.

2. Preliminary

We recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces with variable exponents. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ a measurable function such that } \int_\Omega |u|^{p(x)}dx < \infty \right\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The $p(.)$-modular of the space $L^{p(.)}(\Omega)$ is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [14], [15]) will be used later.

$$\min\left\{|u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{\ |u|^{p^-}_{p(.)};\ |u|^{p^+}_{p(.)}\right\}. \tag{2.1}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1$ for any $x \in \Omega$, we have the Hölder type inequality

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(.)}|v|_{q(.)}. \tag{2.2}$$

If Ω is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous (see [21], Theorem 2.8).

Herein, we need the following anisotropic Sobolev space with variable exponent.

$$W^{1,\overrightarrow{p}(.)}(\Omega) := \left\{u \in L^{p_M(.)}(\Omega) :\ \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega),\ i = 1, ..., N\right\},$$

which is a separable and reflexive Banach space (see [23]) under the norm

$$||u||_{\overrightarrow{p}(.)} = |u|_{p_M(.)} + \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}.$$

We need the following embedding and trace results.

Theorem 2.1 ([14, Corollary 2.1]). . *Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded open set and for all $i = 1, ..., N, p_i \in L^\infty(\Omega), p_i(x) \geq 1$ a.e. $x \in \Omega$. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. $x \in \Omega$ such that*

$$ess \inf_{x\in\Omega} (p_M(x) - q(x)) > 0,$$

we have the compact embedding

$$W^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega). \tag{2.3}$$

We introduce now the numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1} \text{ and } q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q}.$$

The following result is due to Troisi (see [27]).

Theorem 2.2. *Let $p_1, ..., p_N \in [1,\infty)$; $g \in W^{1,(p_1,...,p_N)}(\Omega)$ and*

$$\begin{cases} q = (\overline{p})^* & \text{if } (\overline{p})^* < N, \\ q \in [1,\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, ..., p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left[||g||_{L^{p_M}(\Omega)} + \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right]^{\frac{1}{N}}, \tag{2.4}$$

where $\frac{1}{\overline{p}} = \sum_{i=1}^{N} p_i$ *and* $(\overline{p})^* = \dfrac{N\overline{p}}{N-\overline{p}}$.

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) := meas(\{x \in \Omega : |g(x)| > k\}), \ k \geq 0 \tag{2.5}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{2.6}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$.

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \forall k > 0\}. \tag{2.7}$$

Finally, we introduce some functions that will be frequently used.
For given constant$(k > 0)$, we defined the truncation function T_k as

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.8}$$

It is clear that $\lim_{k\to+\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.
Then, for $r \in \mathbb{R}$ let $r \to r^+ := \max(r, 0)$ and let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by $h_l(r) := \min((l + 1 - |r|)^+, \ 1)$ for each $r \in \mathbb{R}$.
Set $\mathcal{T}^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of the measurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that for any $k > 0$, $T_k(u) \in W^{1,\overrightarrow{p}(.)}(\Omega)$.

We need the following lemma proved in [9].

Lemma 2.3. *Let g be a nonnegative function in $W^{1,\overrightarrow{p}(.)}(\Omega)$. Assume $\overline{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p_M^-} dx + \sum_{i=1}^{N} \int_{\{|g| \leq k\}} \left| \frac{\partial g}{\partial x_i} \right|^{p_i^-} dx \leq C(k+1), \tag{2.9}$$

for every $k > 0$.
Then, there exists a constant D, depending on C such that

$$||g||_{\mathcal{M}^{q^*}(\Omega)} \leq D,$$

where $q^* = \dfrac{N(\overline{p}-1)}{N-\overline{p}}$.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into several steps.

3.1. Approximate problem and a priori estimates.

First we approximate (E,f) for $f \in L^{\infty}(\Omega)$ by problems for which existence can be proved by standard variational arguments. For $0<\epsilon\leq 1$, we consider the Yosida regularization $\beta_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ of β (see [11]), defined by $\beta_\epsilon = \dfrac{1}{\epsilon}(I-(I+\epsilon\beta)^{-1})$.
We introduce the operators
$A_{1,\epsilon} : W_0^{1,\overrightarrow{p}(.)}(\Omega) \longrightarrow (W_0^{1,\overrightarrow{p}(.)}(\Omega))^*$ and $A_{2,\epsilon} : W_0^{1,\overrightarrow{p}(.)}(\Omega) \longrightarrow (W_0^{1,\overrightarrow{p}(.)}(\Omega))^*$ such that

$$\langle A_{1,\epsilon}u,\varphi\rangle \ = \ \langle Au,\varphi\rangle + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\varphi dx, \ \forall \ u,\varphi \ \in W_0^{1,\overrightarrow{p}}(\Omega),$$

where

$$\langle Au,\varphi\rangle \ = \ \int_\Omega \sum_{i=1}^N a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i}dx.$$

and

$$\langle A_{2,\epsilon}u,\varphi\rangle \ = -\int_\Omega F(T_{\frac{1}{\epsilon}}(u)).\nabla\varphi dx, \ \forall \ u,\varphi \ \in W_0^{1,\overrightarrow{p}}(\Omega).$$

$A_{1,\epsilon}$ is monotone, hemicontinuous (see [18, 19]). From the continuity and boundedness of $F\circ T_{\frac{1}{\epsilon}}$ it follows that $A_{2,\epsilon}$ is strongly continuous. Therefore the operator $A_\epsilon := A_{1,\epsilon}+A_{2,\epsilon}$ is pseudomonotone (see [22]). Using the coercivity of $A_{1,\epsilon}$ and the boundary condition on the convection term $-\int_\Omega F(T_{\frac{1}{\epsilon}}(u)).\nabla u dx$, we show that A_ϵ is coercive. Since, $A_{1,\epsilon}$ and $A_{2,\epsilon}$ are bounded A_ϵ is also bounded. Then it follows from [22] (Theorem 2.7) that A_ϵ is surjective.
Therefore, for each $0<\epsilon\leq 1$ and $f\in (W_0^{1,\overrightarrow{p}(.)}(\Omega))^*$, there exists at least one solution $u_\epsilon \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ to the approximated problem

$$\left\{\begin{array}{ll} \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - \displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}a_i\Big(x,\frac{\partial u_\epsilon}{\partial x_i}\Big) - \mathrm{div}F(T_{\frac{1}{\epsilon}}(u_\epsilon)) = f \text{ in } & \Omega \\ u_\epsilon = 0 & \text{on } \ \partial\Omega, \end{array}\right. \tag{3.1}$$

such that

$$\int_\Omega \sum_{i=1}^N a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial \varphi}{\partial x_i}dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))\varphi dx + \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla\varphi dx$$

$$= \int_\Omega f\varphi dx \tag{3.2}$$

holds for all $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega)\cap L^\infty(\Omega)$.
We have the following results.

Lemma 3.1. *For any $k > 0$ and $f \in L^\infty(\Omega)$ let $u_\epsilon \in W_0^{1,\vec{p}(.)}(\Omega)$ be a solution of problem* (3.1). *Then,*

$$\sum_{i=1}^{N} \int_{\{|u_\epsilon| \leq k\}} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx \leq \frac{k\|f\|_\infty}{C_3}, \tag{3.3}$$

$$\int_\Omega \left| \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \right| dx \leq \|f\|_\infty, \tag{3.4}$$

$$\sum_{i=1}^{N} \int_{\{l < |u_\epsilon| < l+k\}} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} dx \leq k \int_{\{|u_\epsilon| > l\}} |f| dx \tag{3.5}$$

and

$$\int_{\{|u_\epsilon| \leq k\}} |\nabla T_k(u_\epsilon)|^{p_m^-} dx \leq C_5. \tag{3.6}$$

Proof. Taking $\varphi = T_k(u_\epsilon)$ as a test function in (3.2) we obtain

$$\sum_{i=1}^{N} \int_{\{|u_\epsilon| \leq k\}} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} dx + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) T_k(u_\epsilon) dx + \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)) \nabla T_k(u_\epsilon) dx$$

$$= \int_\Omega f T_k(u_\epsilon) dx. \tag{3.7}$$

Let us remark that

$$\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) T_k(u_\epsilon) dx \geq 0, \tag{3.8}$$

$$\left| \int_\Omega f T_k(u_\epsilon) dx \right| \leq \int_\Omega \left| f T_k(u_\epsilon) \right| dx \leq k\|f\|_\infty, \tag{3.9}$$

$$\begin{aligned} \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)) \nabla T_k(u_\epsilon) dx &= \int_\Omega F(T_{\frac{1}{\epsilon}}(T_k(u_\epsilon))).\nabla T_k(u_\epsilon)) dx \\ &= \int_\Omega \nabla \left(\int_0^{T_k(u_\epsilon)} F \circ T_{\frac{1}{\epsilon}}(s) ds \right) dx = 0. \end{aligned} \tag{3.10}$$

By using (1.4) in (3.7) we get (3.3).
Taking $\frac{1}{\delta}[T_{k+\delta}(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))) - T_k(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)))]$ as test function in (3.2), passing to the limit with $\delta \to 0$ and choosing $k > \|f\|_\infty$, we obtain (3.4).

For $k, l > 0$ fixed, we take $T_k(u_\epsilon - T_l(u_\epsilon))$ as test function in (3.2). Using

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u_\epsilon - T_l(u_\epsilon)) dx = \sum_{i=1}^{N} \int_{\{l < |u_\epsilon| < l+k\}} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} dx,$$

the fact that the first term and the second term of the left-hand side are non-negative and the convection term vanishes, we get

$$\sum_{i=1}^{N}\int_{\{l<|u_\epsilon|<l+k\}} a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i}dx \le \int_\Omega fT_k(u_\epsilon - T_l(u_\epsilon))dx \le k\int_{\{|u_\epsilon|>l\}}|f|dx.$$

□

Remark 3.2. We have

$$\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \ge C_6\|\nabla u\|^{p_m^-}_{L^{p_m^-}(\Omega)} - Nmeas(\Omega) \tag{3.11}$$

and

$$\sum_{i=1}^{N}\int_{\{l<|u_\epsilon|<l+k\}} a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i}dx \le k\|f\|_\infty|\{|u_\epsilon|\ge l\}| \le C(k)l^{-p_m^-}. \tag{3.12}$$

Indeed, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx &= \sum_{i=1}^{N}\int_{\{|\frac{\partial u}{\partial x_i}|\le 1\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx + \sum_{i=1}^{N}\int_{\{|\frac{\partial u}{\partial x_i}|>1\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\\
&\ge \sum_{i=1}^{N}\int_{\{|\frac{\partial u}{\partial x_i}|>1\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx\\
&\ge \sum_{i=1}^{N}\int_{\{|\frac{\partial u}{\partial x_i}|>1\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_m^-}dx\\
&\ge \sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_m^-}dx - \sum_{i=1}^{N}\int_{\{|\frac{\partial u}{\partial x_i}|\le 1\}}\left|\frac{\partial u}{\partial x_i}\right|^{p_m^-}dx\\
&\ge \sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_m^-}dx - Nmeas(\Omega)\\
&\ge \sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|^{p_m^-}_{L^{p_m^-}(\Omega)} - Nmeas(\Omega)\\
&\ge C_6||\nabla u||^{p_m^-}_{(L^{p_m^-}(\Omega))^N} - Nmeas(\Omega),\ \text{(thanks to Poincaré inequality).}\end{aligned}$$

As $\{|u_\epsilon| \ge l\} = \{|T_l(u_\epsilon)| \ge l\}$, we get

$$
\begin{aligned}
|\{|u_\epsilon| \geq l\}| &\leq \int_{\{|u_\epsilon|\geq l\}} \frac{|T_l(u_\epsilon)|^{p_m^-}}{l^{p_m^-}} dx \\
&\leq C(p_m^-, N) l^{-p_m^-} \|\nabla u_\epsilon\|^{p_m^-}_{L^{p_m^-}(\Omega)} \qquad (3.13)\\
&\leq C(p_m^-, N) l^{-p_m^-} \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i(x)} dx + N.meas(\Omega) \right)
\end{aligned}
$$

where $C(p_m^-, N) > 0$ is a constant from the Poincaré inequality in $W_0^{1,p_m^-}(\Omega)$. Combining (3.3) and (3.13), and setting

$$C(p_m^-, C_3, k, f, \Omega) = C(p_m^-, N)\left(\frac{k\|f\|_\infty}{C_3} + N.meas(\Omega) \right) > 0,$$

we obtain

$$\{|u_\epsilon| \geq l\}| \leq C(p_m^-, C_3, k, f, \Omega) l^{-p_m^-}. \qquad (3.14)$$

So we have

$$\lim_{l \to +\infty} |\{|u_\epsilon| \geq l\}| = 0.$$

Hence (3.14) gives (3.12) with $C(k) = C(p_m^-, C_3, k, f, \Omega) k \|f\|_\infty$.

Lemma 3.3. *(see [9, 18]) For any $k > 0$, there exists some constants $C_7,\ C_8 > 0$ such that.*

(i) $\|u_\epsilon\|_{\mathcal{M}^{q^*}(\Omega)} \leq C_7$;

(ii) $\|\frac{\partial u_\epsilon}{\partial x_i}\|_{\mathcal{M}^{p_i^- \frac{q}{\overline{p}}}(\Omega)} \leq C_8, \quad \forall i = 1, ..., N.$

3.2. **Basic convergence results.**

Lemma 3.4. *For $0 < \epsilon \leq 1$ and $f \in L^\infty(\Omega)$, let $u_\epsilon \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ be a solution of (E_ϵ, f). There exists $u \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ and $b \in L^\infty(\Omega)$ such that for a not relabelled subsequence of $(u_\epsilon)_{0<\epsilon\leq 1}$ as $\epsilon \downarrow 0$;*

$$u_\epsilon \longrightarrow u \text{ in } L^{\overrightarrow{p}(.)}(\Omega) \text{ and a.e. in } \Omega; \qquad (3.15)$$

$$\frac{\partial u_\epsilon}{\partial x_i} \text{ converges in measure to the weak partial gradient of } u; \qquad (3.16)$$

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ in } L^1(\Omega) \text{ a.e. } x \in \Omega; \qquad (3.17)$$

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \text{ in } L^1(\Omega) \text{ and a.e. } x \in \Omega \qquad (3.18)$$

$$\text{and } \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) \rightharpoonup b \text{ weakly-* in } L^\infty(\Omega). \qquad (3.19)$$

Moreover, for any $k > 0$,

$$\frac{\partial T_k(u_\epsilon)}{\partial x_i} \rightharpoonup \frac{\partial T_k(u)}{\partial x_i} \text{ in } L^{p_i(\cdot)}(\Omega), \tag{3.20}$$

$$a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big) \longrightarrow a_i\Big(x, \frac{\partial T_k(u)}{\partial x_i}\Big) \text{ in } L^1(\Omega). \tag{3.21}$$

Proof. The proof of Lemma 3.4 follows the same lines as the proof of Lemma 4.6. in [28]. □

3.3. **Strong convergence and Subdifferential argument.** Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be arbitrary. Taking $h_l(u_\epsilon)h(u)\varphi$ as test function in (3.2), we obtain

$$I^1_{\epsilon,l} + I^2_{\epsilon,l} + I^3_{\epsilon,l} = I^4_{\epsilon,l}, \tag{3.22}$$

where

$$I^1_{\epsilon,l} = \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))h_l(u_\epsilon)h(u)\varphi dx,$$

$$I^2_{\epsilon,l} = \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h_l(u_\epsilon)h(u)\varphi]dx,$$

$$I^3_{\epsilon,l} = \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla[h_l(u_\epsilon)h(u)\varphi]dx,$$

$$I^4_{\epsilon,l} = \int_\Omega f h_l(u_\epsilon)h(u)\varphi dx.$$

Firstly, we let $\epsilon \downarrow 0$ in (3.22).
Using Lemma 3.4, we can immediately calculate the following limits:

$$\lim_{\epsilon\to 0} I^1_{\epsilon,l} = \int_\Omega b h_l(u)h(u)\varphi dx, \tag{3.23}$$

$$\lim_{\epsilon\to 0} I^4_{\epsilon,l} = \int_\Omega f h_l(u)h(u)\varphi dx. \tag{3.24}$$

We wrote $I^2_{\epsilon,l} = I^{2,1}_{\epsilon,l} + I^{2,2}_{\epsilon,l}$ where,

$$I^{2,1}_{\epsilon,l} = \sum_{i=1}^N \int_\Omega h_l'(u_\epsilon)a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}h(u)\varphi dx,$$

$$I^{2,2}_{\epsilon,l} = \sum_{i=1}^N \int_\Omega h_l(u_\epsilon)a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\varphi]dx.$$

Using (3.12), we get the estimate

$$|\lim_{\epsilon\to 0} I^{2,1}_{\epsilon,l}| \le \|h\|_\infty\|\varphi\|_\infty C(1)l^{-p_m^-}. \tag{3.25}$$

Since modular convergence is equivalent to norm convergence in $L^{p_i}(\Omega)$, by Lebesgue's dominated convergence theorem it follows that for $i \in \{1, ..., N\}$, we have

$$h_l(u_\epsilon)\frac{\partial}{\partial x_i}[h(u)\varphi] \longrightarrow h_l(u)\frac{\partial}{\partial x_i}[h(u)\varphi] \quad \text{as } \epsilon \downarrow 0.$$

Keeping in mind that $I_{\epsilon,l}^{2,2} = \sum_{i=1}^{N} \int_\Omega h_l(u_\epsilon) a_i\Big(x, \frac{\partial T_{l+1}(u_\epsilon)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\varphi] dx,$
by (3.21), we get

$$\lim_{\epsilon\to 0} I_{\epsilon,l}^{2,2} = \sum_{i=1}^{N} \int_\Omega h_l(u) a_i\Big(x, \frac{\partial T_{l+1}(u)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\varphi] dx. \tag{3.26}$$

Let us write $I_{\epsilon,l}^{3} = I_{\epsilon,l}^{3,1} + I_{\epsilon,l}^{3,2}$, where

$$I_{\epsilon,l}^{3,1} = \int_\Omega h_l'(u_\epsilon) h(u)\varphi F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla u_\epsilon dx,$$
$$I_{\epsilon,l}^{3,2} = \int_\Omega h_l(u_\epsilon) F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla[h(u)\varphi] dx.$$

For any $l \in \mathbb{N}$, there exists $\epsilon_0(l)$ such that for all $\epsilon < \epsilon_0(l)$,

$$I_{\epsilon,l}^{3,1} = \int_\Omega h_l'(T_{l+1}(u_\epsilon)) h(u)\varphi F(T_{\frac{1}{\epsilon}}(T_{l+1}(u_\epsilon))).\nabla u_\epsilon dx. \tag{3.27}$$

Using Gauss-Green Theorem for Sobolev functions in (3.27), we get

$$I_{\epsilon,l}^{3,1} = -\int_\Omega \int_0^{T_{l+1}(u_\epsilon)} h_l'(r) F(r) dr.\nabla[h(u)\varphi] dx. \tag{3.28}$$

Now, using (3.15) and the Gauss-Green Theorem, after letting $\epsilon \downarrow 0$, we get

$$\lim_{\epsilon\to 0} I_{\epsilon,l}^{3,1} = \int_\Omega h_l'(u) h(u)\varphi F(u).\nabla u dx. \tag{3.29}$$

Choosing ϵ small enough, we can write

$$I_{\epsilon,l}^{3,2} = \int_\Omega h_l(u_\epsilon) F(T_{l+1}(u_\epsilon)).\nabla[h(u)\varphi] dx \tag{3.30}$$

and conclude that

$$\lim_{\epsilon\to 0} I_{\epsilon,l}^{3,2} = \int_\Omega h_l(u) F(u).\nabla[h(u)\varphi] dx. \tag{3.31}$$

Secondly, we pass to the limit with $l \to \infty$ in (3.22) and (3.23)-(3.31).
Combining (3.22) and (3.23)-(3.31), we obtain

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6, \tag{3.32}$$

where

$$I_l^1 = \int_\Omega b h_l(u) h(u)\varphi dx,$$
$$I_l^2 = \sum_{i=1}^{N} \int_\Omega h_l(u) a_i\Big(x, \frac{\partial T_{l+1}(u)}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[h(u)\varphi] dx,$$
$$|I_l^3| \le C(1) l^{-p_m^-} \|h\|_\infty \|\varphi\|_\infty,$$
$$I_l^4 = \int_\Omega h_l(u) F(u).\nabla[h(u)\varphi] dx,$$
$$I_l^5 = \int_\Omega h_l'(u) h(u)\varphi F(u).\nabla u dx,$$

$$I_l^6 = \int_\Omega f h_l(u) h(u) \varphi dx.$$

Obviously, we have

$$\lim_{l \to \infty} I_l^3 = 0. \tag{3.33}$$

Choosing $m > 0$ such that $supph \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, ..., I_l^6$ and $h_l'(u) = h_l'(T_m(u)) = 0$ if $l + 1 > m$, $h_l(u) = h_l(T_m(u)) = 1$, if $l > m$. Therefore, letting $l \to \infty$ and combining (3.32) and (3.33), yield

$$\begin{aligned} &\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u)\xi] dx + \int_\Omega b h(u) \varphi dx \\ &+ \int_\Omega F(u).\nabla[h(u)\varphi] dx = \int_\Omega f h(u) \varphi dx, \end{aligned} \tag{3.34}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$.
Now, we use the subdifferential argument to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for all $x \in \Omega$.
Since β is a maximal monotone graph, there exists a convex, l.s.c and proper function $j : \mathbb{R} \longrightarrow [0, \infty]$ such that $\beta(r) = \partial j(r)$ for all $r \in \mathbb{R}$. According to [11], for $0 < \epsilon \leq 1$ the function $j_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $j_\epsilon(r) = \int_0^r \beta_\epsilon(s) ds$ has the following properties (see [28]):

(i) for any $0 < \epsilon \leq 1$, j_ϵ is convex and differentiable for all $r \in \mathbb{R}$ and such that

$$j_\epsilon'(r) = \beta_\epsilon(r) \quad \text{for all } r \in \mathbb{R};$$

(ii) $j_\epsilon(r) \uparrow j(r)$ pointwise in $\mathbb{R}$ as $\epsilon \to 0$.

Using the same argument as in [28], we can prove that for all $r \in \mathbb{R}$ and almost every $x \in \Omega$, $u \in D(\beta)$ and $b \in \beta(u)$ almost everywhere in Ω.

We conclude the proof of Theorem 1.1 by the following.

Lemma 3.5. *The limit u of the approximate solution u_ϵ of (3.1) satisfies*

$$\lim_{l \to \infty} \sum_{i=1}^N \int_{\{l<|u|<l+1\}} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial u}{\partial x_i} dx = 0. \tag{3.35}$$

Proof. **Step 1** Let us prove that

$$\lim_{\epsilon \to 0} \sup \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial}{\partial x_i}[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))] dx \leq 0. \tag{3.36}$$

Taking $h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))$ as test function in (3.2), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i} h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \\ &\quad + \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \\ &\quad + \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]dx \\ &\quad = \int_\Omega f h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \end{aligned} \tag{3.37}$$

Using $|h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))| \leq 2k\|h\|_\infty$, by Lebesgue's dominated convergence theorem we get

$$\lim_{\epsilon\to 0} \int_\Omega f h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx = 0 \tag{3.38}$$

and

$$\lim_{\epsilon\to 0} \int_\Omega F(T_{\frac{1}{\epsilon}}(u_\epsilon)).\nabla[h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))]dx = 0. \tag{3.39}$$

We have

$$\begin{aligned} &\int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \\ &= \int_\Omega h(u_\epsilon)\Big(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\Big)(T_k(u_\epsilon) - T_k(u))dx \\ &\quad + \int_\Omega h(u_\epsilon)\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))(T_k(u_\epsilon) - T_k(u))dx \end{aligned}$$

and

$$\int_\Omega h(u_\epsilon)\Big(\beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon)) - \beta_\epsilon(T_{\frac{1}{\epsilon}}(u))\Big)(T_k(u_\epsilon) - T_k(u))dx \geq 0.$$

By Lebesgue's dominated convergence theorem we obtain

$$\lim_{\epsilon\to 0} \int_\Omega h(u_\epsilon)\beta_\epsilon(T_{\frac{1}{\epsilon}}(u))(T_k(u_\epsilon) - T_k(u))dx = 0,$$

which allows to write

$$\limsup_{\epsilon\to 0} \int_\Omega \beta_\epsilon(T_{\frac{1}{\epsilon}}(u_\epsilon))h(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx \geq 0.$$

Passing to the limit in (3.37) and using the above results, we obtain (3.36).
Step 2 Now we prove that for every $k > 0$,

$$\limsup_{\epsilon\to 0} \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial}{\partial x_i}[T_k(u_\epsilon) - T_k(u)]dx \leq 0. \tag{3.40}$$

For any $k, l > 0$, we take $h_l(u_\epsilon)(T_k(u_\epsilon) - T_k(u))$ as test function in (3.2). Letting $\epsilon \to 0$, we obtain

$$\limsup_{\epsilon\to 0} \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i} h_l(u_\epsilon)(T_k(u_\epsilon) - T_k(u))dx = E_1 + E_2 + E_2,$$

where

$$E_1 = \sum_{i=1}^{N} \int_{\{|u_\epsilon|\le k\}} h_l(u_\epsilon) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}\Big(T_k(u_\epsilon) - T_k(u)\Big)dx,$$

$$E_2 = \sum_{i=1}^{N} \int_{\{|u_\epsilon|> k\}} h_l(u_\epsilon) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\Big(-\frac{\partial}{\partial x_i}T_k(u)\Big)dx$$

and

$$E_3 = \sum_{i=1}^{N} \int_\Omega h_l'(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}dx.$$

Since $l > k$, on the set $\{|u_\epsilon| \le k\}$ we have $h_l(u_\epsilon) = 1$ so that we can write

$$\limsup_{\epsilon\to 0} E_1 = \limsup_{\epsilon\to 0} \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(T_k(u_\epsilon) - T_k(u))dx.$$

Since

$$h_l(u_\epsilon)\chi_{\{|u_\epsilon|>k\}} a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u) \to h_l(u)\chi_{\{|u|>k\}} a_i\Big(x, \frac{\partial T_k(u)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u) \ \text{ in } L^1(\Omega),$$

we use Lebesgue's dominated convergence theorem in E_2, to get

$$\lim_{\epsilon\to 0} E_2 = -\sum_{i=1}^{N} \int_{\{|u|>k\}} h_l(u) a_i\Big(x, \frac{\partial T_k(u)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}T_k(u)dx = 0.$$

For E_3 we have

$$-\sum_{i=1}^{N} \int_\Omega h_l'(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}dx$$
$$\le 2k \sum_{i=1}^{N} \int_{\{l\le|u_\epsilon|\le l+1\}} a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}dx.$$

From (3.12), we deduce that

$$\limsup_{l\to\infty} \limsup_{\epsilon\to 0} \Big(-\sum_{i=1}^{N} \int_\Omega h_l'(u_\epsilon)(T_k(u_\epsilon) - T_k(u) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial u_\epsilon}{\partial x_i}dx\Big) \le 0.$$

Applying (3.36) with h replaced by $h_l, l > k$, we get

$$\limsup_{\epsilon\to 0} \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[T_k(u_\epsilon) - T_k(u)]dx$$

$$\leq \limsup_{\epsilon\to 0} \Big[-\sum_{i=1}^{N} \int_\Omega h_l'(u_\epsilon)(T_k(u_\epsilon) - T_k(u)) a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big) \frac{\partial u_\epsilon}{\partial x_i} dx \Big].$$

By letting $l \to \infty$ yields (3.40).
As an immediate consequence of (3.40) and (1.3), we have

$$\lim_{\epsilon\to 0} \sum_{i=1}^{N} \int_\Omega \Big[a_i\Big(x, \frac{\partial T_k(u_\epsilon)}{\partial x_i}\Big) - a_i\Big(x, \frac{\partial T_k(u)}{\partial x_i}\Big)\Big] . \Big[\frac{\partial T_k(u_\epsilon)}{\partial x_i} - \frac{\partial T_k(u)}{\partial x_i}\Big] dx = 0. \tag{3.41}$$

For any fixed $l \geq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{l<|u_\epsilon|<l+1\}} a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial u_\epsilon}{\partial x_i} dx \\
&\quad = \sum_{i=1}^{N} \int_{\{l<|u_\epsilon|<l+1\}} a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \Big[\frac{\partial T_{l+1}(u_\epsilon)}{\partial x_i} - \frac{\partial T_l(u)}{\partial x_i}\Big] dx \\
&\quad = \sum_{i=1}^{N} \int_{\{l<|u_\epsilon|<l+1\}} a_i\Big(x, \frac{\partial T_{l+1}(u_\epsilon)}{\partial x_i}\Big) \frac{\partial T_{l+1}(u_\epsilon)}{\partial x_i} dx \\
&\qquad - \sum_{i=1}^{N} \int_{\{l<|u_\epsilon|<l+1\}} a_i\Big(x, \frac{\partial T_l(u_\epsilon)}{\partial x_i}\Big) \frac{\partial T_l(u_\epsilon)}{\partial x_i} dx
\end{aligned}$$

According to (3.41), we pass to the limit as $\epsilon \to 0$ for fixed $l \geq 0$ to obtain

$$\begin{aligned}
\lim_{\epsilon\to 0} \sum_{i=1}^{N} \int_{\{l<|u_\epsilon|<l+1\}} a_i\Big(x, \frac{\partial u_\epsilon}{\partial x_i}\Big) \frac{\partial u_\epsilon}{\partial x_i} dx &= \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial T_{l+1}(u)}{\partial x_i}\Big) \frac{\partial T_{l+1}(u)}{\partial x_i} dx \\
&\quad - \sum_{i=1}^{N} \int_\Omega a_i\Big(x, \frac{\partial T_l(u)}{\partial x_i}\Big) \frac{\partial T_l(u)}{\partial x_i} dx \\
&= \sum_{i=1}^{N} \int_{\{l<|u|<l+1\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial u}{\partial x_i} dx.
\end{aligned} \tag{3.42}$$

Passing to limit in (3.42) as $l \to \infty$ and using (3.12) we show that u satisfies (iii) of Theorem 1.1 and the proof of the lemma is complete. □

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Manuscript received September 24 2021
revised January 25 2022

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