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ON CONDITION SPECTRUM OF TOEPLITZ OPERATOR

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Abstract. Here we investigate $\epsilon-$condition spectrum of Toeplitz matrix T_n and Toeplitz operator T on $\ell^2(\mathbb{Z}_+)$ associated with a symbol of essentially bounded function. We have derived some subset(lower bound) and superset(upper bound) of condition spectrum of Toeplitz operator and Toeplitz matrix. Also we obtained some bettter estimations of the subsets and supersets of condition spectrum of Toeplitz matrix and operator when the associated symbol is an element of Wiener algebra(W).

1. Introduction

Let $\mathcal{A}$ denote a complex unital Banach algebra. By $\mathcal{G}(\mathcal{A})$ we denote the open set of all invertible elements of $\mathcal{A}$ and the spectrum of an element $a \in \mathcal{A}$ is given by

$$\sigma(a) := \{\lambda \in \mathbb{C} : (\lambda 1 - a) \notin \mathcal{G}(\mathcal{A})\}.$$

It is well known that $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$ and is a particular Ransford spectrum [5] with the corresponding Ransford set $\mathcal{G}(\mathcal{A})$. Here we are going to discuss about another special Ransford spectrum namely $\epsilon-$condition spectrum. In this paper, $B(0,r)$ and $B[0,r]$ denote the open ball and closed ball respectively centred at 0 and radius $r > 0$.

Definition 1.1 ([3] $\epsilon-$condition spectrum)**.** Let $0 < \epsilon < 1$. The $\epsilon-$condition spectrum of an element $a \in \mathcal{A}$ is defined by

$$\sigma_\epsilon(a) = \{\lambda \in \mathbb{C} : (\lambda - a) \notin \mathcal{G}(\mathcal{A}) \text{ or } \|\lambda - a\|\|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\}.$$

Note that for any $\epsilon \in (0,1)$, $\sigma(a) \subseteq \sigma_\epsilon(a)$. In the definition of $\epsilon-$condition spectrum $\epsilon \in (0,1)$ is considered intentionally, otherwise for $\epsilon \geq 1$, we get $\sigma_\epsilon(a) = \mathbb{C}$. In [3], $\epsilon-$condition spectrum is shown as Ransford spectrum with the corresponding Ransford set

$$\Omega_\epsilon = \{a \in \mathcal{G}(\mathcal{A}) : \|a\|\|a^{-1}\| < \frac{1}{\epsilon}\}.$$

Hence $\epsilon-$condition spectrum is a compact subset of $\mathbb{C}$. From now onwards we consider $\epsilon \in (0,1)$ and write simply condition spectrum instead of $\epsilon-$condition spectrum.

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In [6], the pseudospectrum of of Toeplitz matrices and operators are discusssed. In this paper we investigate the condition spectrum of Toeplitz matrices and Toeplitz operators. Toeplitz matrices, a special family of matrices have broad applications in integral equations, finite-difference equations, matrix iterations, spline approximation, signal processing etc.(see [4]). Also it is to be noted that the family of Toeplitz matrices is an important class of non-normal matrices. First we define a Toeplitz operator before defining Toeplitz matrix.

Definition 1.2 (Toeplitz operator)**.** Let $\mathcal{H}$ be a seperable Hilbert space. Let $(e_n)_{n=0}^{\infty}$ be a countable orthonormal basis for $\mathcal{H}$. A linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be (formal) Toeplitz operator if $\langle Te_n, e_m \rangle = a_{m-n}$ for some complex sequence $(a_n)_{n=-\infty}^{\infty}$. This means that the matrix with respect to the orthonormal basis $(e_n)_{n=0}^{\infty}$ is constant along each diagonal parallel to the main one. That is, the matrix of the Toeplitz operator(denoted here by T for simplicity) is

$$T = (a_{j-k})_{j,k=0}^{\infty} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

Without loss of generality we take the infinite dimensional seperable *Hilbert* space $\mathcal{H}$ to be $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z}_+)$.

The following well known theorem (Theorem 1.1 in [1]) gives a necessary and sufficient condition for the sequence (a_n) to represent a bounded Toeplitz operator. This theorem also gives the norm of the Toeplitz operator when the function $a \in L^{\infty}(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denotes the unit circle in the complex plane, thanks to *Otto Toeplitz*.

Theorem 1.3 ([1])**.** *(Toeplitz 1911): The matrix T as given above defines a bounded operator on $\ell^2(\mathbb{Z}_+)$ if and only if the numbers (a_n) are Fourier coefficients of some function $a \in L^{\infty}(\mathbb{T})$ expressed as,*

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(\theta) e^{-in\theta} d\theta, \ n \in \mathbb{Z}.$$

In that case, the norm of the operator given by T equals

$$\|a\|_{\infty} := ess \sup_{t \in \mathbb{T}} |a(t)|.$$

A Toeplitz matrix is a finite truncation of Toeplitz operator. Toeplitz matrix of order $n \times n$ is of the form

$$(a_{j-k})_{j,k=0}^{n-1} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}.$$

The Toeplitz operator and Toeplitz matrix corresponding to $a \in L^\infty(\mathbb{T})$ are denoted by $T(a)$ and $T_n(a)$ respectively.

Spectrum of Toeplitz operators and matrices is well studied in [2]. In an attempt to investigate the Question 4 in the Preface of [1], we tried to locate the condition spectrum of the $T(a)$ and $T_n(a)$, for which we approach the set from outside as well as from inside, in terms of superset and subset. We do this systematically from simple to general form. From now onwards we write simply T and T_n instead of $T(a)$ and $T_n(a)$ respectively. Also hereby we identify $L^\infty(\mathbb{T})$ with the space of the 2π- periodic functions $L^\infty_{2\pi}[0, 2\pi]$.

2. Estimation of condition spectrum of some Toeplitz matrices and operators

We start with the simplest case, the diagonal Toeplitz matrix which is just a scalar matrix and it is easy to verify that its spectrum and condition spectrum are the same.

Theorem 2.1 ([3]). *Let $n \in \mathbb{N}$ and T_n be a diagonal $n \times n$ Toeplitz matrix*

$$T_n = \begin{bmatrix} a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_0 \end{bmatrix}.$$

Then the condition spectrum of T_n is given by $\sigma_\epsilon(T_n) = \{a_0\} = \sigma(T_n)$.

In a similar way the condition spectrum of diagonal Toeplitz operator is also found to be singleton set. So we shall discuss about non-diagonal Toeplitz matrices and operators. The inclusion of the condition spectrum is given in terms of the expression ϑ_n(symbol of minimal norm) given in page 135 of [2]. This ϑ_n looks for the best estimation of the norm of the matrix as a truncation of various general operators.

Definition 2.2 ([2]). For $n \in \mathbb{N}$, in a Toeplitz matrix $T_n(a)$, since only Fourier coefficients a_k with $|k| \leq n-1$ enter the matrix, we have $T_n(a) = T_n(\varphi)$ for every $\varphi \in L^\infty(\mathbb{T})$ satisfying $\varphi_k = a_k$ for $|k| \leq n-1$, put

$$\vartheta_n(a) = \inf\{\|\varphi\|_\infty : \varphi_k = a_k, \ for \ |k| \leq n-1\}.$$

The following lemma gives lower and upper bounds for the norm of a Toeplitz matrix.

Lemma 2.3 ([2]). *Let $n \in \mathbb{N}$. For a Toeplitz matrix $T_n(a)$ the following inequality holds*

$$\frac{1}{3}\vartheta_n(a) \leq \|T_n(a)\| \leq \vartheta_n(a) \leq \|a\|_\infty,$$

where $\|T_n(a)\|$ denotes the operator norm of $T_n(a)$.

We need the following well known elementary lemma for estimating set inclusions of condition spectrum of Toeplitz matrix.

Lemma 2.4 ([3]). *Let $\mathcal{A}$ be a complex unital Banach algebra with unit* $\mathbf{1}$ *and $a \in \mathcal{A}$. If $\lambda \in \sigma(a)$ and $\lambda_\epsilon \notin \sigma(a)$ then $|\lambda - \lambda_\epsilon| \geq \frac{1}{\|(\lambda_\epsilon - a)^{-1}\|}$.*

In the following theorem, we get a subset as well as a superset of the condition spectrum of a non-diagonal Toeplitz matrix $T_n(a)$.

Theorem 2.5. *Let $n \in \mathbb{N}$ and $T_n(a)$ be an $n \times n$ non-diagonal Toeplitz matrix, given by*

$$T_n = T_n(a) = (a_{j-k})_{j,k=0}^{n-1} = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \cdots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix},$$

where $a \in L^\infty_{2\pi}[0, 2\pi]$. Then the condition spectrum of T_n, $\sigma_\epsilon(T_n)$ satisfies

$$A_n \subseteq \sigma_\epsilon(T_n) \subseteq B_n,$$

where

$$A_n = \left\{ z \in \mathbb{C} : \frac{3|\lambda - z|}{\vartheta_n(z-a)} \leq \epsilon, \ for\ all\ \lambda \in \sigma(T_n) \right\}$$

and

$$B_n = \left\{ z \in \mathbb{C} : \frac{|z| - \|T_n\|}{\vartheta_n(z-a)} \leq \epsilon \right\}.$$

Proof. Let $\lambda_\epsilon \in \sigma_\epsilon(T_n)$. Since $T_n(a)$ is not a diagonal matrix we have $\vartheta_n(\lambda_\epsilon - a) > 0$. Note that whenever $|\lambda_\epsilon| \leq \|T_n\|$, we have $\lambda_\epsilon \in B_n$. Hence let us consider $|\lambda_\epsilon| > \|T_n\|$.

Then we have $(\lambda_\epsilon - T_n)$ is invertible and also

$$\|(\lambda_\epsilon - T_n)^{-1}\| \leq \frac{1}{|\lambda_\epsilon| - \|T_n\|}.$$

Also from Lemma 2.3 we know that $\|(\lambda_\epsilon - T_n)\| \leq \vartheta_n(\lambda_\epsilon - a)$.

So,

$$\frac{\vartheta_n(\lambda_\epsilon - a)}{|\lambda_\epsilon| - \|T_n\|} \geq \|\lambda_\epsilon - T_n\| \|(\lambda_\epsilon - T_n)^{-1}\|.$$

Since $\lambda_\epsilon \in \sigma_\epsilon(T_n)$, we have $\|\lambda_\epsilon - T_n\| \|(\lambda_\epsilon - T_n)^{-1}\| \geq \frac{1}{\epsilon}$. So $\frac{\vartheta_n(\lambda_\epsilon - a)}{|\lambda_\epsilon| - \|T_n\|} \geq \frac{1}{\epsilon}$ which implies $\frac{|\lambda_\epsilon| - \|T_n\|}{\vartheta_n(\lambda_\epsilon - a)} \leq \epsilon$. So we get $\lambda_\epsilon \in B_n$ and hence $\sigma_\epsilon(T_n) \subseteq B_n$.

Again let $\omega \in A_n$. Let $\lambda \in \sigma(T_n)$ be arbitrary. If $\omega \in \sigma(T_n)$, then $\omega \in \sigma_\epsilon(T_n)$ since $\sigma(T_n) \subseteq \sigma_\epsilon(T_n)$. If $\omega \notin \sigma(T_n)$ then from Lemma 2.4, we have

$$|\lambda - \omega| \geq \frac{1}{\|(\omega - T_n)^{-1}\|}$$

$$\Rightarrow \|(\omega - T_n)^{-1}\| \geq \frac{1}{|\lambda - \omega|}$$

Also from Lemma 2.3, we know

$$\|(\omega - T_n)\| \geq \frac{1}{3}\vartheta_n(\omega - a).$$

So from the above inequalities, we have

$$\|\omega - T_n\|\|(\omega - T_n)^{-1}\| \geq \frac{\vartheta_n(\omega - a)}{3|\lambda - \omega|}.$$

But $\frac{\vartheta_n(\omega-a)}{3|\lambda-\omega|} \geq \frac{1}{\epsilon}$ since $\omega \in A_n$. So $\|\omega - T_n\|\|(\omega - T_n)^{-1}\| \geq \frac{1}{\epsilon}$ which implies $\omega \in \sigma_\epsilon(T_n)$. So $A_n \subseteq \sigma_\epsilon(T_n)$ and hence the result follows. □

For illustration, we would like to give some examples. The Toeplitz matrix in the following example is taken from Example 2.12 of [3].

Example 2.6. Consider the 2×2 matrix truncation of right shift operator $\mathcal{R}_2(x, y) = (0, x)$, which is a Toeplitz operator where the associated symbol function $a : \mathbb{T} \to \mathbb{C}$ is $a(t) = t^{-1}$. Let $0 < \epsilon < 1$. The condition spectrum $\sigma_\epsilon(\mathcal{R}_2)$ is calulated in [3]. Obviously $\sigma(\mathcal{R}_2) = \{0\}$, and $\|\mathcal{R}_2\| = 1$. So the estimated lower bound of $\sigma_\epsilon(\mathcal{R}_2)$ from Theorem 2.5 is

$$\begin{aligned} A_2 &= \left\{ z \in \mathbb{C} : \frac{3|z - \lambda|}{\vartheta_2(z - a)} \leq \epsilon, \forall \lambda \in \sigma(\mathcal{R}_2) \right\} \\ &= \left\{ z \in \mathbb{C} : \frac{3|z - 0|}{\vartheta_2(z - a)} \leq \epsilon \right\} \\ &= \left\{ z \in \mathbb{C} : \frac{|z|}{\vartheta_2(z - a)} \leq \frac{\epsilon}{3} \right\}. \end{aligned}$$

And the upper bound from Theorem 2.5 is

$$\begin{aligned} B_2 &= \left\{ z \in \mathbb{C} : \frac{|z| - \|\mathcal{R}_2\|}{\vartheta_2(z - a)} \leq \epsilon \right\} \\ &= \left\{ z \in \mathbb{C} : \frac{|z| - 1}{\vartheta_2(z - a)} \leq \epsilon \right\}. \end{aligned}$$

For the sake of calculation in view of the inequality $\|z - \mathcal{R}_2\| \leq \vartheta_2(z - a)$, from Lemma 2.3, one can obtain another lower bound of $\sigma_\epsilon(\mathcal{R}_2)$, given by

$$A_2^{'} = \left\{ z \in \mathbb{C} : \frac{|z|}{\|z - \mathcal{R}_2\|} \leq \frac{\epsilon}{3} \right\}.$$

Similarly from the inequality $\vartheta_2(z - a) \leq \|z - a\|$, from Lemma 2.3, another upper bound of $\sigma(\mathcal{R}_2)$ can be obtained which is given by

$$B_2^{'} = \left\{ z \in \mathbb{C} : \frac{|z| - 1}{\|z - a\|} \leq \epsilon \right\}.$$

But these are not better bounds since $A_2^{'} \subseteq A_2$ and $B_2 \subseteq B_2^{'}$.

Similar estimates can be found for the truncation of Left shift operator also.

Next we would like to give the following example, where the Toeplitz matrix is taken from Exercise 8 of page 135 of [2].

Example 2.7. Let us consider the 2×2 Toeplitz matrix $T_2(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with associated symbol $a : \mathbb{T} \to \mathbb{C}$ given by $a(t) = t + t^{-1}$. Let $0 < \epsilon < 1$. Then the

operator norm of T_2, $\|T_2\| = (largest\ eigen\ value\ of\ (T_2^*T_2))^{\frac{1}{2}} = 1$ and spectrum of T_2, $\sigma(T_2) = \{-1, 1\}$. The condition spectrum of T_2 is $\sigma_\epsilon(T_2) = B\big[0, \frac{1+\epsilon}{1-\epsilon}\big]$.

So our estimates of lower bound and upper bounds of $\sigma_\epsilon(T_2)$ from Theorem 2.5 are

$$A_2 = \left\{ z \in \mathbb{C} : \frac{3|z-\lambda|}{\vartheta_2(z-a)} \leq \epsilon\ \forall \lambda \in \sigma(T_2) \right\}$$
$$= \left\{ z \in \mathbb{C} : \frac{3|z-\lambda|}{\vartheta_2(z-a)} \leq \epsilon\ \forall \lambda \in \{-1, 1\} \right\}$$

and

$$B_2 = \left\{ z \in \mathbb{C} : \frac{|z| - \|T_2\|}{\vartheta_2(z-a)} \leq \epsilon \right\}$$
$$= \left\{ z \in \mathbb{C} : \frac{|z| - 1}{\vartheta_2(z-a)} \leq \epsilon \right\}$$

respectively.

For the sake of calculation in view of the inequality $\|z - T_2\| \leq \vartheta_2(z-a)$, from Lemma 2.3, one can obtain another lower bound of $\sigma_\epsilon(T_2)$, given by

$$A_2^{'} = \left\{ z \in \mathbb{C} : \frac{3|z-\lambda|}{\|z-T_2\|} \leq \epsilon\ \forall \lambda \in \{-1, 1\} \right\}.$$

Similarly from the inequality $\vartheta_2(z-a) \leq \|z-a\|$, from Lemma 2.3, another upper bound of $\sigma(T_2)$ can be obtained which is given by

$$B_2^{'} = \left\{ z \in \mathbb{C} : \frac{|z| - 1}{\|z-a\|} \leq \epsilon \right\}.$$

But these are not better bounds since $A_2^{'} \subseteq A_2$ and $B_2 \subseteq B_2^{'}$.

Now we shall see that for any $n \in \mathbb{N}$, the subset A_n of $\sigma_\epsilon(T_n)$ in Theorem 2.5 is also inside an annulus in the following proposition.

Proposition 2.8. *Let $n \in \mathbb{N}$ and A_n be as in Theorem 2.5. Then $A_n \subseteq C_n$, where C_n is the annular region given by*

$$C_n = \left\{ z \in \mathbb{C} : \frac{3|\lambda| - \epsilon\|a\|_\infty}{3+\epsilon} \leq |z| \leq \frac{3|\lambda| + \epsilon\|a\|_\infty}{3-\epsilon},\ for\ all\ \lambda \in \sigma(T_n) \right\}$$

and hence $A_n \subseteq C_n \cap \sigma_\epsilon(T_n)$.

Proof. Let $\omega \in A_n$. Then for all $\lambda \in \sigma(T_n)$, $\frac{3|\lambda-\omega|}{\vartheta_n(\omega-a)} \leq \epsilon$. Now from Lemma 2.3, we have

$$\vartheta_n(\omega - a) \leq \|\omega - a\|_\infty \leq |\omega| + \|a\|_\infty (by\ Triangle\ inequality)$$
$$\Rightarrow \frac{1}{|\omega| + \|a\|_\infty} \leq \frac{1}{\vartheta_n(\omega-a)} (since\ T_n\ is\ non-diagonal,\ 0 \neq \|\omega - T_n\| \leq \vartheta(\omega - a)).$$

Case 1 : Let $|\omega| \geq |\lambda|$. Then by Triangle inequality we have $|\omega| - |\lambda| \leq |\omega - \lambda|$. So we get

$$\frac{3|\omega| - 3|\lambda|}{|\omega| + \|a\|_\infty} \leq \frac{3|\lambda - \omega|}{\vartheta_n(\omega - a)} \leq \epsilon.$$

On simplifying we get

$$|\omega| \leq \frac{3|\lambda| + \epsilon\|a\|_\infty}{3-\epsilon}.$$

Case 2 : Let $|\omega| \leq |\lambda|$. Then by Triangle inequality we have $|\lambda| - |\omega| \leq |\omega - \lambda|$. So we get

$$\frac{3|\lambda| - 3|\omega|}{|\omega| + \|a\|_\infty} \leq \frac{3|\lambda - \omega|}{\vartheta_n(\omega - a)} \leq \epsilon.$$

On simplifying we get

$$|\omega| \geq \frac{3|\lambda| - \epsilon\|a\|_\infty}{3+\epsilon}.$$

Therefore from both the cases we get $\omega \in C_n$. So $A_n \subseteq C_n$. The last part of the assertion follows from Theorem 2.5. □

Remark 2.9. In a closer look of the proof of Theorem 2.5 and Proposition 2.8, we can observe that for any $n \in \mathbb{N}$, $A_n = \cap_{\lambda \in \sigma(T_n)} A_{n,\lambda}$ and $C_n = \cap_{\lambda \in \sigma(T_n)} C_{n,\lambda}$, where

$$A_{n,\lambda} = \left\{ z \in \mathbb{C} : \frac{3|\lambda - z|}{\vartheta_n(z-a)} \leq \epsilon \right\}$$

and

$$C_{n,\lambda} = \left\{ z \in \mathbb{C} : \frac{3|\lambda| - \epsilon\|a\|_\infty}{3+\epsilon} \leq |z| \leq \frac{3|\lambda| + \epsilon\|a\|_\infty}{3-\epsilon} \right\}$$

for some $\lambda \in \sigma(T_n)$.

Remark 2.10. It is to be noted that for any $n \in \mathbb{N}$, A_n coincides with $\sigma(T_n)$ and B_n coicides with the ball $B[0, \|T_n\|]$ as ϵ goes to zero, where A_n and B_n are as in in Theorem 2.5.

Question 2.11. From the above Proposition 2.8, we can realize for any $n \in \mathbb{N}$, $\sigma_\epsilon(T_n) \subseteq C_n \cup G_n$, for some $G_n (\subseteq \sigma_\epsilon(T_n))$. Can we estimate G_n ?

So far we have tried to estimate a subset and superset of the condition spectrum of Toeplitz matrices which are finite truncations of Toeplitz operators. Now for a general Toeplitz operator, the following theorem gives a subset as well as a superset of the condition spectrum on the similar lines of that of Toeplitz matrices .

Theorem 2.12. *Let $T = T(a)$ be a non-diagonal Toeplitz operator where $a \in L^\infty_{2\pi}[0, 2\pi]$. Then the condition spectrum of T can be estimated as $L \subseteq \sigma_\epsilon(T) \subseteq U$, where*

$$L = \left\{ z \in \mathbb{C} : \frac{|z - \lambda|}{\|z - a\|_\infty} \leq \epsilon \ for \ all \ \lambda \in \sigma(T) \right\}$$

and

$$U = \left\{ z \in \mathbb{C} : \frac{|z| - \|T\|}{\|z - a\|_\infty} \leq \epsilon \right\}.$$

Proof. Let $\omega \in L$. If $\omega \in \sigma(T)$ then $\omega \in \sigma_\epsilon(T)$ since $\sigma(T) \subseteq \sigma_\epsilon(T)$ and we are done.

Now let $\omega \notin \sigma(T)$ and $\lambda \in \sigma(T)$ be arbitrary. Then $(\omega - T)$ is invertible. So from Lemma 2.4, we get

$$|\omega - \lambda| \geq \frac{1}{\|(\omega - T)^{-1}\|}$$
$$\Rightarrow \|(\omega - T)^{-1}\| \geq \frac{1}{|\omega - \lambda|}.$$

Also

$$\|\omega - T\| = \|\omega - a\|_\infty > 0 \quad (since\ T\ is\ non-diagonal).$$

From the above inequalities, we get

$$\|\omega - T\|\|(\omega - T)^{-1}\| \geq \frac{\|\omega - a\|_\infty}{|\omega - \lambda|} \geq \frac{1}{\epsilon} \quad (since\ \omega \in L\ and\ \omega \neq \lambda).$$

Therefore $\omega \in \sigma_\epsilon(T)$ which implies $L \subseteq \sigma_\epsilon(T)$.

Now let $\mu \in \sigma_\epsilon(T)$. Since T is non-diagonal we have

$$\|\mu - T\| = \|\mu - a\|_\infty > 0.$$

We have the following two cases.

Note that whenever $|\mu| \leq \|T\|$ we have $\mu \in U$ and we are done. Hence let us consider $|\mu| > \|T\|$. Then we have $(\mu - T)$ is invertible and also

$$\|(\mu - T)^{-1}\| \leq \frac{1}{|\mu| - \|T\|}.$$

So

$$\|\mu - T\|\|(\mu - T)^{-1}\| \leq \frac{\|\mu - a\|_\infty}{|\mu| - \|T\|}.$$

Since $\mu \in \sigma_\epsilon(T)$ we have $\|\mu - T\|\|(\mu - T)^{-1}\| \geq \frac{1}{\epsilon}$. So $\frac{\|\mu - a\|_\infty}{|\mu| - \|T\|} \geq \frac{1}{\epsilon}$ which implies $\frac{|\mu| - \|T\|}{\|\mu - a\|_\infty} \leq \epsilon$. So $\mu \in U$ which implies $\sigma_\epsilon(T) \subseteq U$ and hence the result follows. □

In the following remark we discuss about a better estimate of superset of $\sigma_\epsilon(T)$ than our known estimate.

Remark 2.13. From the above Theorem 2.12, we get $\sigma_\epsilon(T) \subseteq U$. But we also know that $\sigma_\epsilon(T) \subseteq B\left[0, \frac{1+\epsilon}{1-\epsilon}\|T\|\right]$ (see [3]). We shall show that the set U is a better superset approximation of $\sigma_\epsilon(T)$ than the set $B\left[0, \frac{1+\epsilon}{1-\epsilon}\|T\|\right]$ in the sense that $U \subseteq B\left[0, \frac{1+\epsilon}{1-\epsilon}\|T\|\right]$.

Proof. Let $\mu \in U$. Then

$$\frac{|\mu| - \|T\|}{\|\mu - a\|_\infty} \leq \epsilon$$
$$\Rightarrow |\mu| - \|T\| \leq \epsilon\|\mu - a\|_\infty \leq \epsilon(|\mu| + \|a\|_\infty)$$
$$\Rightarrow |\mu| \leq \frac{\|T\| + \epsilon\|a\|_\infty}{1 - \epsilon} = \frac{1 + \epsilon}{1 - \epsilon}\|T\|(\ since\ \|T\| = \|a\|_\infty\ by\ Theorem\ 1.3)$$

$$\Rightarrow \mu \in B\left[0, \frac{1+\epsilon}{1-\epsilon}\|T\|\right].$$

Therefore $U \subseteq B\left[0, \frac{1+\epsilon}{1-\epsilon}\|T\|\right]$. □

In the following example, we shall try to derive the subset and superset of the condition spectrum of right shift operator, the same for the left shift operator can be obtained similarly.

Example 2.14. Let $\mathcal{R}$ be the right shift operator on $\ell^2(\mathbb{N})$. Let $0 < \epsilon < 1$. Then our derived subset and superset from Theorem 2.12 are

$$L_{\mathcal{R}} = \left\{ z \in \mathbb{C} : \frac{|z-\lambda|}{\|z-a\|_\infty} \leq \epsilon \ \forall \lambda \in \sigma(\mathcal{R}) \right\}$$

and

$$U_{\mathcal{R}} = \left\{ z \in \mathbb{C} : \frac{|z| - \|\mathcal{R}\|}{\|z-a\|_\infty} \leq \epsilon \right\}$$

respectively.

Since the associated symbol function for $\mathcal{R}$ is $a : \mathbb{T} \to \mathbb{C}$ is $a(t) = t^{-1}$, we get $\|z-a\|_\infty = 1 + |z|$. Also $\|\mathcal{R}\| = 1$ and $\sigma(\mathcal{R}) = \overline{B(0,1)} = B[0,1]$ (from [3]). Hence

$$L_{\mathcal{R}} = \left\{ z \in \mathbb{C} : \frac{|z-\lambda|}{|z|+1} \leq \epsilon, \forall \lambda \in B[0,1] \right\}.$$

Again

$$\frac{|z| - \|\mathcal{R}\|}{\|z-a\|_\infty} = \frac{|z|-1}{|z|+1}.$$

So

$$\frac{|z| - \|\mathcal{R}\|}{\|z-a\|_\infty} \leq \epsilon \Leftrightarrow \frac{|z|-1}{|z|+1} \leq \epsilon \Leftrightarrow |z| \leq \frac{1+\epsilon}{1-\epsilon}.$$

Hence $U_{\mathcal{R}} = B\left[0, \frac{1+\epsilon}{1-\epsilon}\right]$.

Again from [3] we get the condition spectrum $\sigma_\epsilon(\mathcal{R}) = B\left[0, \frac{1+\epsilon}{1-\epsilon}\|\mathcal{R}\|\right] = B\left[0, \frac{1+\epsilon}{1-\epsilon}\right]$, and we know the fact that $\sigma_\epsilon(\mathcal{R}) \subseteq U_{\mathcal{R}} \subseteq B\left[0, \frac{1+\epsilon}{1-\epsilon}\right]$. So the upper bound $U_{\mathcal{R}}$ coincides with both $\sigma_\epsilon(\mathcal{R})$ and $B\left[0, \frac{1+\epsilon}{1-\epsilon}\right]$. Similar fact is true for the left shift operator.

In the next Proposition, we shall see that the subset L of $\sigma_\epsilon(T)$ given in Theorem 2.12 is also inside an annulus as before it was discussed in Proposition 2.8.

Proposition 2.15. *Let $T(a)$ be a non diagonal Toeplitz operator. Then we have $L \subseteq R$, where*

$$R = \left\{ z \in \mathbb{C} : \frac{|\lambda| - \epsilon\|a\|_\infty}{1+\epsilon} \leq |z| \leq \frac{|\lambda| + \epsilon\|a\|_\infty}{1-\epsilon}, \ for\ all\ \lambda \in \sigma(T) \right\}$$

and hence $L \subseteq \sigma_\epsilon(T) \cap R$.

Proof. Let $\omega \in L$ and $\lambda \in \sigma(T)$ be arbitrary. Since T is non-diagonal, we have

$$\|\omega - a\|_\infty = \|\omega - T\| > 0.$$

So

$$\frac{|\omega-\lambda|}{\|\omega-a\|_\infty} \leq \epsilon.$$

By Triangle inequality we have $\|\omega-a\|_\infty \leq |\omega|+\|a\|_\infty$ which gives

$$\frac{1}{\|\omega-a\|_\infty} \geq \frac{1}{|\omega|+\|a\|_\infty}.$$

We consider the following two cases.

Case 1 : Let $|\omega| \geq |\lambda|$. Then we have $|\omega-\lambda| \geq |\omega|-|\lambda|$.

So we get

$$\frac{|\omega|-|\lambda|}{|\omega|+\|a\|_\infty} \leq \frac{|\omega-\lambda|}{\|\omega-a\|_\infty} \leq \epsilon \Rightarrow (1-\epsilon)|\omega| \leq |\lambda|+\epsilon\|a\|_\infty$$

which gives

$$|\omega| \leq \frac{|\lambda|+\epsilon\|a\|_\infty}{1-\epsilon}.$$

Case 2 : Let $|\omega| \leq |\lambda|$.

Then we have $|\omega-\lambda| \geq |\lambda|-|\omega|$.

So we get

$$\frac{|\lambda|-|\omega|}{|\omega|+\|a\|_\infty} \leq \frac{|\omega-\lambda|}{\|\omega-a\|_\infty} \leq \epsilon \Rightarrow (1+\epsilon)|\omega| \geq |\lambda|-\epsilon\|a\|_\infty$$

which gives

$$|\omega| \geq \frac{|\lambda|-\epsilon\|a\|_\infty}{1+\epsilon}.$$

So from both the cases we get $\omega \in R$ and hence $L \subseteq R$. The last part of the assertion follows from Theorem 2.12. □

Remark 2.16. In the above Theorem 2.12 and Proposition 2.15 we can find that $L = \cap_{\lambda\in\sigma(T)} L_\lambda$ and $R = \cap_{\lambda\in\sigma(T)} R_\lambda$, where

$$L_\lambda = \left\{ z \in \mathbb{C} : \frac{|z-\lambda|}{\|z-a\|_\infty} \leq \epsilon \right\}$$

and

$$R_\lambda = \left\{ z \in \mathbb{C} : \frac{|\lambda|-\epsilon\|a\|_\infty}{1+\epsilon} \leq |z| \leq \frac{|\lambda|+\epsilon\|a\|_\infty}{1-\epsilon} \right\}$$

for some $\lambda \in \sigma(T)$.

Remark 2.17. As before in Remark 2.10, it is to be noted that L coincides with $\sigma(T)$ and U coincides $B[0, \|T\|]$ as ϵ goes to 0, where L and U are as in Theorem 2.12.

Question 2.18. From the above Proposition 2.15, we can realize $\sigma_\epsilon(T) \subseteq S \cup R$, for some
$S(\subseteq \sigma_\epsilon(T))$. Can we estimate S ?

In the following remark, we shall see that the subset estimation of condition spectrum of Toeplitz matrix is sitting inside the same for that of Toeplitz operator.

Remark 2.19. Since for any $n \in \mathbb{N}$, $\frac{1}{3}\vartheta_n(a) \leq \|a\|_\infty$(by Lemma 2.3), it can be derived that $A_n \subseteq L$ where A_n and L are as in Theorem 2.5 and Theorem 2.12 respectively. Since this holds for every $n \in \mathbb{N}$, we can as well conclude that

$$\cup_{n\in\mathbb{N}} A_n \subseteq L.$$

In the following remark, we shall see that the superset estimation of condition spectrum of a Toeplitz matrix is sitting inside the same for that of Toeplitz operator.

Remark 2.20. Since for any $n \in \mathbb{N}$, $\|T_n\| \leq \|T\|$ and $\vartheta_n(a) \leq \|a\|_\infty$(by Lemma 2.3), it can be derived that $B_n \subseteq U$, where B_n and U are as in Theorem 2.5 and Theorem 2.12 respectively. Since this holds for every $n \in \mathbb{N}$, we can as well conclude that

$$\cup_{n\in\mathbb{N}} B_n \subseteq U.$$

In the following remark, we shall see that the estimated annulus containing the subset estimation of condition spectrum of a Toeplitz matrix is sitting inside the same for that of Toeplitz operator.

Remark 2.21. For any $n \in \mathbb{N}$, it can be derived that $C_n \subseteq R$ where C_n and R are as in Proposition 2.8 and Proposition 2.15 respectively. Since it holds for all $n \in \mathbb{N}$, we can conclude that

$$\cup_{n\in\mathbb{N}} C_n \subseteq R.$$

3. Toeplitz operator and matrix with symbol in Wiener algebra

In this section we give estimation of condition spectrum of Toeplitz operator for an element in Wiener algebra.

Definition 3.1 ([2])**.** (The Wiener algebra). The Wiener algebra, denoted by W, is defined as the set of all function $a : \mathbb{T} \to \mathbb{C}$ with absolutely convergent Fourier series that is, as the collection of all function $a : \mathbb{T} \to \mathbb{C}$ which can be represented in the form $a(t) = \sum\limits_{n=-\infty}^{\infty} a_n t^n (t \in \mathbb{T})$ with the norm given by $\|a\|_W := \sum\limits_{n=-\infty}^{\infty} |a_n| < \infty$.

In fact W is properly contained in $L^\infty_{2\pi}[0, 2\pi]$. For $a \in W$, let $\mathcal{R}(a)$ denote the range of a and $conv\mathcal{R}(a)$ be its convex hull.

Proposition 4.12 in [2] gives a sufficient condition of invertibility of $T_n(a)$ and also, Corollary 4.28 in [2] gives location of $\sigma(T_n(a))$ as a subset in terms of $\mathcal{R}(a)$. Brown and Halmos theorem, stated in Theorem 4.29 of [2], gives us a way to get some new subset and superset of condition spectrum of Toeplitz operator and matrix associated with symbol in W which are a little bit different from the general case.

Theorem 3.2 ([2] Brown and Halmos)**.** *Let $a \in W$ and suppose*

$$d := d(0, conv\mathcal{R}(a)) > 0.$$

Then $T(a)$ is invertible on $l^2(\mathbb{N})$ with $\|T^{-1}(a)\|_2 \leq \frac{1}{d}$ and $T_n(a)$ is invertible for all $n \geq 1$ with $\|T_n^{-1}(a)\|_2 \leq \frac{1}{d}$.

Now in the following theorems we shall establish similar results as in (Theorems 2.5 , 2.12). Though the subsets remain same, we have different supersets of the condition spectrum in each of these cases with the help of Brown and Halmos Theorem.

Theorem 3.3. *Let $n \in \mathbb{N}$ and $T_n(a)$ be a non diagonal Toeplitz matrix with symbol $a \in W$. Then the condition spectrum of T_n can be estimated as:*

$$A_n \subseteq \sigma_\epsilon(T_n) \subseteq B_{W,n},$$

where A_n is as in Theorem 2.5 and

$$B_{W,n} = \left\{ z \in \mathbb{C} : \frac{d(0, conv\mathcal{R}(z-a))}{\vartheta_n(z-a)} \leq \epsilon \right\}.$$

Proof. Since every element of W is in $L^\infty_{2\pi}[0, 2\pi]$, the first inclusion follows from Theorem 2.5. Now to prove the next inclusion let $\mu \in \sigma_\epsilon(T_n)$. Since T_n is non-diagonal, we have

$$0 \neq \|\mu - T_n\| \leq \vartheta_n(\mu - a) \quad (\textit{by Lemma } 2.4).$$

Note that whenever $d(0, conv\mathcal{R}(\mu - a)) = 0$ we have $\mu \in B_{W,n}$ and we are done. Hence let us consider $d(0, conv\mathcal{R}(\mu - a)) > 0$. Then by Brown and Halmos Theorem, $(\mu - T_n)$ is invertible and

$$\|(\mu - T_n)^{-1}\| \leq \frac{1}{d(0, conv\mathcal{R}(\mu - a))}.$$

From the above inequalities we get

$$\|\mu - T_n\| \|(\mu - T_n)^{-1}\| \leq \frac{\vartheta_n(\mu - a)}{d(0, conv\mathcal{R}(\mu - a))}.$$

But $\mu \in \sigma_\epsilon(T_n)$. So

$$\begin{aligned} &\|\mu - T_n\| \|(\mu - T_n)^{-1}\| \geq \frac{1}{\epsilon} \\ \Rightarrow &\frac{\vartheta_n(\mu - a)}{d(0, conv\mathcal{R}(\mu - a))} \geq \frac{1}{\epsilon} \\ \Rightarrow &\frac{d(0, conv\mathcal{R}(\mu - a))}{\vartheta_n(\mu - a)} \leq \epsilon. \end{aligned}$$

So $\mu \in B_{W,n}$ which implies $\sigma_\epsilon(T_n) \subseteq B_{W,n}$ and hence the result follows. □

For illustration we would like to give the following example.

Example 3.4. In Example (2.7) let us consider the symbol $a \in W$. Since $a \in W$, $\|a\|_W = 1 + 1 = 2$ and range of a is

$$R(a) = \{a(t) : t \in \mathbb{T}\} = \{t + t^{-1} : t \in \mathbb{T}\} = \{2\cos\theta : \theta \in [0, 2\pi]\} = [-2, 2].$$

So the convex hull of $R(a)$ is $Conv(R(a)) = [-2, 2]$. The estimated upper bound of $\sigma_\epsilon(T_2)$ from Theorem 3.3 is

$$\begin{aligned} B_{W,2} &= \left\{ z \in \mathbb{C} : \frac{d(z, Conv(R(a))}{\vartheta_2(z-a)} \leq \epsilon \right\} \\ &= \left\{ z \in \mathbb{C} : \inf_{\eta \in [-2,2]} \frac{|z - \eta|}{\vartheta_2(z-a)} \leq \epsilon \right\}. \end{aligned}$$

For the sake of calculation in view of the inequality $\|z - T_2\| \leq \vartheta_2(z-a)$, from Lemma 2.3, one can obtain another upper bound of $\sigma_\epsilon(T_2)$, given by

$$B'_{W,2} = \left\{ z \in \mathbb{C} : \inf_{\eta \in [-2,2]} \frac{|z-\eta|}{\|z - T_2\|} \leq \epsilon \right\}.$$

But it is not better bound since $B_{W,2} \subseteq B'_{W,2}$.

In the next Proposition we shall see that the subset A_n of $\sigma_\epsilon(T_n)$ given in Theorem 3.3 is also inside an annulus as before in Proposition 2.8.

Proposition 3.5. *For some $n \in \mathbb{N}$, let A_n be as in Theorem 3.3. Then $A_n \subseteq C_n$, where C_n is as in Proposition 2.8 and hence $A_n \subseteq \sigma_\epsilon(T_n) \cap C_n$.*

Proof. Since every element of W is in $L^\infty_{2\pi}[0, 2\pi]$, the result follows from Proposition 2.8. □

Since $W \subseteq L^\infty_{2\pi}[0, 2\pi]$, we can observe that the statement of Remark 2.9 is applicable here also. Not only that Question 2.11 can be asked here as well.

Remark 3.6. As before in Remark 2.10 and Remark 2.17, it is to be noted that for any $n \in \mathbb{N}$, $B_{W,n}$ coincides with $conv\mathcal{R}(a)$ as ϵ goes to 0, where $B_{W,n}$ is as in Theorem 3.3.

Analogously as before we shall obtain subset and superset of $\sigma_\epsilon(T)$ for Toeplitz operator with a symbol in Wiener algebra.

Theorem 3.7. *Let $T = T(a)$ be a non diagonal Toeplitz operator with symbol $a \in W$. Then the condition spectrum of T can be estimated as: $L \subseteq \sigma_\epsilon(T) \subseteq U_W$, where L is as in Theorem 2.12 and*

$$U_W = \left\{ z \in \mathbb{C} : \frac{d(0, conv\mathcal{R}(z-a))}{\|(z-a)\|_\infty} \leq \epsilon \right\}.$$

Proof. Since every element of W is in $L^\infty_{2\pi}[0, 2\pi]$, the first inclusion follows from Theorem 2.12.

Now let $\mu \in \sigma_\epsilon(T)$. Note that whenever $d(0, conv\mathcal{R}(\mu - a)) = 0$, we have $\mu \in U_W$ and we are done. Hence let us consider $d(0, conv\mathcal{R}(\mu - a)) > 0$.

Then by Brown and Halmos Theorem, $\mu - T$ is invertible and

$$\|(\mu - T)^{-1}\| \leq \frac{1}{d(0, conv\mathcal{R}(\mu - a))}$$

$$\Rightarrow \frac{1}{d(0, conv\mathcal{R}(\mu - a))} \geq \|(\mu - T)^{-1}\|.$$

And also

$$\|\mu - a\|_\infty = \|\mu - T\| \neq 0 \quad (since\ T\ is\ non-diagonal).$$

From the above inequalities we get

$$\frac{\|\mu - a\|_\infty}{d(0, conv\mathcal{R}(\mu - a))} \geq \|\mu - T\| \|(\mu - T)^{-1}\| \geq \frac{1}{\epsilon} \quad (since\ \mu \in \sigma_\epsilon(T)).$$

So $\mu \in U_W$ which implies $\sigma_\epsilon(T) \subseteq U_W$ and hence the result follows.

□

For illustration, let us consider the following example where the associated symbol, a of the Toeplitz operator, $T(a)$ is taken in W.

Example 3.8. For the full operator $T(a)$ in Example 2.7,

$$\|z-a\|_\infty = \|z-a\|_W = |z|+2.$$

So the lower bound L remains same as before while the upper bound changes to U_W, where

$$U_W = \left\{ z \in \mathbb{C} : \frac{d(z, Conv(R(a))}{\|z-a\|_\infty} \leq \epsilon \right\}$$
$$= \left\{ z \in \mathbb{C} : \inf_{\eta \in [-2,2]} \frac{|z-\eta|}{|z|+2} \leq \epsilon \right\}.$$

Next we shall see that as before in Proposition 2.15, the subset L of $\sigma_\epsilon(T)$ given in Theorem 3.7 is also inside an annulus in the following Proposition.

Proposition 3.9. *Let L be as in Theorem 3.7. Then $L \subseteq R$, where R is as in Proposition 2.15 and hence $L \subseteq \sigma_\epsilon(T) \cap R$.*

Proof. Since every element of W is in $L^\infty_{2\pi}[0, 2\pi]$, the result follows from Proposition 2.15. □

As before we noted in Remark 2.20, for $a \in W$ also we will see that the superset estimation of Toeplitz matrix is sitting inside the same of that of Toeplitz operator.

Remark 3.10. It is to be noted that since for any $n \in \mathbb{N}$, $\|T_n\| \leq \|T\|$ and $\vartheta_n(a) \leq \|a\|_\infty$(by Lemma 2.3), it can be easily derived that $B_{W,n} \subseteq U_W$, where $B_{W,n}$ and U_W are as in Theorem 3.3 and Theorem 3.7 respectively. Since this holds for every $n \in \mathbb{N}$, we can as well conclude that

$$\cup_{n\in\mathbb{N}} B_{W,n} \subseteq U_W.$$

Since $W \subseteq L^\infty_{2\pi}[0, 2\pi]$, we can observe that the statements of Remark 2.16 2.19 and 2.21 are applicable here also and Question 2.18 can be asked here as well.

Remark 3.11. As before in Remark 3.6, it is to be noted that U_W coincides with $conv\mathcal{R}(a)$ as ϵ goes to 0, where U_W is as in Theorem 3.7.

Being a finite truncation of a Toeplitz operator, a Toeplitz matrix associated with symbol in $L^\infty_{2\pi}[0, 2\pi]$ can also be considered as a Toeplitz matrix with symbol in W. So the corresponding superset estimations B_n(in Theorem 2.5) and $B_{W,n}$(in Theorem 3.3) can be swapped. But for a general Toeplitz operator it is not so.

For $a \in L^\infty_{2\pi}[0, 2\pi]$ we get that the condition spectrum of Toeplitz operator $\sigma_\epsilon(T) \subseteq U$ by Theorem 2.12 whereas when $a \in W$ we get $\sigma_\epsilon(T) \subseteq U_W$ by Theorem 3.7. Hence it is very natural to ask that between U_W and U, which would be a better estimate of upper bound of $\sigma_\epsilon(T)$. In the following proposition we shall show that U_W is a better estimate of the upper bound of $\sigma_\epsilon(T)$.

Proposition 3.12. *Let T be a Toeplitz operator with symbol $a \in W$, and let U_W and U be as in Theorem 3.7 and Theorem 2.12 respectively. Then $U_W \subseteq U$.*

Proof. Let $w \in U_W$. Then

$$\frac{d(0, conv\mathcal{R}(w-a))}{\|(w-a)\|_\infty} \leq \epsilon. \tag{3.1}$$

But $d(0, conv\mathcal{R}(w-a)) = d(w, conv\mathcal{R}(a))$. So from (3.1), we get

$$d(w, conv\mathcal{R}(a)) \leq \epsilon\|(w-a)\|_\infty. \tag{3.2}$$

Note that whenever $|w| \leq \|T\|$, we have $w \in U$ and we are done. Hence let us consider the case when $|w| > \|T\|$.

By definition,

$$d(w, conv\mathcal{R}(a)) = \inf\{|w-l| : l \in conv\mathcal{R}(a)\}.$$

Since $a : \mathbb{T} \to \mathbb{C}$ is continuous and $\mathbb{T}$ is a compact set in $\mathbb{C}$, we get $\mathcal{R}(a)$ is compact in $\mathbb{C}$. This implies $conv\mathcal{R}(a)$ is compact in $\mathbb{C}$. So, $|w-l|$ will attain its infimum at some point of $conv\mathcal{R}(a)$, say at $l = \sum_{i=1}^k \lambda_i a(t_i)$, where $t_i \in \mathbb{T},\ 0 \leq \lambda_i \leq 1,\ i = 1, \cdots, k$ and $\sum_{i=1}^k \lambda_i = 1$. So from the above inequality (3.2) we get

$$|w - \sum_{i=1}^k \lambda_i a(t_i)| \leq \epsilon\|w-a\|_\infty. \tag{3.3}$$

But

$$\begin{aligned}
|w - \sum_{i=1}^k \lambda_i a(t_i)| &\geq |w| - |\sum_{i=1}^k \lambda_i a(t_i)| (by\ Triangle\ inequality) \\
&\geq |w| - \sum_{i=1}^k \lambda_i |a(t_i)| \\
&\geq |w| - \sum_{i=1}^k \lambda_i \|a\|_\infty (since\ |a(t_i)| \leq \|a\|_\infty) \\
&= |w| - \|a\|_\infty (since \sum_{i=1}^k \lambda_i = 1).
\end{aligned}$$

Combining with (3.3), we get

$$|w| - \|a\|_\infty \leq \epsilon\|w-a\|_\infty.$$

Since $\|T\| = \|a\|_\infty$ by Theorem 1.3, we have

$$\frac{|w| - \|T\|}{\|w-a\|_\infty} \leq \epsilon.$$

Hence $w \in U$ and the result follows. □

It is to be noted that in Example 2.14, if we consider the symbol a of Right shift operator $\mathcal{R}$ to be an Wiener algebra element, then the upper bound (superset) U_W of $\sigma_\epsilon(\mathcal{R})$ will coincide with the closed ball $B\left[0, \frac{1+\epsilon}{1-\epsilon}\right]$ since

$$B\left[0, \frac{1+\epsilon}{1-\epsilon}\right] = \sigma_\epsilon(\mathcal{R}) \subseteq U_W \subseteq U = B\left[0, \frac{1+\epsilon}{1-\epsilon}\right].$$

Conclusion

Here in this paper we have estimated theoritically some subset and superset of condition spectrum of Toeplitz operators and matrices associated with the symbols of essentially bounded functions and Wiener algebra. As a consequence of our discussion, it is also very natural to ask the following question.

Question 3.13. Is it possible to get a concrete estimation of the condition spectrum of Toeplitz operator in terms of the associated symbol ? If not in general, then is there any sufficient condition to do so ?

The condition spectrum of other kind of operators are also quite interesting. So as an immediate spin-off of our discussion it is quite relevant to ask the following question.

Question 3.14. Can we estimate the condition spectrum of Hankel operators and other kind of operators ?

Again since Toeplitz operators can act on Hardy space and other kind of spaces we would like to pose this question.

Question 3.15. Is there any way to estimate condition spectrum of Toeplitz operators acting on Hardy spaces and other nice spaces in terms of the symbol associated with it ?

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