Pure and Applied Functional Analysis
Volume 8, Number 1, 2023, 215–241

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON A CLASS OF FORCED ACTIVE SCALAR EQUATIONS WITH SMALL DIFFUSIVE PARAMETERS

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ABSTRACT. Many equations that model fluid behaviour are derived from systems that encompass multiple physical forces. When the equations are written in non dimensional form appropriate to the physics of the situation, the resulting partial differential equations often contain several small parameters. We study a general class of such PDEs called active scalar equations which in specific parameter regimes produce certain well known models for fluid motion. We address various mathematical questions relating to well-posedness, regularity and long time behaviour of the solutions to this general class including vanishing limits of several diffusive parameters.

1. Introduction

Active scalar equations belong to a class of partial differential equations where the evolution in time of a scalar quantity is governed by the motion of the fluid where the velocity itself varies with this scalar quantity. They have been a topic of considerable study in recent years, in part because they arise in many physical models and in part because they present challenging nonlinear PDEs. Various active scalar equations, such as surface quasi-geostrophic equation (SQG) and drift-diffusion equations, have received considerable attention in the past decade because of the challenging nature of the delicate balance between the nonlinear term and the dissipative term. The physics of an active scalar equation is encoded in the constitutive law that relates the transport velocity vector u with a scalar field θ. This law produces a differential operator that when applied to the scalar field determines the velocity. The singular or smoothing properties of this operator are closely connected with the mathematics of the nonlinear advection equation for θ.

More precisely, we are interested in an abstract class of active scalar equations in $\mathbb{T}^d \times (0,\infty) = [0,2\pi]^d \times (0,\infty)$ with $d \in \{2,3\}$ [1] of the following form

$$\left\{\begin{array}{l} \partial_t\theta + u\cdot\nabla\theta = -\kappa\Lambda^\gamma\theta + S, \\ u_j[\theta] = \partial_{x_i}T^\nu_{ij}[\theta], \theta(x,0) = \theta_0(x) \end{array}\right. \tag{1.1}$$

2020 *Mathematics Subject Classification.* 76D03, 35Q35, 76W05.
Key words and phrases. Active scalar equations, vanishing viscosity limit, Gevrey-class solutions, global attractors.
*S. Friedlander is supported by NSF DMS-1613135.
†A. Suen is supported by Hong Kong General Research Fund (GRF) grant project number 18300720.

[1]We point out that, most of the results given in our work hold for $d \geq 2$.

where $\nu \geq 0$, $\kappa \geq 0$, $\gamma \in (0,2]$ and $\Lambda := \sqrt{-\Delta}$. Here θ_0 is the initial datum and $S = S(x)$ is a given function that represents the forcing of the system. We assume that [2]

$$\int_{\mathbb{T}^d} \theta_0(x)dx = \int_{\mathbb{T}^d} S(x) = 0 \tag{1.2}$$

which immediately implies that θ obeys

$$\int_{\mathbb{T}^d} \theta(x,t)dx = 0, \qquad \forall t \geq 0. \tag{1.3}$$

$\{T^\nu_{ij}\}_{\nu\geq 0}$ is a sequence of operators which satisfy:

(A1) For all $\nu \geq 0$, $\partial_i\partial_j T^\nu_{ij} f = 0$ for any smooth functions f.

(A2) There exists a constant $C > 0$ independent of ν, such that for all $i, j \in \{1,\ldots,d\}$,

$$\sup_{\nu\in(0,1]} \sup_{\{k\in\mathbb{Z}^3\}} |\widehat{T}^\nu_{ij}(k)| \leq C; \ \sup_{\{k\in\mathbb{Z}^3\}} |\widehat{T}^0_{ij}(k)| \leq C, \text{ where } T^0_{ij} = T^\nu_{ij}\Big|_{\nu=0}.$$

(A3) For each $\nu > 0$, there exists a constant $C_\nu > 0$ such that for all $1 \leq i, j \leq d$,

$$|\widehat{T}^\nu_{ij}(k)| \leq C_\nu |k|^{-3}, \forall k \in \mathbb{Z}^d.$$

(A4) $T^\nu_{ij} : L^\infty \to BMO$ are bounded operators for all $\nu \geq 0$.

(A5) For each $1 \leq j \leq d$ and $g \in L^2$, $\lim_{\nu\to 0} \sum_{k\in\mathbb{Z}^3} |\widehat{T^\nu_{ij}}(k) - \widehat{T^0_{ij}}(k)|^2 |\widehat{g}(k)|^2 = 0$.

Our motivation for addressing such a class of active scalar equations mainly comes from several different physical systems, all of them take the form (1.1) under particular parameter regimes:

1. The first example comes from a model proposed by Moffatt and Loper [49], Moffatt [47] for magenetostrophic turbulence in the Earth's fluid core. Under the postulates in [49], the governing equation becomes a 3D active scalar equation for a temperature field θ

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta + S, \\ u = M^\nu[\theta], \theta(x,0) = \theta_0(x). \end{cases} \tag{1.4}$$

The expressions for the Fourier multiplier symbol $\widehat{M}^\nu$ are explicitly given by

$$\begin{cases} \widehat{M}^\nu_1(k) = [k_2k_3|k|^2 - k_1k_3(k_2^2 + \nu|k|^4)]D(k)^{-1}, \\ \widehat{M}^\nu_2(k) = [-k_1k_3|k|^2 - k_2k_3(k_2^2 + \nu|k|^4)]D(k)^{-1}, \\ \widehat{M}^\nu_3(k) = [(k_1^2 + k_2^2)(k_2^2 + \nu|k|^4)]D(k)^{-1}, \end{cases} \tag{1.5}$$

where $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ is the Fourier variable and $D(k) = |k|^2k_3^2 + (k_2^2 + \nu|k|^4)^2$. The nonlinear equation (1.4) with u related to θ via the operator M^ν is known as the magnetogeostrophic (MG$^\nu$) equation (or simply MG equation) and its mathematical properties have been addressed in a series of papers which include [28], [29], [30], [31], [32], [33], [35], [36]. The behaviour

[2] Such mean zero assumption is common in many physical models which include SQG equation and MG equation; see [16] and [29] for example.

of the MG^ν equation is strikingly different when the parameters ν and κ are present (i.e. positive) or absent (i.e. zero). As the Fourier multiplier symbols $\widehat{M}^0$ given by (1.5) with $\nu = 0$ are *not* bounded in all regions of Fourier space [32], when $\nu = 0$ the relation between u and θ is given by a *singular* operator of order 1. The implications of such fact for the inviscid MG^0 equation are summarized in the survey article by Friedlander, Rusin and Vicol [35]. In particular, when $\kappa > 0$ the inviscid but thermally dissipative MG^0 equation is globally well-possed [32]; on the other hand when $\nu = 0$ *and* $\kappa = 0$, the singular inviscid MG^0 equation is *ill-possed* in the sense of Hadamard in any Sobolev space [33]. In [28], Friedlander and Suen first addressed the system (1.4)-(1.5) for $\nu > 0$ and obtained well-posedness results in Sobolev space. In a series of papers [29]- [30] the authors further examined the limit of vanishing viscosity as $\nu \to 0$ in the case when $\kappa > 0$ and $\kappa = 0$ and the long-time behaviour of solutions. They proved global existence of classical solutions to the forced MG^ν equations and obtained convergences of solutions as ν vanishes. Moreover, it was shown in [31] that the equations (1.4)-(1.5) possess global attractors with various interesting properties.

2. The second example of a physical system comes from incompressible flow in a porous medium. It can be modeled by an active scalar equation where a small smoothing parameter enters into the constitutive law. Different from the usual incompressible porous media (IPM) equation, the incompressible porous media Brinkman (IPMB^ν) equation with an "effective viscosity" ν is derived via a modified Darcy's Law as suggested by Brinkman [2]. The IPM equation becomes the limiting case for IPMB^ν equation when $\nu = 0$. The 2D equation relating the velocity u, the density θ and the pressure P is given in non-dimensional form by

$$u = -\nabla P - e_2\theta + \nu\Delta u \tag{1.6}$$

$$\nabla \cdot u = 0 \tag{1.7}$$

which produces the constitutive law

$$\begin{aligned} u &= (1-\nu\Delta)^{-1}[-\nabla\cdot(-\Delta)^{-1}e_2\cdot\nabla\theta - e_2\theta] \\ &= (1-\nu\Delta)^{-1}R^\perp R_1\theta = M^\nu[\theta] \end{aligned} \tag{1.8}$$

where $R = (R_1, R_2)$ is the vector of Riesz transforms and $e_2 = (0, 1)$. The corresponding active scalar equation is thus given by

$$\begin{cases} \partial_t\theta^\nu + (u^\nu\cdot\nabla)\theta^\nu = 0, \\ u^\nu = M^\nu[\theta^\nu], \theta^\nu(x,0) = \theta_0(x), \end{cases} \tag{1.9}$$

where the 2D components of the Fourier multiplier symbol corresponding to $\widehat{M}^\nu$ as in (1.9) are

$$\frac{1}{1+\nu(k_1^2+k_2^2)}\left(\frac{k_1k_2}{k_1^2+k_2^2}\quad,\quad\frac{-k_1^2}{(k_1^2+k_2^2)}\right) \tag{1.10}$$

Similar to the case of $\mathrm{MG}^{\kappa,\nu}$ equation, there is a noticeable difference in the operator M^ν between the two cases $\nu > 0$ and $\nu = 0$: the operator is smoothing of order 2 when $\nu > 0$, while for $\nu = 0$ the operator is singular of order zero.

The well known IPM equations, i.e. (1.8)-(1.10) without the effective viscosity ν, have been studied in a number of papers (see [24], [25] and the reference therein). As we pointed out before, when $\nu = 0$ the operator in (1.8) becomes a singular integral operator of order zero, which is similar to the case for the SQG equations. Yet there is a crucial difference between the two operators: the SQG operator is *odd* while the IPM operator is *even.* Implications for well/ill posedness due to the odd/even structure of the operator in an active scalar equation are further explored in [26], [36], [42]. In a recent work, Friedlander and Suen [30] studied the system (1.8)-(1.10) in the limit of vanishing viscosity as $\nu \to 0$ and obtained convergence results in Sobolev spaces, which are *not* available for the case of MG equation when $\kappa = 0$. The foremost difference is that the MG^0 operator is singular of order 1 where as the IPM operator is singular of order zero. In view of such difference, for the " smoother" IPM case the convergence results are valid in Sobolev spaces rather than the analytic convergence results for the MG equation.

3. The third physical example comes from the *modified* surface quasi-geostrophic ($\mathrm{SQG}^{\kappa,\nu}$) equation. It relates the potential temperature $\theta^{\kappa,\nu}$ and the flow velocity $u^{\kappa,\nu}$ as follows:

$$\begin{cases} \partial_t \theta^{\kappa,\nu} + u^{\kappa,\nu} \cdot \nabla \theta^{\kappa,\nu} = -\kappa(-\Delta)^\gamma \theta^{\kappa,\nu} + S, \\ u^{\kappa,\nu} = M^\nu[\theta^{\kappa,\nu}], \theta^{\kappa,\nu}(x,0) = \theta_0(x), \end{cases} \tag{1.11}$$

where the Fourier multiplier symbol for $\widehat{M}^\nu$ is given by

$$\frac{1}{1+\nu(k_1^2+k_2^2)}\left(\frac{k_2}{\sqrt{k_1^2+k_2^2}} \quad , \quad \frac{-k_1}{\sqrt{k_1^2+k_2^2}}\right). \tag{1.12}$$

When $\nu = 0$, the system (1.11)-(1.12) reduces to the well-known SQG equation which has been investigated by many researchers [5], [7], [8], [10], [14], [17], [22], [55]. For $\kappa > 0$ and $\gamma = \frac{1}{2}$, it can be used as a model for studying the temperature distribution θ on the 2D boundary of a rapidly rotating fluid with small Rossby and Ekman numbers [19], while for $\kappa = 0$, the inviscid model can be applied for studying frontogenesis in meteorology [43]. It is also worth mentioning that the 3D analog of (1.11)-(1.12) is widely used as a testing model for the vorticity evolution of the 3D Navier-Stokes equations [8].

We point out that, in view of the physical models as mentioned above, conditions (A1)–(A5) are both mathematically and physically important for studying the active scalar equations (1.1)-(1.3), which can be explained as follows:

- Condition (A1) implies the drift velocity u in (1.1) is *divergence-free*, which is compatible with the *incompressibility* of the fluid described by those physical models. On the other hand, condition (A2) requires a *uniform* bound (independent of ν) on the Fourier multiplier symbol $\widehat{T}^\nu$. This conditions implies that the operator $\partial_{x_i} T^\nu_{ij}$ is at most *singular* of order 1 for $\nu \geq 0$, which is consistent with the cases for $\widehat{M}^\nu$ given in the MG$^{\kappa,\nu}$ equation. We remark here that, however, for the cases of IPMB$^\nu$ and SQG$^{\kappa,\nu}$ equation, one can replace the condition (A2) by the following condition:

(A2*) There exists a constant $C_0 > 0$ independent of ν, such that for all $i, j \in \{1, \ldots, d\}$,

$$\sup_{\nu\in(0,1]} \sup_{\{k\in\mathbb{Z}^3\}} |\widehat{\partial_{x_i} T^\nu_{ij}}(k)| \leq C_0; \ \sup_{\{k\in\mathbb{Z}^3\}} |\widehat{\partial_{x_i} T^0_{ij}}(k)| \leq C_0, \text{ where } T^0_{ij} = T^\nu_{ij}\Big|_{\nu=0}.$$

Condition (A2*) requires that the operators $\partial_{x_i} T^\nu_{ij}$ are *less* singular than those given by condition (A2), which allows us to obtain better regularity results on IPMB$^\nu$ and SQG$^{\kappa,\nu}$ equation.

- Condition (A4) imposes a minimal regularity requirement on the operators T^ν_{ij}, while condition (A3) describes the *smoothing effect* given by the parameter ν. Condition (A3) implies that the operators $\partial_{x_i} T^\nu_{ij}$ are smoothing of degree two for $\nu > 0$, which is crucial for proving well-posedness for the system (1.1) especially when $\kappa = 0$ (refer to [29] and [32] for the striking differences between the cases $\nu > 0$ and $\nu = 0$).
- For condition (A5), we notice that the Fourier multiplier symbols $\widehat{M}^\nu$ given in either (1.5), (1.10) or (1.12) do satisfy condition (A5), which implies that the drift velocity u converges to $u^{\kappa,0}$ strongly in L^2. Such behaviour of T^ν_{ij} allows strong convergence of solutions of (1.1) (refer to Theorem 7.4 in [29]), despite the fact that T^ν_{ij} are all matrices of zero-order pseudo-differential operators as required by condition (A4).

In section 2, we state the results proved in [29] for the diffusive active scalar equations, which are the equations (1.1)-(1.3) for $\kappa > 0$, $\nu \geq 0$ and $\gamma = 2$. In this case, the system (1.1)-(1.3) is globally well-posed. To prove this fact, we control the term $\|\theta(t)\|_{L^\infty}$ which can be done by using De Giorgi techniques. Having established the global well-posedness of (1.1)-(1.3), we proceed to address the convergence of solutions as $\nu \to 0$ which is based on some uniform estimates on θ which are independent of ν. Moreover, we define a weak solution to the MG^0 equation which we call a "*vanishing viscosity*" *solution* and prove the existence of a compact global attractor in $L^2(\mathbb{T}^3)$ for the MG$^\nu$ equations (1.4)-(1.5). We further obtain the upper semicontinuity of the global attractor as ν vanishes.

In section 3, we examine the non-diffusive active scalar equations (1.1)-(1.3) by considering $\kappa = 0$ and $\nu \geq 0$ (cf [30]). For the case of $\nu > 0$, the operators $\partial_x T^\nu$ are smoothing order 2 which give rise to well-posedness for (1.1)-(1.3) with $\kappa = 0$, $\nu > 0$ and $\theta_0 \in W^{s,d}$ for $s \geq 0$ and smooth forcing term S, and the results will be discussed in subsection 3.1. On the other hand, for the case when $\kappa = 0$ and $\nu = 0$, in general the system (1.1)-(1.3) fails to be well-posed in Sobolev spaces. In [33], It was proved that the singular inviscid MG^0 equation is *ill-possed* in the sense of Hadamard in any Sobolev space. Yet it is possible to obtain the local existence and uniqueness of solutions to (1.1)-(1.3) with $\kappa = \nu = 0$ in spaces of real-analytic functions, owing to the fact that the derivative loss in the nonlinearity $u \cdot \nabla\theta$ is of order at most one. We thus prove the local-in-time existence of analytic solutions which are summarised in subsection 3.2. We further address the convergence of solutions as $\nu \to 0$ and apply the claimed results to inviscid MG$^\nu$ equation (1.4)-(1.5) with $\kappa = 0$ and IPMB$^\nu$ equation (1.8)-(1.10).

In section 4, we discuss the result obtained in [31] by investigating the properties of (1.1)-(1.3) in the full range $\gamma \in [0, 2]$. More precisely we prove the existence and convergence of solutions in various space for the cases $\nu > 0$ and $\nu = 0$, which are applicable for the critical SQG equation (1.11)-(1.12) with $\nu = 0$. We then address the long-time behaviour for solutions when κ, $\nu > 0$ and obtain global attractors in H^1. We further prove some additional properties of the attractors which will be given in subsection 4.3. The results on the global attractors can be applied to MG$^\nu$ equation (1.4)-(1.5) which are related to those discussed in section 2.

2. Diffusive active scalar equations

In this section, we discuss the global existence of classical solutions to (1.1)-(1.3) for $\kappa > 0$, $\nu \geq 0$ and $\gamma = 2$, and study the convergence of solutions as ν vanishes. Specifically, we consider the following abstract system:

$$\left\{ \begin{array}{l} \partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S, \\ u_j[\theta] = \partial_{x_i} T^\nu_{ij}[\theta], \theta(x,0) = \theta_0(x) \end{array} \right. \tag{2.1}$$

with $\nu \geq 0$, and S, θ_0 satisfy the zero mean assumptions (1.2)-(1.3). The results can then be applied to MG$^\nu$ equations, which allows us to obtain the existence of a compact global attractor $\{\mathcal{A}^\nu\}_{\nu \geq 0}$ in $L^2(\mathbb{T}^3)$ including the critical equation where $\nu = 0$.

2.1. Existence of smooth solutions and convergence as $\nu \to 0$.

Friedlander and Vicol [32] analyzed the unforced $S = 0$ system (2.1) with the viscosity parameter ν set to zero, i.e. the unforced MG^0 equation. In this situation the drift diffusion equation (2.1) is critical in the sense of the derivative balance between the advection and the diffusion term. They used De Giorgi techniques to obtain global well-posedness results for the unforced critical MG^0 equation in a similar manner to the proof of global well-possedness given by Caffarelli and Vaseur [20] for the critical SQG equation. Following this work, we verify that the technical details of the De Giorgi techniques are, in fact, valid for drift diffusion equations with a smooth force. More precisely, the result is given by

Theorem 2.1 (Friedlander and Suen [29])**.** *Let $\theta_0 \in L^2$, $S \in C^\infty$ and $\kappa > 0$ be given, and assume that $\{T^\nu_{ij}\}_{\nu \geq 0}$ satisfy conditions* (A1)–(A5)*. For $\gamma = 2$, there exists a classical solution $\theta^\nu(t,x) \in C^\infty((0,\infty) \times \mathbb{T}^d)$ of* (2.1)*, evolving from θ_0 for all $\nu \geq 0$.*

Furthermore, for any $\nu \geq 0$, we prove that the smooth solutions θ^ν obtained in Theorem 2.1 satisfy a uniform bound which is independent of ν:

Theorem 2.2 (Friedlander and Suen [29])**.** *Assume that the hypotheses and notations of Theorem 2.1 are in force. Then given $0 < t_1 < t_2$ and $s \geq 0$, there exists a positive constant C which depends on C_0, t_1, t_2, s, d, κ, S, $\|\theta_0\|_{L^2}$ but independent of ν such that*

$$\sup_{t \in [t_1, t_2]} \|\theta^\nu(t,\cdot)\|_{H^s} + \int_{t_1}^{t_2} \|\theta^\nu(t,\cdot)\|^2_{H^{s+1}} dt \leq C(C_0, t_1, t_2, s, d, \kappa, S, \|\theta_0\|_{L^2}), \tag{2.2}$$

where $C_0 > 0$ is the constant as stated in condition (A5)*.*

Using Theorem 2.2, we obtain the convergence of θ^ν as $\nu \to 0$:

Theorem 2.3 (Friedlander and Suen [29])**.** *Assume that the hypotheses and notations of Theorem 2.1 are in force. For $\gamma = 2$, if θ^ν, θ are C^∞ smooth classical solutions of the system* (2.1) *for $\nu > 0$ and $\nu = 0$ respectively with initial data θ_0, then given $\tau > 0$, for all $s \geq 0$, we have*

$$\lim_{\nu \to 0} \|(\theta^\nu - \theta)(t, \cdot)\|_{H^s} = 0, \tag{2.15}$$

whenever $t \geq \tau$.

2.2. **The MG equations and existence of compact global attractor.** We apply the results obtained in subsection 2.1 to the case for MG equations given by (1.4)-(1.5). We write

$$u_j^\nu = M_j^\nu[\theta^\nu] := \partial_i T_{ij}^\nu,$$

where we have denoted

$$T_{ij}^\nu := -\partial_i(-\Delta)^{-1} M_j^\nu \text{ for } \nu \geq 0$$

and M^ν is defined by the inverse Fourier transform of (1.5). Using [29, Lemma 5.1], one can verify that there are constants $C_1, C_2 > 0$ independent of ν such that, for all $1 \leq i, j \leq 3$,

$$\sup_{\nu \in (0,1]} \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} |\widehat{T}_{ij}^\nu(k)| \leq \sup_{\nu \in (0,1]} \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \frac{|\widehat{M}^\nu(k)|}{|k|} \leq C_1,$$

$$\sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} |\widehat{T}_{ij}^0(k)| \leq \sup_{\{k \in \mathbb{Z}^3 : k \neq 0\}} \frac{|\widehat{M}^0(k)|}{|k|} \leq C_2.$$

Hence conditions (A1)–(A5) are satisfied. Theorem 2.1 and Theorem 2.2 can therefore be applied to the MG equations in order to obtain the global-in-time existence and convergence of smooth solutions:

Theorem 2.4 (Friedlander and Suen [29])**.** *Let $\theta_0 \in L^2$, $S \in C^\infty$ and $\kappa > 0$ be given. There exists a classical solution $\theta^\nu(t, x) \in C^\infty((0, \infty) \times \mathbb{T}^3)$ of* (1.4)-(1.5)*, evolving from θ_0 for all $\nu \geq 0$.*

Theorem 2.5 (Friedlander and Suen [29])**.** *Let $\theta_0 \in L^2$, $S \in C^\infty$ and $\kappa > 0$ be given. Then if θ^ν, θ are C^∞ smooth classical solutions of* (1.4)-(1.5) *for $\nu > 0$ and $\nu = 0$ respectively with initial data θ_0, then given $\tau > 0$, for all $s \geq 0$, we have*

$$\lim_{\nu \to 0} \|(\theta^\nu - \theta)(t, \cdot)\|_{H^s} = 0,$$

whenever $t \geq \tau$.

With the results of Theorems 2.4 and Theorem 2.5 in place, we define a weak solution to the MG^0 equation which we call a "*vanishing viscosity*" *solution.* We use this concept to prove the existence of a compact global attractor in $L^2(\mathbb{T}^3)$ for the MG^ν equations (1.4)-(1.5) including the critical equation where $\nu = 0$, and we further obtain the upper semicontinuity of the global attractor as ν vanishes.

Definition 2.6. A *weak solution* to (1.4)-(1.5) with $\nu = 0$ is a function $\theta \in C_w([0,T]; L^2(\mathbb{T}^3))$ with zero spatial mean that satisfies (1.4) in a distributional sense. That is, for any $\phi \in C_0^\infty((0,T) \times \mathbb{T}^3)$,

$$-\int_0^T \langle \theta, \phi_t \rangle dt - \int_0^T \langle u\theta, \nabla\phi \rangle dt + \kappa \int_0^T \langle \nabla\theta, \nabla\phi \rangle dt$$
$$= \langle \theta_0, \phi(0,x) \rangle + \int_0^T \langle S, \phi \rangle dt,$$

where $u = u\Big|_{\nu=0}$. A weak solution $\theta(t)$ to (1.4)-(1.5) on $[0,T]$ with $\nu = 0$ is called a "*vanishing viscosity*" *solution* if there exist sequences $\nu_n \to 0$ and $\{\theta^{\nu_n}\}$ such that $\{\theta^{\nu_n}\}$ are smooth solutions to (1.4)-(1.5) as given by Theorem 2.4 and $\theta^{\nu_n} \to \theta$ in $C_w([0,T]; L^2)$ as $\nu_n \to 0$.

We prove that the system (1.4)-(1.5) driven by a force S possesses a compact global attractor in $L^2(\mathbb{T}^3)$ which is *upper semicontinuous* at $\nu = 0$. More precisely, we have

Theorem 2.7 (Friedlander and Suen [29])**.** *Assume $S \in C^\infty$. Then the system (1.4)-(1.5) with $\nu = 0$ possesses a compact global attractor $\mathcal{A}$ in $L^2(\mathbb{T}^3)$, namely*

$$\mathcal{A} = \{\theta_0 : \theta_0 = \theta(0) \text{ for some bounded complete "vanishing viscosity" solution } \theta(t)\}.$$

For any bounded set $\mathcal{B} \subset L^2(\mathbb{T}^3)$, and for any $\varepsilon, T > 0$, there exists t_0 such that for any $t_1 > t_0$, every "vanishing viscosity" solution $\theta(t)$ with $\theta(0) \in \mathcal{B}$ satisfies

$$\|\theta(t) - x(t)\|_{L^2} < \varepsilon, \forall t \in [t_1, t_1 + T],$$

for some complete trajectory $x(t)$ on the global attractor ($x(t) \in \mathcal{A}, \forall t \in (-\infty, \infty)$). Furthermore, for $\nu \in [0,1]$, there exists a compact global attractor $\mathcal{A}^\nu \subset L^2$ for (1.4)-(1.5) such that $\mathcal{A}^0 = \mathcal{A}$ and $\mathcal{A}^\nu$ is upper semicontinuous at $\nu = 0$, which means that

$$\sup\nolimits_{\phi \in \mathcal{A}^\nu} \inf\nolimits_{\psi \in \mathcal{A}} \|\phi - \psi\|_{L^2} \to 0 \text{ as } \nu \to 0. \tag{2.16}$$

We give a brief discussion of the proof of Theorem 2.7, and we refer the interested reader to [29] for full details. Roughly speaking, Theorem 2.7 can be divided into two parts, namely:

1. existence of global attractors $\mathcal{A}^\nu$ for (1.4)-(1.5) with $\nu \geq 0$; and
2. upper semicontinuity of $\mathcal{A}^\nu$ at $\nu = 0$.

For the existence of global attractors $\mathcal{A}^\nu$, it can be proved in the following several steps:

I. First let $\theta(t)$ be a "vanishing viscosity" solution of (1.4)-(1.5) on $[0,\infty)$ with $\theta(0) \in L^2$. Then $\theta(t)$ satisfies the following energy equality:

$$\frac{1}{2}\|\theta(t)\|_{L^2}^2 + \kappa \int_{t_0}^t \|\nabla\theta(s,\cdot)\|_{L^2}^2 ds = \frac{1}{2}\|\theta(t_0)\|_{L^2}^2 + \int_{t_0}^t \int_{\mathbb{T}^3} S\theta dx ds, \tag{2.17}$$

for all $0 \leq t_0 \leq t$. The energy equality implies that:

– Every "vanishing viscosity" solution to (1.4)-(1.5) is strongly continuous in time t.
– There exists an absorbing ball $\mathcal{Y}$ for (1.4)-(1.5) given by

$$\mathcal{Y} = \{\theta \in L^2 : \|\theta\|_{L^2} \leq R\}, \tag{2.18}$$

where R is any number larger than $\kappa^{-1}\|S\|_{H^{-1}(\mathbb{T}^3)}$.

II. Next we define $\pi^\nu : L^2 \to L^2$ as the map $\pi^\nu\theta_0 = \theta^\nu$, where θ^ν is the solution to (1.4)-(1.5) given by Theorem 2.4. Then using [29, Lemma 6.7], for $t > 0$, $\pi^\nu(t)\theta_0$ is continuous in ν, uniformly for θ_0 in compact subsets of L^2.

III. We denote the weak distance on $L^2(\mathbb{T}^3)$ by

$$d_w(\phi, \psi) = \sum_{k\in\mathbb{Z}^3} \frac{1}{2^{|k|}} \frac{|\hat{\phi}_k - \hat{\psi}_k|}{1 + |\hat{\phi}_k - \hat{\psi}_k|},$$

where $\hat{\phi}_k$ and $\hat{\psi}_k$ are the Fourier coefficients of ϕ and ψ. If

$$\mathcal{E}[T,\infty) = \{\theta(\cdot) : \theta(\cdot) \text{ is a "vanishing viscosity" solution of (1.4)-(1.5)} \\ \text{on } [T,\infty) \text{ and } \theta \in \mathcal{Y} \text{ for all } t \in [T,\infty)\},$$

$$\mathcal{E}(-\infty,\infty) = \{\theta(\cdot) : \theta(\cdot) \text{ is a "vanishing viscosity" solution of (1.4)-(1.5)} \\ \text{on } (-\infty,\infty) \text{ and } \theta \in \mathcal{Y} \text{ for all } t \in (-\infty,\infty)\},$$

then $\mathcal{E}$ is an *evolutionary system* (see [6] and [11] for the definition), so by [11, Theorem 4.5], there exists a weak global attractor $\mathcal{A}_w$ to $\mathcal{E}$ with

$$\mathcal{A}_w = \{\theta_0 : \theta_0 = \theta(0) \text{ for some } \theta \in \mathcal{E}((-\infty,\infty))\}.$$

Applying the arguments given in [11], $\mathcal{E}$ satisfies all the following properties:
– $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty);\mathcal{Y}_w)$, here $\mathcal{Y}_w$ refers to the metric space $(\mathcal{Y}, d_w)$;
– for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\theta \in \mathcal{E}([0,\infty))$ and $t > 0$,

$$\|\theta(t)\|_{L^2} \leq \|\theta(t)\|_{L^2} + \varepsilon,$$

for t_0 a.e. in $(t-\delta, t) \cap [0,\infty)$;
– if $\theta_n \in \mathcal{E}([0,\infty))$ and $\theta_n \to \theta \in \mathcal{E}([0,\infty))$ in $C([0,\infty);\mathcal{Y}_w)$ for some $T > 0$, then $\theta_n(t) \to \theta(t)$ strongly a.e. in $[0,T]$.

Together with [11, Theorem 4.5], it implies that the strong global attractor $\mathcal{A}_s$ for (1.4)-(1.5) with $\nu = 0$ exists, it is strongly compact and $\mathcal{A}^0 := \mathcal{A}_s = \mathcal{A}_w$. The case for (1.4)-(1.5) with $\nu > 0$ is just similar.

On the other hand, to prove the upper semicontinuity of $\mathcal{A}^\nu$ at $\nu = 0$, we note that the absorbing ball $\mathcal{Y}$ as given by (2.18) has radius which is independent of ν. Hence for all $\nu \geq 0$, $\mathcal{A}^\nu$ satisfies

- $\pi^0(t)\mathcal{A}^\nu = \mathcal{A}^\nu$ for all $t \in \mathbb{R}$;
- for any bounded set $\mathcal{B}$, $\sup_{\phi\in\pi^0(t)\mathcal{B}} \inf_{\psi\in\mathcal{A}} d_w(\phi,\psi) \to 0$ as $t \to 0$.

We also have that $\mathcal{E}([0,\infty))$ is a compact set in $C([0,\infty);\mathcal{Y}_w)$ such that $\mathcal{A}^\nu \subset \mathcal{K}$ for every $\nu \in [0,1]$. Together with the continuity of $\pi^\nu(t)\theta_0$ in ν, the result from [38] implies the *weak* upper semicontinuity, namely

$$\sup_{\phi\in\mathcal{A}^\nu} \inf_{\psi\in\mathcal{A}} d_w(\phi,\psi) \to 0 \text{ as } \nu \to 0. \tag{2.19}$$

Moreover, for any $\phi^{\nu_j} \in \mathcal{A}^{\nu_j}$ and $\psi_j \in \mathcal{A}$, if

$$\lim_{j\to\infty} d_w(\phi^{\nu_j},\psi_j) = 0,$$

for some sequence $\nu_j \to 0$, then

$$\lim_{j\to 0} \|\phi^{\nu_j} - \psi_j\|_{L^2} = 0.$$

In other words, the *weak* upper semicontinuity implies the *strong* upper semicontinuity, and hence (2.19) further implies the *strong* upper semicontinuity of $\mathcal{A}^\nu$ at $\nu = 0$.

3. Non-diffusive active scalar equations

We switch our attention to the non-diffusive active scalar equations (1.1)-(1.3) for $\kappa = 0$ and $\nu \geq 0$, namely

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta = S, \\ u_j[\theta] = \partial_{x_i}T^\nu_{ij}[\theta], \theta(x,0) = \theta_0(x) \end{cases} \tag{3.1}$$

with $\nu \geq 0$, and S, θ_0 satisfy the zero mean assumptions (1.2)-(1.3). We address the well-posedness of (3.1) and convergence of solutions as $\nu \to 0$. The results can then be applied to the thermally non diffusive MG$^\nu$ equation (1.4)-(1.5) (with $\kappa = 0$) as well as the IPMB$^\nu$ equation (1.8)-(1.10).

3.1. Well-posedness in various spaces for $\nu > 0$.

In [33], Friedlander and Vicol showed that the singular inviscid MG^0 equation is *ill-possed* in the sense of Hadamard in any Sobolev space. One of the main reasons for the ill-posedness is that, the Fourier multiplier symbols $\widehat{M}^0$ given by (1.5) is an *even* function in k. When we perform energy estimates on (1.4), we may have trouble on controlling the following term

$$R := \int (-\Delta)^{\frac{s}{2}} u \cdot \nabla\theta (-\Delta)^{\frac{s}{2}}\theta.$$

Without the diffusive term $\kappa\Delta\theta$, we loss control on the term R and the only hope to treat it would be to discover a commutator structure. If the operator $\widehat{\partial_{x_i}T^\nu_{ij}}(k)$ is odd, then there is an extra *cancellation* which allows us to close the estimates (at the level of Sobolev spaces) by the Coifman-Meyer type commutator estimate. On the other hand, for the case when the operator $\widehat{\partial_{x_i}T^\nu_{ij}}(k)$ is even, such argument fails and one has to seek for some other type of estimates. In the case of $\nu > 0$, however, the operators $\partial_x T^\nu$ are smoothing order 2 which give rise to well-posedness for (1.1) with $\theta_0 \in W^{s,d}$ for $s \geq 0$ and smooth forcing term S. The results are summarised as follows:

Theorem 3.1 (Friedlander and Suen [30]). *Let $\theta_0 \in W^{s,d}$ for $s \geq 0$ and S be a C^∞-smooth source term. Then for each $\nu > 0$, we have:*

- *if* $s = 0$*, there exists unique global weak solution to* (3.1) *such that*

$$\theta^\nu \in BC((0,\infty); L^d),$$
$$u^\nu \in C((0,\infty); W^{2,d}).$$

  In particular, $\theta^\nu(\cdot,t) \to \theta_0$ *weakly in* L^d *as* $t \to 0^+$*. Here BC stands for bounded continuous functions.*
- *if* $s > 0$*, there exists a unique global-in-time solution* θ^ν *to* (3.1) *such that* $\theta^\nu(\cdot,t) \in W^{s,d}$ *for all* $t \geq 0$*. Furthermore, for* $s = 1$*, we have the following single exponential growth in time on* $\|\nabla\theta^\nu(\cdot,t)\|_{L^d}$*:*

$$\|\nabla\theta^\nu(\cdot,t)\|_{L^d} \leq C\|\nabla\theta_0\|_{L^d} \exp\left(C\left(t\|\theta_0\|_{W^{1,d}} + t^2\|S\|_{L^\infty} + t\|S\|_{W^{1,d}}\right)\right),$$

  where $C > 0$ *is a constant which depend only on* ν *and the spatial dimension* d*.*

The proof of Theorem 3.1 for the case of $s = 0$ relies on the existence and uniqueness of the flow map, which is essential for Euler system as well [4]. We briefly sketch here and the full details can be found in [30].

In view of condition (A3), by applying Fourier multiplier theorem (see Stein [53]), given $p > 1$, there exists some constant $C = C(\nu, p, d) > 0$ such that

$$\|u^\nu(\cdot,t)\|_{W^{2,p}} \leq C\|\theta^\nu(\cdot,t)\|_{L^p}. \tag{3.2}$$

Together with (3.2) and embedding theorems, one can show that the *Log-Lipschitzian* norm of u^ν given by $\|u^\nu(\cdot,t)\|_{L.L.}$ is bounded in terms of θ_0 and S:

$$\|u^\nu(\cdot,t)\|_{L.L.} \leq C\left(\|\theta_0\|_{L^d} + t\|S\|_{L^\infty}\right). \tag{3.3}$$

Next, we consider the standard mollifier $\rho \in C_0^\infty$, and we set $\theta_{(n),0} = \rho_n * \theta_0$ for $n \in \mathbb{N}$ and $\rho_n(x) = n^d\rho(nx)$. By a standard argument, given $\nu > 0$, we can obtain a sequence of global smooth solution $(\theta^\nu_{(n)}, u^\nu_{(n)})$ to (3.1) with $\theta^\nu_{(n)}(x,0) = \theta_{(n),0}$ and $u^\nu_{(n)} = \partial_{x_i} T^\nu_{ij}[\theta^\nu_{(n)}]$. We define $\psi_n(x,t)$ to be the flow map given by

$$\partial_t\psi_n(x,t) = u^\nu_{(n)}(\psi_n(x,t),t),$$

then ψ_n satisfies

$$\|\psi_n(\cdot,t)\|_* \leq C\exp\left(\int_0^t \|u^\nu_{(n)}(\cdot,\tilde{t})\|_{L.L.} d\tilde{t}\right), \tag{3.4}$$

where the norm $\|\cdot\|_*$ is given by

$$\|\psi\|_* = \sup_{x\neq y} \Phi(|\psi(x)-\psi(y)|, |x-y|)$$

with

$$\Phi(r,s) = \begin{cases} \max\{\frac{1+|\log(s)|}{1+|\log(r)|}, \frac{1+|\log(r)|}{1+|\log(s)|}\}, \text{ if } (1-s)(1-r) \geq 0, \\ (1+|\log(s)|)(1+|\log(r)|), \text{ if } (1-s)(1-r) \leq 0. \end{cases}$$

Using (3.3) and (3.4) (with u^ν replaced by $u^\nu_{(n)}$), we obtain

$$|\psi_n(x_1,t) - \psi_n(x_2,t)| \leq \alpha(t)|x_1 - x_2|^{\beta(t)} \tag{3.5}$$

for all $(x_1,t),(x_2,t)\in\mathbb{R}^d\times\mathbb{R}^+$, where $\alpha(t),\beta(t)$ are some continuous functions which depend on θ_0 and S. Furthermore, for $t_1,t_2\geq 0$, using (3.2) (with u^ν replaced by $u^\nu_{(n)}$),

$$|\psi_n(x,t_1)-\psi_n(x,t_2)|\leq C|t_2-t_1|(\|\theta_0\|_{L^p}+\max\{t_1,t_2\}\|S\|_{L^\infty}). \tag{3.6}$$

The estimates (3.5) and (3.6) imply that the family $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded and equicontinuous on every compact set in $\mathbb{R}^d\times\mathbb{R}^+$. By the Arzela-Ascoli theorem, there exists a limiting trajectory $\psi(x,t)$ as $n\to\infty$. Performing the same analysis for $\{\psi_n^{-1}\}$, where ψ_n^{-1} is the inverse of ψ_n, we see that $\psi(x,t)$ is a Lebesgue measure preserving homeomorphism as well. Define $\theta^\nu(x,t)=\theta_0(\psi^{-1}(x,t))$ and $u^\nu=\partial_{x_i}T^\nu_{ij}[\theta^\nu]$. Then one can show that (θ^ν,u^ν) is a weak solution to (1.1). To show that (θ^ν,u^ν) is unique, let $T>0$ and $\nu>0$, and suppose that $(\theta^{\nu,1},u^{\nu,1})$ and $(\theta^{\nu,2},u^{\nu,2})$ solve (1.1) on $\mathbb{T}^d\times[0,T]$ with $\theta^{\nu,1}(\cdot,0)=\theta^{\nu,2}(\cdot,0)=\theta_0$. Then there exists a constant $C>0$ such that for all $\delta\in(0,1)$ and $k\in\{-1\}\cup\mathbb{N}$, we have, for all $t\in[0,T]$ that,

$$\begin{aligned}&\|\Delta_k(\theta^{\nu,1}-\theta^{\nu,2})(\cdot,t)\|_{L^\infty}\\ &\leq 2^{k\delta}(k+1)C(\|u^{\nu,1}(\cdot,t)\|_{\overline{L.L.}}+\|u^{\nu,2}(\cdot,t)\|_{\overline{L.L.}})\|(\theta^{\nu,1}-\theta^{\nu,2})(\cdot,t)\|_{B^{-\delta}_{\infty,\infty}},\end{aligned}$$

where $\|\cdot\|_{\overline{L.L.}}=\|\cdot\|_{L^\infty}+\|\cdot\|_{L.L.}$ and Δ_k's are the dyadic blocks for $k\in\{-1\}\cup\mathbb{N}$. We define

$$\bar{t}=\sup\left\{t\in[0,T]:C\int_0^t(\|u^{\nu,1}(\cdot,\tilde{t})\|_{\overline{L.L.}}+\|u^{\nu,2}(\cdot,\tilde{t})\|_{\overline{L.L.}})d\tilde{t}\leq\frac{1}{2}\right\},$$

then by the bounds (3.2) and (3.3), $\bar{t}$ is well-defined. We let

$$\delta_{\bar{t}}=C\int_0^{\bar{t}}(\|u^{\nu,1}(\cdot,\tilde{t})\|_{\overline{L.L.}}+\|u^{\nu,2}(\cdot,\tilde{t})\|_{\overline{L.L.}})d\tilde{t}.$$

Using [3, Theorem 3.28], for all $k\geq -1$ and $t\in[0,\bar{t}]$,

$$2^{-k\delta_{\bar{t}}}\|\Delta_k(\theta^{\nu,1}-\theta^{\nu,2})(\cdot,t)\|_{L^\infty}\leq\frac{1}{2}\sup_{t\in[0,\bar{t}]}\|(\theta^{\nu,1}-\theta^{\nu,2})(\cdot,t)\|_{B^{-\delta_{\bar{t}}}_{\infty,\infty}}.$$

Summing over k and taking supremum over $[0,\bar{t}]$, we conclude that $\theta^{\nu,1}=\theta^{\nu,2}$ on $[0,\bar{t}]$. By repeating the argument a finite number of times, we obtain the uniqueness on the whole interval $[0,T]$. This concludes our sketch of the proof of Theorem 3.1.

Next we study the Gevrey-class s solutions to (3.1) for $\nu>0$ when the initial datum θ_0 and forcing term S are in the same Gevrey-class. We prove that there exists a unique global-in-time Gevrey-class s solution θ^ν with radius of convergence bounded below by some positive function $\tau(t)$ for all $t\in[0,\infty)$. More precisely, we have:

Theorem 3.2 (Friedlander and Suen [30])**.** *Fix $s\geq 1$. Let θ_0 and S be of Gevrey-class s with radius of convergence $\tau_0>0$. Then there exists a unique Gevrey-class s solution θ^ν to* (3.1) *on $\mathbb{T}^d\times[0,\infty)$ with radius of convergence at least $\tau=\tau(t)$ for all $t\in[0,\infty)$, where τ is a decreasing function satisfying*

$$\tau(t)\geq\tau_0 e^{-C\left(\|e^{\tau_0\Lambda^{\frac{1}{s}}}\theta_0\|_{L^2}+2\|e^{\tau_0\Lambda^{\frac{1}{s}}}S\|_{L^2}\right)t}. \tag{3.7}$$

Here $C > 0$ is a constant which depends on ν but independent of t.

The Gevrey-class s is given by

$$\bigcup_{\tau>0} \mathcal{D}(\Lambda^r e^{\tau\Lambda^{\frac{1}{s}}}),$$

for any $r \geq 0$, where

$$\|\Lambda^r e^{\tau\Lambda^{\frac{1}{s}}} f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}_*^d} |k|^{2r} e^{2\tau|k|^{\frac{1}{s}}} |\hat{f}(k)|^2,$$

where $\tau = \tau(t) > 0$ denotes the radius of convergence and $\Lambda = (-\Delta)^{\frac{1}{2}}$. We point out that for the case when $s = 1$, it gives the space of analytic functions.

By taking L^2-inner product of $(3.1)_1$ with $e^{2\tau\Lambda^{\frac{1}{s}}}\theta^\nu$ and applying Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2 - \dot{\tau}\|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2 \\
&\leq \Big| - \langle u^\nu\cdot\nabla\theta^\nu, e^{2\tau\Lambda^{\frac{1}{s}}}\theta^\nu\rangle\Big| + \|e^{\tau\Lambda^{\frac{1}{s}}} S\|_{L^2}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}.
\end{aligned}$$

The key step for proving Theorem 3.2 is to estimate the term $\Big| - \langle u^\nu\cdot\nabla\theta^\nu, e^{2\tau\Lambda^{\frac{1}{s}}}\theta^\nu\rangle\Big|$. Using a Cauchy-Kowalewski-type argument and together with condition (A3), it can be showed in [30, Lemma 4.1] that

$$\Big|-\langle e^{\tau\Lambda^{\frac{1}{s}}}(u^\nu\cdot\nabla\theta^\nu), e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\rangle\Big| \leq C\tau\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}\|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2,$$

and we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2 - \dot{\tau}\|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2 + \kappa\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2 \\
&\leq C\tau\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}\|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2}^2.
\end{aligned}$$

By choosing $\tau > 0$ such that

$$\dot{\tau} + C\tau\|e^{\tau\Lambda^{\frac{1}{s}}}\theta^\nu\|_{L^2} = 0,$$

we have

$$\|e^{\tau(t)\Lambda^{\frac{1}{s}}}\theta^\nu(t)\|_{L^2} \leq \|e^{\tau_0\Lambda^{\frac{1}{s}}}\theta_0\|_{L^2} + 2\|e^{\tau_0\Lambda^{\frac{1}{s}}} S\|_{L^2}.$$

and τ satisfies the lower bound (3.7). We remark that, for the diffusive case given by the system (2.1), one can obtain global-in-time existence of solution Gevrey class $s \geq 1$ with lower bound on $\tau(t)$ that does not vanish as $t \to \infty$, refer to [30, Remark 4.3] for more details.

3.2. **Well-posedness in various spaces for $\nu = 0$.** In this subsection we study the non-diffusive equations (3.1) for $\nu = 0$:

$$\left\{\begin{array}{l}
\partial_t\theta^0 + u^0\cdot\nabla\theta^0 = S, \\
u_j^0 = \partial_{x_i} T_{ij}^0[\theta^0], \theta^0(x,0) = \theta_0(x).
\end{array}\right. \tag{3.8}$$

When $\nu = 0$ and condition (A2) is imposed, as it was proved in [33], the equation (3.8) is *ill-posed* in the sense of Hadamard, which means that the solution map associated to the Cauchy problem for (3.8) is not Lipschitz continuous with respect to perturbations in the initial datum around a specific steady profile θ_0, in the topology of a certain Sobolev space X. Nevertheless, as pointed out in [33], it is possible to obtain the local existence and uniqueness of solutions to (3.8) in spaces of real-analytic functions, owing to the fact that the derivative loss in the nonlinearity $u^0 \cdot \nabla\theta^0$ is of order at most one (both in u^0 and in $\nabla\theta^0$). In [30], we extended the results of [33] which are summarised as follows:

Theorem 3.3 (Friedlander and Suen [30])**.** *Fix $r > \frac{d}{2} + \frac{3}{2}$ and $K_0 > 0$. Let $\theta^0(\cdot, 0) = \theta_0$ and S be analytic with radius of convergence $\tau_0 > 0$ and satisfy*

$$\|\Lambda^r e^{\tau_0\Lambda}\theta^0(\cdot,0)\|_{L^2} \leq K_0, \qquad \|\Lambda^r e^{\tau_0\Lambda}S\|_{L^2} \leq K_0. \tag{3.9}$$

For $\nu = 0$, under the condition (A2)*, there exists $\bar{T}, \bar{\tau} > 0$ and a unique analytic solution θ^0 to* (3.8) *defined on $\mathbb{T}^d \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$. In particular, there exists a constant $C = C(K_0) > 0$ such that for all $t \in [0, \bar{T}]$,*

$$\|\Lambda^r e^{\bar{\tau}\Lambda}\theta^0(\cdot,t)\|_{L^2} \leq C. \tag{3.10}$$

The bound (3.10) *also applies on θ^ν for $\nu > 0$.*

In contrast, when $\nu = 0$ and condition (A2*) is in force, the operator $\partial_x T^0$ becomes a zero order operator with $\partial_x T^0 : L^2 \to L^2$ being bounded. Following the idea given in [35], we show that the equation (3.8) is locally well-posed in Sobolev space H^s for $s > \frac{d}{2} + 1$:

Theorem 3.4 (Friedlander and Suen [30])**.** *For $d \geq 2$, we fix $s > \frac{d}{2} + 1$. Assume that $\theta_0, S \in H^s(\mathbb{T}^d)$ have zero-mean on $\mathbb{T}^d$. Then for $\nu = 0$, under the condition* (A2*)*, there exists a $T > 0$ and a unique smooth solution θ^0 to* (3.8) *such that*

$$\theta^0 \in L^\infty(0, T; H^s(\mathbb{T}^d)).$$

The proof of Theorem 3.4 consists of three steps, which can be briefly outline as follows (details can be found in [30]):

I. We first construct a sequence of approximations $\{\theta_n\}_{n\geq 1}$ given by the solutions of

$$\begin{aligned} \partial_t\theta_1 &= S \\ \theta_1(\cdot,0) &= \theta_0. \end{aligned}$$

and for $n > 1$,

$$\begin{aligned} \partial_t\theta_n + u_{n-1}\cdot\nabla\theta_n &= S \\ \theta_{n-1} &= \partial_x T^0[\theta_{n-1}] \\ \theta_n(\cdot,0) &= \theta_0. \end{aligned} \tag{3.11}$$

Then by applying [35, Theorem A1], one can show that $\theta_n \in L^\infty(0, T; H^s)$ for all $n \in \mathbb{N}$.

II. Next, by induction on n, we prove that $\|\Lambda^s\theta_n(\cdot,t)\|_{L^2}$ is bounded on $[0, T]$ for some $T > 0$. Assume that

$$\|\Lambda^s\theta_j\|_{L^\infty(0,T;L^2)} \leq C\|\Lambda^s\theta_0\|_{L^2},$$

for $1 \leq j \leq n-1$. We apply Λ^s on (3.11) and take inner product with $\Lambda^s\theta_n$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^d}|\Lambda^s\theta_n|^2 + \int_{\mathbb{T}^d}\Lambda^s\theta_n\cdot\Lambda^s(u_{n-1}\cdot\nabla\theta_n) = \int_{\mathbb{T}^d}\Lambda^s\theta_n\cdot\Lambda^s S. \tag{3.12}$$

Using commutator estimates (see [35] for example), the term involving u_{n-1} can be bounded in terms of θ_n and θ_{n-1}:

$$\left|\int_{\mathbb{T}^d}\Lambda^s\theta_n\cdot\Lambda^s(u_{n-1}\cdot\nabla\theta_n)\right| \leq C\|\Lambda^s\theta_n\|_{L^2}(\|\Lambda^s\theta_{n-1}\|_{L^2}\|\Lambda^s\theta_n\|_{L^2}).$$

Hence by integrating the identity (3.12) over t, choosing T small enough and applying the induction hypothesis, $\|\Lambda^s\theta_n(\cdot,t)\|_{L^2}$ is bounded on $[0,T]$ as well.

III. Finally, we show that $\{\theta_n\}_{n\geq 0}$ is a Cauchy sequence. This can be done by considering the difference $\tilde{\theta}_n = \theta_n - \theta_{n-1}$ and one can prove that

$$\frac{d}{dt}\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} \leq C(\|\Lambda^s\theta_0\|_{L^2}\|\Lambda^{s-1}\tilde{\theta}_n\|_{L^2} + \|\Lambda^{s-1}\tilde{\theta}_{n-1}\|_{L^2}\|\Lambda^s\theta_0\|_{L^2}).$$

Integrating the above over t and choosing T small enough, we obtain

$$\sup_{t\in[0,T]}\|\Lambda^{s-1}\tilde{\theta}_n(\cdot,t)\|_{L^2} \leq \frac{1}{2}\sup_{t\in[0,T]}\|\Lambda^{s-1}\tilde{\theta}_{n-1}(\cdot,t)\|_{L^2}.$$

Thus θ_n is Cauchy in $L^\infty(0,T;H^{s-1})$ with θ_n converges strongly to θ^0 in $L^\infty(0,T,H^{s-1})$. Since we assume that $s > \frac{d}{2}+1$, this also implies that the strong convergence occurs in a Hölder space relative to x as $n\to\infty$, hence the limiting function θ^0 is a solution of (3.8). Uniqueness of θ^0 follows by the same argument given in [35] and we omit the details.

3.3. **Convergence of solutions as $\nu \to 0$.** In this subsection, we address the convergence of solutions to (3.1) as $\nu\to 0$. Depending on the conditions (A2) and (A2*), we can address the convergence of solutions in two cases respectively:

3.3.1. *Analytic solutions:* We focus on the case for analytic solutions θ^ν to (3.1) when (A2) is in force. By Theorem 3.2 and Theorem 3.3, given analytic initial datum θ_0 and forcing S, there exists $\bar{T},\bar{\tau} > 0$ and a unique analytic solution θ^ν to (3.1) defined on $[0,\bar{T}]$ with radius of convergence at least $\bar{\tau}$ for all $\nu \geq 0$. In particular, the analytic solutions θ^ν converges to θ^0 as $\nu\to 0$ and the results are summarised in the following theorem:

Theorem 3.5 (Friedlander and Suen [30])**.** *Under the condition* (A2)*, if θ^ν and θ^0 are analytic solutions to* (3.1) *for $\nu > 0$ and $\nu = 0$ respectively with initial datum θ_0 on $\mathbb{T}^d\times[0,\bar{T}]$ with radius of convergence at least $\bar{\tau}$ as described in Theorem 3.3, then there exists $T < \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t\in[0,T]$, we have*

$$\lim_{\nu\to 0}\|(\Lambda^r e^{\tau\Lambda}\theta^\nu - \Lambda^r e^{\tau\Lambda}\theta^0)(\cdot,t)\|_{L^2} = 0. \tag{3.13}$$

The proof of Theorem 3.5 relies on the estimates of the difference $\phi^\nu := \theta^\nu - \theta^0$, and it can be shown that ϕ^ν satisfies

$$\frac{1}{2}\frac{d}{dt}\|\phi^\nu\|^2_{\tau,r} = \dot{\tau}\|\Lambda^{\frac{1}{2}}\phi^\nu\|^2_{\tau,r} + \mathcal{R}_1 + \mathcal{R}_2,$$

where the terms $\mathcal{R}_1$ and $\mathcal{R}_2$ can be bounded as follows:

$$\mathcal{R}_1 \le C\|\Lambda^{\frac{1}{2}}\theta^0\|_{\tau,r}\|\Lambda^{\frac{1}{2}}\phi^\nu\|_{\tau,r}\|\phi^\nu\|_{\tau,r} + C\|\Lambda^{\frac{1}{2}}\theta^0\|_{\tau,r}\|\Lambda^{\frac{1}{2}}\phi^\nu\|_{\tau,r}\left(\sum_{j\in\mathbb{Z}^d_*}|j|^{d+3}|\widehat{\theta^0}(j)|^2 e^{2\tau|j|}|(\widehat{T^\nu}-\widehat{T^0})(j)|^2\right)^{\frac{1}{2}},$$

$$\mathcal{R}_2 \le C\|\Lambda^{\frac{1}{2}}\phi^\nu\|^2_{\tau,r}\|\theta^\nu\|_{\tau,r}.$$

By choosing $\tau = \tau(t) \le \bar{\tau}$ such that

$$\begin{cases} \dot{\tau} + C\|\theta^\nu\|_{\tau,r} + C\|\Lambda^{\frac{1}{2}}\theta^0\|^2_{\tau,r} < 0, \\ \tau < \bar{\tau}, \end{cases}$$

and applying the bound (3.10) to conclude that

$$\frac{d}{dt}\|\phi^\nu\|^2_{\tau,r} \le C\|\phi^\nu\|^2_{\tau,r} + C\sum_{j\in\mathbb{Z}^d_*}|j|^{d+3}|\widehat{\theta^0}(j)|^2 e^{2\tau|j|}|(\widehat{T^\nu}-\widehat{T^0})(j)|^2.$$

Integrating the above with respect to t and using the condition (A2), we have $\lim_{\nu\to 0}\|\phi^\nu\|_{\tau,r} = 0$ and (3.13) follows.

3.3.2. *H^s solutions:* When condition (A2*) is in force, by Theorem 3.4, the equation (3.1) for $\nu = 0$ is locally well-posed in Sobolev space H^s for $s > \frac{d}{2}+1$. For sufficiently smooth initial data θ_0 and forcing term S, one can show that $\|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^s} \to 0$ as $\nu \to 0$ for $s > \frac{d}{2}+1$ and $t \in [0,T]$. Such result is parallel to the one proved in [29], in which the authors proved that if θ^ν, θ^0 are C^∞ smooth classical solutions of the diffusive system (2.1) for $\nu > 0$ and $\nu = 0$ respectively with initial datum $\theta_0 \in L^2$ and forcing term $S \in C^\infty$, then $\|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^s} \to 0$ as $\nu \to 0$ for $s \ge 0$ and $t > 0$. The convergence results are summarised below:

Theorem 3.6 (Friedlander and Suen [30])**.** *Under the condition* (A2*)*, we have*

$$\lim_{\nu\to 0}\|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^{s-1}} = 0, \tag{3.14}$$

and for $d \ge 2$ and $s > \frac{d}{2}+1$ and $t \in [0,T]$, we have

$$\lim_{\nu\to 0}\|(\theta^\nu - \theta^0)(\cdot,t)\|_{H^{s-1}} = 0. \tag{3.15}$$

It suffices to consider the case for the convergence in L^2 given by (3.14), since the case for (3.15) follows by Gagliardo-Nirenberg interpolation inequality [37] and [50]. The key step of the proof is to estimate $\|(u^\nu - u^0)(\cdot,t)\|_{L^2}$, which can be bounded by $\|\phi^\nu(\cdot,t)\|^2_{L^2} + I(\nu,t)$ with $I(\nu,t)$ becoming zero as ν vanishes, see [30] for further details.

3.4. **Applications to physical models.** We now apply our results discussed previous subsections to some physical models, namely the thermally non diffusive magnetogeostrophic (MG$^\nu$) equations (1.4)-(1.5) with $\kappa = 0$ and the incompressible porous media Brinkman equations (IPMB$^\nu$) (1.8)-(1.10). The results are summarised in the following theorems (also refer to [30] for details):

Theorem 3.7 (Well-posedness in Sobolev space for the MG$^\nu$ equations)**.** *Let $\theta_0 \in W^{s,3}$ for $s \geq 0$ and S be a C^∞-smooth source term. Then for each $\nu > 0$, we have:*

- *if $s = 0$, there exists unique global weak solution to* (1.4)-(1.5) *with $\kappa = 0$ such that*
$$\theta^\nu \in BC((0,\infty); L^3),$$
$$u^\nu \in C((0,\infty); W^{2,3}).$$
In particular, $\theta^\nu(\cdot,t) \to \theta_0$ weakly in L^3 as $t \to 0^+$.
- *if $s > 0$, there exists a unique global-in-time solution θ^ν to* (1.4)-(1.5) *with $\kappa = 0$ such that $\theta^\nu(\cdot,t) \in W^{s,3}$ for all $t \geq 0$. Furthermore, for $s = 1$, we have the following single exponential growth in time on $\|\nabla\theta^\nu(\cdot,t)\|_{L^3}$:*
$$\|\nabla\theta^\nu(\cdot,t)\|_{L^3} \leq C\|\nabla\theta_0\|_{L^3} \exp\left(C\left(t\|\theta_0\|_{W^{1,3}} + t^2\|S\|_{L^\infty} + t\|S\|_{W^{1,3}}\right)\right),$$
where $C > 0$ is a constant which depends only on some dimensional constants.

Theorem 3.8 (Analytic and Gevrey-class well-posedness for the MG$^\nu$ equations)**.** *Fix $s \geq 1$. Let θ_0 and S be of Gevrey-class s with radius of convergence $\tau_0 > 0$. Then for each $\nu > 0$, there exists a unique Gevrey-class s solution θ^ν to* (1.4)-(1.5) *with $\kappa = 0$ on $\mathbb{T}^3 \times [0,\infty)$ with radius of convergence at least $\tau = \tau(t)$ for all $t \in [0,\infty)$, where τ is a decreasing function satisfying*
$$\tau(t) \geq \tau_0 e^{-C\left(\|e^{\tau_0\Lambda^{\frac{1}{s}}}\theta_0\|_{L^2} + 2\|e^{\tau_0\Lambda^{\frac{1}{s}}}S\|_{L^2}\right)t}.$$
Here $C > 0$ is a constant which depends on ν but independent of t. For the singular case when $\nu = 0$, if θ_0 and S are analytic with radius of convergence $\tau_0 > 0$, then there exists $\bar\tau \in (0,\tau_0]$, $\bar T > 0$ and a unique analytic solution θ^0 to (1.4)-(1.5) *for $\kappa = 0$ defined on $\mathbb{T}^3 \times [0,\bar T]$ with radius of convergence at least $\bar\tau$.*

Theorem 3.9 (Convergence of solutions as $\nu \to 0$ for the MG$^\nu$ equations)**.** *Fix $s \geq 1$, $r > 3$ and $K_0 > 0$. Let θ_0 and S be analytic with radius of convergence $\tau_0 > 0$ and satisfy the assumptions given in Theorem 3.8. If θ^ν and θ^0 are analytic solutions to* (1.4)-(1.5) *with $\kappa = 0$ for $\nu > 0$ and $\nu = 0$ respectively with initial datum θ_0 on $\mathbb{T}^3 \times [0,\bar T]$ with radius of convergence at least $\bar\tau$ as described in Theorem 3.8, then there exists $T < \bar T$ and $\tau = \tau(t) < \bar\tau$ such that, for $t \in [0,T]$, we have*
$$\lim_{\nu\to 0} \|(\Lambda^r e^{\tau\Lambda}\theta^\nu - \Lambda^r e^{\tau\Lambda}\theta^0)(\cdot,t)\|_{L^2} = 0.$$

Theorem 3.10 (Well-posedness in Sobolev space for the IPMB$^\nu$ equations)**.** *Let $\theta_0 \in W^{s,2}$ for $s \geq 0$. Then for each $\nu > 0$, we have:*

- *if $s = 0$, there exists unique global weak solution to* (1.8)-(1.10) *such that*
$$\theta^\nu \in BC((0,\infty); L^2),$$
$$u^\nu \in C((0,\infty); W^{2,2}).$$
In particular, $\theta^\nu(\cdot,t) \to \theta_0$ weakly in L^2 as $t \to 0^+$.

- *if $s > 0$, there exists a unique global-in-time solution θ^ν to* (1.8)-(1.10) *such that $\theta^\nu(\cdot, t) \in W^{s,2}$ for all $t \geq 0$. Furthermore, for $s = 1$, we have the following single exponential growth in time on $\|\nabla\theta^\nu(\cdot, t)\|_{L^2}$:*

$$\|\nabla\theta^\nu(\cdot, t)\|_{L^2} \leq C\|\nabla\theta_0\|_{L^2} \exp\left(Ct\|\theta_0\|_{W^{1,2}}\right),$$

where $C > 0$ is a constant which depends only on some dimensional constants.

Theorem 3.11 (Gevrey-class global well-posedness for the IPMB$^\nu$ equations)**.** *Fix $s \geq 1$. Let θ_0 be of Gevrey-class s with radius of convergence $\tau_0 > 0$. Then for each $\nu > 0$, there exists a unique Gevrey-class s solution θ^ν to* (1.8)-(1.10) *on $\mathbb{T}^2 \times [0, \infty)$ with radius of convergence at least $\tau = \tau(t)$ for all $t \in [0, \infty)$, where τ is a decreasing function satisfying*

$$\tau(t) \geq \tau_0 e^{-Ct\|e^{\tau_0\Lambda^{\frac{1}{s}}}\theta_0\|_{L^2}}.$$

Here $C > 0$ is a constant which depends on ν but independent of t.

Theorem 3.12 (Local well-posedness and convergence of solutions in Sobolev space for the IPMB$^\nu$ equations)**.** *Fix $s > 2$ and assume that $\theta_0 \in H^s(\mathbb{T}^2)$ has zero-mean on $\mathbb{T}^2$. Then there exists a positive time T and a unique smooth solution θ^0 to* (1.8)-(1.10) *with $\nu = 0$ such that*

$$\theta^0 \in L^\infty(0, T; H^s(\mathbb{T}^2)).$$

Moreover, for $t \in [0, T]$, we have

$$\lim_{\nu\to 0} \|(\theta^\nu - \theta^0)(\cdot, t)\|_{H^{s-1}} = 0.$$

4. Fractionally diffusive active scalar equations

In this section, we investigate the properties of the family of active scalar equations (1.1)-(1.3) in the context of the fractional Laplacian. The results can be applied to the modified surface quasi-geostrophic (SQG$^{\kappa,\nu}$) equation (1.11)-(1.12) and MG$^\nu$ equation (1.4)-(1.5).

4.1. **Existence and convergence of H^s-solutions when $\nu > 0$.** When the parameter ν is taken to be positive, the ensuing smoothing properties of T^ν_{ij} permits existence and convergence in Sobolev space H^s as κ goes to zero. The global-in-time existence theorem is given as follows:

Theorem 4.1 (Friedlander and Suen [31])**.** *Fix $\nu > 0$, $s \geq 0$ and $\gamma \in (0, 2]$, and let $\theta_0 \in H^s$ and $S \in H^s \cap L^\infty$ be given.*

- *For any $\kappa > 0$, there exists a global-in-time solution to* (1.1) *such that*

$$\theta^\kappa \in C([0, \infty); H^s) \cap L^2([0, \infty); H^{s+\frac{\gamma}{2}}).$$

- *For $\kappa = 0$, if we further assume that $\theta_0 \in L^\infty$, then exists a global-in-time solution to* (1.1)-(1.3) *such that $\theta^0(\cdot, t) \in H^s$ for all $t \geq 0$.*

In view of the case when $\kappa > 0$, the most subtle part for proving Theorem 4.1 is to estimate the L^∞-norm of $\theta^\kappa(\cdot, t)$ when θ_0 is *not* necessarily in L^∞. In achieving our goal, we apply De Giorgi iteration method (see [31, Lemma 4.5]) and obtain

$$\|\theta(t)\|_{L^\infty} \leq C\Big[\Big(\frac{2}{t}+1\Big)^{\frac{d+1-\gamma}{2\gamma}}\Big(\|\theta_0\|_{L^2} + \frac{\|S\|_{L^2}}{c_0^{\frac{1}{2}}\kappa^{\frac{1}{2}}}\Big) + \|S\|_{L^\infty}\Big],$$

for some constant $C = C(d) > 0$ which only depends on the dimension d. Once Theorem 4.1 is proved, we can show the convergence of H^s solutions which are summarised as follows:

Theorem 4.2 (Friedlander and Suen [31])**.** *Let $\nu > 0$ and $\gamma \in (0, 2]$ be given in* (1.1)*, and let $\theta_0, S \in C^\infty$ be the initial datum and forcing term respectively which satisfy* (1.2)*. If θ^κ and θ^0 are smooth solutions to* (1.1)-(1.3) *for $\kappa > 0$ and $\kappa = 0$ respectively, then*

$$\lim_{\kappa\to 0}\|(\theta^\kappa - \theta^0)(\cdot, t)\|_{H^s} = 0,$$

for all $s \geq 0$ and $t \geq 0$.

4.2. **Existence and convergence of analytic solutions when $\nu = 0$.** In contrast to the case for $\nu > 0$, when the parameter ν is set to zero, condition (A2) implies that $\partial_{x_i}T^\nu_{ij}$ is a singular operator. In this case the existence and convergence results for (1.1)-(1.3) are restricted to analytic solutions which are summarised in the following theorem:

Theorem 4.3 (Friedlander and Suen [31])**.** *Let $\kappa \geq 0$ and $\gamma \in (0, 2]$ be fixed, and let θ_0 and S be the initial datum and forcing term respectively. Fix $K_0 > 0$. Suppose θ_0 and S are both analytic with radius of convergence $\tau_0 > 0$ and*

$$\|\Lambda^r e^{\tau_0\Lambda}\theta^\kappa(\cdot, 0)\|_{L^2} \leq K_0, \qquad \|\Lambda^r e^{\tau_0\Lambda}S\|_{L^2} \leq K_0,$$

where $r > \frac{d}{2} + \frac{3}{2}$. Then there exists $T_* = T_*(\tau_0, K_0) > 0$ and a unique analytic solution on $[0, T_*)$ to the initial value problem associated to (1.1)-(1.3) *with $\nu = 0$. Furthermore, if θ^κ, θ^0 are analytic solutions to* (1.1)-(1.3) *with $\nu = 0$ for $\kappa > 0$ and $\kappa = 0$ respectively with initial datum θ_0 on $\mathbb{T}^d \times [0, \bar{T}]$ with radius of convergence at least $\bar{\tau}$, then there exists $T \leq \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t \in [0, T]$, we have:*

$$\lim_{\kappa\to 0}\|(\Lambda^r e^{\tau\Lambda}\theta^\kappa - \Lambda^r e^{\tau\Lambda}\theta^0)(\cdot, t)\|_{L^2} = 0.$$

For $\kappa > 0$ and $\gamma \in [1, 2]$, under a smallness assumption on the initial data, it can be proved that the analytic solutions obtained in Theorem 4.3 exist for all time:

Theorem 4.4 (Friedlander and Suen [31])**.** *Let $\kappa > 0$ and $\gamma \in [1, 2]$, and suppose that θ_0 and S are both analytic functions. There exists $\varepsilon > 0$ depending on κ such that, if θ_0 and S satisfy*

$$\|\theta_0\|^\beta_{L^2}\|\theta_0\|^{1-\beta}_{H^\alpha} + \|\theta_0\|^\beta_{L^2}\|S\|^{1-\beta}_{H^{\alpha-\frac{\gamma}{2}}} \leq \varepsilon, \tag{4.1}$$

and

$$\|\Lambda^\alpha\theta_0\|^2_{L^2} + \frac{2}{\kappa^2}\|S\|^2_{H^{\alpha-\frac{\gamma}{2}}} \leq \varepsilon,$$

where $\alpha > \frac{1}{2}(d+2)+(1-\gamma)$ *and* $\beta = 1-\frac{1}{\alpha}\Big[\frac{1}{2}(d+2)+(1-\gamma)\Big]$*, then the local-in-time analytic solution* θ^κ *as claimed by Theorem 4.3 can be extended to all time.*

As a by-product of Theorem 4.4, for the case when $S \in H^{s-\frac{\gamma}{2}}(\mathbb{T}^d)$ and $\theta_0 \in H^\alpha(\mathbb{T}^d)$ with $\gamma \in [1,2]$ and $\alpha > \frac{1}{2}(d+2)+(1-\gamma)$, under the smallness assumption (4.1), the system (1.1)-(1.3) possesses a global-in-time H^α solution:

Theorem 4.5 (Friedlander and Suen [31])**.** *Let* $\kappa > 0$*,* $\gamma \in [1,2]$ *and* $S \in H^{s-\frac{\gamma}{2}}(\mathbb{T}^d)$*, and let* $\theta_0 \in H^\alpha(\mathbb{T}^d)$ *have zero mean on* $\mathbb{T}^d$*, where* $\alpha > \frac{d+2}{2}+(1-\gamma)$*. There exists a small enough constant* $\varepsilon > 0$ *depending on* κ*, such that if* θ_0 *satisfies* (4.1)*, then there exists a unique global-in-time* H^α*-solution to* (1.1)-(1.3) *with* $\nu = 0$*. In particular, for all* $t > 0$*, we have the following bound on* θ^κ*:*

$$\|\Lambda^\alpha\theta^\kappa(\cdot,t)\|^2_{L^2} \le \|\Lambda^\alpha\theta_0\|^2_{L^2} + \frac{2}{\kappa^2}\|S\|^2_{H^{\alpha-\frac{\gamma}{2}}}.$$

The proofs of Theorem 4.3-4.5 can be found in [31]. We point out that all the abstract results obtained in Theorem 4.3-4.5 can be applied to the critical SQG equation, which is a special example of (1.1)-(1.3) with $\nu = 0$ and $\gamma = 1$.

4.3. **Long time behaviour for solutions when $\nu > 0$ and $\kappa > 0$.** In this subsection, we study the long time behaviour for solutions to the active scalar equations (1.1) when $\nu > 0$ and $\kappa > 0$. Based on the global-in-time existence results established in Theorem 4.1, for fixed $\nu > 0$ and $\kappa > 0$, we can define a solution operator $\pi^\nu(t)$ for the initial value problem (1.1) via

$$\pi^\nu(t) : H^1 \to H^1, \qquad \pi^\nu(t)\theta_0 = \theta(\cdot,t), \qquad t \ge 0. \tag{4.2}$$

We study the long-time dynamics of $\pi^\nu(t)$ on the phase space H^1. Specifically, we establish the existence of global attractors for $\pi^\nu(t)$ in H^1 and address some properties for the attractors. The following theorem first gives the existence of global attractors:

Theorem 4.6 (Friedlander and Suen [31])**.** *Let* $S \in L^\infty \cap H^1$*. For* ν*,* $\kappa > 0$ *and* $\gamma \in (0,2]$*, the solution map* $\pi^\nu(t) : H^1 \to H^1$ *associated to* (1.1) *possesses a unique global attractor* $\mathcal{G}^\nu$*. Moreover, there exists* $M_{\mathcal{G}^\nu}$ *which depends only on* ν*,* κ*,* γ*,* $\|S\|_{L^\infty\cap H^1}$ *and universal constants, such that if* $\theta_0 \in \mathcal{G}^\nu$*, we have that*

$$\|\theta(\cdot,t)\|_{H^{1+\frac{\gamma}{2}}} \le M_{\mathcal{G}^\nu}, \qquad \forall t \ge 0, \tag{4.3}$$

and

$$\frac{1}{T}\int_t^{t+T} \|\theta(\cdot,\tau)\|_{H^{1+\gamma}} d\tau \le M_{\mathcal{G}^\nu}, \qquad \forall t \ge 0 \text{ and } T > 0, \tag{4.4}$$

where $\theta(\cdot,t) = \pi^\nu(t)\theta_0$*.*

Details of the proof of Theorem 4.6 can be found in [31, Subsection 6.1], we also refer to [18] for the case of SQG equations. The steps of proof can be outlined as follows:

I. By standard energy method (see for example [16] for the case $\gamma = 1$), one can show that for all $t \geq 0$, θ^κ satisfies

$$\|\theta^\kappa(\cdot, t)\|_{L^2}^2 + \kappa \int_0^t \|\Lambda^{\frac{\gamma}{2}} \theta^\kappa(\cdot, \tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2 + \frac{t}{c_0 \kappa} \|S\|_{L^2}^2, \tag{4.5}$$

where $c_0 > 0$ is a universal constant which depends only on the dimension d. Moreover, by [31, Lemma 6.2], the set

$$B_\infty = \left\{ \phi \in L^\infty \cap H^1 : \|\phi\|_{L^\infty} \leq \frac{2}{c_0 \kappa} \|S\|_{L^\infty} \right\}$$

is an absorbing set for $\pi^\nu(t)$ and

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_\infty} \|\pi^\nu(t) \theta_0\|_{L^\infty} \leq \frac{3}{c_0 \kappa} \|S\|_{L^\infty}. \tag{4.6}$$

II. Next by [31, Lemma 6.3], we obtain the necessary *a priori* estimate in C^α-space with some appropriate exponent $\alpha \in (0, 1)$. Furthermore, as pointed out in [16], we can see that the solutions to (1.1) emerging from data in a bounded subset of H^1 are absorbed in finite time by B_∞. Hence if $\theta_0 \in H^1 \cap L^\infty$ and fix ν, $\kappa > 0$, then there exists $\alpha \in (0, \frac{\gamma}{3+\gamma}]$ which depends on $\|\theta_0\|_{L^\infty}$, $\|S\|_{L^\infty}$, ν, κ, γ such that

$$\|\theta(\cdot, t)\|_{C^\alpha} \leq C(K_\infty + \bar{K}_\infty), \qquad \forall t \geq t_\alpha := \frac{3}{2\gamma(1-\alpha)}, \tag{4.7}$$

where $C > 0$ is a positive constant, K_∞ and $\bar{K}_\infty$ are given respectively by

$$K_\infty := \|\theta_0\|_{L^\infty} + \frac{1}{c_0 \kappa} \|S\|_{L^\infty},$$
$$\bar{K}_\infty := \kappa^{-\frac{1}{\gamma}} K_\infty + \|S\|_{L^\infty}^{\frac{2+\gamma}{2(1+\gamma)}} \kappa^{-\frac{1}{2(1+\gamma)}} K_\infty^{\frac{\gamma}{2(1+\gamma)}} + \kappa^{-\frac{1}{\gamma}} K_\infty^{\frac{6+\gamma}{4}}.$$

With the help of (4.7), we obtain the following result which can be regarded as an improvement of the regularity of the absorbing set B_∞, namely there exists $\alpha \in (0, \frac{\gamma}{3+\gamma}]$ and a constant $C_\alpha = C_\alpha(\|S\|_{L^\infty}, \alpha, \nu, \kappa, \gamma, K_\infty, \bar{K}_\infty) \geq 1$ such that the set

$$B_\alpha = \left\{ \phi \in C^\alpha \cap H^1 : \|\phi\|_{C^\alpha} \leq C_\alpha \right\}$$

is an absorbing set for $\pi^\nu(t)$.

III. As in [31, Lemma 6.7], by establishing an *a priori* estimate for initial data in $H^1 \cap C^\alpha$, we can show that there exists a bounded absorbing set for $\pi^\nu(t)$ in H^1. More precisely, there exists $\alpha \in (0, \frac{\gamma}{3+\gamma}]$ and a constant $R_1 = R_1(\|S\|_{L^\infty \cap H^1}, \alpha, \nu, \kappa, \gamma) \geq 1$ such that the set

$$B_1 = \{ \phi \in C^\alpha \cap H^1 : \|\phi\|_{H^1}^2 + \|\phi\|_{C^\alpha}^2 \leq R_1^2 \}$$

is an absorbing set for $\pi^\nu(t)$. Moreover, we have

$$\sup_{t \geq 0} \sup_{\theta_0 \in B_1} \left[\|\pi^\nu(t)\theta_0\|_{H^1}^2 + \|\pi^\nu(t)\theta_0\|_{C^\alpha}^2 + \int_t^{t+1} \|\pi^\nu(\tau)\theta_0\|_{H^{1+\frac{\gamma}{2}}}^2 d\tau \right]$$
$$\leq 2R_1^2. \tag{4.8}$$

The bound (4.8) turns out to be crucial for improving the regularity of the absorbing set B_1 to $H^{1+\frac{\gamma}{2}}$, which allows us to obtain an absorbing set $B_{1+\frac{\gamma}{2}}$ for $\pi^\nu(t)$ given by

$$B_{1+\frac{\gamma}{2}} = \left\{\phi \in H^{1+\frac{\gamma}{2}} : \|\phi\|_{H^{1+\frac{\gamma}{2}}} \leq R_{1+\frac{\gamma}{2}}\right\}$$

for some constant $R_{1+\frac{\gamma}{2}} \geq 1$ which depends on $\|S\|_{L^\infty\cap H^1}$, ν, κ, γ. The existence and regularity of the global attractor claimed by Theorem 4.6 now follows by applying the argument given in [9, Proposition 8] and the bound (4.8).

After we have obtained the global attractors as described in Theorem 4.6, we prove some additional properties on the attractors under the assumption that $\gamma \in [1, 2]$ (also refer to [31, Subsection 6.2] for details):

Theorem 4.7 (Friedlander and Suen [31])**.** *Let $S \in L^\infty \cap H^1$. For ν, $\kappa > 0$, assume that the exponent $\gamma \in [1, 2]$. Then the global attractor $\mathcal{G}^\nu$ of $\pi^\nu(t)$ further enjoys the following properties:*

- *$\mathcal{G}^\nu$ is fully invariant, namely*
$$\pi^\nu(t)\mathcal{G}^\nu = \mathcal{G}^\nu, \qquad \forall t \geq 0.$$
- *$\mathcal{G}^\nu$ is maximal in the class of H^1-bounded invariant sets.*
- *$\mathcal{G}^\nu$ has finite fractal dimension.*

To prove the invariance and the maximality of the attractor $\mathcal{G}^\nu$, we observe that the solution map $\pi^\nu(t)$ is indeed continuous in the H^1-topology, in other words for every $t > 0$, the solution map $\pi^\nu(t) : B_{1+\frac{\gamma}{2}} \to H^1$ is continuous in the topology of H^1. The key ingredient for the proof of continuity is the bound

$$\|u\|_{L^\infty} \leq C_\nu \|\Lambda\theta\|_{L^2},$$

where $C_\nu > 0$ is a constant depending on $\nu > 0$, and such bound comes from the condition (A3) and the assumption that $d = 2$ or 3. Following the argument given in [16, Proposition 5.5] and using the log-convexity method introduced by [1], we can also prove that the solution map π^ν is injective on the absorbing set $B_{1+\frac{\gamma}{2}}$. Hence by applying [18, Proposition 6.4], we can obtain the invariance and the maximality of the attractor $\mathcal{G}^\nu$ stated in Theorem 4.7.

It remains to address the fractal dimensions for the global attractors $\mathcal{G}^\nu$. In order to prove that $\dim_f(\mathcal{G}^\nu)$ is finite, we need to show that the solution map π^ν is *uniform differentiable* (refer to [31, Definition 6.18] for the definition for being uniform differentiable). And by [31, Lemma 6.21], the large-dimensional volume elements which are carried by the flow of $\pi^\nu(t)\theta_0$, with $\theta_0 \in \mathcal{G}^\nu$, actually have exponential decay in time. The argument in [13, pp. 115–130, and Chapter 14] can then be applied which shows that $\dim_f(\mathcal{G}^\nu)$ is finite.

4.4. **Applications to magneto-geostrophic equations.** We now apply our results claimed in subsections 4.1-4.3 to MG$^\nu$ equation (1.4)-(1.5), which can be summarised in the following theorems (see also [31, Subsection 7.1]. We point out that, Theorem 4.8 strengthens and generalises the results obtained in [28] in which the authors showed weak convergence as $\kappa \to 0$.

Theorem 4.8 (H^s-convergence as $\kappa \to 0$ for MG$^\nu$ equations)**.** *Let $\nu > 0$ be given as in* (1.4)*, and let $\theta_0, S \in C^\infty$ be the initial datum and forcing term respectively. If θ^κ and θ^0 are smooth solutions to* (1.4)-(1.5) *for $\kappa > 0$ and $\kappa = 0$ respectively, then*

$$\lim_{\kappa\to 0} \|(\theta^\kappa - \theta^0)(\cdot,t)\|_{H^s} = 0,$$

for all $s \geq 0$ and $t \geq 0$.

Theorem 4.9 (Analytic convergence as $\kappa \to 0$ for MG equations)**.** *Let $\nu = 0$ be given as in* (1.4)*, and let θ_0, S the initial datum and forcing term respectively. Suppose that θ_0 and S are both analytic functions with zero mean. Then if θ^κ, θ^0 are analytic solutions to* (1.4)-(1.5) *for $\kappa > 0$ and $\kappa = 0$ respectively with initial datum θ_0 and with radius of convergence at least $\bar{\tau}$, then there exists $T \leq \bar{T}$ and $\tau = \tau(t) < \bar{\tau}$ such that, for $t \in [0,T]$, we have:*

$$\lim_{\kappa\to 0} \|(\Lambda^r e^{\tau\Lambda}\theta^\kappa - \Lambda^r e^{\tau\Lambda}\theta^0)(\cdot,t)\|_{L^2} = 0.$$

Theorem 4.10 (Existence of global attractors for MG$^\nu$ equations)**.** *Let $S \in L^\infty \cap H^1$. For $\nu, \kappa > 0$, let $\pi^\nu(t)$ be solution operator for the initial value problem* (1.4) *via* (4.2)*. Then the solution map $\pi^\nu(t) : H^1 \to H^1$ associated to* (1.4)-(1.5) *possesses a unique global attractor $\mathcal{G}^\nu$ for all $\nu > 0$. In particular, for each $\nu > 0$, the global attractor $\mathcal{G}^\nu$ of $\pi^\nu(t)$ enjoys the following properties:*

- *$\mathcal{G}^\nu$ is fully invariant, namely*

$$\pi^\nu(t)\mathcal{G}^\nu = \mathcal{G}^\nu, \qquad \forall t \geq 0.$$

- *$\mathcal{G}^\nu$ is maximal in the class of H^1-bounded invariant sets.*
- *$\mathcal{G}^\nu$ has finite fractal dimension.*

We recall from subsection 2.2 that there exists a compact global attractor $\mathcal{A}$ in $L^2(\mathbb{T}^3)$ for the MG^0 equations, namely the equations (1.4)-(1.5) when $\kappa > 0$, $\nu = 0$ and $S \in L^\infty \cap H^1$. When ν is varying, we relate the global attractors $\mathcal{G}^\nu$ with $\mathcal{A}$ and further obtain the following theorem:

Theorem 4.11 (Friedlander and Suen [31])**.** *Let $\kappa > 0$ be fixed in* (1.4)*. Then we have:*

(1) *If $\mathcal{G}^\nu$ are the global attractors for the MG$^\nu$ equations* (1.4)-(1.5) *as obtained by Theorem 4.10, then $\mathcal{G}^\nu$ and $\mathcal{A}$ satisfy*

$$\sup_{\phi\in\mathcal{G}^\nu}\inf_{\psi\in\mathcal{A}} \|\phi - \psi\|_{L^2} \to 0 \text{ as } \nu \to 0. \tag{4.9}$$

(2) *Let $\nu^* > \nu_* > 0$ be arbitrary. For each $\nu_0 \in [\nu_*, \nu^*]$, the collection $\{\mathcal{G}^\nu\}_{\nu\in[\nu_*,\nu^*]}$ is upper semicontinuous at ν_0 in the following sense:*

$$\sup_{\phi\in\mathcal{G}^\nu}\inf_{\psi\in\mathcal{G}^{\nu_0}} \|\phi - \psi\|_{H^1} \to 0 \text{ as } \nu \to \nu_0. \tag{4.10}$$

To prove the convergence (4.9), we recall from Theorem 2.7 that for $\kappa > 0$, $\nu \in [0,1]$ and $S \in L^\infty \cap H^2$, there exists global attractor $\mathcal{A}^\nu$ in L^2 generated by the solution map $\tilde{\pi}^\nu$ via

$$\tilde{\pi}^\nu(t) : L^2 \to L^2, \qquad \tilde{\pi}^\nu(t)\theta_0 = \theta(\cdot,t), \qquad t \geq 0,$$

and $\mathcal{A}^\nu$ is upper semicontinuous at $\nu = 0$. Since $\tilde{\pi}^\nu\Big|_{H^1} = \pi^\nu$ and $\mathcal{G}^\nu \subset \mathcal{A}^\nu$ for all $\nu \in (0,1]$, the convergence (4.9) follows immediately from (2.16). On the other hand, to prove the convergence (4.10), we write $I^* = [\nu_*, \nu^*]$ and show that

I. there is a compact subset $\mathcal{U}$ of H^1 such that $\mathcal{G}^\nu \subset \mathcal{U}$ for every $\nu \in I^*$; and
II. for $t > 0$, $\pi^\nu\theta_0$ is continuous in I^*, uniformly for θ_0 in compact subsets of H^1.

The key for showing Step I. and Step II. is the following bound, namely for any $\nu \in I^*$, $s \in [0,2]$ and $f \in L^p$ with $p > 1$, we have

$$\|\Lambda^s u^\nu[f]\|_{L^p} \leq C_*\|f\|_{L^p}, \tag{4.11}$$

where C_* is a positive constant which depends only on p, ν_* and ν^*. The bound can be used for obtaining a bounded set B_2 in H^2 given by

$$B_2 = \left\{\phi \in H^2 : \|\phi\|_{H^2} \leq R_2\right\}$$

where $R_2 \geq 1$ is a constant which depends only on ν_*, ν^*, κ, $\|S\|_{L^\infty\cap H^1}$, and B_2 enjoys the following properties:

- B_2 is a compact set in H^1 which depends only on ν_*, ν^*, κ, $\|S\|_{L^\infty\cap H^1}$;
- $\mathcal{G}^\nu \subset B_2$ for all $\nu \in I^*$.

For the uniform continuity stated in Step II., with the help of the bound (4.11), we can obtain the necessary H^1-estimates [31, Lemma 7.11]: Define $\mathcal{U} = \{\phi \in H^1 : \|\phi\|_{H^1}^2 \leq R_\mathcal{U}\}$ where $R_\mathcal{U} > 0$, then for any $\theta_0 \in \mathcal{U}$ and $\nu \in I^*$, if $\theta^\nu(t) = \pi^\nu(t)\theta_0$, then $\theta^\nu(t)$ satisfies

$$\sup_{0\leq\tau\leq t}\|\theta^\nu(\cdot,\tau)\|_{H^1}^2 + \int_0^t \|\theta^\nu(\cdot,\tau)\|_{H^2}^2 d\tau \leq M_*(t), \qquad \forall t > 0, \tag{4.12}$$

where $M_*(t)$ is a positive function in t which depends only on t, κ, ν_*, ν^*, $\|S\|_{H^1}$ and $R_\mathcal{U}$. The uniform continuity stated in Step II. then follows by energy-type estimates on the difference $\theta^{\nu_1} - \theta^{\nu_2}$ with ν_1, $\nu_2 \in I^*$, which completes the proof of Theorem 4.11.

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Manuscript received November 15 2020
revised December 2 2020

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