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POLLICOTT–RUELLE RESOLVENT AND SOBOLEV REGULARITY

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Abstract. In this note we compute the threshold regularity for meromorphic continuation of the Pollicott–Ruelle resolvent of an Anosov flow as an operator on anisotropic Sobolev spaces, in the setting of lifts to general vector bundles. These thresholds are related to the Sobolev regularity needed for the decay of correlations.

1. Introduction

Let M be a compact d-dimensional C^∞ manifold (without boundary) and $X \in C^\infty(M;TM)$ be a nonvanishing vector field. This field generates a flow

$$\varphi^t := \exp(tX) : M \to M, \quad t \in \mathbb{R}.$$

A fundamental topic in dynamical systems is the study of the behavior of *correlations*

$$\rho_{f,g}(t) := \int_M (f \circ \varphi^{-t})g, \quad t \in \mathbb{R}$$

where f is an L^2 function and g is an L^2 density on M. (Here the line bundle of densities on M is used because the integral of a density over M is invariantly defined; we do not assume a priori that the flow preserves a smooth volume form.) In particular, one is interested in *mixing* (when $\rho_{f,g}(t)$ has a limit as $t \to \infty$) and also in the stronger property of *exponential mixing* (when the remainder in the mixing property decays exponentially fast as $t \to \infty$). We note that even when exponential mixing is known, it does not hold for all $f, g \in L^2$, instead one has to restrict to more regular functions.

In this paper we focus on the case when φ^t is an *Anosov flow*, that is the tangent spaces to M decompose into the flow, stable, and unstable directions – see §2.1 for a precise definition. There are many examples of such flows, including geodesic flows on manifolds of negative curvature (see §2.2). A key tool in studying long time asymptotics of correlations is the *Pollicott–Ruelle resolvent*

$$R_X(\lambda)f = \int_0^\infty e^{-\lambda t}(f \circ \varphi^{-t})\,dt, \quad \operatorname{Re}\lambda > 0, \quad f \in C^\infty(M)$$

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with the integral converging in the space of continuous functions $C^0(M)$. The integral $\int_M (R_X(\lambda)f)g$ is the Fourier–Laplace transform of the correlation $\rho_{f,g}(t)$ at λ.

Since X is a smooth vector field, differentiation along it defines a first order differential operator which we also denote by X. This operator acts in particular on the space of smooth functions $C^\infty(M)$ and on the space of distributions $\mathcal{D}'(M)$. Now, $R_X(\lambda)$ is an inverse of $X+\lambda$ in the following sense:

$$R_X(\lambda)(X+\lambda)f = (X+\lambda)R_X(\lambda)f = f \quad \text{for all} \quad f\in C^\infty(M), \quad \operatorname{Re}\lambda > 0. \tag{1.1}$$

A fundamental property of $R_X(\lambda)$ is that it continues meromorphically to the entire complex plane:

Theorem 1.1. *Assume that X is an Anosov flow. Then $R_X(\lambda)$ admits a meromorphic extension*

$$R_X(\lambda) : C^\infty(M) \to \mathcal{D}'(M), \quad \lambda \in \mathbb{C}.$$

The poles of the extended family $R_X(\lambda)$, called the *Pollicott–Ruelle resonances* of φ^t, are the complex characteristic frequencies governing the decay of correlations. They also appear as singularities (zeroes and/or poles) of dynamical zeta functions.

A typical proof of Theorem 1.1 is to use (1.1) and construct the meromorphic continuation of $R_X(\lambda)$ as the inverse of $X+\lambda$ acting between two Banach spaces of distributions $D \to H$ which are carefully designed so that $X+\lambda : D \to H$ is a Fredholm operator. This gives the continuation to a half-plane $\operatorname{Re}\lambda > -c$ where the value of the constant c depends on the choice of the spaces, and it is possible to choose D, H to make c arbitrarily large.

The present paper establishes a version of Theorem 1.1 in the more general setting of a smooth vector bundle $\mathcal{E}$ over M and an arbitrary lift $\mathbf{X} : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathcal{E})$ of X – see §2.3.1 for details and §2.3.2 for examples. It is already known that such an extension holds, however in this paper we compute the needed regularity for the spaces on which Fredholm property holds. This can be used in particular to better understand the regularity assumptions for exponential decay of correlations as well as regularity of resonant states.

We use an anisotropic Sobolev space $H^{\mathfrak{m}}(M;\mathcal{E})$ associated to a weight function $\mathfrak{m} \in C^\infty(T^*M \setminus 0; \mathbb{R})$ which is homogeneous of degree 0. This function needs to satisfy natural dynamical assumptions (see §4.1), in particular to it correspond two numbers

$$m_u \le 0 \le m_s$$

such that $H^{m_s}(M;\mathcal{E}) \subset H^{\mathfrak{m}}(M;\mathcal{E}) \subset H^{m_u}(M;\mathcal{E})$. See Adam–Baladi [1, §3.3] for the threshold regularity computation for the case of trivial one-dimensional bundles, giving (1.2) in that case (see also Guillarmou–Poyferré–Bonthonneau [16, Appendix A]), Wang [27] for radial estimates giving regularity in the more general Besov spaces in the scalar case, and Bonthonneau–Lefeuvre [5] for a related result giving the regularity threshold in the case of general bundles for Hölder–Zygmund spaces. For an estimate of the regularity threshold in anisotropic Banach spaces in the related case of Anosov maps, see [7, Theorem 4.1] or [2, Theorem 6.12].

The main result of this paper, Theorem 4.1 in §4.1.1, shows meromorphic continuation of the Pollicott–Ruelle resolvent $R_{\mathbf{X}}(\lambda)$ associated to the lift $\mathbf{X}$ to a half-plane which is explicitly described in terms of m_u, m_s and the dynamics of the flow φ^t. More precisely, the condition on λ is that there exists $\varepsilon > 0$ and a constant C such that for all $x \in M$ and $t \geq 0$

$$\begin{aligned} |\det d\varphi^t(x)|^{\frac{1}{2}} \cdot \|\mathscr{T}^t_{\mathbf{X}}(x)\| \cdot \big\|d\varphi^t(x)|_{E_s}\big\|^{-m_u} &\leq Ce^{(\operatorname{Re}\lambda-\varepsilon)t}, \\ |\det d\varphi^t(x)|^{\frac{1}{2}} \cdot \|\mathscr{T}^t_{\mathbf{X}}(x)\| \cdot \big\|d\varphi^t(x)^{-1}|_{E_u}\big\|^{m_s} &\leq Ce^{(\operatorname{Re}\lambda-\varepsilon)t}. \end{aligned} \tag{1.2}$$

Here E_u, E_s are the unstable/stable spaces of the flow and $\mathscr{T}^t_{\mathbf{X}}(x) : \mathcal{E}(x) \to \mathcal{E}(\varphi^t(x))$ is the parallel transport associated to the lift $\mathbf{X}$. See §3.3 and the statement of Theorem 4.1 for details and §3.3.1 for examples.

The use of anisotropic Hölder and Sobolev spaces to prove Theorem 1.1 and an analogous statement in the related setting of Anosov maps has a long tradition, see in particular the works of Blank–Keller–Liverani [3], Liverani [22,23], Gouezel–Liverani [17], Baladi–Tsujii [6], and Butterley–Liverani [4]. We use the *microlocal approach* originating in the papers of Faure–Roy–Sjöstrand [14] and Faure–Sjöstrand [15]. See the review of Zworski [29, §4] for a comprehensive introduction to this microlocal approach. Our proof is similar in structure to the one in the paper of Dyatlov–Zworski [12] on dynamical zeta functions. (See also the work of Dyatlov–Guillarmou [9,10] for the more general setting of basic sets of Axiom A flows.) The main difference between the present paper and [12] is the precise analysis of what regularity is needed for radial estimates – see §§3.2.3, 3.3, and 4.2.3.

We also address a minor mistake present in [9,12]: when the vector bundle $\mathcal{E}$ is not trivial, it is not possible to extend pseudodifferential operators on $C^\infty(M)$ canonically to operators on $C^\infty(M;\mathcal{E})$. Thus all the pseudodifferential cutoffs $A, B, B_1, \dots$ used in the propagation estimates in [9,12] should be taken to be principally scalar operators rather than operators on $C^\infty(M)$.

For applications of anisotropic spaces to exponential mixing for contact flows, see the works of Liverani [22], Tsujii [26], and Nonnenmacher–Zworski [24]. We note that the latter paper [24] uses the microlocal approach and thus could be potentially combined with the present result to yield exponential mixing for more general bundles, however in the case when $\mathbf{X}^* \neq -\mathbf{X}$ more adjustments would be needed to the argument there.

2. Anosov flows

2.1. **Definition.** As in the introduction, we assume that X is a nonvanishing vector field on a compact manifold M and $\varphi^t = \exp(tX)$ is the corresponding flow.

Definition 2.1. We say that φ^t is an *Anosov flow* if there exists a splitting of tangent spaces into the *flow/unstable/stable spaces*

$$T_xM = E_0(x) \oplus E_u(x) \oplus E_s(x), \qquad x \in M \tag{2.1}$$

such that:

- $E_0(x) = \mathbb{R}X(x)$;
- $E_u(x), E_s(x)$ depend continuously on x and are invariant under the flow:

$$d\varphi^t(x)E_u(x) = E_u(\varphi^t(x)), \quad d\varphi^t(x)E_s(x) = E_s(\varphi^t(x));$$

- we have the exponential contraction property under the differential of the flow,

$$|d\varphi^t(x)v| \le Ce^{-\theta|t|}|v| \quad \text{if} \quad \begin{cases} v \in E_u(x), & t \le 0 \quad \text{or} \\ v \in E_s(x), & t \ge 0. \end{cases} \tag{2.2}$$

Here $C, \theta > 0$ are some constants and we fix an arbitrary Riemannian metric on M; C depends on the choice of the metric but θ does not.

Remark 2.2. The dependence of $E_u(x), E_s(x)$ on the base point x is Hölder continuous but typically not C^∞, see for example [18].

In this paper we always assume that φ^t is an Anosov flow. It is sometimes useful to make additional assumptions, given by

Definition 2.3. Let X be a nonvanishing vector field on a manifold M. We say that the flow $\varphi^t = \exp(tX)$ is:

- a *volume preserving* flow, if there exists a C^∞ density μ on M which is invariant under pullback by φ^t;
- a *contact* flow, if $d = \dim M$ is odd and there exists a 1-form $\alpha \in C^\infty(M; T^*M)$ such that $\alpha \wedge (d\alpha)^{\frac{d-1}{2}}$ is nonvanishing, $\iota_X\alpha = 1$, and $\iota_X d\alpha = 0$.

Remark 2.4. For contact flows, the form α is called a *contact form* and X is called the *Reeb vector field* associated to α. The manifold M is oriented by requiring that $d\,\mathrm{vol}_\alpha := \alpha \wedge (d\alpha)^{\frac{d-1}{2}}$ be positive. Moreover, $d\,\mathrm{vol}_\alpha$ is invariant under the flow φ^t, so contact flows are always volume preserving.

2.2. **Examples.** We now give a few standard examples of Anosov flows.

2.2.1. *Geodesic flows.* Assume that (Σ, g) is a compact Riemannian manifold. We let M be the sphere bundle of Σ:

$$M = S\Sigma := \{(y, w) \in T\Sigma \colon |w|_g = 1\}.$$

The geodesic flow φ^t is the flow on M defined as follows: if $(y, w) \in S\Sigma$ and $\gamma : \mathbb{R} \to \Sigma$ is the geodesic such that $\gamma(0) = y$, $\dot\gamma(0) = w$, then $\varphi^t(y, w) = (\gamma(t), \dot\gamma(t))$. The flow φ^t is a contact flow, where the contact 1-form α on $S\Sigma$ is defined as follows:

$$\alpha_{(y,w)}(\xi) = \langle d\pi_{(y,w)}\xi, w\rangle_g$$

where $\pi : S\Sigma \to \Sigma$ is the projection map – see for example [25, §1.3.3].

Proposition 2.5. *If* (Σ, g) *has everywhere negative sectional curvature, then the geodesic flow* φ^t *on* $M = S\Sigma$ *is Anosov.*

For the proof, see for example [21, Theorem 3.9.1].

2.2.2. *Suspensions of Anosov maps.* An Anosov map is a discrete time analog of an Anosov flow:

Definition 2.6. Let $\widetilde{M}$ be a compact manifold and $T : \widetilde{M} \to \widetilde{M}$ be a diffeomorphism. We say that T is an *Anosov map* if the tangent spaces to $\widetilde{M}$ admit a decomposition $T_x\widetilde{M} = E_u(x) \oplus E_s(x)$ which is invariant under T, depends continuously on x, and satisfies the following exponential contraction property for some constants $C, \theta > 0$ and a Riemannian metric on $\widetilde{M}$:

$$|dT^k(x)v| \leq Ce^{-\theta|k|}|v| \quad \text{if} \quad \begin{cases} v \in E_u(x), & k \leq 0 \quad \text{or} \\ v \in E_s(x), & k \geq 0. \end{cases}$$

Basic examples of Anosov maps are the toric automorphisms

$$T : \mathbb{T}^d \to \mathbb{T}^d, \quad T(x) = Ax \bmod \mathbb{Z}^d$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the d-dimensional torus and the matrix $A \in \mathrm{GL}(d, \mathbb{Z})$, $|\det A| = 1$, has no eigenvalues on the unit circle.

To make an Anosov map into an Anosov flow, we use suspensions. Let $T : \widetilde{M} \to \widetilde{M}$ be an Anosov map and $\tau : \widetilde{M} \to (0, \infty)$ be a smooth function, called the *roof function* of the suspension. Let M be the manifold obtained by taking the cylinder $\{(x, s) \mid x \in \widetilde{M},\ 0 \leq s \leq \tau(x)\}$ and gluing its two ends by identifying $(x, \tau(x))$ with $(T(x), 0)$. Alternatively, we may define M as the quotient of $\widetilde{M} \times \mathbb{R}$ by the action of $\mathbb{Z}$ generated by the map $(x, s + \tau(x)) \mapsto (T(x), s)$. Now, the vector field $X := \partial_s$ is well-defined on M and generates an Anosov flow called the *suspension* of T with roof function τ. Here the Anosov property is easy to check when τ is constant and the general case is obtained by a time change, which does not change the Anosov property – see for example [20, Proposition 17.4.5].

2.3. **Operators and resolvents.** Let $\varphi^t = \exp(tX)$ be an Anosov flow on a manifold M. The vector field X defines a first order differential operator $X : C^\infty(M) \to C^\infty(M)$. For $t \in \mathbb{R}$, define the operator

$$e^{-tX} : C^\infty(M) \to C^\infty(M), \quad e^{-tX}f := f \circ \varphi^{-t}.$$

The notation e^{-tX} is justified as follows: for each $f \in C^\infty(M)$ we have

$$\partial_t(e^{-tX}f) = -e^{-tX}Xf = -Xe^{-tX}f. \tag{2.3}$$

Now, for a complex number λ such that $\operatorname{Re}\lambda > 0$ we define the Pollicott–Ruelle resolvent

$$R_X(\lambda)f := \int_0^\infty e^{-\lambda t}e^{-tX}f\,dt. \tag{2.4}$$

Here the integral converges exponentially fast in the sup-norm when f is continuous.

We have the identity (1.1). Indeed, take $f \in C^\infty(M)$ and assume that $\operatorname{Re}\lambda > 0$. Then

$$R_X(\lambda)(X + \lambda)f = (X + \lambda)R_X(\lambda)f = -\int_0^\infty \partial_t(e^{-\lambda t}e^{-tX}f)\,dt = f$$

where in the second equality we consider $X + \lambda$ as a differential operator on distributions.

2.3.1. *More general operators.* We now extend the definition of Pollicott–Ruelle resolvent to more general operators. Let $\mathcal{E}$ be a (finite dimensional complex) C^∞ vector bundle over M. Denote by $C^\infty(M;\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

Definition 2.7. An operator $\mathbf{X} : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathcal{E})$ is called a *lift* of the vector field X to $\mathcal{E}$ if

$$\mathbf{X}(f\mathbf{u}) = (Xf)\mathbf{u} + f(\mathbf{X}\mathbf{u}) \quad \text{for all} \quad f \in C^\infty(M;\mathbb{C}), \quad u \in C^\infty(M;\mathcal{E}). \tag{2.5}$$

If we fix a local frame $\mathbf{e}_1, \dots, \mathbf{e}_n \in C^\infty(U;\mathcal{E})$ on $\mathcal{E}$, where $U \subset M$ is an open set, then lifts of X have the form

$$\mathbf{X}\sum_{j=1}^n f_j(x)\mathbf{e}_j(x) = \sum_{j=1}^n \Big(Xf_j(x) + \sum_{k=1}^n A_{jk}(x)f_k(x)\Big)\mathbf{e}_j(x), \quad x \in U \tag{2.6}$$

for all $f_1, \dots, f_n \in C^\infty(M;\mathbb{C})$ where $(A_{jk}(x))$ is an $n \times n$ complex matrix with entries which are smooth functions on U.

We next define parallel transport on $\mathcal{E}$. Let $x_0 \in M$ and define the curve $x(t) := \varphi^t(x_0)$. Assume that $\mathbf{v}(t) \in \mathcal{E}(x(t))$, $t \in \mathbb{R}$, is a smooth section of the pullback of $\mathcal{E}$ to the curve $x(t)$. We define the derivative $D_\mathbf{X}\mathbf{v}(t) \in \mathcal{E}(x(t))$ by requiring that

$$D_\mathbf{X}(\mathbf{u}(x(t))) = \mathbf{X}\mathbf{u}(x(t)) \quad \text{for all} \quad \mathbf{u} \in C^\infty(M;\mathcal{E}).$$

In a local frame we can write

$$D_\mathbf{X}\sum_{j=1}^n f_j(t)\mathbf{e}_j(x(t)) = \sum_{j=1}^n \Big(\dot f_j(t) + \sum_{k=1}^n A_{jk}(x(t))f_k(t)\Big)\mathbf{e}_j(x(t)). \tag{2.7}$$

We say that $\mathbf{v}(t)$ is *parallel* if $D_\mathbf{X}\mathbf{v}(t) = 0$ for all t. Using the coordinate expression (2.7) and the existence/uniqueness theorem for linear systems of ODEs, we see that for each $\mathbf{v}_0 \in \mathcal{E}(x(0))$ there exists a unique parallel field $\mathbf{v}(t)$ such that $\mathbf{v}(0) = \mathbf{v}_0$. We then define the *parallel transport operator*

$$\mathscr{T}^t_\mathbf{X}(x) : \mathcal{E}(x) \to \mathcal{E}(\varphi^t(x)), \quad t \in \mathbb{R} \tag{2.8}$$

such that for any parallel field $\mathbf{v}(t)$ we have $\mathscr{T}^t_\mathbf{X}(x(0))\mathbf{v}(0) = \mathbf{v}(t)$.

We now define the family of operators

$$e^{-t\mathbf{X}} : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathcal{E}), \quad t \in \mathbb{R}$$

so that the evolution equation (2.3) holds with X replaced by $\mathbf{X}$. In terms of parallel transport it can be described as follows: for each $\mathbf{u} \in C^\infty(M;\mathcal{E})$ and $x \in M$ we have

$$e^{-t\mathbf{X}}\mathbf{u}(x) = \mathscr{T}^t_\mathbf{X}(\varphi^{-t}(x))\mathbf{u}(\varphi^{-t}(x)). \tag{2.9}$$

We now want to define the Pollicott–Ruelle resolvent of $\mathbf{X}$ similarly to (2.4). For that fix an inner product on the fibers of $\mathcal{E}$ and take constants $C_\mathbf{X}, C_1$ such that

$$\|\mathscr{T}^t_\mathbf{X}(x)\|_{\mathcal{E}(x)\to\mathcal{E}(\varphi^t(x))} \le C_1 e^{C_\mathbf{X} t} \quad \text{for all} \quad t \ge 0, \quad x \in M.$$

Note that the constant C_1 depends on the choice of the inner product but $C_\mathbf{X}$ does not. Now we define

$$R_\mathbf{X}(\lambda)\mathbf{u} := \int_0^\infty e^{-\lambda t}e^{-t\mathbf{X}}\mathbf{u}\,dt \quad \text{for} \quad \operatorname{Re}\lambda > C_\mathbf{X}, \quad \mathbf{u} \in C^\infty(M;\mathcal{E}). \tag{2.10}$$

The integral converges in the space of continuous functions $C^0(M;\mathcal{E})$. We have the identities similar to (1.1):
(2.11)

$$R_{\mathbf{X}}(\lambda)(\mathbf{X}+\lambda)\mathbf{u} = (\mathbf{X}+\lambda)R_{\mathbf{X}}(\lambda)\mathbf{u} = \mathbf{u} \quad \text{for all} \quad \mathbf{u}\in C^\infty(M;\mathcal{E}), \quad \operatorname{Re}\lambda > C_{\mathbf{X}}.$$

2.3.2. *Examples.* We now give several natural examples of lifts $\mathbf{X}$. First of all, if $\mathcal{E} = M\times\mathbb{C}$ is the trivial line bundle over M, then lifts of X have the form

$$\mathbf{X} = X + V \quad \text{for some potential} \quad V\in C^\infty(M;\mathbb{C}).$$

The operator $e^{-t\mathbf{X}}$ is given by

$$e^{-t\mathbf{X}}u(x) = \exp\Big(-\int_0^t V(\varphi^{-s}(x))\,ds\Big)u(\varphi^{-t}(x)).$$

The next example is given by the bundles of differential forms

$$\Omega^k := \wedge^k T^*M$$

and $\mathbf{X} := \mathcal{L}_X$ is the Lie derivative. In this case the operator $e^{-t\mathbf{X}}$ is the pullback of differential forms by φ^{-t}.

One can also consider the smaller bundle of perpendicular forms

$$\Omega_0^k := \{\mathbf{u}\in\Omega^k \mid \iota_X\mathbf{u} = 0\}$$

with the same operator $\mathbf{X} = \mathcal{L}_X$, which is important for the analysis of the Ruelle zeta function (see for example [12]).

We can consider a more general setting by taking a complex vector bundle $\mathcal{V}$ over M equipped with a flat connection ∇, considering the bundle $\mathcal{E} := \Omega^k\otimes\mathcal{V}$, and putting

$$\mathbf{X} := \mathcal{L}_{X,\nabla} = d^\nabla\iota_X + \iota_X d^\nabla$$

where $d^\nabla : C^\infty(\Omega^k\otimes\mathcal{V}) \to C^\infty(\Omega^{k+1}\otimes\mathcal{V})$ is the twisted exterior derivative associated to ∇. The resulting Pollicott–Ruelle resonances have important applications to *Fried's conjecture* relating dynamical zeta functions and torsion – see for example Dang–Guillarmou–Rivière–Shen [11, §3.3].

A special case of the flat connection example above is when $\mathcal{E}$ is the orientation bundle of the bundle E_s. This bundle can be used to generalize known results on meromorphic continuation of dynamical zeta functions to the case of nonorientable E_s – see Borns-Weil–Shen [8].

3. Microlocal framework and the lifted flow

In this and the next section we assume that $\varphi^t = e^{tX}$ is an Anosov flow on a compact manifold M, $\mathcal{E}$ is a vector bundle over M, and $\mathbf{X} : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathcal{E})$ is a lift of X in the sense of Definition 2.7. (In particular, this includes the special scalar case when $\mathcal{E} = M\times\mathbb{C}$ and $\mathbf{X} = X$.)

We henceforth fix a density ρ_0 on M and an Hermitian inner product $\langle\bullet,\bullet\rangle_{\mathcal{E}}$ on the fibers of $\mathcal{E}$, which together fix the inner product on the space $L^2(M;\mathcal{E})$.

We use the semiclassically rescaled version of $\mathbf{X}$,

$$\mathbf{P} := -ih\mathbf{X}.$$

Here $h \in (0,1]$ is a small number called the *semiclassical parameter*. In the present paper the semiclassical rescaling is a technical tool useful in the proof of the meromorphic continuation of the Pollicott–Ruelle resolvent, and h will be ultimately fixed small enough (so that the $\mathcal{O}(h^\infty)$ remainders in semiclassical estimates can be removed and Lemma 4.3 holds). In applications to spectral gaps (such as the work of Nonnenmacher–Zworski [24]) one has $h \approx |\operatorname{Re}\lambda|^{-1}$ and studies the limit $h \to 0$.

3.1. **Semiclassical analysis.** We discuss the behavior of $\mathbf{P}$ from the point of view of microlocal analysis, more precisely its semiclassical version. We refer the reader to the book of Zworski [28] for an introduction to semiclassical analysis and to the book of Dyatlov–Zworski [13, Appendix E] (which builds on [28]) for some of the more advanced tools used here.

For $m \in \mathbb{R}$, denote by $S^m_h(T^*M)$ the class of h-dependent *Kohn–Nirenberg symbols* of order m on the cotangent bundle T^*M, consisting of h-dependent functions $a(x,\xi;h) \in C^\infty(T^*M)$ satisfying the derivative bounds for all multiindices α, β

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|} \quad \text{for all} \quad (x,\xi) \in T^*M, \quad 0 < h \leq 1.$$

Here $\langle\xi\rangle := \sqrt{1+|\xi|^2}$. This is the class used in [28, §14.2.2]. The estimates from the book [13], on which this paper relies, use instead the smaller class of *polyhomogeneous symbols* with expansions in powers of ξ and h, see [13, Definition E.3]. We will apply these estimates to the conjugated operator $\widetilde{\mathbf{P}}$ (see §4.1.2) which is not polyhomogeneous and we explain below why the results of [13] still hold.

Denote by $\Psi^m_h(M)$ the class of semiclassical pseudodifferential operators with symbols in $S^m_h(T^*M)$. These are h-dependent families of operators on $C^\infty(M)$ and on the space of distributions $\mathcal{D}'(M)$. We refer to [28, §14.2.2] and [13, §E.1.7] for details. We use the *semiclassical principal symbol* isomorphism

$$\sigma_h : \frac{\Psi^m_h(M)}{h\Psi^{m-1}_h(M)} \ \to\ \frac{S^m_h(T^*M)}{hS^{m-1}_h(T^*M)}. \tag{3.1}$$

The space T^*M is not compact because ξ is allowed to go to infinity. We will use the *fiber-radial compactification* $\overline{T^*M}$ obtained by adding to T^*M a sphere at infinity. See for example [13, §E.1.3] for details.

3.1.1. *Operators on sections of vector bundles.* We now discuss the class of semiclassical pseudodifferential operators $\Psi^m_h(M;\operatorname{End}(\mathcal{E}))$ acting on the space of sections $C^\infty(M;\mathcal{E})$ of the vector bundle $\mathcal{E}$. If $\mathcal{E}$ is trivial and $n = \dim\mathcal{E}$, then operators on $C^\infty(M;\mathcal{E})$ are identified with $n \times n$ matrices of operators on $C^\infty(M)$. We say such a matrix is in $\Psi^m_h(M;\operatorname{End}(\mathcal{E}))$ if all of its entries are in $\Psi^m_h(M)$. This class does not depend on the choice of a (smooth) trivialization of $\mathcal{E}$ since composition with multiplication operators maps $\Psi^m_h(M)$ into itself. Since pseudodifferential operators are smoothing and rapidly decaying in h away from the diagonal, one can use the above definition locally to make sense of $\Psi^m_h(M;\operatorname{End}(\mathcal{E}))$ for a general bundle $\mathcal{E}$. See [19, Definition 18.1.32] for more details (in the related nonsemiclassical setting). Any element of $\Psi^m_h(M;\operatorname{End}(\mathcal{E}))$ is bounded uniformly in h in operator norm $H^s_h(M;\mathcal{E}) \to H^{s-m}_h(M;\mathcal{E})$ where $H^s_h(M;\mathcal{E})$ denotes the semiclassical Sobolev space defined similarly to [13, Definition E.20].

For $\mathbf{A} \in \Psi_h^m(M; \mathrm{End}(\mathcal{E}))$, we use the above procedure and the map (3.1) to define the semiclassical principal symbol

$$\sigma_h(\mathbf{A}) \in \frac{S_h^m(T^*M; \mathrm{End}(\pi^*\mathcal{E}))}{hS_h^{m-1}(T^*M; \mathrm{End}(\pi^*\mathcal{E}))}.$$

Here $\pi : T^*M \to M$ is the projection map, $\pi^*\mathcal{E}$ is the pullback of $\mathcal{E}$ to a vector bundle over T^*M, and $\mathrm{End}(\pi^*\mathcal{E})$ is the bundle of homomorphisms from $\pi^*\mathcal{E}$ to itself. Note that σ_h is surjective and $\sigma_h(\mathbf{A}) = 0$ if and only if $\mathbf{A} \in h\Psi_h^{m-1}(M; \mathrm{End}(\mathcal{E}))$.

We say $\mathbf{A}$ is *principally scalar* if $\sigma_h(\mathbf{A})$ is scalar, that is there exists $a \in S_h^m(T^*M)$ such that $\sigma_h(\mathbf{A}) = aI_{\pi^*\mathcal{E}}$ modulo $hS_h^{m-1}(T^*M; \mathrm{End}(\pi^*\mathcal{E}))$. In this case we treat $\sigma_h(\mathbf{A})$ as a scalar function on T^*M by identifying it with (the equivalence class of) a.

Using the standard algebraic properties of the scalar calculus $\Psi_h^m(M)$ (see for instance [28, Theorem 14.1] and [13, Proposition E.17]) we obtain the following properties of the calculus $\Psi_h^m(M; \mathrm{End}(\mathcal{E}))$:

- Product Rule: if $\mathbf{A} \in \Psi_h^m(M; \mathrm{End}(\mathcal{E}))$ and $\mathbf{B} \in \Psi_h^\ell(M; \mathrm{End}(\mathcal{E}))$, then

$$\mathbf{AB} \in \Psi_h^{m+\ell}(M; \mathrm{End}(\mathcal{E})), \qquad \sigma_h(\mathbf{AB}) = \sigma_h(\mathbf{A})\sigma_h(\mathbf{B}) \tag{3.2}$$

  where the right-hand side is understood as composition of sections of $\mathrm{End}(\pi^*\mathcal{E})$.
- Commutator Rule: if $\mathbf{A} \in \Psi_h^m(M; \mathrm{End}(\mathcal{E}))$, $\mathbf{B} \in \Psi_h^\ell(M; \mathrm{End}(\mathcal{E}))$ are both *principally scalar*, then, with $\{\bullet, \bullet\}$ denoting the Poisson bracket on T^*M,

$$[\mathbf{A}, \mathbf{B}] \in h\Psi_h^{m+\ell-1}(M; \mathrm{End}(\mathcal{E})), \qquad \sigma_h(h^{-1}[\mathbf{A}, \mathbf{B}]) = -i\{\sigma_h(\mathbf{A}), \sigma_h(\mathbf{B})\}. \tag{3.3}$$

- Adjoint Rule: if $\mathbf{A} \in \Psi_h^m(M; \mathrm{End}(\mathcal{E}))$, then its formal adjoint $\mathbf{A}^*$ satisfies

$$\mathbf{A}^* \in \Psi_h^m(M; \mathrm{End}(\mathcal{E})), \qquad \sigma_h(\mathbf{A}^*) = \sigma_h(\mathbf{A})^*$$

  where the right-hand side is defined using the adjoint operation on $\mathrm{End}(\pi^*\mathcal{E})$ induced by the inner product $\langle \bullet, \bullet \rangle_{\mathcal{E}}$.

We next discuss the *wavefront set* and the *elliptic set* of an operator $\mathbf{A} \in \Psi_h^m(M; \mathcal{E})$. The wavefront set $\mathrm{WF}_h(\mathbf{A})$ is a compact subset of $\overline{T^*M}$ giving the essential support of the full symbol of $\mathbf{A}$. In terms of the wavefront set of scalar pseudodifferential operators (see for example [13, Definition E.27]), we define $\mathrm{WF}_h(\mathbf{A})$ as the union of the wavefront sets of the entries of $\mathbf{A}$ as an $n \times n$ matrix of operators, with respect to any trivialization of $\mathcal{E}$.

The elliptic set $\mathrm{ell}_h(\mathbf{A})$ is the open subset of $\overline{T^*M}$ on which the principal symbol $\sigma_h(\mathbf{A})$ is essentially invertible (as an endomorphism of $\pi^*\mathcal{E}$). More precisely, a point $(x_0, \xi_0) \in \overline{T^*M}$ lies in $\mathrm{ell}_h(\mathbf{A})$ if there exists a constant C such that we have $\|(\sigma_h(\mathbf{A})(x, \xi))^{-1}\| \le C\langle \xi \rangle^{-m}$ for all sufficiently small h and all (x, ξ) in some neighborhood of (x_0, ξ_0) in $\overline{T^*M}$.

Finally, we give the following version of sharp Gårding inequality for pseudodifferential operators on vector bundles. It is an analog of [13, Proposition E.34] but we restrict a simpler case, putting $B := 0$ and considering a special subclass of nonnegative symbols in $C^\infty(T^*M; \mathrm{End}(\pi^*\mathcal{E}))$.

Lemma 3.1. *Assume that* $\mathbf{A} \in \Psi_h^{2m}(M; \operatorname{End}(\mathcal{E}))$ *and* $\mathbf{B}_1 \in \Psi_h^0(M; \operatorname{End}(\mathcal{E}))$ *satisfy* $\operatorname{WF}_h(\mathbf{A}) \subset \operatorname{ell}_h(\mathbf{B}_1)$. *Assume moreover that the principal symbol* $\sigma_h(\mathbf{A})$ *has the form*

$$\sigma_h(\mathbf{A}) = \chi \mathbf{a}, \quad \chi \in C^\infty(\overline{T^*M}), \quad \chi \geq 0, \quad \mathbf{a} \in S^{2m}(T^*M; \operatorname{End}(\pi^*\mathcal{E})) \tag{3.4}$$

where $\chi, \mathbf{a}$ *are* h*-independent and* $\operatorname{Re}\mathbf{a}$ *is uniformly positive definite on some neighborhood* $V \subset \overline{T^*M}$ *of* $\operatorname{supp}\chi$, *that is there exists a constant* $c > 0$ *such that*

$$\operatorname{Re}\langle \mathbf{a}(x,\xi)\mathbf{v}, \mathbf{v}\rangle_{\mathcal{E}(x)} \geq c\langle\xi\rangle^{2m}\|\mathbf{v}\|^2_{\mathcal{E}(x)} \quad \text{for all} \quad (x,\xi) \in V, \quad \mathbf{v} \in \mathcal{E}(x).$$

Then there exists a constant C *such that for each* N, *all small* h, *and all* $\mathbf{u} \in H^m(M; \mathcal{E})$

$$\operatorname{Re}\langle \mathbf{A}\mathbf{u}, \mathbf{u}\rangle_{L^2(M;\mathcal{E})} \geq -Ch\|\mathbf{B}_1\mathbf{u}\|^2_{H_h^{m-\frac{1}{2}}} - \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}. \tag{3.5}$$

Remark 3.2. In fact (3.4) can be replaced by the weaker and more natural assumption that $\operatorname{Re}\sigma_h(\mathbf{A})$ is nonnegative everywhere, see [19, Remark 2 on p.79] for the nonsemiclassical case. Instead of establishing a semiclassical version of this result here, we choose to make the stronger assumption (3.4) which allows us to use the scalar sharp Gårding inequality as a black box.

Proof. If $\sigma_h(\mathbf{A}) = 0$, then $\mathbf{A} \in h\Psi_h^{2m-1}(M; \operatorname{End}(\mathcal{E}))$ so (3.5) holds by the elliptic estimate (see §4.2.1 below) since $\operatorname{WF}_h(\mathbf{A}) \subset \operatorname{ell}_h(\mathbf{B}_1)$. Therefore, we may replace $\mathbf{A}$ with any other operator with the same principal symbol and wavefront set contained in $\operatorname{ell}_h(\mathbf{B}_1)$. Moreover, from the Adjoint Rule above we see that one may replace $\mathbf{a}$ by $\operatorname{Re}\mathbf{a} := \frac{1}{2}(\mathbf{a} + \mathbf{a}^*)$. We thus henceforth assume that $\mathbf{a}$ is self-adjoint. Since $\operatorname{WF}_h(\mathbf{A}) \subset \operatorname{ell}_h(\mathbf{B}_1)$, we may also assume that $\operatorname{supp}\chi \subset \operatorname{ell}_h(\mathbf{B}_1)$.

Since $\mathbf{a}$ is positive definite on $V \Supset \operatorname{supp}\chi$, we may write

$$\mathbf{a} = \mathbf{f}^*\mathbf{f} \quad \text{near } \operatorname{supp}\chi \quad \text{for some} \quad \mathbf{f} \in S^m(T^*M; \operatorname{End}(\pi^*\mathcal{E})), \quad \operatorname{supp}\mathbf{f} \subset \operatorname{ell}_h(\mathbf{B}_1).$$

For example, we may take $\chi' \in C^\infty(\overline{T^*M})$ such that $\chi' = 1$ near $\operatorname{supp}\chi$ and $\operatorname{supp}\chi' \subset V \cap \operatorname{ell}_h(\mathbf{B}_1)$, and put $\mathbf{f} := \chi'\sqrt{\mathbf{a}}$.

Using a partition of unity on χ, we reduce to a case when χ is supported in some open set over which $\mathcal{E}$ is trivialized by some orthonormal frame. Using that frame, we may consider the pseudodifferential operator $\operatorname{Op}_h(\chi) \in \Psi_h^0(M)$ as an operator on sections of $\mathcal{E}$. Take $\mathbf{F} \in \Psi_h^m(T^*M; \operatorname{End}(\mathcal{E}))$ with principal symbol $\mathbf{f}$ and $\operatorname{WF}_h(\mathbf{F}) \subset \operatorname{ell}_h(\mathbf{B}_1)$, then $\sigma_h(\mathbf{A}) = \sigma_h(\mathbf{F}^*\operatorname{Op}_h(\chi)\mathbf{F})$, so we may assume that $\mathbf{A} = \mathbf{F}^*\operatorname{Op}_h(\chi)\mathbf{F}$. Now

$$\begin{aligned}\langle \mathbf{A}\mathbf{u}, \mathbf{u}\rangle_{L^2(M;\mathcal{E})} = \langle \operatorname{Op}_h(\chi)\mathbf{F}\mathbf{u}, \mathbf{F}\mathbf{u}\rangle_{L^2(M;\mathcal{E})} &\geq -Ch\|\mathbf{F}\mathbf{u}\|^2_{H_h^{-\frac{1}{2}}} \\ &\geq -Ch\|\mathbf{B}_1\mathbf{u}\|^2_{H_h^{m-\frac{1}{2}}} - \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}\end{aligned}$$

Here in the first inequality we use that $\chi \geq 0$ and apply the scalar sharp Gårding inequality [13, Proposition E.23] for the operator $\operatorname{Op}_h(\chi)$. In the last inequality we use the elliptic estimate. $\square$

3.2. **Semiclassical properties of P.** The operator $\mathbf{P} = -ih\mathbf{X}$ is a semiclassical differential operator in the class $\Psi_h^1(M; \mathrm{End}(\mathcal{E}))$, as follows from (2.6). It is principally scalar with the principal symbol given by

$$p(x,\xi) := \langle \xi, X(x)\rangle, \quad x \in M, \quad \xi \in T_x^*M.$$

Note that p is real valued and homogeneous of degree 1 in ξ.

3.2.1. *The lifted flow.* For semiclassical estimates, it is important to understand the characteristic surface $\{p=0\} \subset \overline{T^*M}$ and the Hamiltonian flow e^{tH_p} on this surface. For that we introduce the *dual flow/unstable/stable decomposition* of the fibers of the cotangent bundle T^*M:

$$T_x^*M = E_0^*(x) \oplus E_u^*(x) \oplus E_s^*(x), \quad x \in M \tag{3.6}$$

which is defined in terms of the original flow/unstable/stable decomposition (2.1) as follows:

$$E_0^* := (E_u \oplus E_s)^\perp, \qquad E_u^* := (E_0 \oplus E_u)^\perp, \qquad E_s^* := (E_0 \oplus E_s)^\perp.$$

Any continuous subbundle of T^*M can be considered as a closed subset of $\overline{T^*M}$, and the characteristic surface of p is

$$\{p=0\} = \{(x,\xi) \in \overline{T^*M} \mid \langle \xi, X(x)\rangle = 0\} = E_u^* \oplus E_s^*.$$

Next, the Hamiltonian flow of p has the form

$$e^{tH_p}(x,\xi) = (\varphi^t(x), d\varphi^t(x)^{-T}\xi)$$

and extends to a smooth flow on $\overline{T^*M}$. Here $d\varphi^t(x)^{-T} : T_x^*M \to T_{\varphi^t(x)}^*M$ is the inverse of the transpose of $d\varphi^t(x) : T_xM \to T_{\varphi^t(x)}M$.

Following [13, (E.1.11)], denote by

$$\kappa : T^*M \setminus 0 \to \partial\overline{T^*M}$$

the canonical projection to fiber infinity $\partial\overline{T^*M}$. Then $\kappa(E_u^*), \kappa(E_s^*)$ are compact subsets of $\partial\overline{T^*M}$ invariant under the flow e^{tH_p}.

The Anosov property (2.2) carries over to the decomposition (3.6) as follows: if $|\bullet|$ denotes some smooth norm on the fibers of T^*M, then

$$|e^{tH_p}(x,\xi)| \le Ce^{-\theta|t|}|\xi| \quad \text{if} \quad \begin{cases} \xi \in E_u^*(x), & t \le 0 \quad \text{or} \\ \xi \in E_s^*(x), & t \ge 0. \end{cases}$$

Moreover, if $\xi \in E_0^*(x)$, then $|e^{tH_p}(x,\xi)| \le C|\xi|$ for all t. This implies the following global dynamical properties of the flow e^{tH_p} on $\overline{T^*M}$:

- if $(x,\xi) \in \overline{T^*M} \setminus (E_0^* \oplus E_s^*)$, then as $t \to \infty$, $e^{tH_p}(x,\xi)$ converges to $\kappa(E_u^*)$ (in the topology of $\overline{T^*M}$) and $|e^{tH_p}(x,\xi)| \to \infty$ exponentially fast;
- if $(x,\xi) \in \overline{T^*M} \setminus (E_0^* \oplus E_u^*)$, then as $t \to -\infty$, $e^{tH_p}(x,\xi)$ converges to $\kappa(E_s^*)$ and $|e^{tH_p}(x,\xi)| \to \infty$ exponentially fast.

Indeed, to show for example the first statement we may write $\xi = \xi_0 + \xi_u + \xi_s$ where $\xi_0 \in E_0^*(x)$, $\xi_u \in E_u^*(x)$, $\xi_s \in E_s^*(x)$ and $\xi_u \ne 0$. Then as $t \to \infty$, $e^{tH_p}(x, \xi_0 + \xi_s)$ stays bounded while $e^{tH_p}(x,\xi_u)$ grows exponentially and thus is the dominant component of $e^{tH_p}(x,\xi)$.

The above statements are locally uniform in (x,ξ). They imply in particular that $\kappa(E_u^*)$ is a *radial sink* and $\kappa(E_s^*)$ is a *radial source* for the flow e^{tH_p} in the sense of [13, Definition E.50]. They also give the following statement about the flow on the characteristic set:

Lemma 3.3. *Fix arbitrary neighborhoods V_u, V_s, V_0 of $\kappa(E_u^*)$, $\kappa(E_s^*)$, and the zero section in $\overline{T^*M}$. Let $(x,\xi)\in\{p=0\}\subset\overline{T^*M}$. Then:*

- *if $(x,\xi)\notin\kappa(E_s^*)$, then there exists $t\geq 0$ such that $e^{tH_p}(x,\xi)\in V_u\cup V_0$;*
- *if $(x,\xi)\notin\kappa(E_u^*)$, then there exists $t\leq 0$ such that $e^{tH_p}(x,\xi)\in V_s\cup V_0$.*

Proof. We only show the first statement. If $(x,\xi)\notin E_s^*$ then, since $(x,\xi)\in\{p=0\}$, we have $(x,\xi)\notin E_0^*\oplus E_s^*$, so $e^{tH_p}(x,\xi)$ converges to $\kappa(E_u^*)$ as $t\to\infty$. Thus $e^{tH_p}(x,\xi)\in V_u$ for $t\geq 0$ large enough. Now, if $(x,\xi)\in E_s^*$ and ξ is finite, then $e^{tH_p}(x,\xi)$ converges to the zero section as $t\to\infty$. Thus $e^{tH_p}(x,\xi)\in V_0$ for $t\geq 0$ large enough. $\square$

3.2.2. *Weight functions.* The dynamical properties of the flow e^{tH_p} discussed in §3.2.1 make it possible to construct weight functions decaying along this flow, which are used later to define the anisotropic Sobolev spaces:

Lemma 3.4. *Fix some real numbers $m_u\leq m_0\leq m_s$ and conic neighborhoods $V_u, V_s\subset T^*M\setminus 0$ of E_u^*, E_s^*. Then there exists a function $\mathfrak{m}\in C^\infty(T^*M\setminus 0;\mathbb{R})$ such that:*

- *$\mathfrak{m}(x,\xi)$ is positively homogeneous of degree 0 in ξ;*
- *$m_u\leq\mathfrak{m}\leq m_s$ everywhere;*
- *$\mathfrak{m}=m_u$ in some conic neighborhood of E_u^*;*
- *$\mathfrak{m}=m_s$ in some conic neighborhood of E_s^*;*
- *$\mathfrak{m}=m_0$ outside of $V_u\cup V_s$;*
- *$H_p\mathfrak{m}\leq 0$ everywhere.*

Remark 3.5. A more refined version of Lemma 3.4 can be found in [15, Lemma 1.2]. In the present paper we do not use that $\mathfrak{m}=m_0$ outside of $V_u\cup V_s$, but it is a convenient property to have for wavefront set analysis, see [15, Theorem 1.7].

Proof. A positively homogeneous function of degree 0 on $T^*M\setminus 0$ is the pullback by κ of a function on the fiber infinity $\partial\overline{T^*M}$, and V_u, V_s are preimages by κ of some neighborhoods of $\kappa(E_u^*),\kappa(E_s^*)$. Moreover, the flow e^{tH_p} commutes with κ. Thus we will construct $\mathfrak{m}$ as a function on $\partial\overline{T^*M}$, consider V_u, V_s as open subsets of $\partial\overline{T^*M}$, and work with the flow e^{tH_p} restricted to $\partial\overline{T^*M}$.

We now construct dynamically adapted cutoffs on V_u, V_s following a standard argument presented for example in [13, Lemma E.53]. We shrink V_u if necessary so that it does not intersect $\kappa(E_0^*\oplus E_s^*)$. Take $\psi_u\in C_c^\infty(V_u;[0,1])$ such that $\psi_u=1$ near $\kappa(E_u^*)$. Since $\kappa(E_u^*)$ is a radial sink for the flow e^{tH_p}, there exists $T>0$ such that

$$e^{tH_p}(\operatorname{supp}\psi_u)\subset\{\psi_u=1\}\quad\text{for all}\quad t\geq T. \tag{3.7}$$

Put

$$\chi_u:=\frac{1}{T}\int_T^{2T}\psi_u\circ e^{-tH_p}\,dt\ \in\ C^\infty(\partial\overline{T^*M};[0,1]).$$

Then $\operatorname{supp}\chi_u \subset V_u$ (as follows from (3.7)) and $\chi_u = 1$ near $\kappa(E_u^*)$. Moreover

$$H_p\chi_u = -\frac{1}{T}\int_T^{2T} \partial_t(\psi_u \circ e^{-tH_p})\, dt = \frac{\psi_u \circ e^{-TH_p} - \psi_u \circ e^{-2TH_p}}{T} \geq 0$$

where we again use (3.7): for each $(x,\xi) \in \partial\overline{T^*M}$ we have $\psi_u(e^{-2TH_p}(x,\xi)) = 0$ or $\psi_u(e^{-TH_p}(x,\xi)) = 1$.

A similar argument gives a function $\chi_s \in C_c^\infty(V_s;[0,1])$ such that $\chi_s = 1$ near $\kappa(E_s^*)$ and $H_p\chi_s \leq 0$ everywhere. It remains to put

$$\mathfrak{m} := (m_u - m_0)\chi_u + (m_s - m_0)\chi_s + m_0. \quad \square$$

3.2.3. *Computing the adjoint-commutator.* We now give the following lemma which computes the expression in the positive commutator argument for the radial estimates in §4.2.3 below:

Lemma 3.6. *Assume that* $\mathbf{W} \in \Psi_h^{2m}(M;\operatorname{End}(\mathcal{E}))$ *and* $\mathbf{W}^* = \mathbf{W}$. *Then there exists* $\mathbf{Z} \in \Psi_h^{2m}(M;\operatorname{End}(\mathcal{E}))$ *such that* $\mathbf{Z}^* = \mathbf{Z}$, $\operatorname{WF}_h(\mathbf{Z}) \subset \operatorname{WF}_h(\mathbf{W})$, *and for each* $\lambda \in \mathbb{C}$ *and* $\mathbf{u} \in C^\infty(M;\operatorname{End}(\mathcal{E}))$

$$\operatorname{Im}\langle(\mathbf{P} - ih\lambda)\mathbf{u}, \mathbf{W}\mathbf{u}\rangle_{L^2(M;\mathcal{E})} = h\big\langle(\mathbf{Z} - (\operatorname{Re}\lambda)\mathbf{W})\mathbf{u}, \mathbf{u}\big\rangle_{L^2(M;\mathcal{E})}. \tag{3.8}$$

Moreover, the semiclassical principal symbol of $\mathbf{Z}$ *is given by*

$$\sigma_h(\mathbf{Z}) = \frac{1}{2}H_{\mathbf{X}}\sigma_h(\mathbf{W}) \tag{3.9}$$

where $H_{\mathbf{X}} : C^\infty(T^*M;\operatorname{End}(\pi^*\mathcal{E})) \to C^\infty(T^*M;\operatorname{End}(\pi^*\mathcal{E}))$ *is a lift of the vector field* H_p *(see Definition 2.7 which can be applied to any vector field).*

Finally, the evolution group $e^{tH_{\mathbf{X}}}$ *is described in terms of the parallel transport from* (2.8)*:*

$$e^{tH_{\mathbf{X}}}\mathbf{w}(x,\xi) = |\det d\varphi^t(x)|(\mathscr{T}_{\mathbf{X}}^t(x))^*\mathbf{w}(e^{tH_p}(x,\xi))\mathscr{T}_{\mathbf{X}}^t(x) \tag{3.10}$$

for all $\mathbf{w} \in C^\infty(T^*M;\operatorname{End}(\pi^*\mathcal{E}))$. *Here the adjoint is taken with respect to the inner product* $\langle\bullet,\bullet\rangle_{\mathcal{E}}$ *on* $\mathcal{E}$ *and the determinant is taken with respect to the density* ρ_0 *fixed in the beginning of §3.*

Proof. 1. A direct computation shows that (3.8) holds with

$$\mathbf{Z} := \frac{i}{2h}(\mathbf{P}^*\mathbf{W} - \mathbf{W}\mathbf{P}).$$

Since $\mathbf{P} \in \Psi_h^1(M;\operatorname{End}(\mathcal{E}))$ is principally scalar with real-valued principal symbol, the principal symbol of $\mathbf{P}^*\mathbf{W} - \mathbf{W}\mathbf{P}$ is equal to 0. Thus $\mathbf{Z} \in \Psi_h^{2m}(M;\operatorname{End}(\mathcal{E}))$. From the definition of $\mathbf{Z}$ we see also that $\mathbf{Z}^* = \mathbf{Z}$ and $\operatorname{WF}_h(\mathbf{Z}) \subset \operatorname{WF}_h(\mathbf{W})$.

2. Fix a frame $\mathbf{e}_1,\dots,\mathbf{e}_n$ on $\mathcal{E}$ over some open set $U \subset M$ which is orthonormal with respect to the inner product $\langle\bullet,\bullet\rangle_{\mathcal{E}}$. The operator $\mathbf{X}$ is given by (2.6) for some matrix $A(x) = (A_{jk}(x))_{j,k=1}^n$ depending on $x \in U$, so the operator $\mathbf{P}$ is given by

$$\mathbf{P}\sum_{j=1}^n f_j\mathbf{e}_j = -ih\sum_{j=1}^n\Big(Xf_j + \sum_{k=1}^n A_{jk}f_k\Big)\mathbf{e}_j.$$

Denoting by $\operatorname{div}_{\rho_0} X := \rho_0^{-1}\mathcal{L}_X\rho_0$ the divergence of the vector field X with respect to the density ρ_0, we compute the adjoint operator:

$$\mathbf{P}^*\sum_{j=1}^n f_j\mathbf{e}_j = -ih\sum_{j=1}^n\Big((X+\operatorname{div}_{\rho_0}X)f_j - \sum_{k=1}^n \overline{A_{kj}}f_k\Big)\mathbf{e}_j.$$

Using this we see that (3.9) holds with

$$H_{\mathbf{X}}\mathbf{w}(x,\xi) = H_p\mathbf{w}(x,\xi) + \operatorname{div}_{\rho_0}X(x)\mathbf{w}(x,\xi) - A(x)^*\mathbf{w}(x,\xi) - \mathbf{w}(x,\xi)A(x) \tag{3.11}$$

where we identify sections of $\operatorname{End}(\pi^*\mathcal{E})$ with $n\times n$ matrices using the frame $\mathbf{e}_1,\dots,\mathbf{e}_n$ and H_p on the right-hand side acts on each matrix entry separately.

3. The operator defined by (3.10) forms a group in t, so it suffices to check that for each $\mathbf{w}\in C^\infty(T^*M;\operatorname{End}(\pi^*\mathcal{E}))$ we have

$$\partial_t|_{t=0}\big(|\det d\varphi^t(x)|(\mathscr{T}^t_{\mathbf{X}}(x))^*\mathbf{w}(e^{tH_p}(x,\xi))\mathscr{T}^t_{\mathbf{X}}(x)\big) = H_{\mathbf{X}}\mathbf{w}(x,\xi). \tag{3.12}$$

We argue in a local frame as in Step 2 above. Using this frame we view $\mathscr{T}^t_{\mathbf{X}}(x)$ as an $n\times n$ matrix. Using the definition of parallel transport (see (2.8)) and the formula (2.7) we see that

$$\partial_t|_{t=0}\mathscr{T}^t_{\mathbf{X}}(x) = -A(x).$$

We also have $\partial_t|_{t=0}\det d\varphi^t(x) = \operatorname{div}_{\rho_0}X(x)$. Using these two identities and (3.11), we verify that (3.12) holds. $\square$

3.3. **The threshold conditions and existence of multipliers.** We now introduce the threshold regularity conditions needed for the proof of the Fredholm property of $\mathbf{P}-ih\lambda$; more specifically, they are used in the proofs of the radial estimates in §4.2.3 below. We start with the following

Definition 3.7. Assume that $m_u\le 0\le m_s$ are given constants. Define the *growth factors* $r_u(m_u), r_s(m_s)\in\mathbb{R}$ as the smallest numbers such that for each $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that for all $x\in M$ and $t\ge 0$

$$\begin{aligned}|\det d\varphi^t(x)|^{\frac12}\cdot\|\mathscr{T}^t_{\mathbf{X}}(x)\|\cdot\big\|d\varphi^t(x)^T|_{E_u^*}\big\|^{-m_u} &\le C_\varepsilon e^{(r_u(m_u)+\varepsilon)t},\\ |\det d\varphi^t(x)|^{\frac12}\cdot\|\mathscr{T}^t_{\mathbf{X}}(x)\|\cdot\big\|d\varphi^t(x)^{-T}|_{E_s^*}\big\|^{m_s} &\le C_\varepsilon e^{(r_s(m_s)+\varepsilon)t}.\end{aligned} \tag{3.13}$$

Remark 3.8. The constants $r_u(m_u), r_s(m_s)$ do not depend on the choice of the inner product on $\mathcal{E}$, the metric on M, and the density ρ_0 used to define the norms in (3.13).

Remark 3.9. The bounds (3.13) can be reformulated in terms of the action of $d\varphi^t$ on the spaces E_u, E_s: for all $x\in M$ and $t\ge 0$

$$\begin{aligned}|\det d\varphi^t(x)|^{\frac12}\cdot\|\mathscr{T}^t_{\mathbf{X}}(x)\|\cdot\big\|d\varphi^t(x)|_{E_s}\big\|^{-m_u} &\le C_\varepsilon e^{(r_u(m_u)+\varepsilon)t},\\ |\det d\varphi^t(x)|^{\frac12}\cdot\|\mathscr{T}^t_{\mathbf{X}}(x)\|\cdot\big\|d\varphi^t(x)^{-1}|_{E_u}\big\|^{m_s} &\le C_\varepsilon e^{(r_s(m_s)+\varepsilon)t}.\end{aligned} \tag{3.14}$$

We see that

$$r_u(m_u)\le C_1+\theta m_u,\quad r_s(m_s)\le C_1-\theta m_s \tag{3.15}$$

for some constant C_1 depending only on the lift $\mathbf{X}$, where $\theta>0$ is the constant in the exponential contraction property (2.2).

The next lemma introduces the threshold regularity conditions and constructs the multipliers used in the proofs of the radial estimates:

Lemma 3.10. *Assume that $m_u \le 0 \le m_s$ and $\lambda \in \mathbb{C}$ satisfy the threshold condition*

$$r_u(m_u) < \operatorname{Re}\lambda, \qquad r_s(m_s) < \operatorname{Re}\lambda. \tag{3.16}$$

Then there exist $\mathbf{w}_u, \mathbf{w}_s \in C^\infty(T^*M \setminus 0; \operatorname{End}(\pi^*\mathcal{E}))$ such that:

- *$\mathbf{w}_u^* = \mathbf{w}_u$, $\mathbf{w}_s^* = \mathbf{w}_s$, and $\mathbf{w}_u, \mathbf{w}_s$ are positive definite everywhere;*
- *$\mathbf{w}_u$, $\mathbf{w}_s$ are positively homogeneous of degrees $2m_u, 2m_s$, that is for each $(x,\xi) \in T^*M \setminus 0$ and $\tau > 0$*

$$\mathbf{w}_u(x,\tau\xi) = \tau^{2m_u}\mathbf{w}_u(x,\xi), \qquad \mathbf{w}_s(x,\tau\xi) = \tau^{2m_s}\mathbf{w}_s(x,\xi);$$

- *if $H_{\mathbf{X}}$ is the operator defined in Lemma 3.6, then*

$$(H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_u(x,\xi), \qquad (H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_s(x,\xi)$$

  are self-adjoint, positively homogeneous of degrees $2m_u$, $2m_s$ respectively, and negative definite for all (x,ξ) in $E_u^* \setminus 0$ and $E_s^* \setminus 0$ respectively.

Proof. For two self-adjoint elements $\mathbf{a}, \mathbf{b} \in \operatorname{End}(\mathcal{E}(x))$, we write $\mathbf{a} < \mathbf{b}$ if $\mathbf{b} - \mathbf{a}$ is positive definite.

1. Fix a metric on M and define the sections $\mathbf{w}_u^0, \mathbf{w}_s^0 \in C^\infty(T^*M \setminus 0; \operatorname{End}(\pi^*\mathcal{E}))$ by

$$\mathbf{w}_u^0(x,\xi) := |\xi|^{2m_u} I_{\mathcal{E}(x)}, \quad \mathbf{w}_s^0(x,\xi) := |\xi|^{2m_s} I_{\mathcal{E}(x)}$$

where $I_{\mathcal{E}(x)}$ is the identity map in $\operatorname{End}(\mathcal{E}(x))$. We claim that under the threshold condition (3.16) we have for all $t > 0$ large enough

$$\begin{aligned} e^{tH_{\mathbf{X}}}\mathbf{w}_u^0(x,\xi) &< e^{2\operatorname{Re}\lambda t}\mathbf{w}_u^0(x,\xi) \quad \text{for all} \quad (x,\xi) \in E_u^* \setminus 0, \\ e^{tH_{\mathbf{X}}}\mathbf{w}_s^0(x,\xi) &< e^{2\operatorname{Re}\lambda t}\mathbf{w}_s^0(x,\xi) \quad \text{for all} \quad (x,\xi) \in E_s^* \setminus 0. \end{aligned} \tag{3.17}$$

We show the first statement in (3.17), with the second one proved similarly. Let $(x,\xi) \in E_u^* \setminus 0$ and $\mathbf{v} \in \mathcal{E}(x)$. Using the formula (3.10) for $e^{tH_{\mathbf{X}}}$ we compute

$$\begin{aligned} \langle e^{tH_{\mathbf{X}}}\mathbf{w}_u^0(x,\xi)\mathbf{v}, \mathbf{v}\rangle_{\mathcal{E}} &= |\det d\varphi^t(x)| \cdot |d\varphi^t(x)^{-T}\xi|^{2m_u} \cdot \|\mathscr{T}_{\mathbf{X}}^t(x)\mathbf{v}\|^2_{\mathcal{E}(\varphi^t(x))}, \\ \langle \mathbf{w}_u^0(x,\xi)\mathbf{v}, \mathbf{v}\rangle_{\mathcal{E}} &= |\xi|^{2m_u}\|\mathbf{v}\|^2_{\mathcal{E}(x)}. \end{aligned}$$

We have $m_u \le 0$ and $|d\varphi^t(x)^{-T}\xi| \ge \|d\varphi^t(x)^T|_{E_u^*}\|^{-1} \cdot |\xi|$, so

$$|d\varphi^t(x)^{-T}\xi|^{2m_u} \le \|d\varphi^t(x)^T|_{E_u^*}\|^{-2m_u} \cdot |\xi|^{2m_u}.$$

Now (3.17) follows from the bound

$$|\det d\varphi^t(x)| \cdot \|d\varphi^t(x)^T|_{E_u^*}\|^{-2m_u}\|\mathscr{T}_{\mathbf{X}}^t(x)\|^2 < e^{2\operatorname{Re}\lambda t}$$

which holds for $t > 0$ large enough by (3.16) since the left-hand side is $\mathcal{O}_\varepsilon(e^{(2r_u(m_u)+\varepsilon)t})$ for any $\varepsilon > 0$.

2. Fix $t_0 > 0$ such that (3.17) holds with $t := t_0$. We define

$$\mathbf{w}_u := \int_0^{t_0} e^{-2\operatorname{Re}\lambda t}e^{tH_{\mathbf{X}}}\mathbf{w}_u^0\,dt, \qquad \mathbf{w}_s := \int_0^{t_0} e^{-2\operatorname{Re}\lambda t}e^{tH_{\mathbf{X}}}\mathbf{w}_s^0\,dt.$$

It is straightforward to check using (3.10) that $\mathbf{w}_u, \mathbf{w}_s$ are self-adjoint, positively homogeneous of degrees $2m_u, 2m_s$ respectively, and positive definite. Since $e^{tH_{\mathbf{X}}}$ is the evolution group associated to $H_{\mathbf{X}}$, we have

$$(H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_u = \int_0^{t_0} \partial_t\big(e^{-2\operatorname{Re}\lambda t} e^{tH_{\mathbf{X}}}\mathbf{w}_u^0\big)\,dt = e^{-2\operatorname{Re}\lambda t_0} e^{t_0 H_{\mathbf{X}}}\mathbf{w}_u^0 - \mathbf{w}_u^0$$

and similarly $(H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_s = e^{-2\operatorname{Re}\lambda t_0}e^{t_0H_{\mathbf{X}}}\mathbf{w}_s^0 - \mathbf{w}_s^0$. We see that $(H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_u$, $(H_{\mathbf{X}} - 2\operatorname{Re}\lambda)\mathbf{w}_s$ are self-adjoint and positively homogeneous of degrees $2m_u$, $2m_s$ respectively. Moreover, by (3.17) these sections are negative definite on $E_u^* \setminus 0, E_s^* \setminus 0$ respectively. □

3.3.1. *Examples.* We now compute the growth factors $r_u(m_u)$, $r_s(m_s)$ from Definition 3.7 in a couple of special cases of the examples considered in §2.3.2. More precisely, we study the threshold regularity condition $\operatorname{Re}\lambda > \max(r_u(m_u), r_s(m_s))$ given in (3.16).

We start with the basic case when $\mathcal{E} = M \times \mathbb{C}$ is trivial, $\mathbf{X} = X$, and φ^t is volume preserving. In this case the condition (3.16) becomes

$$\operatorname{Re}\lambda > \max(\theta_s m_u, -\theta_u m_s) \tag{3.18}$$

where $\theta_s, \theta_u > 0$ are the largest numbers such that for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $t \geq 0$

$$\|d\varphi^t|_{E_s}\| \leq C_\varepsilon e^{-(\theta_s - \varepsilon)t}, \qquad \|d\varphi^{-t}|_{E_u}\| \leq C_\varepsilon e^{-(\theta_u-\varepsilon)t}.$$

We next discuss the case when X is the generator of the geodesic flow on an $n+1$-dimensional compact hyperbolic manifold (Σ, g) and $\mathbf{X} = \mathcal{L}_X$ acts on sections of the bundle of perpendicular differential k-forms Ω_0^k. In this case φ^t is volume preserving, $\dim E_u = \dim E_s = n$, and for the correct choice of metric on M (the Sasaki metric) we have

$$|d\varphi^t(x)v| = \begin{cases} |v|, & v \in E_0(x); \\ e^t|v|, & v \in E_u(x); \\ e^{-t}|v|, & v \in E_s(x). \end{cases}$$

It follows that the parallel transport $\mathscr{T}_{\mathbf{X}}^t(x)$ has norm $e^{\min(k,2n-k)t}$ for $t \geq 0$, and the condition (3.16) becomes

$$\operatorname{Re}\lambda > \max(m_u, -m_s) + \min(k, 2n-k). \tag{3.19}$$

4. Meromorphic continuation

In this section we state and prove the main result of this paper, Theorem 4.1 (see §4.1.1).

4.1. **Anisotropic Sobolev spaces and statement of the result.** We first introduce the spaces on which meromorphic continuation holds. We fix a function

$$\mathfrak{m} \in C^\infty(T^*M \setminus 0; \mathbb{R})$$

which satisfies the following conditions:

- $\mathfrak{m}$ is positively homogeneous of degree 0, that is $\mathfrak{m}(x, \tau\xi) = \mathfrak{m}(x, \xi)$ for all $(x, \xi) \in T^*M \setminus 0$ and $\tau > 0$;

- there exist constants $m_u \le 0 \le m_s$ such that $m_u \le \mathfrak{m} \le m_s$ everywhere and
$$\mathfrak{m} = m_u \quad \text{near } E_u^* \setminus 0, \qquad \mathfrak{m} = m_s \quad \text{near } E_s^* \setminus 0$$
where the dual unstable/stable spaces $E_u^*, E_s^* \subset T^*M$ were introduced in (3.6);
- $H_p\mathfrak{m} \le 0$ everywhere, where the vector field H_p is introduced in §3.2.1; equivalently, $\mathfrak{m}(\varphi^t(x), d\varphi^t(x)^{-T}\xi) \le \mathfrak{m}(x,\xi)$ for all $(x,\xi) \in T^*M \setminus 0$ and $t \ge 0$.

Such $\mathfrak{m}$ exists for any choice of $m_u \le 0 \le m_s$ by Lemma 3.4.

Given $\mathfrak{m}$, we fix a semiclassical pseudodifferential operator $\mathbf{F}_\mathfrak{m}$ such that:

- $\mathbf{F}_\mathfrak{m}$ lies in $\Psi_h^{0+}(M;\mathrm{End}(\mathcal{E})) := \bigcap_{\varepsilon>0} \Psi_h^\varepsilon(M;\mathrm{End}(\mathcal{E}))$ and $\mathbf{F}_\mathfrak{m}^* = \mathbf{F}_\mathfrak{m}$;
- $\mathbf{F}_\mathfrak{m}$ is principally scalar and, for some fixed choice of Riemannian metric on M,
$$\sigma_h(\mathbf{F}_\mathfrak{m})(x,\xi) = \mathfrak{m}(x,\xi)\log|\xi| \quad \text{when} \quad |\xi| \ge 1.$$

For $t \ge 0$ we can define the exponential operators

$$e^{t\mathbf{F}_\mathfrak{m}} \in \Psi_h^{tm_s+}(M;\mathrm{End}(\mathcal{E})), \qquad e^{-t\mathbf{F}_\mathfrak{m}} \in \Psi_h^{-tm_u+}(M;\mathrm{End}(\mathcal{E})). \tag{4.1}$$

See [28, Theorem 8.6] for the case of scalar operators and Weyl quantization on $\mathbb{R}^n$ (with Beals's theorem for the Kohn–Nirenberg calculus given in [28, Theorem 9.12]); the proof adapts to the case of manifolds and vector bundles studied here. Alternatively, see [14, Appendix A].

We now define the semiclassical *anisotropic Sobolev space* $H_h^\mathfrak{m}(M;\mathcal{E})$ similarly to [28, §8.3.1]:

$$H_h^\mathfrak{m}(M;\mathcal{E}) := e^{-\mathbf{F}_\mathfrak{m}}L^2(M;\mathcal{E}), \qquad \|\mathbf{u}\|_{H_h^\mathfrak{m}} := \|e^{\mathbf{F}_\mathfrak{m}}\mathbf{u}\|_{L^2}.$$

The spaces $H_h^\mathfrak{m}(M;\mathcal{E})$ for different values of h are all equivalent, with constants in the norm equivalency bounds depending on h. Therefore, we may use the notation $H^\mathfrak{m}(M;\mathcal{E})$ when the choice of norm is not important. We have

$$H^{m_s}(M;\mathcal{E}) \subset H^\mathfrak{m}(M;\mathcal{E}) \subset H^{m_u}(M;\mathcal{E}) \tag{4.2}$$

and the space $C^\infty(M;\mathcal{E})$ is dense in $H^\mathfrak{m}(M;\mathcal{E})$.

Fix open subsets

$$\widetilde{V}_u, \widetilde{V}_s \subset \overline{T^*M} \setminus 0, \quad \kappa(E_u^*) \subset \widetilde{V}_u, \quad \kappa(E_s^*) \subset \widetilde{V}_s, \tag{4.3}$$

such that $\mathfrak{m} = m_u$ on $\widetilde{V}_u$ and $\mathfrak{m} = m_s$ on $\widetilde{V}_s$. Then the space $H_h^\mathfrak{m}(M;\mathcal{E})$ is equivalent to the usual Sobolev space $H_h^{m_u}(M;\mathcal{E})$ microlocally on $\widetilde{V}_u$, that is for each $\mathbf{A} \in \Psi_h^0(M;\mathrm{End}(\mathcal{E}))$ with $\mathrm{WF}_h(\mathbf{A}) \subset \widetilde{V}_u$, there exists a constant C such that for each $\mathbf{u} \in C^\infty(M;\mathcal{E})$ and each N

$$\begin{aligned} \|\mathbf{Au}\|_{H_h^\mathfrak{m}} &\le C\|\mathbf{u}\|_{H_h^{m_u}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}, \\ \|\mathbf{Au}\|_{H_h^{m_u}} &\le C\|\mathbf{u}\|_{H_h^\mathfrak{m}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \end{aligned} \tag{4.4}$$

Similarly, $H_h^\mathfrak{m}(M;\mathcal{E})$ is equivalent to the space $H^{m_s}(M;\mathcal{E})$ microlocally on $\widetilde{V}_s$.

4.1.1. *Statement of the result.* We can now state the main result of this paper, which gives meromorphic continuation of the Pollicott–Ruelle resolvent on anisotropic Sobolev spaces to a specific half-plane:

Theorem 4.1. *Let X be the generator of an Anosov flow φ^t on a compact manifold M, $\mathcal{E}$ be a smooth vector bundle over M, and $\mathbf{X} : C^\infty(M;\mathcal{E}) \to C^\infty(M;\mathcal{E})$ be a lift of X (see Definition 2.7).*

Assume that the function $\mathfrak{m} \in C^\infty(T^*M \setminus 0;\mathbb{R})$ satisfies the conditions in the beginning of §4.1, for some constants $m_u \le 0 \le m_s$. Let $H^{\mathfrak{m}}(M;\mathcal{E})$ be the corresponding anisotropic Sobolev space.

Then the Pollicott–Ruelle resolvent $R_{\mathbf{X}}(\lambda)$ defined in (2.10) admits a meromorphic continuation as a family of operators $H^{\mathfrak{m}}(M;\mathcal{E}) \to H^{\mathfrak{m}}(M;\mathcal{E})$ to the half-plane

$$\operatorname{Re}\lambda > \max(r_u(m_u), r_s(m_s)) \tag{4.5}$$

where $r_u(m_u), r_s(m_s)$ were introduced in Definition 3.7.

Remark 4.2. By (3.15), if we fix λ then for $-m_u, m_s$ large enough the condition (4.5) holds. Since $C^\infty(M;\mathcal{E}) \subset H^{\mathfrak{m}}(M;\mathcal{E}) \subset \mathcal{D}'(M;\mathcal{E})$, we see that $R_{\mathbf{X}}(\lambda)$ continues meromorphically as a family of operators $C^\infty(M;\mathcal{E}) \to \mathcal{D}'(M;\mathcal{E})$ to $\lambda \in \mathbb{C}$.

4.1.2. *The conjugated operator.* The action of $\mathbf{P} = -ih\mathbf{X}$ on $H^{\mathfrak{m}}_h(M;\mathcal{E})$ is equivalent to the action on $L^2(M;\mathcal{E})$ of the conjugated operator

$$\widetilde{\mathbf{P}} := e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{P}e^{-\mathbf{F}_{\mathfrak{m}}}. \tag{4.6}$$

Using Taylor's formula with integral remainder for the family of operators $e^{t\mathbf{F}_{\mathfrak{m}}}\mathbf{P}e^{-t\mathbf{F}_{\mathfrak{m}}}$, $t \in [0,1]$, we see that for any $N \in \mathbb{N}$, we can expand $\widetilde{\mathbf{P}}$ as follows:

$$\widetilde{\mathbf{P}} = \sum_{j=0}^{N-1} \frac{\operatorname{ad}^j_{\mathbf{F}_{\mathfrak{m}}}\mathbf{P}}{j!} + \int_0^1 (1-t)^{N-1} e^{t\mathbf{F}_{\mathfrak{m}}} \frac{\operatorname{ad}^N_{\mathbf{F}_{\mathfrak{m}}}\mathbf{P}}{(N-1)!} e^{-t\mathbf{F}_{\mathfrak{m}}}\,dt \tag{4.7}$$

where $\operatorname{ad}_{\mathbf{F}_{\mathfrak{m}}}\mathbf{A} = [\mathbf{F}_{\mathfrak{m}}, \mathbf{A}]$ for any operator $\mathbf{A}$ on $C^\infty(M;\mathcal{E})$.

Since $\mathbf{F}_{\mathfrak{m}} \in \Psi^{0+}_h(M;\operatorname{End}(\mathcal{E}))$ is principally scalar, we have $\operatorname{ad}_{\mathbf{F}_{\mathfrak{m}}} : \Psi^{m+}_h(M;\operatorname{End}(\mathcal{E})) \to h\Psi^{m-1+}_h(M;\operatorname{End}(\mathcal{E}))$ for all m. Therefore, the j-th term in the sum in (4.7) is in $h^j\Psi^{1-j+}_h$; using (4.1), we see that the remainder is in $h^N\Psi^{1-N+m_s-m_u+}_h$. Since N can be chosen arbitrarily large, we in particular get the expansion

$$\widetilde{\mathbf{P}} = \mathbf{P} + [\mathbf{F}_{\mathfrak{m}}, \mathbf{P}] + \mathcal{O}(h^2)_{\Psi^{-1+}_h(M;\operatorname{End}(\mathcal{E}))}.$$

It follows that $\widetilde{\mathbf{P}}$ lies in $\Psi^1_h(M;\operatorname{End}(\mathcal{E}))$ and is principally scalar with

$$\sigma_h(\widetilde{\mathbf{P}}) = p + ih(H_p\mathfrak{m})\log|\xi|, \tag{4.8}$$

where we used that $H_p \log|\xi| \in S^0$ for $|\xi| \ge 1$.

An expansion of the form (4.7) is valid for any pseudodifferential operator in place of $\mathbf{P}$. In particular, we get

$$\mathbf{A} \in \Psi^0_h(M;\operatorname{End}(\mathcal{E})) \implies e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{A}e^{-\mathbf{F}_{\mathfrak{m}}} \in \Psi^0_h(M;\operatorname{End}(\mathcal{E})) \tag{4.9}$$

and the wavefrontset / elliptic set of $\mathbf{A}$ coincide with those of $e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{A}e^{-\mathbf{F}_{\mathfrak{m}}}$.

4.2. **Invertibility of the perturbed operator.** We now state the key estimate for the proof of Theorem 4.1, which gives invertibility for the operator $\mathbf{P} = -ih\mathbf{X}$ on the anisotropic Sobolev space $H_h^{\mathfrak{m}}(M;\mathcal{E})$ when modified by a complex absorbing operator. Consider the dual space of $H_h^{\mathfrak{m}}(M;\mathcal{E})$ (with respect to the L^2 inner product), given by

$$H_h^{-\mathfrak{m}}(M;\mathcal{E}) := e^{\mathbf{F}_{\mathfrak{m}}} L^2(M;\mathcal{E}).$$

Fix a principally scalar pseudodifferential operator

$$\mathbf{Q} \in \Psi_h^0(M;\operatorname{End}(\mathcal{E})), \quad \sigma_h(\mathbf{Q}) \geq 0$$

such that $\operatorname{WF}_h(\mathbf{Q})$ does not intersect the fiber infinity $\partial\overline{T^*M}$ and the elliptic set $\operatorname{ell}_h(\mathbf{Q})$ contains the zero section of T^*M. For technical reasons we also assume that

$$\operatorname{WF}_h(\mathbf{Q}) \cap \widetilde{V}_u = \operatorname{WF}_h(\mathbf{Q}) \cap \widetilde{V}_s = \emptyset \tag{4.10}$$

where $\widetilde{V}_u, \widetilde{V}_s \subset \overline{T^*M} \setminus 0$ were introduced in (4.3).

Lemma 4.3. *Let $\mathfrak{m}$ satisfy the conditions in the beginning of §4.1 and assume that $\Omega \subset \mathbb{C}$ is a compact set such that*

$$\operatorname{Re}\lambda > \max(r_u(m_u), r_s(m_s)) \quad \textit{for all} \quad \lambda \in \Omega. \tag{4.11}$$

Then we have the following estimates for h small enough, all $\lambda \in \Omega$, and all $\mathbf{u} \in C^\infty(M;\mathcal{E})$, with the constants independent of $h, \lambda, \mathbf{u}$:

$$\|\mathbf{u}\|_{H_h^{\mathfrak{m}}} \leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathfrak{m}}}, \tag{4.12}$$

$$\|\mathbf{u}\|_{H_h^{-\mathfrak{m}}} \leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})^*\mathbf{u}\|_{H_h^{-\mathfrak{m}}}. \tag{4.13}$$

We will only give the proof of the direct estimate (4.12). The adjoint estimate (4.13) follows from the direct estimate for the operator $\mathbf{X}^*$ which is a lift of the vector field $-X$. Note that $-(\mathbf{P} - ih\lambda - i\mathbf{Q})^* = -ih\mathbf{X}^* - ih\bar{\lambda} - i\mathbf{Q}^*$. The associated flow is φ^{-t} and the stable/unstable spaces are switched places. The constants m_u, m_s are replaced by $-m_s, -m_u$ and the weight $\mathfrak{m}$ is replaced by $-\mathfrak{m}$. Using (3.13), we see that the threshold condition (4.11) gives the analogous condition for the operator $\mathbf{X}^*$. (Here the parallel transport corresponding to $\mathbf{X}^*$ can be computed using (2.9), as $(e^{-t\mathbf{X}})^* = e^{-t\mathbf{X}^*}$.)

The proof of (4.12) is broken into several components. Throughout this section we assume that h is small, $\lambda \in \Omega$, and $\mathbf{u} \in C^\infty(M;\mathcal{E})$. The constants in the estimates below are independent of h, and the Sobolev exponent N in the remainders can be chosen arbitrarily.

4.2.1. *Elliptic estimate.* We first state the elliptic estimate:

Lemma 4.4. *Assume that $\mathbf{A} \in \Psi_h^0(M;\operatorname{End}(\mathcal{E}))$ and*

$$\operatorname{WF}_h(\mathbf{A}) \ \subset \ \operatorname{ell}_h(\mathbf{P}) \cup \operatorname{ell}_h(\mathbf{Q}).$$

Then

$$\|\mathbf{A}\mathbf{u}\|_{H_h^{\mathfrak{m}}} \leq C\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathfrak{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.14}$$

To prove Lemma 4.4, we first reduce it to an estimate in the space L^2 for the conjugated operator $\widetilde{\mathbf{P}} - ih\lambda - \widetilde{\mathbf{Q}}$ where

$$\widetilde{\mathbf{Q}} := e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{Q}e^{-\mathbf{F}_{\mathfrak{m}}}. \tag{4.15}$$

Denote $\widetilde{\mathbf{A}} := e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{A}e^{-\mathbf{F}_{\mathfrak{m}}}$. Then (4.14) follows from the estimate

$$\|\widetilde{\mathbf{A}}\mathbf{v}\|_{L^2} \leq C\|(\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}})\mathbf{v}\|_{L^2} + \mathcal{O}(h^\infty)\|\mathbf{v}\|_{H_h^{-N}} \tag{4.16}$$

where we put $\mathbf{v} := e^{\mathbf{F}_{\mathfrak{m}}}\mathbf{u} \in C^\infty(M;\mathcal{E})$.

Since $\mathrm{WF}_h(\mathbf{Q})$ does not intersect the fiber infinity $\partial\overline{T^*M}$, using the expansion (4.7) for $\mathbf{Q}$ in place of $\mathbf{P}$ we see that $\widetilde{\mathbf{Q}} = \mathbf{Q} + \mathcal{O}(h)_{\Psi_h^{-\infty}(M;\mathrm{End}(\mathcal{E}))}$. Moreover, by (4.9) the operator $\widetilde{\mathbf{A}} \in \Psi_h^0(M;\mathrm{End}(\mathcal{E}))$ has the same wavefront set as $\mathbf{A}$. It follows that $\mathrm{WF}_h(\widetilde{\mathbf{A}}) \subset \mathrm{ell}_h(\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}})$. Now (4.16) follows from the standard elliptic estimate [13, Theorem E.33] whose proof adapts directly to the case of operators on vector bundles.

4.2.2. *Propagation of singularities.* Our next estimate is propagation of singularities:

Lemma 4.5. *Assume that* $\mathbf{A},\mathbf{B},\mathbf{B}_1 \in \Psi_h^0(M;\mathrm{End}(\mathcal{E}))$ *and the following control condition holds:*

$$\text{for all } (x,\xi) \in \mathrm{WF}_h(\mathbf{A}) \quad \text{there exists } T \geq 0 \quad \text{such that}$$
$$e^{-TH_p}(x,\xi) \in \mathrm{ell}_h(\mathbf{B}) \quad \text{and} \quad e^{-tH_p}(x,\xi) \in \mathrm{ell}_h(\mathbf{B}_1) \quad \text{for all } t \in [0,T].$$

Then

$$\|\mathbf{A}\mathbf{u}\|_{H_h^{\mathfrak{m}}} \leq C\|\mathbf{B}\mathbf{u}\|_{H_h^{\mathfrak{m}}} + Ch^{-1}\|\mathbf{B}_1(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathfrak{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

Similarly to §4.2.1, Lemma 4.5 can be reduced to an estimate in the space L^2 for the conjugated operator $\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}}$. The latter estimate is proved using the same positive commutator estimate as standard scalar propagation of singularities [13, Theorem E.47], using a principally scalar multiplier $\mathbf{G}$, given that:

- $\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}} \in \Psi_h^1(M;\mathrm{End}(\mathcal{E}))$ is principally scalar;
- $\operatorname{Re}\sigma_h(\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}}) = p$;
- $\operatorname{Im}\sigma_h(\widetilde{\mathbf{P}} - ih\lambda - i\widetilde{\mathbf{Q}}) \leq 0$. Indeed, $\operatorname{Im}\sigma_h(\widetilde{\mathbf{P}}) = h(H_p\mathfrak{m})\log|\xi|$ by (4.8) and $H_p\mathfrak{m} \leq 0$ as required in the beginning of §4.1. Moreover, $\sigma_h(\widetilde{\mathbf{Q}}) = \sigma_h(\mathbf{Q}) \geq 0$;
- the sharp Gårding inequality applies to principally scalar operators in $\Psi_h^{2m}(M;\mathrm{End}(\mathcal{E}))$ with nonnegative principal symbol, as follows for example from Lemma 3.1.

4.2.3. *Radial estimates.* We now prove the two radial estimates that are crucial in the proof of Lemma 4.3. This is the place in the argument where the threshold regularity condition (4.11) is important. Recall the sets $\widetilde{V}_u, \widetilde{V}_s \subset \overline{T^*M}$ introduced in (4.3).

We start with the high regularity radial estimate at the set $\kappa(E_s^*) \subset \partial\overline{T^*M}$.

Lemma 4.6. *There exist operators*

$$\mathbf{A}_s, \mathbf{B}_{1,s} \in \Psi_h^0(M; \mathrm{End}(\mathcal{E})), \quad \kappa(E_s^*) \subset \mathrm{ell}_h(\mathbf{A}_s), \quad \mathrm{WF}_h(\mathbf{A}_s) \cup \mathrm{WF}_h(\mathbf{B}_{1,s}) \subset \widetilde{V}_s$$

such that the following estimate holds:

$$\|\mathbf{A}_s\mathbf{u}\|_{H_h^{\mathfrak{m}}} \leq Ch^{-1}\|\mathbf{B}_{1,s}(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathfrak{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.17}$$

Proof. 1. Since $H_h^{\mathfrak{m}}$ is equivalent to $H_h^{m_s}$ microlocally on $\widetilde{V}_s$ (see (4.4)) and $\mathrm{WF}_h(\mathbf{Q}) \cap \widetilde{V}_s = \emptyset$ (see (4.10)), it suffices to show the estimate

$$\|\mathbf{A}_s\mathbf{u}\|_{H_h^{m_s}} \leq Ch^{-1}\|\mathbf{B}_{1,s}(\mathbf{P} - ih\lambda)\mathbf{u}\|_{H_h^{m_s}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.18}$$

2. We now follow the proof of [13, Theorem E.52], indicating the necessary changes. Since the threshold condition (4.11) holds, Lemma 3.10 applies to give a section $\mathbf{w}_s \in C^\infty(T^*M \setminus 0; \mathrm{End}(\pi^*\mathcal{E}))$ which is positive definite everywhere, positively homogeneous of degree $2m_s$, and satisfies (where '< 0' means 'negative definite')

$$(H_{\mathbf{X}} - 2\,\mathrm{Re}\,\lambda)\mathbf{w}_s(x, \xi) < 0 \quad \text{for all} \quad \lambda \in \Omega, \quad (x, \xi) \in E_s^* \setminus 0. \tag{4.19}$$

Fix an open set $U_s \subset \widetilde{V}_s$ such that $\kappa(E_s^*) \subset U_s$ and there exists $\delta > 0$ such that

$$(\tfrac{1}{2}H_{\mathbf{X}} - \mathrm{Re}\,\lambda + \delta)\mathbf{w}_s(x, \xi) < 0 \quad \text{for all} \quad \lambda \in \Omega, \quad (x, \xi) \in U_s. \tag{4.20}$$

Arguing as in the proof of Lemma 3.4 (see also [13, Lemma E.53]), we construct a function

$$\chi_s \in C_{\mathrm{c}}^\infty(U_s; [0, 1]), \qquad \chi_s = 1 \quad \text{near} \quad \kappa(E_s^*), \qquad H_p\chi_s \leq 0. \tag{4.21}$$

Denote by $\sqrt{\mathbf{w}_s}$ the square root of $\mathbf{w}_s$, which is a positive definite section in $C^\infty(T^*M \setminus 0; \mathrm{End}(\pi^*\mathcal{E}))$ and positively homogeneous of degree m_s. Define

$$\mathbf{g}_s := \chi_s\sqrt{\mathbf{w}_s} \;\in\; C^\infty(T^*M; \mathrm{End}(\pi^*\mathcal{E}))$$

and note that $\mathbf{g}_s$ lies in the symbol class S^{m_s}.
3. Take a pseudodifferential operator

$$\mathbf{G}_s \in \Psi_h^{m_s}(M; \mathrm{End}(\mathcal{E})), \quad \mathrm{WF}_h(\mathbf{G}_s) \subset U_s, \quad \sigma_h(\mathbf{G}_s) = \mathbf{g}_s.$$

Note that $\mathbf{G}_s$ is elliptic on $\kappa(E_s^*)$. Fix also operators $\mathbf{A}_s, \mathbf{B}_{2,s} \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ such that

$$\kappa(E_s^*) \subset \mathrm{ell}_h(\mathbf{A}_s),\ \mathrm{WF}_h(\mathbf{A}_s) \subset \mathrm{ell}_h(\mathbf{G}_s),\ \mathrm{WF}_h(\mathbf{G}_s) \subset \mathrm{ell}_h(\mathbf{B}_{2,s}),\ \mathrm{WF}_h(\mathbf{B}_{2,s}) \subset U_s.$$

By Lemma 3.6, we have

$$h^{-1}\,\mathrm{Im}\langle(\mathbf{P} - ih\lambda)\mathbf{u}, \mathbf{G}_s^*\mathbf{G}_s\mathbf{u}\rangle_{L^2} + \delta\|\mathbf{G}_s\mathbf{u}\|_{L^2}^2 = \langle\mathbf{Z}_s\mathbf{u}, \mathbf{u}\rangle_{L^2} \tag{4.22}$$

where

$$\mathbf{Z}_s \in \Psi^{2m_s}(M; \mathrm{End}(\mathcal{E})), \quad \mathbf{Z}_s^* = \mathbf{Z}_s, \quad \mathrm{WF}_h(\mathbf{Z}_s) \subset \mathrm{ell}_h(\mathbf{B}_{2,s})$$

has principal symbol

$$\sigma_h(\mathbf{Z}_s) = (\tfrac{1}{2}H_{\mathbf{X}} - \mathrm{Re}\,\lambda + \delta)(\chi_s^2\mathbf{w}_s) = \chi_s(H_p\chi_s)\mathbf{w}_s + \chi_s^2(\tfrac{1}{2}H_{\mathbf{X}} - \mathrm{Re}\,\lambda + \delta)\mathbf{w}_s. \tag{4.23}$$

By (4.20)–(4.21), each of the two summands on the right-hand side of (4.23) is the product of a nonnegative function in $C_{\mathrm{c}}^\infty(U_s)$ and a self-adjoint section of $\mathrm{End}(\pi^*\mathcal{E})$

which is positively homogeneous of degree $2m_s$ and negative definite on U_s. Thus the version of the sharp Gårding inequality given in Lemma 3.1 gives

$$\langle \mathbf{Z}_s\mathbf{u},\mathbf{u}\rangle_{L^2}\leq Ch\|\mathbf{B}_{2,s}\mathbf{u}\|^2_{H_h^{m_s-\frac12}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}.$$

Together with (4.22) this implies

$$\|\mathbf{G}_s\mathbf{u}\|^2_{L^2}\leq Ch^{-1}\|\mathbf{B}_{2,s}(\mathbf{P}-ih\lambda)\mathbf{u}\|_{H_h^{m_s}}\cdot\|\mathbf{G}_s\mathbf{u}\|_{L^2}+Ch\|\mathbf{B}_{2,s}\mathbf{u}\|^2_{H_h^{m_s-\frac12}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}$$

which gives the estimate

(4.24)

$$\|\mathbf{G}_s\mathbf{u}\|_{L^2}\leq Ch^{-1}\|\mathbf{B}_{2,s}(\mathbf{P}-ih\lambda)\mathbf{u}\|_{H_h^{m_s}}+Ch^{\frac12}\|\mathbf{B}_{2,s}\mathbf{u}\|_{H_h^{m_s-\frac12}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

4. We now argue similarly to step 2 of the proof of [13, Theorem E.52]. By the elliptic estimate we can replace $\|\mathbf{G}_s\mathbf{u}\|_{L^2}$ on the left-hand side of (4.24) by $\|\mathbf{A}_s\mathbf{u}\|_{H_h^{m_s}}$. If the set U_s is chosen small enough, then propagation of singularities gives

(4.25)

$$\|\mathbf{B}_{2,s}\mathbf{u}\|_{H_h^{m_s-\frac12}}\leq C\|\mathbf{A}_s\mathbf{u}\|_{H_h^{m_s-\frac12}}+Ch^{-1}\|\mathbf{B}_{1,s}(\mathbf{P}-ih\lambda)\mathbf{u}\|_{H_h^{m_s}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}$$

for some $\mathbf{B}_{1,s}\in\Psi_h^0(M;\operatorname{End}(\mathcal{E}))$ such that

$$\operatorname{WF}_h(\mathbf{B}_{2,s})\subset\operatorname{ell}_h(\mathbf{B}_{1,s}),\quad \operatorname{WF}_h(\mathbf{B}_{1,s})\subset\widetilde{V}_s.$$

Combining (4.24) and (4.25) and taking h small enough, we get (4.18). □

We next give the low regularity radial estimate at the set $\kappa(E_u^*)$:

Lemma 4.7. *There exist operators*

$$\mathbf{A}_u,\mathbf{B}_u,\mathbf{B}_{1,u}\in\Psi_h^0(M;\operatorname{End}(\mathcal{E})),\quad \kappa(E_u^*)\subset\operatorname{ell}_h(\mathbf{A}_u),$$
$$\operatorname{WF}_h(\mathbf{A}_u)\cup\operatorname{WF}_h(\mathbf{B}_{1,u})\subset\widetilde{V}_u,\quad \operatorname{WF}_h(\mathbf{B}_u)\subset\widetilde{V}_u\setminus\kappa(E_u^*)$$

such that the following estimate holds:

(4.26) $$\|\mathbf{A}_u\mathbf{u}\|_{H_h^{\mathbf{m}}}\leq C\|\mathbf{B}_u\mathbf{u}\|_{H_h^{\mathbf{m}}}+Ch^{-1}\|\mathbf{B}_{1,u}(\mathbf{P}-ih\lambda-i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

Proof. 1. We argue similarly to the proof of [13, Theorem E.54], making changes similar to the proof of Lemma 4.6. Since $H_h^{\mathbf{m}}$ is equivalent to $H_h^{m_u}$ microlocally on $\widetilde{V}_u$, it suffices to show the estimate

(4.27) $$\|\mathbf{A}_u\mathbf{u}\|_{H_h^{m_u}}\leq C\|\mathbf{B}_u\mathbf{u}\|_{H_h^{m_u}}+Ch^{-1}\|\mathbf{B}_{1,u}(\mathbf{P}-ih\lambda)\mathbf{u}\|_{H_h^{m_u}}+\mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.$$

2. Since the threshold condition (4.11) holds, Lemma 3.10 applies to give a section $\mathbf{w}_u\in C^\infty(T^*M\setminus 0;\operatorname{End}(\pi^*\mathcal{E}))$ which is positive definite everywhere, positively homogeneous of degree $2m_u$, and satisfies

$$(H_{\mathbf{X}}-2\operatorname{Re}\lambda)\mathbf{w}_u(x,\xi)<0\quad\text{for all}\quad\lambda\in\Omega,\quad(x,\xi)\in E_u^*\setminus 0.$$

Fix an open set $U_u\subset\widetilde{V}_u$ such that $\kappa(E_u^*)\subset U_u$ and there exists $\delta>0$ such that

(4.28) $$(\tfrac12 H_{\mathbf{X}}-\operatorname{Re}\lambda+\delta)\mathbf{w}_u(x,\xi)<0\quad\text{for all}\quad\lambda\in\Omega,\quad(x,\xi)\in U_u.$$

Take an arbitrary cutoff

$$\chi_u\in C_c^\infty(U_u;[0,1]),\quad \chi_u=1\quad\text{near}\quad\kappa(E_u^*)$$

and define

$$\mathbf{g}_u := \chi_u \sqrt{\mathbf{w}_u} \ \in\ C^\infty(T^*M; \mathrm{End}(\pi^*\mathcal{E}))$$

which lies in the class S^{m_u}.

3. Take a pseudodifferential operator

$$\mathbf{G}_u \in \Psi_h^{m_u}(M; \mathrm{End}(\mathcal{E})), \quad \mathrm{WF}_h(\mathbf{G}_u) \subset U_u, \quad \sigma_h(\mathbf{G}_u) = \mathbf{g}_u.$$

Note that $\mathbf{G}_u$ is elliptic on $\kappa(E_u^*)$. Fix a cutoff function

$$\psi_u \in C_{\mathrm{c}}^\infty(U_u \setminus \kappa(E_u^*)) \quad \text{such that} \quad \chi_u(H_p\chi_u) \le |\psi_u|^2 \quad \text{everywhere} \tag{4.29}$$

and an operator $\mathbf{E}_u \in \Psi_h^{m_u}(M; \mathrm{End}(\mathcal{E}))$ such that

$$\mathrm{WF}_h(\mathbf{E}_u) \subset U_u \setminus \kappa(E_u^*), \quad \sigma_h(\mathbf{E}_u) = \psi_u \sqrt{\mathbf{w}_u}.$$

Now, fix $\mathbf{A}_u, \mathbf{B}_u \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ such that, putting $\mathbf{B}_{1,u} := \mathbf{A}_u^*\mathbf{A}_u + \mathbf{B}_u^*\mathbf{B}_u$,

$$\begin{gathered}\kappa(E_u^*) \subset \mathrm{ell}_h(\mathbf{A}_u), \quad \mathrm{WF}_h(\mathbf{A}_u) \subset \mathrm{ell}_h(\mathbf{G}_u), \quad \mathrm{WF}_h(\mathbf{E}_u) \subset \mathrm{ell}_h(\mathbf{B}_u),\\ \mathrm{WF}_h(\mathbf{B}_u) \subset U_u \setminus \kappa(E_u^*), \quad \mathrm{WF}_h(\mathbf{G}_u) \subset \mathrm{ell}_h(\mathbf{A}_u) \cup \mathrm{ell}_h(\mathbf{B}_u) \subset \mathrm{ell}_h(\mathbf{B}_{1,u}).\end{gathered}$$

By Lemma 3.6, we have

$$h^{-1}\,\mathrm{Im}\langle(\mathbf{P} - ih\lambda)\mathbf{u}, \mathbf{G}_u^*\mathbf{G}_u\mathbf{u}\rangle_{L^2} - \|\mathbf{E}_u\mathbf{u}\|_{L^2}^2 + \delta\|\mathbf{G}_u\mathbf{u}\|_{L^2}^2 = \langle \mathbf{Z}_u\mathbf{u}, \mathbf{u}\rangle_{L^2} \tag{4.30}$$

where

$$\mathbf{Z}_u \in \Psi^{2m_u}(M; \mathrm{End}(\mathcal{E})), \quad \mathbf{Z}_u^* = \mathbf{Z}_u, \quad \mathrm{WF}_h(\mathbf{Z}_u) \subset \mathrm{ell}_h(\mathbf{B}_{1,u})$$

has principal symbol

$$\sigma_h(\mathbf{Z}_u) = (\chi_u(H_p\chi_u) - |\psi_u|^2)\mathbf{w}_u + \chi_u^2(\tfrac12 H_{\mathbf{X}} - \mathrm{Re}\,\lambda + \delta)\mathbf{w}_u. \tag{4.31}$$

By (4.28)–(4.29), each of the two summands on the right-hand side of (4.31) is the product of a nonnegative function in $C_{\mathrm{c}}^\infty(U_u)$ and a self-adjoint section of $\mathrm{End}(\mathcal{E})$ which is positively homogeneous of degree $2m_u$ and negative definite on U_u. Thus Lemma 3.1 gives

$$\langle \mathbf{Z}_u\mathbf{u}, \mathbf{u}\rangle_{L^2} \le Ch\|\mathbf{B}_{1,u}\mathbf{u}\|^2_{H_h^{m_u-\frac12}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|^2_{H_h^{-N}}$$

which together with (4.30) implies

$$\begin{aligned}\|\mathbf{G}_u\mathbf{u}\|_{L^2} \le{} & C\|\mathbf{E}_u\mathbf{u}\|_{L^2} + Ch^{-1}\|\mathbf{B}_{1,u}(\mathbf{P} - ih\lambda)\mathbf{u}\|_{H_h^{m_u}}\\ & + Ch^{\frac12}\|\mathbf{B}_{1,u}\mathbf{u}\|_{H_h^{m_u-\frac12}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}.\end{aligned} \tag{4.32}$$

4. By the elliptic estimate, we can replace $\|\mathbf{G}_u\mathbf{u}\|_{L^2}$ on the left-hand side of (4.32) by $\|\mathbf{A}_u\mathbf{u}\|_{H_h^{m_u}}$. Similarly we may replace $\|\mathbf{E}_u\mathbf{u}\|_{L^2}$ on the right-hand side of (4.32) by $\|\mathbf{B}_u\mathbf{u}\|_{H_h^{m_u}}$. Finally, recalling the definition of $\mathbf{B}_{1,u}$ we see that

$$\|\mathbf{B}_{1,u}\mathbf{u}\|_{H_h^{m_u-\frac12}} \le C\big(\|\mathbf{A}_u\mathbf{u}\|_{H_h^{m_u-\frac12}} + \|\mathbf{B}_u\mathbf{u}\|_{H_h^{m_u-\frac12}}\big).$$

Taking h small enough in (4.32), we now obtain (4.27). $\square$

4.2.4. *Proof of Lemma 4.3.* We are now ready to finish the proof of Lemma 4.3, following the proof of [12, Proposition 3.4]. Let $\mathbf{A}_s, \mathbf{A}_u, \mathbf{B}_u \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ be the operators from Lemmas 4.6–4.7. We first combine ellipticity, propagation of singularities, and the high regularity radial estimate to get

Lemma 4.8. *Let* $\mathbf{A} \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ *satisfy* $\mathrm{WF}_h(\mathbf{A}) \cap \kappa(E_u^*) = \emptyset$. *Then*

$$\|\mathbf{A}\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.33}$$

Proof. Fix an operator $\widetilde{\mathbf{Q}} \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ such that $\mathrm{WF}_h(\widetilde{\mathbf{Q}}) \subset \mathrm{ell}_h(\mathbf{Q})$ and $\mathrm{ell}_h(\widetilde{\mathbf{Q}})$ contains the zero section of T^*M. Define the open set $\mathcal{U} \subset \overline{T^*M}$ as follows:

$$\mathcal{U} := \{(x, \xi) \in \overline{T^*M} \mid \exists t \geq 0 : \ e^{-tH_p}(x, \xi) \in \mathrm{ell}_h(\widetilde{\mathbf{Q}}) \cup \mathrm{ell}_h(\mathbf{A}_s)\}.$$

Since $\mathrm{ell}_h(\mathbf{A}_s)$ contains $\kappa(E_s^*)$, by Lemma 3.3 we have $\mathrm{WF}_h(\mathbf{A}) \cap \{p = 0\} \subset \mathcal{U}$, that is $\mathrm{WF}_h(\mathbf{A}) \subset \mathrm{ell}_h(\mathbf{P}) \cup \mathcal{U}$. Using a microlocal partition of unity, we write

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2, \quad \mathbf{A}_1, \mathbf{A}_2 \in \Psi_h^0(M; \mathrm{End}(\mathcal{E})), \quad \mathrm{WF}_h(\mathbf{A}_1) \subset \mathrm{ell}_h(\mathbf{P}), \quad \mathrm{WF}_h(\mathbf{A}_2) \subset \mathcal{U}.$$

By the elliptic estimate, Lemma 4.4, we have

$$\|\mathbf{A}_1\mathbf{u}\|_{H_h^{\mathbf{m}}} + \|\widetilde{\mathbf{Q}}\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq C\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.34}$$

Next, by propagation of singularities, Lemma 4.5, with $\mathbf{B} := \mathbf{A}_s + \widetilde{\mathbf{Q}}$, $\mathrm{ell}_h(\mathbf{B}) = \mathrm{ell}_h(\mathbf{A}_s) \cup \mathrm{ell}_h(\widetilde{\mathbf{Q}})$, we have

$$\begin{aligned} \|\mathbf{A}_2\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq{} & C\|\mathbf{A}_s\mathbf{u}\|_{H_h^{\mathbf{m}}} + C\|\widetilde{\mathbf{Q}}\mathbf{u}\|_{H_h^{\mathbf{m}}} \\ & + Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \end{aligned} \tag{4.35}$$

Finally, recall that by the high regularity radial estimate, Lemma 4.6,

$$\|\mathbf{A}_s\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.36}$$

Putting together (4.34)–(4.36), we get (4.33). $\square$

Now, recall that the low regularity radial estimate, Lemma 4.7, gives

$$\|\mathbf{A}_u\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq C\|\mathbf{B}_u\mathbf{u}\|_{H_h^{\mathbf{m}}} + Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.37}$$

Take $\mathbf{A} \in \Psi_h^0(\mathbf{M}; \mathrm{End}(\mathcal{E}))$ such that

$$\overline{T^*M} \setminus \mathrm{ell}_h(\mathbf{A}_u) \subset \mathrm{ell}_h(\mathbf{A}), \quad \mathrm{WF}_h(\mathbf{A}) \subset \overline{T^*M} \setminus \kappa(E_u^*).$$

Since $\mathrm{WF}_h(\mathbf{B}_u) \cap \kappa(E_u^*) = \emptyset$, Lemma 4.8 applies to both $\mathbf{A}$ and $\mathbf{B}_u$ to give

$$\|\mathbf{A}\mathbf{u}\|_{H_h^{\mathbf{m}}} + \|\mathbf{B}_u\mathbf{u}\|_{H_h^{\mathbf{m}}} \leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \tag{4.38}$$

Since $\mathbf{A}^*\mathbf{A} + \mathbf{A}_u^*\mathbf{A}_u \in \Psi_h^0(M; \mathrm{End}(\mathcal{E}))$ is elliptic on the entire $\overline{T^*M}$, we can use the elliptic estimate to derive from (4.38) and (4.37) the bound

$$\begin{aligned} \|\mathbf{u}\|_{H_h^{\mathbf{m}}} &\leq C\|\mathbf{A}\mathbf{u}\|_{H_h^{\mathbf{m}}} + C\|\mathbf{A}_u\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|u\|_{H_h^{-N}} \\ &\leq Ch^{-1}\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H_h^{\mathbf{m}}} + \mathcal{O}(h^\infty)\|\mathbf{u}\|_{H_h^{-N}}. \end{aligned}$$

For h small enough, we may remove the last term on the right-hand side, obtaining (4.12) and finishing the proof of Lemma 4.3.

4.3. **Meromorphic continuation.** We finally give the proof of Theorem 4.1, following [12, §§3.3–3.4]. Let $\Omega \subset \mathbb{C}$ be a compact set satisfying the threshold regularity condition (4.11). Fix $h > 0$ small enough so that Lemma 4.3 applies. We henceforth suppress the subscript h in the notation $H^{\mathfrak{m}}_h$. Define the space

$$D^{\mathfrak{m}} := \{\mathbf{u} \in H^{\mathfrak{m}} \mid \mathbf{P}\mathbf{u} \in H^{\mathfrak{m}}\}$$

with the Hilbert norm

$$\|\mathbf{u}\|^2_{D^{\mathfrak{m}}} := \|\mathbf{u}\|^2_{H^{\mathfrak{m}}} + \|\mathbf{P}\mathbf{u}\|^2_{H^{\mathfrak{m}}}.$$

Since $\mathrm{WF}_h(\mathbf{Q})$ does not intersect the fiber infinity, $\mathbf{Q}$ is a smoothing operator. In particular, $\mathbf{Q}$ maps $H^{\mathfrak{m}}$ to itself. Therefore,

$$\mathbf{P} - ih\lambda - i\mathbf{Q} : D^{\mathfrak{m}} \to H^{\mathfrak{m}} \tag{4.39}$$

is a holomorphic family of bounded operators.

The space $C^\infty(M;\mathcal{E})$ is dense in $D^{\mathfrak{m}}$ as follows from [13, Lemma E.45] applied to the conjugated operator $\widetilde{\mathbf{P}}$ from §4.1.2 (whose proof adapts directly to the case of operators on vector bundles). Therefore, the estimates of Lemma 4.3 show that there exists a constant C such that for all $\lambda \in \Omega$

$$\begin{aligned} \|\mathbf{u}\|_{D^{\mathfrak{m}}} &\le C\|(\mathbf{P} - ih\lambda - i\mathbf{Q})\mathbf{u}\|_{H^{\mathfrak{m}}} \quad \text{for all} \quad \mathbf{u} \in D^{\mathfrak{m}}, \\ \|\mathbf{v}\|_{D^{-\mathfrak{m}}} &\le C\|(\mathbf{P} - ih\lambda - i\mathbf{Q})^*\mathbf{v}\|_{H^{-\mathfrak{m}}} \quad \text{for all} \quad \mathbf{v} \in D^{-\mathfrak{m}}. \end{aligned} \tag{4.40}$$

Here C and $\mathbf{Q}$ depend on h, however we already fixed h small enough above.

By a standard argument from functional analysis (see for example the proof of [13, Theorem 5.30]), the estimates (4.40) imply that the operator (4.39) is invertible for all $\lambda \in \Omega$. Since $\mathbf{Q}$ is smoothing, it is a compact operator $D^{\mathfrak{m}} \to H^{\mathfrak{m}}$. It follows that $\mathbf{P} - ih\lambda : D^{\mathfrak{m}} \to H^{\mathfrak{m}}$ is a Fredholm operator of index 0 for all $\lambda \in \Omega$. Recalling that $\mathbf{P} = -ih\mathbf{X}$ and Ω is an arbitrary compact subset of

$$\Omega_{\mathfrak{m}} := \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \max(r_u(m_u), r_s(m_s))\},$$

we get the following Fredholm property:

$$\mathbf{X} + \lambda : D^{\mathfrak{m}} \to H^{\mathfrak{m}} \quad \text{is a Fredholm operator of index 0 for all } \lambda \in \Omega_{\mathfrak{m}}. \tag{4.41}$$

Recall from (2.10) that the Pollicott–Ruelle resolvent $R_{\mathbf{X}}(\lambda)$ was defined for $\operatorname{Re}\lambda > C_{\mathbf{X}}$ by

$$R_{\mathbf{X}}(\lambda)\mathbf{f} := \int_0^\infty e^{-\lambda t} e^{-t\mathbf{X}}\mathbf{f}\, dt \quad \text{for} \quad \mathbf{f} \in C^\infty(M;\mathcal{E}). \tag{4.42}$$

The operator $e^{-t\mathbf{X}}$ is bounded on the space $H^{m_s}(M;\mathcal{E})$ locally uniformly in t. Since $e^{-t\mathbf{X}}$ forms a group in t, we see that there exists a constant $C_{\mathbf{X}}(m_s) \ge C_{\mathbf{X}}$ such that

$$\|e^{-t\mathbf{X}}\|_{H^{m_s} \to H^{m_s}} = \mathcal{O}(e^{C_{\mathbf{X}}(m_s)t}) \quad \text{as} \quad t \to \infty.$$

For $\operatorname{Re}\lambda > C_{\mathbf{X}}(m_s)$ and $\mathbf{f} \in C^\infty$, the integral (4.42) converges in the space H^{m_s} and thus (recalling (4.2)) in the larger space $H^{\mathfrak{m}}$. Thus $\mathbf{u} := R_{\mathbf{X}}(\lambda)\mathbf{f}$ lies in $H^{\mathfrak{m}}$ and (recalling (2.11)) satisfies $(\mathbf{X} + \lambda)\mathbf{u} = \mathbf{f}$. It follows that the range of the operator (4.41) contains $C^\infty(M;\mathcal{E})$ and is thus dense in $H^{\mathfrak{m}}$. From the Fredholm property we then see that when $\operatorname{Re}\lambda > \max(C_{\mathbf{X}}(m_s), r_u(m_u), r_s(m_s))$, the operator (4.41) is invertible and its inverse coincides on $C^\infty(M;\mathcal{E})$ with the Pollicott–Ruelle resolvent $R_{\mathbf{X}}(\lambda)$. Now by Analytic Fredholm Theory [13, Theorem C.8] we see that

$$(\mathbf{X}+\lambda)^{-1} : H^{\mathfrak{m}} \to D^{\mathfrak{m}}, \quad \lambda \in \Omega_{\mathfrak{m}}$$

is meromorphic with poles of finite rank. This operator gives the meromorphic continuation of the Pollicott–Ruelle resolvent, which finishes the proof of Theorem 4.1.

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