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FRACTIONAL SOBOLEV SPACES OF SYMMETRIC FUNCTIONS AND APPLICATIONS TO HAMILTONIAN ELLIPTIC SYSTEMS

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Abstract. In this paper we study compact embeddings of fractional Sobolev spaces of symmetric functions into weighted L^p spaces in situations above the Sobolev critical exponent. The proof combines a compact embedding of a Sobolev space of symmetric functions into a weighted L^p space with an interpolation result by Persson. The result is applied to prove existence of solutions for a class of non autonomous Hamiltonian systems.

In memory of Louis Nirenberg

1. Introduction

When $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain, the Sobolev space $H_0^1(\Omega)$ is compactly embedded into $L^p(\Omega)$ for $p \in [1, 2^*)$, $2^* = \frac{2N}{N-2}$. In [14], by using an analogue of Strauss' radial estimate [17], Ni proved the compact compact embedding into $L^p(\Omega, |x|^\alpha)$ holds for all $p \in [1, 2^* + \frac{2\alpha}{N-2})$ for all $\alpha > 0$, when one considers radially symmetric functions $u(x) = v(|x|) \in H_0^1(\Omega)$ on the unit ball $\Omega = B$ centered at the origin. By considering partially symmetric functions in

$$H_{0,\ell}^1(B) := \{u \in H_0^1(B) : u(x) = u(y,z) = v(|y|,|z|),$$
$$x = (y,z) \in \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}\},\ 2 \leq N - \ell \leq \ell\,,$$

Badialle and Serra proved in [1] that $H_{0,\ell}^1(B)$ is compactly embedded into $L_\ell^p(\Omega, |x|^\alpha)$, for $\alpha > N+2$, when $p \in [1, \frac{2(N-1)}{N-3})$, $N \geq 4$. We recall that $L_\ell^q(B, |x|^\alpha)$ is the weighted L^q space endowed with the norm

$$||u||_{q,\alpha} = (\int_B |x|^\alpha |u|^q dx)^{1/q}\,.$$

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Also, notice that $H^1_{0,\ell}(B)$ is a closed subspace of the Hilbert space $H^1_0(B)$, and consequently it is also a Hilbert space. Indeed, $H^1_{0,\ell}(B)$ is the set of the fixed points of the group $O(l) \times O(N-l)$ that acts isometrically on $H^1_0(B)$.

Now, consider the space

$$H^2_\ell(B) \cap H^1_0(B) = \{u \in H^2(B) \cap H^1_0(B) : \ u(x) = u(y,z) = v(|y|,|z|)\},$$

endowed with the norm

$$\|u\|_{H^2} = \left(\int_B |\Delta u|^2 dx\right)^{1/2}, \quad u \in H^2_\ell(B) \cap H^1_0(B),$$

which is compactly embedded into $L^2(B) \hookrightarrow L^2_l(B,|x|^\alpha)$; throughout this paper $\hookrightarrow$ represents continuous embedding. Given $f \in L^2_l(B,|x|^\alpha) \hookrightarrow L^2_l(B,|x|^{2\alpha})$, since $f|x|^\alpha \in L^2(B)$, then

$$-\Delta u = f|x|^\alpha \ \text{ in } \ B, \quad u = 0 \ \text{ on } \ \partial B, \tag{1.1}$$

has a unique solution in $H^2_l(B) \cap H^1_0(B)$. Therefore, the linear operator

$$\begin{array}{rccl} T_\alpha : & L^2_\ell(B,|x|^\alpha) & \longrightarrow & L^2_\ell(B,|x|^\alpha) \\ & f & \longmapsto & u = (-\Delta)^{-1}(f|x|^\alpha) \end{array}$$

is compact. Moreover, it is symmetric. Indeed, for all $f, g \in L^2_\ell(B,|x|^\alpha)$,

$$\begin{aligned} (T_\alpha f, g) &= \int_B T_\alpha f \, g|x|^\alpha dx = \int_B (-\Delta)^{-1}(f|x|^\alpha)\, g|x|^\alpha dx \\ &= \int_B (-\Delta)^{-1}(f|x|^\alpha)\,(-\Delta)((-\Delta)^{-1}(g|x|^\alpha))dx \\ &= \int_B (-\Delta)((-\Delta)^{-1}(f|x|^\alpha))(-\Delta)^{-1}(g|x|^\alpha)dx \\ &= \int_B f|x|^\alpha(-\Delta)^{-1}(g|x|^\alpha)dx = \int_B f(-\Delta)^{-1}(g|x|^\alpha)|x|^\alpha dx \\ &= (f, T_\alpha g). \end{aligned}$$

Consequently, T_α has a sequence of eigenfunctions and a corresponding sequence of eigenvalues, denoted by (ψ_n) and (μ_n^{-1}), respectively, such that (ψ_n) is a complete orthonormal system in $L^2_\ell(B,|x|^\alpha)$ and, since $(T_\alpha f, f) > 0$ for all $f \neq 0$,

$$0 < \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_n \to +\infty, \text{ as } n \to \infty\,.$$

Moreover, the identity $T_\alpha \psi_n = \mu_n^{-1}\psi_n$ reads

$$-\Delta\psi_n = \mu_n \psi_n |x|^\alpha \ \text{ in } \ B, \quad \psi_n = 0 \ \text{ on } \ \partial B.$$

We consider $E^2_\ell = \{u = \sum_{n=1}^\infty a_n\psi_n \in L^2_\ell(B,|x|^\alpha); \sum_{n=1}^\infty |a_n|^2\mu_n^2 < \infty\}$ endowed with the norm

$$|||u||| := (\sum_{n=1}^\infty |a_n|^2\mu_n^2)^{1/2}.$$

It follows that $E^2_\ell \hookrightarrow H^2_\ell(B) \cap H^1_0(B)$, and that E^2_ℓ is the domain of the operator T_α^{-1}; see Lemmas 3.1 and 3.2 ahead.

For $0 \leq t \leq 2$, we define fractional Sobolev spaces, as in [11], since T_α^{-1} is an accretive operator, by setting

$$E_\ell^t \equiv D(T_\alpha^{-t/2}) = \{u = \sum_{n=1}^{\infty} a_n \psi_n \in L_\ell^2(B, |x|^\alpha); \sum_{n=1}^{\infty} |a_n|^2 \mu_n^t < \infty\}.$$

Then, writing $A^t = T_\alpha^{-t/2}$, we have for $u = \sum_{n=1}^{\infty} a_n \psi_n$, that

$$\begin{array}{rccl} A^t : & E_\ell^t & \longrightarrow & L_\ell^2(B, |x|^\alpha) \\ & u & \longmapsto & A^t u = \displaystyle\sum_{n=1}^{\infty} \mu_n^{t/2} a_n \psi_n. \end{array}$$

We observe that E_ℓ^t is a Hilbert space with inner product and norm given by

$$(u, v)_{E_\ell^t} := \int_B A^t u A^t v |x|^\alpha dx \quad \text{and} \quad ||u||_{E_\ell^t} := (\int_B |A^t u|^2 |x|^\alpha dx)^{1/2},$$

for $u, v \in E_\ell^t$, and the Poincaré's type inequality

$$||u||_{E_\ell^t} = (\int_B |A^t u|^2 |x|^\alpha dx)^{1/2} \geq \mu_1^{t/2} ||u||_{2,\alpha}, \ \forall u \in E_\ell^t,$$

holds, whence we infer that E_ℓ^t is continuously embedded into $L_\ell^2(B, |x|^\alpha)$.

Following [12], we define the fractional Sobolev space $H_\ell^s(B)$ as the interpolation space

$$H_\ell^s(B) = [H_\ell^2(B) \cap H_0^1(B), L_\ell^2(B)]_\theta,$$

where $0 < \theta < 1$, $s = 2(1-\theta)$ and we refer to [11] for results regarding this space. In particular, it is proved in [11, Theorem 1], with $\alpha = 0$, that

$$E_\ell^t = D(A^t) \subset H_\ell^t(B), \ 0 \leq t \leq 2\,.$$

Also, for $0 < \theta < 1$, we consider the interpolation spaces given by

$$L_\ell^r(B, |x|^\alpha) = [L_\ell^q(B, |x|^\alpha), L_\ell^2(B, |x|^\alpha)]_\theta, \quad \frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{2}.$$

One of our main result is the following

Theorem 1.1. *The embedding* $E_\ell^t \subset L_\ell^r(B, |x|^\alpha)$ *is compact for* $2 \leq r < \frac{2(N-1)}{N-1-2t}$, $\alpha > 0$ *large*, $0 \leq t \leq 2$.

As an application, we consider the following non autonomous Hamiltonian system with weights

$$\left\{ \begin{array}{rl} -\Delta u = |x|^\beta |v|^{q-1} v + g(x, v) & \text{in} \quad B, \\ -\Delta v = |x|^\alpha |u|^{p-1} u + f(x, u) & \text{in} \quad B, \\ u, v = 0 & \text{on} \quad \partial B, \\ u, v > 0 & \text{in} \quad B, \end{array} \right. \qquad (P)$$

where α, β are positive constants, and $p, q > 1$ are such that (p, q) lies below the α, β critical hyperbola, that is,

$$\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} > N-2, \qquad N > 2. \tag{1.2}$$

Regarding the functions $f, g : B \times \mathbb{R} \longrightarrow \mathbb{R}$ we make the following assumptions:

(H_1) $$f, g \in C(B \times \mathbb{R}, \mathbb{R}),\ f(x,0) = g(x,0) = 0\ \forall x \in B;$$

(H_2) $$\lim_{|u|\to\infty} \frac{f(x,u)}{|u|^\tau},\ \lim_{|v|\to\infty} \frac{g(x,v)}{|v|^\sigma} < +\infty,\ \text{uniformly in } x \in B,$$

where τ, σ satisfy

$$\frac{N}{\tau+1} + \frac{N}{\sigma+1} > N-2, \qquad N > 2; \tag{1.3}$$

(H_3) $$\lim_{|u|\to 0} \frac{f(x,u)}{|u|} = \lim_{|v|\to 0} \frac{g(x,v)}{|v|} = 0,\ \text{uniformly in } x \in B;$$

(H_4) There exist $\gamma > 2$ and $\eta > 0$ such that

$$0 < \gamma F(x,u) \le uf(x,u),\ 0 < \gamma G(x,v) \le ug(x,v),\ \text{for } |u|, |v| \ge \eta,$$

uniformly in $x \in B$, where

$$F(x,s) = \int_0^s f(x,t)\,dt \text{ and } G(x,s) = \int_0^s g(x,t)\,dt.$$

When $f = g = 0$ and under the above conditions, it is proved in [5] a non-existence result of classical solutions in $C^2(B) \cap C^1(\overline{B})$ for (p,q) lying *above* (and on) the α, β critical hyperbola, that is,

$$\frac{N+\alpha}{p+1} + \frac{N+\beta}{q+1} \le N-2, \qquad N > 2. \tag{1.4}$$

In the same work, it is proved the existence of radial solutions for (p,q) lying *below* the α, β critical hyperbola, i.e., (p,q) verifying (1.2). In [2, 3], the authors studied radial and foliated Schwarz symmetric solutions for (P) with $f = g = 0$, when $p.q > 1$ lies below the critical hyperbola, namely

$$\frac{N}{p+1} + \frac{N}{q+1} > N-2, \qquad N > 2. \tag{1.5}$$

We recall that definitions of such hyperbola appeared independently in [6] and [18], and they were considered by several authors, including [13] and [7, 15]. We would also like to mention that these types of systems have been considered before in [3, 5, 8–10] with $f = g = 0$.

Our next goal is to show existence of a solution (u,v) for the problem (P).

Theorem 1.2. *Assume $(H_1) - (H_4)$, $\alpha, \beta > 0$ are sufficiently large, and $p > 1$, $q > 1$ verify*

$$\frac{N-1}{p+1} + \frac{N-1}{q+1} > N-3, \qquad N > 3.$$

In addition, assume that $f(x,s) = f((|y|,|z|),s)$ and $g(x,s) = g((|y|,|z|),s)$, $x = (y,z) \in R^\ell \times \mathbb{R}^{N-\ell}, \ell \ge 2$ and $N - l \ge 2$. Then the system (P) possesses at least one nontrivial positive solution (u,v).

2. Interpolation spaces

In this section we will establish a compact embedding result for fractional Sobolev spaces into weighted L^p spaces. For that, we will use an abstract theorem due to Persson [16], which involves a result on compact linear mappings between interpolation spaces.

Let us start by giving some definitions. A pair E_0, E_1 of Banach spaces is called an interpolation pair if E_0 and E_1 are continuously embedded in some separated topological linear space $\mathbf{E}$. Let A_0, A_1 and E_0, E_1 be interpolation pairs. A_θ and E_θ are called interpolation spaces of exponent $\theta \in (0,1)$, with respect to A_0, A_1 and E_0, E_1, if we have the topological inclusions

$$A_0 \cap A_1 \subset A_\theta \subset A_0 + A_1, \qquad E_0 \cap E_1 \subset E_\theta \subset E_0 + E_1,$$

and if each linear mapping T from a separated topological linear space $\mathbf{A}$ into $\mathbf{E}$, which maps A_i continuously into E_i $(i = 0,1)$, also maps A_θ continuously into E_θ in such a way that

$$M \leq M_0^{1-\theta} M_1^{\theta},$$

where M denotes the norm of $T : A_\theta \longrightarrow E_\theta$ and M_i the norm of $T : A_i \longrightarrow E_i, i = 0, 1$.

For the interpolation pair E_0, E_1 we shall consider the following condition:

(H) To each compact set $K \subset E_0$ there exists a constant $C > 0$ and a set P of linear operators $P : \mathbf{E} \to \mathbf{E}$ which map E_i into $E_0 \cap E_1$ $i = 0, 1$, and are such that

$(H)(i)$ $\;||P||_{L(E_i,E_i)} = \sup_{||x||_{E_i} \leq 1} ||Tx||_{E_i} \leq C,\ i = 0, 1;$

$(H)(ii)$ Furthermore, for each $\epsilon > 0$ there is $P_0 \in \mathrm{P}$ so that $||P_0x - x||_{E_i} < \epsilon$ for all $x \in K$.

We now recall the following result due to Persson [16].

Theorem 2.1. (Persson) *Let A_0, A_1 and E_0, E_1 be interpolation pairs, and suppose that A_θ and E_θ are interpolation spaces of exponent $\theta \in (0,1)$ with respect to A_0, A_1 and E_0, E_1. Suppose also that $A_\theta \subset \overline{A_\theta}$ and E_0, E_1 satisfy (H). Then, if $T_0 : A_0 \to E_0$ is compact and $T_1 : A_1 \to E_1$ is bounded, it follows that $T_\theta : A_\theta \to E_\theta$ is compact.*

3. Proof of Theorem 1.1

We start with two basic lemmas.

Lemma 3.1. *$E^2_\ell \hookrightarrow H^2_\ell(B) \cap H^1_0(B)$.*

Proof. Given $u \in E_l^2$, with $u = \sum_{n=1}^{\infty} a_n \psi_n$, define $u_k = \sum_{n=1}^{k} a_n \psi_n$. Then, for every $m, k \geq 1$,

$$\int_B (\Delta(u_{k+m} - u_k))^2\,dx = \int_B \left(\Delta \sum_{n=k+1}^{k+m} a_n \psi_n\right)^2$$

$$= \int_B \left(\sum_{n=k+1}^{n+m} a_n \mu_n \psi_n |x|^\alpha\right)^2 dx \leq \int_B \left(\sum_{n=k+1}^{n+m} a_n \mu_n \psi_n\right)^2 |x|^\alpha dx$$

$$= \sum_{n=k+1}^{n+m} a_n^2 \mu_n^2.$$

This argument shows that (u_k) is a Cauchy sequence in $H_\ell^2(B) \cap H_0^1(B)$ and that $u_k \to u$ in $H_\ell^2(B) \cap H_0^1(B)$. Moreover,

$$\|u\|_{H^2} \leq |||u|||, \quad \forall\, u \in E_l^2.$$

Therefore, the continuous embedding $E_\ell^2 \hookrightarrow H_\ell^2(B) \cap H_0^1(B)$ holds. □

Observe that

$$u = T_\alpha f = (-\Delta)^{-1}(f|x|^\alpha) \Longleftrightarrow -\Delta u = f|x|^\alpha \Longleftrightarrow |x|^{-\alpha}(-\Delta u) = f$$

$$\Longleftrightarrow T_\alpha^{-1} u = (-\Delta u)|x|^{-\alpha}.$$

From this remark we obtain the following characterization.

Lemma 3.2. *E_ℓ^2 is the domain of the operator T_α^{-1}.*

Proof. Given $u = \sum_{n=1}^{\infty} a_n \psi_n = T_\alpha f$. Then,

$$+\infty > \int_B (T_\alpha^{-1} u)^2 |x|^\alpha dx = \int_B (\Delta u)^2 |x|^{-2\alpha} |x|^\alpha dx$$

$$= \int_B \left(\sum_{n=1}^{\infty} a_n \mu_n \psi_n |x|^\alpha\right)^2 |x|^{-\alpha} dx = \int_B \left(\sum_{n=1}^{\infty} a_n \mu_n \psi_n\right)^2 |x|^\alpha dx = \sum_{n=1}^{\infty} a_n^2 \mu_n^2.$$

This proves that $T_\alpha(L_\ell^2(B, |x|^\alpha)) \subset E_l^2$. On the other hand, given $u = \sum_{n=1}^{\infty} a_n \psi_n \in E_l^2 \subset H_l^2(B) \cap H_0^1(B)$, set $f = \sum_{n=1}^{\infty} a_n \mu_n \psi_n \in L_\ell^2(B, |x|^\alpha)$. Then $-\Delta u = \sum_{n=1}^{\infty} a_n \mu_n \psi_n |x|^\alpha = f|x|^\alpha$, that is, $u = T_\alpha f$. Therefore, E_ℓ^2 is the domain of T_α^{-1}. □

Proof of Theorem 1.1. Note that, in our setting, $E_0 = L_\ell^q(B, |x|^\alpha)$, $E_1 = L_\ell^2(B, |x|^\alpha)$, $A_0 = H_\ell^2$, $A_1 = L_\ell^2$, and we have that

(a): $T_0 : A_0 \to E_0$ is compact for $2 \leq q < \frac{2(N-1)}{N-5}$ and α large enough,
(b): $T_1 : A_1 \to E_1$ is bounded, with $T_1 = $ *identity*.

In order to conclude that $T_\theta : A_\theta \to E_\theta$ is compact it is sufficient to prove the following lemma.

Lemma 3.3. *The interpolation pair $E_0 = L_\ell^q(B, |x|^\alpha)$, $E_1 = L_\ell^2(B, |x|^\alpha)$ satisfies condition (H).*

Proof Actually, as in [16], we will show the following condition, which is stronger than (H):

$(\widetilde{H})$ There exist a constant $C > 0$ and a set D of linear operators $P : \mathbf{E} \to \mathbf{E}$ with $P(E_i) \subset E_0 \cap E_1$, $i = 0, 1$, such that $(H)(i)$ is satisfied and so that, to every $\epsilon > 0$ and every finite set $x_1, x_2, \ldots, x_m$ in E_0, we can find P in D verifying

$(H)(iii)$ $\qquad ||Px_k - x_k||_{E_0} \le \epsilon\,, \quad k = 1, 2, \ldots, m\,.$

We claim that the pair $E_0 = L^q_\ell(B, |x|^\alpha), E_1 = L^2_\ell(B, |x|^\alpha)$ satisfies $(\widetilde{H})$.

Some arguments in this proof were borrowed from [10, lemma 2.1], where $\alpha, \beta \le 0$ or the radial case were considered.

First of all we identify the space $L^q_\ell(B, |x|^\alpha)$ with $L^q_\ell(B, \mu)$, where $\mu = |x|^\alpha dx\,, \alpha > 0$, is a σ finite measure. Then $C^\infty_0(B)$ is dense in $L^q_\ell(B, \mu)$ and, in fact, $C^\infty_{0,\ell}(B)$ is dense in $L^q_\ell(B, \mu)$ (see [4]).

Now, let $f_j\,(j = 1, 2, \ldots, m)$ be given functions in $C^\infty_{0,\ell}(B) \subset L^q_\ell(B, \mu)$ and take a compact set K in B such that $f_j(x) = 0,\ \forall x \in K^c = \mathbb{R}^N \backslash K$ and $j = 1, 2, \ldots, m$. Also, given $\epsilon > 0$, pick $\eta > 0$ so that $\eta\mu(K)) < \epsilon$, where $\mu(K)$ denotes the measure of K. We then construct a partition (K_n) of K consisting of a set K_0 with measure zero and measurable sets $K_1, K_2, \ldots$, with $\mu(K_n) > 0$, such that $\sup\limits_{x \ne y \in K_n} |f_j(x) - f_j(y)| < \eta$, for all $j = 1, 2, \ldots, m$.

Next, define

$$Pf = \sum_{n=1}^{\infty} (\mu(K_n)^{-1} \int_B f\phi_n d\mu)\phi_n,$$

where ϕ_n denotes the characteristic function of K_n, $n = 1, 2, \ldots$.

We claim that

(i): P satisfies $(H)(i)$, i.e.,
$||P||_{L(E_i,E_i)} = \sup_{||x||_{E_i} \le 1} ||Px||_{E_i} \le C,\ i = 0, 1,\ C > 0,$
$P : E_i \to E_0 \cap E_1\,(i = 0, 1)\,.$

(ii): P satisfies $(H)(iii)$, i.e.,
for every finite set $x_1, x_2, \ldots, x_m$ in E_0 we can find P such that
$||Px_k - x_k||_{E_0} \le \epsilon,\ k = 1, 2, \ldots, m.$

We start noticing that, for every $q \ge 2$, we have by Hölder inequality that

$$\begin{aligned}
\int_B |Pf|^q d\mu &= \sum_{n=1}^{\infty} (\mu(K_n)^{-1} \int_{K_n} f\phi_n d\mu)^q \int_{K_n} \phi_n^q d\mu \\
&\le \sum_{n=1}^{\infty} (\mu(K_n)^{-1} (\int_{K_n} |f|^q d\mu)^{1/q} (\int_{K_n} |\phi_n|^{\frac{q}{q-1}} d\mu)^{(q-1)/q})^q \int_{K_n} |\phi_n|^q d\mu \\
&\le \sum_{n=1}^{\infty} ((\mu(K_n)^{-q} (\mu(K_n)^{(q-1)/q})^q |f|^q_{L^q_{\ell,\mu}} \mu(K_n) = |f|^q_{L^q_{\ell,\mu}}\,.
\end{aligned} \tag{3.1}$$

Verification of (i): From (3.1) it follows that $P : E_i \to E_i\,(i = 0, 1)$ is bounded. And, since $E_0 \subset E_1$ we have that $P : E_0 \to E_0 = E_0 \cap E_1$. Now we show that

$P(E_1) \subset E_0$, hence we also have $P : E_1 \to E_0 \cap E_1$:

$$\int_B |Pf|^2 d\mu = \sum_{n=1}^{\infty} (\mu(K_n)^{-1} \int_{K_n} f\phi_n d\mu)^2 \int_{K_n} \phi_n^2 d\mu$$
$$\leq \sum_{n=1}^{\infty} \mu(K_n)^{-2} ((\int_{K_n} |f|^q d\mu)^{1/q} (\int_{K_n} |\phi_n|^{\frac{q}{q-1}} d\mu)^{(q-1)/q})^2 \mu(K_n)$$
$$= \sum_{n=1}^{\infty} \mu(K_n)^{-2} |f|^2_{L^q_{\ell,\mu}} \mu(K_n)^{2(q-1)/q} \mu(K_n)$$
$$= \sum_{n=1}^{\infty} \mu(K_n)^{(q-2)/q} |f|^2_{L^q_{\ell,\mu}} \leq C|f|^2_{L^q_{\ell,\mu}} .$$

Verification of (ii): We note that

$$Pf_k = \sum_{n=1}^{\infty} (\mu(K_n)^{-1} \int_B f_k(y)\phi_n(y) d\mu)\phi_n$$

and

$$f_k(x) = \sum_{n=1}^{\infty} (\mu(K_n)^{-1} f_k(x) \int_B \phi_n(y) d\mu)\phi_n .$$

Then, by construction of the K_n's and our choice of $\eta = \eta(\epsilon, K)$, we conclude that

$$||Pf_k - f_k||^q_{L^q_{\ell,\mu}} \leq \int_B \sum_{n=1}^{\infty} (\mu(K_n)^{-q} (\int_B |f_k(y) - f_k(x)|\phi_n) d\mu)^q |\phi_n|^q d\mu$$
$$\leq \eta \sum_{n=1}^{\infty} \mu(K_n) = \eta\mu(K) < \epsilon,$$

and this finishes the proof of Lemma 3.3.

Now, we continue the proof of Theorem 1.1. Since we have checked (**i**), (**ii**), we now apply Persson result. Indeed, since $\frac{1}{r} = \frac{1-\theta}{q} + \frac{\theta}{2}$ and $H^2_\ell(B)$ is compactly embedded into $L^q_\ell(B, |x|^\alpha)$ when $2 \leq q < \frac{2(N-1)}{N-5}$ and α is large enough, it follows that

$$\frac{1}{r} > \frac{1-\theta}{2} - \frac{2(1-\theta)}{N-1} + \frac{\theta}{2}.$$

And, recalling that $1 - \theta = s/2$, we get

$$r < \frac{2(N-1)}{N-1-2s}, \ 0 \leq s \leq 2, \ \alpha \text{ for sufficiently large.} \tag{3.2}$$

This completes the proof of Theorem 1.1. □

Corollary 3.4. *Let $p, q > 0$ and suppose that*

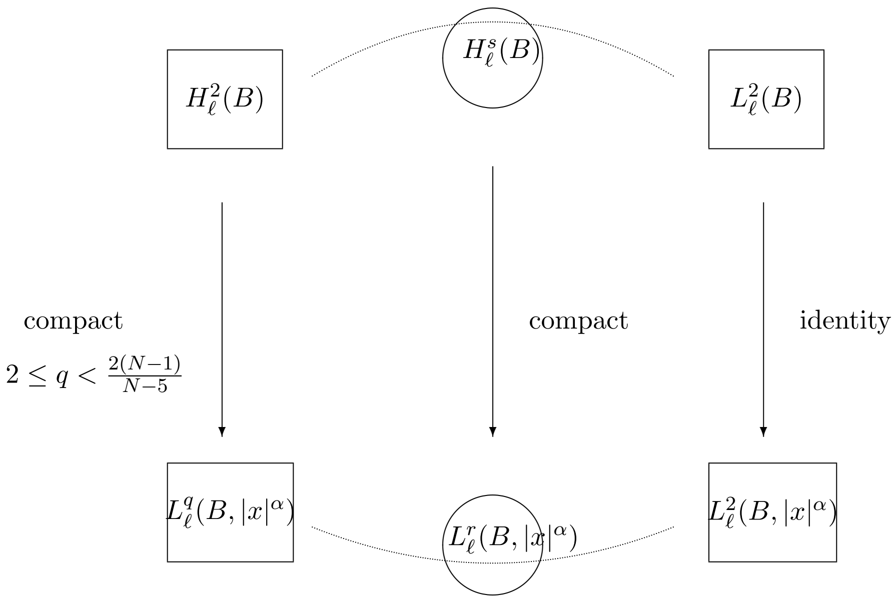
$$\frac{N-1}{p+1} + \frac{N-1}{q+1} > (N-1) - 2 = N - 3,$$

that is, (p,q) is below the (N-1)- critical hyperbola. Then, there exist $s,t>0$ such that $s+t=2$, $p+1<\frac{2(N-1)}{N-1-2s}$, $q+1<\frac{2(N-1)}{N-1-2t}$ and the compact embeddings

$$E^s_\ell \subset L^{p+1}_\ell(B,|x|^\alpha), \qquad E^t_\ell \subset L^{q+1}_\ell(B,|x|^\beta),$$

for α and β sufficiently large, hold.

We include below a diagram illustrating Theorem 1.1:



4. Proof of Theorem 1.2

We are going to search for the critical points of the functional $I_\ell : E^s_\ell \times E^t_\ell \longrightarrow \mathbb{R}$ given by

$$I_\ell(u,v)=\int_B A^s u A^t v-\int_B\Big(\frac{|x|^\alpha|u|^{p+1}}{p+1}+\frac{|x|^\beta|v|^{q+1}}{q+1}\Big)-\int_B(F(x,u)+G(x,v)),$$

which are precisely the (classical) solutions of (P).

Indeed, we have that I_ℓ is of class C^1 with (Fréchet) derivative given by

$$I'_\ell(u,v)(\phi,\psi)=\int_B A^s u A^t\psi+\int_B A^s\phi A^t v-\int_B(|x|^\alpha|u|^{p-1}u\phi+|x|^\beta|v|^{q-1}v\psi)$$
$$-\int_B(f(x,u)\phi+g(x,v)\psi).$$

Recall that $A^s = T^{-s/2} : E^s_\ell \to L^2_\ell(|x|^\alpha)$ [where we are setting $L^2_\ell(B, |x|^\alpha) = L^2_\ell(|x|^\alpha)$ for simplicity] and $E^s_\ell = D(T^{-s/2})$, $0 \le s \le 2$, is endowed with the equivalent norm $||A^s u||$, which satisfies

$$||A^s u|| \ge ||u||, \text{ for all } u \in E^s_\ell \,, \tag{4.1}$$

so that we set

$$\begin{aligned} &< u, v >_{E^s_\ell} := < A^s u, A^s v > = \textstyle\int_B A^s u A^s v dx \quad \text{and} \\ &||u||_{E^s_\ell} := ||A^s u|| = (\textstyle\int_B |A^s u|^2 dx)^{1/2} \;\; \forall u \in E^s_\ell \,. \end{aligned} \tag{4.2}$$

From (4.1), $A^s : E^s_\ell \to L^2_\ell(|x|^\alpha)$ is an isomorphism and we denote by A^{-s} the inverse of A^s.

Also, for given $s, t > 0$ with $s + t = 2$, we denote by E the Hilbert space $E^s_\ell \times E^t_\ell$, and define the symmetric bilinear form $B : E \times E \to \mathbb{R}$ by the formula

$$B((u, v), (\phi, \psi)) = \int_B (A^s u A^t \psi + A^s \phi A^t v) dx \,.$$

From (4.2) and applying the Cauchy-Schwartz inequality, we have that B is continuous, i.e.

$$|B((u, v), (\phi, \psi))| \le ||A^s u||_{E^s_\ell} ||A^t \psi||_{E^t_\ell} + ||A^s \phi||_{E^s_\ell} ||A^t v||_{E^t_\ell} \,,$$

so that B induces a selfadjoint bounded linear operator $L : E \to E$ satisfying

$$B(z, \eta) = < Lz, \eta >_E, \quad \text{for all } z, \eta \in E \,.$$

In addition, we can easily verify that

$$L(u, v) = (A^{-s} A^t v, A^{-t} A^s u), \text{ for } z = (u, v) \in E \,. \tag{4.3}$$

Next, consider the eigenvalue problem

$$Lz = \lambda z \quad \text{in} \quad E \,. \tag{4.4}$$

From (4.3) the above problem is equivalent to

$$A^{-s} A^t v = \lambda u, \quad \text{and} \quad A^{-t} A^s u = \lambda v, \quad z = (u, v).$$

Since the operators A^s and A^t are isomorphisms onto $L^2_\ell(|x|^\alpha)$, λ cannot be zero, and we obtain from the above that

$$v = \lambda^{-2} v \,.$$

Therefore $\lambda = \pm 1$, with corresponding eigenspaces

$$E^- = \{(u, -A^{-t} A^s u) : \; u \in E^s_\ell\} \;\; \text{for} \;\; \lambda = -1 \,, \tag{4.5}$$

$$E^+ = \{(u, A^{-t} A^s u) : \; u \in E^s_\ell\} \;\; \text{for} \;\; \lambda = 1 \,. \tag{4.6}$$

And we have the direct sum decomposition

$$E = E^- \bigoplus E^+ \,,$$

where the spaces E^+ and E^- are orthogonal with respect to the bilinear form B, that is,

$$B(z^+, z^-) = 0 \text{ for all } \; z^+ \in E^+, z^- \in E^- \,. \tag{4.7}$$

We also have

$$\frac{1}{2}||z||_E^2 = Q(z^+) - Q(z^-) \text{ for all } z = z^+ + z^-,\ z^+ \in E^+, z^- \in E^-\,, \tag{4.8}$$

where Q is the quadratic form associated with the bilinear form B:

$$Q(z) = \frac{1}{2}B(z,z) = \int_B A^s u A^t v\, dx,\ z = (u,v) \in E\,.$$

Now, let $\{e_j\}$ $(j = 1, 2, \ldots)$ be a complete orthogonal system in E_ℓ^s and let E_n denote the finite dimensional subspace of E_ℓ^s spanned by $\{e_j\},\ j = 1, 2, \ldots$,n. Since $A^s : E_\ell^s \to L_\ell^2(|x|^\alpha)$ and $A^t : E_\ell^t \to L_\ell^2(|x|^\alpha)$ are isomorphisms, we can assume that $\{\hat{e}_j\},\ j = 1, 2, \ldots,$ where $\hat{e}_j := A^{-t}A^s e_j$, is a complete orthogonal system in E_ℓ^t. We let $\widehat{E}_n$ denote the finite dimensional subspace of E_ℓ^t spanned by $\{\hat{e}_j\},\ j = 1, 2, \ldots$,n. In addition, for each $n \in \mathbb{N}$, we introduce the following subspaces of E^+ and E^-, respectively:

$$E_n^+ = span\{(e_j, \hat{e}_j) \in E^+ \,|\, j = 1, 2, \ldots, n\} \text{ and}$$
$$E_n^- = span\{(e_j, -\hat{e}_j) \in E^- \,|\, j = 1, 2, \ldots, n\},$$

as well as $E_n := E_n^+ \bigoplus E_n^-$. The rest of the proof follows as in [10]. For the sake of completeness, we will sketch some of its parts.

Proposition 4.1. *The functional I_ℓ has a local linking at* 0, *that is,*
(i): $I_\ell(z) \geq 0$, *for* $z \in E^+$, $||z|| \leq r$,
(ii): $I_\ell(z) \leq 0$, *for* $z \in E^-$, $||z|| \leq r$.

Proof. For $z = (u,v) \in E^+$, one shows that there exist $C > 0$ and $r_0 > 2$ such that

$$I_\ell(u,v) \geq ||z||_E^2 - C||z||_E^{r_0}.$$

Hence, there is $r > 0$ such that

$$I_\ell(z) \geq 0, \text{ for } z \in E^+,\ ||z||_E \leq r.$$

Similarly, for $z = (u,v) \in E^-$, one also shows there exist some $C, D > 0$ and $r_0 > 2$ such that

$$I_\ell(u,v) \leq -C||z||_E^2 + D|z||_E^{r_o}.$$

Hence, there is also some $r > 0$ such that

$$I_\ell(z) \leq 0, \text{ for } z \in E^-,\ ||z||_E \leq r. \quad \square$$

Proposition 4.2. *The functional I_ℓ satisfies the $(PS)^*$ condition with respect to $\{E_n\}$, that is,*

$(PS)^*$ *If a sequence $\{z_n\} \subset E_n$ is such that $|I_\ell(z_n)| \leq C$, $| < \nabla_n I_\ell(u_n), \eta > | \leq \epsilon_n ||\eta||_E$, with $\epsilon_n \to 0$, for some $C > 0$ and all $\eta \in E_n$, then $\{z_n\}$ possesses a subsequence converging to a critical point of I_ℓ.*

Here, ∇_n denotes the gradient of I_ℓ restricted to E_n.

Proof. In view of Theorem 1.1, it is sufficient to prove the uniform boundedness of the sequence $\{z_n = (u_n, v_n)\}$, that is, that there exists a constant $C > 0$ verifying $||(z_n)|| = ||(u_n, v_n)||_{E_n} \leq C \; \forall \; n \in \mathbb{N}$.

The argument is standard. Since

$$\begin{aligned} I_\ell(u_n, v_n) &= \int_B A^s u_n A^t v_n - \int_B \left(\frac{|x|^\alpha |u_n|^{p+1}}{p+1} + \frac{|x|^\beta |v_n|^{q+1}}{q+1}\right) \\ &\quad - \int_B (F(x, u_n) + (G(x, v_n)) \end{aligned}$$

and

$$\begin{aligned} I'_\ell(u_n, v_n)(u_n, v_n) = \int_B A^s u_n A^t v_n + \int_B A^s u_n A^t v_n \\ - \int_B (|x|^\alpha |u_n|^{p+1} + |x|^\beta |v_n|^{q+1}) - \int_B (f(x, u_n) u_n + g(x, v_n) v_n)\,, \end{aligned}$$

it follows that

$$\begin{aligned} C + \epsilon_n ||z_n||_E &\geq I_\ell(z_n) - \frac{1}{2} I'_\ell(z_n)(z_n) \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_B |x|^\alpha |u_n|^{p+1} dx + \left(\frac{1}{2} - \frac{1}{q+1}\right) \int_B |x|^\beta |v_n|^{q+1} dx \\ &\quad + \left(\frac{1}{2} - \frac{1}{\gamma}\right) \int_B (f(x, u_n) u_n + g(x, v_n) v_n)\,. \end{aligned} \tag{4.9}$$

On the other hand, recalling that

$$< L(u,v), \eta >= B((u,v), \eta) = \int_B (A^s u A^t \eta_2 + A^s \eta_1 A^t v) dx\,, \qquad \forall \eta = (\eta_1, \eta_2)\,,$$

and writing $z_n^\pm = (u_n^\pm, v_n^\pm)$, we get

$$\begin{aligned} ||z_n^\pm||^2 - \epsilon |||z_n^\pm||_E &\leq |< L z_n, z_n^\pm > - I'_\ell(z_n)(z_n^\pm)| \\ &= |\int_B |x|^\alpha |u_n|^{p-1} u_n u_n^\pm dx + \int_B |x|^\beta |v_n|^{p-1} v_n v_n^\pm) dx \\ &\quad + \int_B (f(x, u_n) u_n^\pm + g(x, v_n) v_n^\pm)| \end{aligned} \tag{4.10}$$

and we will now estimate each term in the r.h.s of above.

From Hölder inequality we get

$$\begin{aligned} |\int_B |x|^\alpha |u_n|^{p-1} u_n u_n^\pm dx| &\leq \left(\int_B |x|^\alpha |u_n|^{p+1} dx\right)^{\frac{p}{p+1}} \left(\int_B |x|^\alpha |u_n^\pm|^{p+1} dx\right)^{\frac{1}{p+1}} \\ &\leq \left(\int_B |x|^\alpha |u_n|^{p+1} dx\right)^{p/(p+1)} ||u_n^\pm||_{E^s_\ell} \end{aligned} \tag{4.11}$$

and

$$|\int_B |x|^\beta |v_n|^{q-1} v_n v_n^\pm dx| \leq \left(\int_B |x|^\beta |v_n|^{q+1} dx\right)^{q/(q+1)} ||v_n^\pm||_{E^t_\ell}\,. \tag{4.12}$$

On other hand, noticing that $(H_1)-(H_4)$ gives the estimate $f(s)s^{1/\tau} \leq C(1+f(s)s),\ \forall s \in \mathbb{R},\ \tau > 1$, and using Hölder inequality, we infer that

$$
\begin{aligned}
(4.13)\quad & |\int_B (f(x,u_n)u_n^{\pm} + g(x,v_n)v_n^{\pm})| \\
& \leq \int_B |f(x,u_n)|^{\frac{\tau+1}{\tau}})^{\frac{\tau}{\tau+1}} ||u_n^{\pm}||_{L_\ell^{\tau+1}} + \int_B |g(x,v_n)|^{\frac{\sigma+1}{\sigma}})^{\frac{\sigma}{\sigma+1}} ||v_n^{\pm}||_{L_\ell^{\sigma+1}} \\
& \leq (\int_B |f(x,u_n)||f(x,u_n)|^{\frac{1}{\tau}})^{\frac{\tau}{\tau+1}} ||z_n^{\pm}||_E + (\int_B |g(x,v_n)||g(x,v_n)|^{\frac{1}{\sigma}})^{\frac{\sigma}{\sigma+1}} ||z_n^{\pm}||_E \\
& \leq C(1 + (\int_B |u_n||f(x,u_n)|)^{\frac{\tau}{\tau+1}} + (\int_B |v_n)||g(x,v_n)|)^{\frac{\sigma}{\sigma+1}}))||z_n^{\pm}||_E
\end{aligned}
$$

for some constant $C > 0$.

Therefore, combining (4.11)-(4.12)-(4.13) with (4.10), we obtain

$$
\begin{aligned}
||z_n^{\pm}||_E - \epsilon \quad & \leq \quad (\int_B |x|^\alpha |u_n|^{p+1} dx)^{p/(p+1)} + (\int_B |x|^\beta |v_n|^{q+1} dx)^{q/(q+1)} \\
(4.14)\quad & \leq \quad (C + \epsilon_n ||z_n||)^{p/(p+1)} + (C + \epsilon_n ||z_n||)^{q/(q+1)} \\
& \quad +C(1 + (C + \epsilon_n ||z_n||)^{\tau/(\tau+1)} + (C + \epsilon_n ||z_n||)^{\sigma/(\sigma+1)}).
\end{aligned}
$$

This implies that $||z_n||_E$ is uniformly bounded in n. □

Proposition 4.3. *For each $n \in \mathbb{N}$, one has $I_\ell(z) \to -\infty$ as $||z||_E \to \infty$, $z \in E_n^+ \bigoplus E^-$.*

Proof. Let $n \in \mathbb{N}$ be fixed and let $z_n \in E_n^+ \bigoplus E^-$ be such that $||z_n||_E \to \infty$. Writing $z = z^+ + z^-$ for $z = (u,v)$, we have that

$$
\begin{aligned}
(4.15)\quad I_\ell(u,v) \leq ||z^+||_E^2 - ||z^-||_E^2 - \int_B \frac{|x|^\alpha |u|^{p+1}}{p+1} + \frac{|x|^\beta |v|^{q+1}}{q+1} \\
- \int_B (F(x,u_n) + G(x,v_n)).
\end{aligned}
$$

And letting $z^{\pm} = (u^{\pm}, v^{\pm})$ we have that $u^- = \eta u^+ + \hat{u}$, where $\hat{u}$ is orthogonal to u^+ in $L_\ell^2(|x|^\alpha)$. Similarly, $v^- = \nu v^+ + \hat{v}$, where $\hat{v}$ is orthogonal to v^+ in $L_\ell^2(|x|^\alpha)$.

Notice that either ν or η is positive. Supposing $\nu > 0$, we have the following estimate, where $1/\gamma + 1/(\gamma') = 1$ and $\gamma > 1$:

$$
(1+\nu)\int_B |x|^\delta |u^+|^2 dx = \int_B |x|^\delta ((1+\nu)u^+ + \hat{u})u^+ dx \leq |u|_{L_\ell^\gamma(|x|^\delta)} |u^+|_{L_\ell^{\gamma'}(|x|^\delta)}.
$$

Since the norms in E_n^+ are equivalent, we get, for a positive constant $C > 0$:

$$
(1+\nu)|u^+|_{L_\ell^2(|x|^\delta)} \leq C|u|_{L_\ell^\gamma(|x|^\delta)}.
$$

Then, using this inequality in (4.15) with $\delta = \alpha$, $\gamma = p+1$ (resp. $\delta = \beta$, $\gamma = q+1$), we get

$$
I_\ell(z) \leq ||z^+||_E^2 - ||z^-||_E^2 - C(|u^+|^{p+1}_{L_\ell^{p+1}(|x|^\alpha)} + C|v^+|^{q+1}_{L_\ell^{q+1}(|x|^\beta)}),
$$

which implies that

$$
I_\ell(z) \to -\infty, \text{ as } ||z|| \to \infty, \text{ because } p,q > 1.
$$

□

Finally, the proof of Theorem 2.1 is complete by applying the following version of Rabinowitz Linking Theorem (see [19]):

Theorem 4.4. *Suppose that $I \in C^1(E,\mathbb{R})$ satisfies the following conditions:*
(a): *I has a local linking at 0.*
(b): *I satisfies $(PS)^*$.*
(c): *I maps bounded sets into bounded sets.*
(d): *For every $n \in \mathbb{N}$, $I(z) \to -\infty$, as $||z|| \to \infty$, $z \in E_n^+ \oplus E^-$.*
Then I has a nontrivial critical point.

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