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NONNEGATIVE SOLUTIONS OF THE POROUS MEDIUM EQUATION WITH CONTINUOUS LATERAL BOUNDARY DATA

KAI-SENG CHOU AND YING CHUEN KWONG

Abstract. It is shown that given any nonnegative, continuous function g on the lateral boundary of a cylinder, a Radon measure μ satisfying (1.4) on the bottom and a Radon measure λ at the corner of the bottom, there is a unique continuous very weak solution to the porous medium equation in the slow diffusion case which is continuous up to the boundary for positive time. Moreover, it is equal to g along the lateral boundary, and takes (μ, λ) as its initial trace.

1. Introduction

Consider the porous medium equation in the slow diffusion case

$$u_t = \Delta u^m \ , \quad m > 1, \tag{1.1}$$

in the cylinder $Q_T \equiv \Omega \times (0, T)$ where Ω is a smooth, bounded domain in $\mathbb{R}^n$. In the previous work [4] we have shown that every nonnegative, continuous very weak solution of (1.1) admits a Radon measure as its lateral trace. Moreover, when $\|u^m\|_{L^1(Q_T)}$ is finite, it has an initial trace consisting of a Radon measure on Ω and a Radon measure at $\partial\Omega$. Thus every nonnegative, continuous very weak solution of (1.1) admits a trace triple consisting of bottom, corner and lateral traces. This fact leads naturally to the initial-boundary value problem for this equation with prescribed trace triple. Given a triple of Radon measures in appropriate spaces of measures, it has been shown that the initial-boundary value problem for (1.1) admits a non-negative, continuous very weak solution in Q_T whose trace triple is the prescribed one. When (1.1) is considered in $\mathbb{R}^n \times (0, T)$, there are no lateral and corner traces and the initial trace problem is solved in [1] and [3].

To proceed further, it is necessary to describe things in analytical terms. A nonnegative function $u \in C(Q_T)$ is called a *continuous very weak solution* of (1.1) if for every $\eta \in C^\infty(\overline{\Omega'} \times (0, T))$ vanishing on $\partial\Omega' \times (0, T)$, where Ω' is a smooth

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domain compactly contained in Ω, and $t_1 < t_2,\ t_1, t_2 \in (0,T)$,

$$\begin{aligned}&\int_{\Omega'} u(x,t_2)\eta(x,t_2)\,dx - \int_{\Omega'} u(x,t_1)\eta(x,t_1)\,dx \\ =\ &\int_{t_1}^{t_2}\int_{\Omega'} (u^m\Delta\eta + u\eta_t)\ dxdt - \int_{t_1}^{t_2}\int_{\partial\Omega'} u^m\frac{\partial\eta}{\partial\nu}\,dsdt\ .\end{aligned} \tag{1.2}$$

A Radon measure ν on $\partial\Omega\times(0,T)$ is called the *lateral trace* of a function $F \in C(Q_T)$ if

$$\lim_{\varepsilon\to 0^+}\int_0^T\int_{\partial\Omega_\varepsilon} Fh\,dsdt = \int_{\partial\Omega\times(0,T)} h\,d\nu\ , \tag{1.3}$$

where $\Omega_\varepsilon = \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ and h is a continuous function defined in a tubular neighborhood of $\partial\Omega\times(0,T)$ vanishing near $t = 0, T$. For a continuous very weak solution u, u^m always admits a lateral trace. On the other hand, consider the pair (μ,λ) where μ is a Radon measure on Ω satisfying

$$\int_\Omega \rho(x)\,d\mu(x) < \infty\ , \tag{1.4}$$

where ρ is the distance to the boundary of Ω, and λ is a Radon measure on $\partial\Omega$. This pair is called the *initial trace* of the continuous very weak solution u if for all smooth φ vanishing on $\partial\Omega$,

$$\lim_{t\to 0^+}\int_\Omega u(x,t)\varphi(x)\,dx = \int_\Omega \varphi\,d\mu - \int_{\partial\Omega}\frac{\partial\varphi}{\partial\nu}\,d\lambda\ . \tag{1.5}$$

The study of these solutions goes in two directions according to whether its "total mass" is finite or not. Indeed, when a continuous very weak solution satisfies the condition

$$\int_0^T\int_\Omega u^m(x,t)\,dxdt < \infty\ , \tag{1.6}$$

we show in [4] that it admits an initial trace. Furthermore, the lateral trace of its m-power is a finite measure. Conversely, given a triple (μ,λ,ν) where μ is a Radon measure on Ω satisfying (1.4), λ is a finite Radon measure on $\partial\Omega$, and ν is a finite Radon measure on $\partial\Omega\times(0,T)$, there is a continuous very weak solution u taking (μ,λ) as its initial trace and whose m-th power taking ν as its lateral trace. Indeed, the continuous very weak solution satisfies the identity

$$\begin{aligned}\int_\Omega u(x,t)\eta(x,t)\,dx\ &=\ \int_\Omega \eta(x,0)\,d\mu - \int_{\partial\Omega}\frac{\partial\eta}{\partial\nu}(x,0)\,d\lambda \\ &+\ \int_{Q_t}(u^m\Delta\eta + u\eta_t)\,dxdt - \int_{\partial\Omega\times(0,t)}\frac{\partial\eta}{\partial\nu}\,d\nu\ ,\end{aligned} \tag{1.7}$$

for all $\eta \in C^\infty(\overline{\Omega}\times[0,T]), \eta = 0$ on $\partial\Omega\times[0,T)$, and *a.e.* $t \in (0,T)$. The proofs of all these facts can be found in [4].

In this note we will restrict our attention to the case where the lateral trace is given by a continuous function. Previously, rather complete results were obtained in [6] when the lateral trace vanishes identically. Now we extend their results to the nonhomogeneous case. First, we have

Theorem 1.1. *Let μ be a Radon measure on Ω satisfying (1.4), λ a Radon measure at $\partial\Omega$, and a Radon measure ν on $\partial\Omega \times (0,T)$ given by $d\nu = g^m dsdt$ where g is a nonnegative, continuous function on $\partial\Omega \times [0,T]$. There is a unique continuous very weak solution u of (1.1) in Q_T belonging to $C(\overline{\Omega} \times (0,T])$, $u = g$ on $\partial\Omega \times (0,T]$, admitting (μ, λ) as its initial trace. Indeed, it satisfies (1.7) for all $t \in (0,T)$.*

Next, we study the case when (1.6) is not satisfied. In the homogeneous case, it is known that there is only one continuous very weak solution satisfies the condition

$$\int_0^T \int_\Omega u^m(x,t)\, dxdt = \infty \ . \tag{1.8}$$

This solution, called the friendly giant, tends to ∞ uniformly in each compact subset of Ω as time goes to 0. We show that the same result remains valid in the nonhomogeneous case.

Theorem 1.2. (a) *Let g be a nonnegative, continuous function on $\partial\Omega \times [0,T]$. There is a unique continuous very weak solution of (1.1) in Q_T, u, belonging to $C(\overline{\Omega} \times (0,T])$, $u = g$ on $\partial\Omega \times (0,T]$, which tends to ∞ uniformly in every compact subset of Ω as $t \to 0^+$.*

(b) *Any continuous very weak solution u of (1.1) in Q_T belonging to $C(\overline{\Omega} \times (0,T])$ satifies (1.8) is a friendly giant, that is, it is the continuous very weak solution described in (a).*

2. Initial-boundary value problem

In this section we prove Theorem 1.1. Our proof does not rely on a regularity theory applying to the continuous very weak solution constructed in our previous paper. Instead, we re-examine the proof therein by incorporating a boundary regularity result of DiBenedetto [5]. There is another boundary regularity result by Ziemer [10] which covers the same equation. However, only the result in [5] provides a modulus of continuity which is essential to our argument. To adapt this result to our context, let us first recall that a modulus of continuity for a function defined on a set $E \subset \mathbb{R}^n \times [0,T]$ is an increasing function $\omega : [0,\infty) \to [0,\infty)$, which is continuous at 0 satisfying $\omega(0) = 0$, such that

$$|f(x_1,t_1) - f(x_2,t_2)| \le \omega(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ , \quad \forall (x_1,t_1),(x_2,t_2) \in E.$$

Every uniformly continuous function f on a set E admits a modulus of continuity defined by

$$\omega(r) = \sup \left\{ |f(x_1,t_1) - f(x_2,t_2)| : \ |x_1 - x_2| + |t_1 - t_2|^{1/2} \le r,\ \forall (x_1,t_1),(x_2,t_2) \in E \right\} .$$

The following result is taken from [5].

Theorem 2.1. *Let $g \in C(\partial\Omega \times [0,T]), g \ge 0$, and u a bounded, H^1-solution of (1.1) in Q_T satisfying $u = g$ on $\partial\Omega \times (0,T]$ (in the sense of Sobolev trace). For each modulus of continuity ω_g of g and $\tau \in (0,T)$, there associates a modulus of continuity for u, ω, so that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ , \quad (x_i,t_i) \in \Omega \times [\tau,T] \ , \quad i = 1,2 \ .$$

The function ω only depends on $\|u\|_{L^\infty(Q_T)}, \omega_g$ and τ.

We will need the following result taken from (3.8) in [4].

Lemma 2.2. *Let u be a continuous very weak solution of (1.1) in some Q_T with trace triple (μ, λ, ν) where μ is a Radon measure on Ω satisfying (1.4), λ a Radon measure at $\partial\Omega$ and ν is a finite Radon measure on $\partial\Omega \times (0, T)$. For each $\varepsilon > 0$, there is some $t_1 \in (0, T)$ such that*

$$\int_0^{t_1} \int_\Omega u^m \, dxdt < \varepsilon,$$

where t_1 depends only on $\varepsilon, \|\rho\|_{L^1(\mu)}, \lambda(\partial\Omega)$ and $\nu(\partial\Omega \times (0, T))$.

Now we prove Theorem 1.1. First, we construct a continuous very weak solution with trace triple $(\mu, 0, \nu)$ where $d\mu = f dx, f \geq 0,\ f \in C_c^\infty(\Omega),\ d\nu = g^m dsdt$, and $g \in C(\partial\Omega \times [0, T]),\ g \geq 0$. We may assume that g has been extended to be a nonnegative, uniformly continuous function $\overline{g}$ in $\mathbb{R}^{n+1}$. Let $\omega_{\overline{g}}$ be a modulus of continuity for $\overline{g}$. Fix a bump function Ψ in the unit ball in $\mathbb{R}^{n+1}$ with $\|\Psi\|_{L^1} = 1$ and, for each $\varepsilon > 0$, define a smooth function g_ε in $\partial\Omega \times [0, T]$ by $g_\varepsilon = \Psi_\varepsilon * \overline{g}$ where $\Psi_\varepsilon(x) = \varepsilon^{-n}\Psi(x/\varepsilon)$. Then g_ε converges to g uniformly as ε tends to 0. Moreover, since we have

$$|\overline{g}(x_1, t_1) - \overline{g}(x_2, t_2)| \leq \omega_{\overline{g}}(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ , \quad (x_i, t_i) \in \mathbb{R}^{n+1} \ , \ i = 1, 2,$$

we have

$$\begin{aligned} & |g_\varepsilon(x_1, t_1) - g_\varepsilon(x_2, t_2)| \\ = \ & \left| \frac{1}{\varepsilon^n} \int \Psi((y, \tau)/\varepsilon) \left(\overline{g}(x_1 - y, t_1 - \tau) - \overline{g}(x_2 - y, t_2 - \tau) \right) ds(y) d\tau \right| \\ \leq \ & \omega_{\overline{g}}(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ , \quad (x_i, t_i) \in \partial\Omega \times [0, T], \ i = 1, 2 \ . \end{aligned}$$

Hence $\omega_{\overline{g}}$ is a modulus of continuity for all g_ε.

Now, for each k, we fix a nonnegative, smooth function ξ_k which is equal to 1 on $[1/k, T]$ and 0 on $[0, 1/2k]$ and let $h_k = \xi_k g_{1/k}$. Each h_k coincides with $g_{1/k}$ in $\partial\Omega \times [1/k, T]$ and vanishes near $t = 0$. Now, we solve (1.1) using $f + 1/k$ as the initial value and $h_k + 1/k$ as the lateral value to obtain a positive, classical solution u_k in Q_T. The existence of u_k can be established by a routine argument. On the other hand, it is shown in section 4 in [4] that there is a supersolution of the form $W(x, t) = t^{-\alpha}\varphi(x),\ \alpha > 0,\ \varphi > 0$ in $\overline{\Omega}$, of which lateral value is always greater than $\|g\|_{L^\infty} + 1$. By the comparison principle, all u_k are bounded by W for all large k. Hence, for each $\tau \in (0, T]$,

$$0 \leq u_k(x, t) \leq M_\tau \ , \quad k \geq 1, \quad (x, t) \in \Omega \times [\tau, T] \ , \tag{2.1}$$

where $M_\tau = \tau^{-\alpha} \sup_\Omega W$. Moreover, since $\omega_{\overline{g}}$ is a modulus of continuity for all g_ε, appealing to Theorem 2.1, for each $\tau \in (0, T)$, there exists a modulus function ω_τ depending only on τ, M_τ and $\omega_{\overline{g}}$ such that for all $k \geq 1/\tau,\ (x_1, t_1), (x_2, t_2) \in [\tau, T]$,

$$|u_k(x_1, t_1) - u_k(x_2, t_2)| \leq \omega_\tau(|x_1 - x_2| + |t_1 - t_2|^{1/2}) \ . \tag{2.2}$$

In view of (2.1) and (2.2), we can apply Ascoli's theorem to select a subsequence from $\{u_k\}$, still denoted by $\{u_k\}$, which converges uniformly to some $v \in C(\overline{\Omega} \times (0, T])$ on each $\Omega \times (\tau, T],\ \tau \in (0, T)$, as $k \to \infty$.

We now verify that v is a continuous very weak solution of (1.1). First of all, v is clearly equal to g on the lateral boundary. Next, let η be a smooth function in $\Omega' \times [0,T]$ vanishing on $\partial\Omega \times (0,T)$ where Ω' is a smooth subdomain of Ω. By multiplying the equation satisfied by u_k with η and then integrating over Ω', we have

$$\begin{aligned} &\int_{\Omega'} u_k(x,t_2)\eta(x,t_2)\,dx - \int_{\Omega'} u_k(x,t_1)\eta(x,t_1)\,dx \\ = &\int_{t_1}^{t_2}\int_{\Omega'} (u_k^m \Delta\eta + u_k\eta_t)\,dxdt - \int_{t_1}^{t_2}\int_{\partial\Omega'} u_k^m \frac{\partial\eta}{\partial\nu}\,dsdt\ . \end{aligned} \tag{2.3}$$

Note that (2.3) is more general than (1.2) since here Ω' is not necessarily compactly contained in Ω. Letting $k \to \infty$, we see that (2.3) also holds for v. In particular, it shows that v is a continuous very weak solution of (1.1). By the continuity of v in $\overline{\Omega} \times (\tau, T]$, it is clear that

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\partial\Omega_\varepsilon} v^m(x,t)h(x,t)\,dsdt = \int_0^T\int_{\partial\Omega} h(x,t)g^m(x,t)\,dsdt\ , \tag{2.4}$$

for all continuous h vanishing near 0 and T. Finally, to verify the initial condition, we observe that

$$\begin{aligned} &\int_\Omega u_k(x,t)\eta(x,t)\,dx - \int_\Omega \left(f_k(x) + \frac{1}{k}\right)\eta(x,0)\,dx \\ = &\int_0^t\int_\Omega (u_k^m\Delta\eta + u_k\eta_t)\,dxdt - \int_0^t\int_{\partial\Omega}\left(g_k + \frac{1}{k}\right)^m \frac{\partial\eta}{\partial\nu}\,dsdt\ . \end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned} &\int_\Omega v(x,t)\eta(x,t)\,dx - \int_\Omega f(x)\eta(x,0)\,dx \\ = &\int_0^t\int_\Omega (v^m\Delta\eta + v\eta_t)\,dxdt - \int_0^t\int_{\partial\Omega} g^m\frac{\partial\eta}{\partial\nu}\,dsdt\ . \end{aligned} \tag{2.5}$$

where we have applied Lemma 2.2 to the first integral on the right hand side. Consequently,

$$\lim_{t\to 0^+}\int_\Omega v(x,t)\eta(x,t)\,dx = \int_\Omega f(x)\eta(x,0)\,dx\ .$$

By comparing with (1.5), we see that $(\mu, 0)$ where $d\mu = fdx$ is the initial trace of v. We have shown that v is a continuous weak solution of (1.1) whose trace triple is given by $(\mu, 0, \nu)$ as asserted. Moreover, (2.1) and (2.2) continue to hold for v.

Next, we solve (1.1) for $(\mu, 0, \nu)$ where μ is a Radon measure compactly supported in Ω and $d\nu = g^m dsdt, g \in C(\partial\Omega \times [0,T]), g \geq 0$. It suffices to fix a sequence of nonnegative functions $\{f_j\}$ in $C_c^\infty(\Omega)$ which converges weakly to μ. Denote the solution of (1.1) as constructed in the first step with initial value f_j and lateral value g by v_j. Since (2.1) and (2.2) hold for all v_j, by Ascoli's theorem again it contains a subsequence, which is still denoted by $\{v_j\}$, converging uniformly on each $\Omega \times [\tau, T]$ to some $w \in C(\overline{\Omega} \times (0,T])$. By passing limit in (2.3) (replacing u_k by v_j), it is readily seen that w is a continuous very weak solution of (1.1). Furthermore, since $w \in C(\overline{\Omega} \times [\tau, T])$ and is equal to g along the lateral boundary, (2.4) holds for w. Finally, each v_j and f_j satisfy (2.5) (replacing v and f by v_j and f_j respectively). By Lemma 2.2 and the fact that $\{v_j\}$ converges uniformly to w in each $\Omega \times [\tau, T], \tau \in (0,T)$, by letting $j \to \infty$, (2.5) implies

$$\begin{aligned} &\int_\Omega w(x,t)\eta(x,t)\,dx - \int_\Omega \eta(x,0)\,d\mu \\ (2.6) \qquad = &\int_0^t \int_\Omega (w^m \Delta\eta + w\eta_t)\ dxdt - \int_0^t \int_{\partial\Omega} g^m \frac{\partial \eta}{\partial \nu}\, dsdt\ . \end{aligned}$$

Letting $t \to 0^+$, by Lemma 2.2 again, the initial trace of w is equal to $(\mu, 0)$. We note that (2.1) and (2.2) hold for w.

Finally, let (μ, λ, ν), $d\nu = g^m dsdt$, be the general case. We may follow [DK] to construct a sequence of Radon measures $\{\mu_i\}$ compactly contained in Ω that satisfies

$$(2.7) \qquad \lim_{i\to\infty} \int_\Omega \varphi\, d\mu_i = \int_\Omega \varphi\, d\mu - \int_{\partial\Omega} \frac{\partial\varphi}{\partial\nu}\, d\lambda\ , \quad \forall \varphi \in C^\infty(\overline{\Omega}),\quad \varphi = 0 \text{ on } \partial\Omega\ .$$

Let w_i be the continuous very weak solution for the trace triple $(\mu_i, 0, \nu)$ constructed in the last paragraph. As (2.1) and (2.2) hold for all $\{w_i\}$, we can extract a subsequence, still denoted by $\{w_i\}$, converging uniformly on each $\Omega \times [\tau, T]$ to some $u \in C(\overline{\Omega} \times (0,T])$. Arguing as before, u is a continuous very weak solution of (1.1) whose lateral value is g. Moreover, by passing limit in (2.6) (for w_i and μ_i) and using (2.7), we see that (μ, λ) is the initial trace of u, and (1.7) holds for u.

We have completed the proof of the existence part of Theorem 1.1. The uniqueness assertion will be established in Section 4.

Remark 2.1. The proof above in fact has shown that the continuous very weak solution satisfies (1.2) in every smooth subdomain Ω' of Ω.

Remark 2.2. When the initial-boundary value is given by a continuous function h on $\overline{\Omega} \times \{0\} \bigcup \partial\Omega \times (0,T]$. According to the main theorem in [D], one can estimate the modulus of continuity of the very weak solution in Q_T. It follows that in this case the continuous very weak solution constructed above in fact belongs to $C(\overline{Q_T})$ and is equal to h on its parabolic boundary.

3. Comparison Principles

The results of this section will be used in the next section to establish the uniqueness part of Theorem 1.1 and Theorem 1.2.

The Green's potential was first used in the study of the porous medium equation in [P]. Pierre's maximum principle was subsequently employed in [6] to establish the uniqueness of the friendly giant under the homogeneous lateral boundary condition. Here we extend it to the nonhomogeneous case.

We start with a comparison principle for continuous very weak solutions. The proof is by modifying a standard argument [ACP].

Lemma 3.1. *Let $u_i, i = 1, 2$, be continuous very weak solutions of (1.1) in Q_T belonging to $C(\overline{Q_T})$. Suppose that $u_1 \leq u_2$ on $\partial_p Q_T$, the parabolic boundary of Q_T. Then $u_1 \leq u_2$ in $\overline{Q_T}$.*

Proof. We note the relation $u_1^m - u_2^m = a(u_1 - u_2)$ where

$$a = m \int_0^1 (u_2 + s(u_1 - u_2))^{m-1} ds \geq 0 \ .$$

Pick a sequence of positive $a_k \in C^\infty(\overline{Q_T})$ satisfying $1/k \leq a_k \leq \|a\|_{L^\infty} + 1/k$ satisfying and

$$\lim_{k \to \infty} \left\| \frac{(a_k - a)}{\sqrt{a_k}} \right\|_{L^2(Q_T)} = 0 \ . \tag{3.1}$$

For a fixed $t, 0 < t < T$, and a given a smooth, non-negative function θ in Ω vanishing on $\partial\Omega$, we solve the initial-boundary problem

$$\begin{cases} \eta_t + a_k \Delta \eta = 0, \quad \text{in } \Omega \times (0, t) \ , \\ \eta(x, t) = \theta(x) \ , \\ \eta(x, \tau) = 0, \quad (x, \tau) \in \partial\Omega \times (0, t) \ , \end{cases}$$

to obtain a non-negative, smooth solution η_k. As $u_i, i = 1, 2$, are continuous up to $t = 0$, from (1.5) we see that their corner traces vanish. Using η_k as a test function in (1.7) and noting that $d\mu_i = u_i\, dx$ and $\lambda_i = 0$ at $t = 0$, $\delta u \equiv u_1 - u_2$ satisfies

$$\begin{aligned} \int_\Omega \delta u(x, t)\theta(x)\, dx &= \int_\Omega \delta u(x, 0)\eta_k(x, 0)\ dx \\ &\quad + \int_0^t \int_\Omega (a - a_k)\Delta\eta_k\ \delta u\, dx d\tau - \int_0^t \int_{\partial\Omega} (u_1^m - u_2^m) \frac{\partial \eta_k}{\partial \nu}\, ds dt \\ &\leq \int_0^t \int_\Omega (a - a_k)\Delta\eta_k\ \delta u\, dx dt \ . \end{aligned} \tag{3.2}$$

where $\delta u = u_1 - u_2$. To estimate the right hand side of (3.2), we multiply $\Delta\eta_k$ to the equation satisfied by η_k and integrate to get

$$\int_0^t \int_\Omega a_k (\Delta\eta_k)^2 \delta u\, dx d\tau \leq \frac{1}{2} \int_\Omega |\nabla\theta|^2\, dx \ .$$

By Cauchy-Schwarz inequality

$$\begin{aligned} &\left| \int_0^t \int_\Omega (a - a_k)\Delta\eta_k \delta u\, dx d\tau \right| \\ &\qquad \leq \|\delta u\|_{L^\infty} \left(\int_0^t \int_\Omega \frac{(a - a_k)^2}{a_k}\, dx dt \right)^{1/2} \left(\frac{1}{2} \int_\Omega |\nabla\theta|^2\, dx \right)^{1/2} . \end{aligned}$$

Using (3.1) and passing limit, we arrive at

$$\int_\Omega \delta u(x, t)\theta(x)\, dx \leq 0 \ ,$$

which implies $\delta u(x, t) \leq 0$. The desired conclusion follows. □

Given a continuous function u in $\overline{\Omega}$, we solve the equation $-\Delta U = u$, $U = 0$ on $\partial\Omega$, to obtain the the Green's potential of u. It assumes the form

$$U(x) = \int_\Omega G(x,y)u(y)\,dy\ ,$$

where G is the Green's function of the Laplacian under the Dirichlet condition [H]. In general, since $C(\overline{\Omega}) \subset L^p(\Omega)$ for all $p \geq 1$, by elliptic theory U belongs to the Sobolev space $W^{2,p}(\Omega)$ for all $p \geq 1$. In particular, it implies that U is continuously differentiable. However, stronger regularity is required to establish the following result, so an approximate argument is needed.

Lemma 3.2. *Let u be a continuous weak solution of (1.1) in Q_T belonging to $C(\overline{\Omega} \times (0,T])$. For $0 < t_1 < t_2 \leq T$,*

$$U(x,t_2) - U(x,t_1) \leq -\int_{t_1}^{t_2}\int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x,y)u^m\,dsdt\ ,$$

where $U(\cdot,t)$ is the Green's potential of $u(\cdot,t)$.

Proof. For $t_1 < t_2$ in $(0,T)$, fix some $t_0 \in (0,t_1)$. We pick a sequence of positive, smooth functions $\{h_k\}$ decreasing to u uniformly on the parabolic boundary of $\Omega \times [t_0,T]$ and let v_k be the continuous very weak solution of (1.1) taking h_k as its initial-boundary data. Each $v_k(\cdot,t)$ is positive, smooth and its Green's potential $V_k(\cdot,t)$ is smooth too. We have

$$\begin{aligned}
\frac{\partial V_k}{\partial t}(x,t) &= \int_\Omega G(x,y)v_{kt}(y,t)\,dy \\
&= \int_\Omega G(x,y)\Delta v_k^m(y,t)\,dy \\
&= -v_k^m(x,t) - \int_{\partial\Omega}\frac{\partial G}{\partial \nu_y}(x,y)h_k^m(y,t)\,ds \\
&\leq -\int_{\partial\Omega}\frac{\partial G}{\partial \nu_y}(x,y)h_k^m(y,t)\,ds\ .
\end{aligned}$$

Therefore,

$$V_k(x,t_2) - V_k(x,t_1) \leq -\int_{t_1}^{t_2}\int_{\partial\Omega}\frac{\partial G}{\partial \nu_y}(x,y)h_k^m(y,t)\,dsdt\ . \tag{3.3}$$

By the weak maximum principle,

$$\|v_k - v_j\|_{L^\infty(\Omega\times(t_0,T]))} \leq \|h_k - h_j\|_{L^\infty(\partial_p(\Omega\times[t_0,T])}\ ,$$

hence $\{v_k\}$ converges to some continuous very weak solution uniformly in $\Omega \times [t_0,T]$, and this solution takes u as its initial-boundary value. By Lemma 3.1, this solution coincides with u. We conclude that $\{v_k\}$ converges to u uniformly in $\Omega \times [t_0,T]$. According to elliptic theory, $V_k(\cdot,t)$ converges to $U(\cdot,t)$ in $W^{2,p}(\Omega)$ for all $t \in (0,T)$ and $p > n/2$. By Sobolev's inequality, in particular, $V_k(\cdot,t)$ converges to $U(\cdot,t)$ uniformly. The lemma follows by passing limit in (3.3). □

Lemma 3.3. *Let $u_i, i = 1, 2$, be two continuous very weak solutions of (1.1) belonging to $C(\overline{\Omega} \times (0, T])$. Let $\varphi \in C^\infty(\overline{\Omega})$ vanish on $\partial\Omega$ and $\tau < t$, $t, \tau \in (0, T)$, be fixed. There are Radon measures m_τ on Ω and Σ_τ on $\partial\Omega \times [\tau, t]$ satisfying*

$$\int_\Omega \delta U(x,t)\varphi(x)\,dx = \int_\Omega \delta U(x,\tau)\,dm_\tau(x) + \int_{\partial\Omega\times[\tau,t]} (u_1^m - u_2^m)\,d\Sigma_\tau \ . \tag{3.4}$$

Moreover,

$$m_\tau(\Omega) \le \|\varphi\|_{L^1}, \quad \Sigma_\tau(\partial\Omega \times (\tau, t)) \le t\|\varphi\|_{L^1} \ . \tag{3.5}$$

The measures m_τ and Σ_τ also depend on u_i, t and φ. Nevertheless, in the following we will let τ tend to 0 while all other quantities are fixed. Therefore, we only put τ in the subscripts of these measures.

Proof. Let θ be the function obtained by solving $-\Delta\theta = \varphi$ in Ω and $\theta = 0$ on $\partial\Omega$. Use this θ to determine η as in the proof of Lemma 3.1. According to (3.2) (replacing 0 by τ),

$$\begin{aligned} &\int_\Omega \delta U(x,t)\varphi(x)\,dx = \int_\Omega \delta U(x,\tau)\Delta\eta_k(x,\tau)\,dx + \\ &\int_\tau^t \int_\Omega (a - a_k)\Delta\eta_k\ \delta u\,dxdt + \int_\tau^t \int_{\partial\Omega} (u_1^m - u_2^m)\left|\frac{\partial\eta_k}{\partial\nu}\right| dsdt \ , \end{aligned} \tag{3.6}$$

where $\delta U \equiv U_1 - U_2$. Observing that $\Delta\eta_k$ satisfies the equation $(\Delta\eta_k)_t + \Delta(a_k\Delta\eta_k) = 0$ and $\Delta\eta_k = -a_k^{-1}\eta_{kt} = 0$ on $\partial\Omega \times (0, T)$, $\Delta\eta_k \le 0$ by the maximum principle. We have

$$\frac{d}{dt}\int_\Omega \Delta\eta_k = -\int_\Omega \frac{\partial}{\partial\nu}(a_k\Delta\eta_k)\,dx \le 0 \ .$$

It follows that

$$\int_\Omega |\Delta\eta_k(x,\tau')|dx = \int_{\partial\Omega} \left|\frac{\partial\eta_k}{\partial\nu}(x,\tau')\right| \ ds \le \int_\Omega \varphi(x)\,dx \ ,$$

for all $\tau' \in [\tau, t]$. As $\{\Delta\eta_k\}$ is uniformly bounded in $L^1(\Omega)$, we can find a subsequence $\{\eta_{k_i}\}$ and a Radon measure m_τ such that $|\Delta\eta_{k_i}|\,dx$ converges to dm_τ weakly. In particular,

$$\int_\Omega \delta U(x,\tau)|\Delta\eta_{k_i}(x,\tau)|\,dx \to \int_\Omega \delta U(x,\tau)\,dm_\tau(x) \ .$$

On the other hand, the L^1-norm of $\partial\eta_k/\partial\nu$ over $\partial\Omega \times [\tau, t]$ is uniformly bounded by $t\|\varphi\|_{L^1}$, we may assume that $|\partial\eta_{k_i}/\partial\nu|\,dsdt$ also converges weakly to some Radon measure Σ_τ. We have

$$\int_\tau^t \int_{\partial\Omega} (u_1^m - u_2^m)\left|\frac{\partial\eta_{k_i}}{\partial\nu}\right| dsdt \to \int_{\partial\Omega\times[\tau,t]} (u_1^m - u_2^m)\,d\Sigma_\tau \ .$$

The desired result follows by letting $k_i \to \infty$ in (3.6). □

Here is a version of Pierre's maximum principle.

Proposition 3.4. *Let $u_i, i = 1, 2$, be two continuous very weak solutions of (1.1) belonging to $\partial\Omega \times (0, T]$. Suppose that u_i coincide with g_i on $\partial\Omega \times (0, T]$ where $g_i \in C(\partial\Omega \times [0, T])$. Let $U_i(\cdot, t)$ be the Green's function of $u_i(\cdot, t)$. Assume that U_1 satisfies (a) $U_1(\cdot, t) \in C_0(\Omega)$, $t \in [0, T)$, and (b) $U_1(\cdot, t)$ converges to $U_1(\cdot, 0)$ uniformly as $t \to 0^+$. For each non-negative $\varphi \in C^\infty(\overline{\Omega})$ vanishing on $\partial\Omega$ and a fixed $t \in (0, T)$, there are Radon measures m on Ω and Σ on $\partial\Omega \times [0, t]$ such that*

$$\int_\Omega \delta U(x, t)\varphi(x)\ dx \le \int_\Omega \left(U_1(x, 0) - \lim_{t\to 0^+} U_2(x, t) \right)\ dm + \int_{\partial\Omega\times[0,t]} (g_1^m - g_2^m)\ d\Sigma\ .$$

Moreover,

$$m(\Omega) \le \|\varphi\|_{L^1}, \quad \Sigma(\partial\Omega \times [0, t]) \le t\|\varphi\|_{L^1}. \tag{3.7}$$

By Lemma 3.2, for continuous very weak solutions u_i, $i = 1, 2$, with Green's potential U_i, the functions

$$\tilde{U_i}(x, t) \equiv U_i(x, t) + \int_t^T h_i(x, \tau)d\tau\ ,$$

where

$$h_i(x, \tau) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x, y) g_i^m(y, \tau)\ ds(y)\ ,$$

is decreasing in t. Here $h_i(\cdot, t)$ is the harmonic function taking $g_i^m(\cdot, t)$ as its boundary value. As the lateral trace $d\nu = g_i^m\, dsdt$ a finite measure, for each $x \in \Omega$, both limits $\lim_{t\to 0^+} U_i(x, t)$ and $\lim_{t\to 0^+} \tilde{U_i}(x, t)$ exist in $[0, \infty]$ and

$$\lim_{t\to 0^+} U_i(x, t) = \lim_{t\to 0^+} \tilde{U_i}(x, t) - \int_0^T h_i(x, \tau)d\tau\ ,$$

holds.

Proof. In view of (3.5), we can pick subsequences $\tau_j \downarrow 0$ such that $m_{\tau_j}, \Sigma_{\tau_j}$ converge weakly to some m, Σ on Ω and $\partial\Omega \times [0, t]$ respectively. (We may extend Σ_τ from $\partial\Omega \times [\tau, t]$ to $\partial\Omega \times [0, t]$ trivially.)

For $k \le j$,

$$\begin{aligned}
\int_\Omega U_2(x, \tau_j)\ dm_{\tau_j} &= \int_\Omega \tilde{U_2}(x, \tau_j)\, dm_{\tau_j} - \int_\Omega \int_{\tau_j}^T h_2(x, \tau)\, d\tau dm_{\tau_j} \\
&\ge \int_\Omega \tilde{U_2}(x, \tau_k)\, dm_{\tau_j} - \int_\Omega \int_{\tau_j}^T h_2\, d\tau dm_{\tau_j} \\
&= \int_\Omega U_2(x, \tau_k)\, dm_{\tau_j} + \int_\Omega \int_{\tau_k}^{\tau_j} h_2\, d\tau dm_{\tau_j}\ .
\end{aligned}$$

Applying the maximum principle to the harmonic function h_2, $|h_2| \le \|g\|_{L^\infty}^m$. We have

$$\left| \int_\Omega \int_0^{\tau_j} h_2\, d\tau dm_{\tau_j} \right| \le \|g\|_{L^\infty}^m \tau_j \|\varphi\|_{L^1}\ ,$$

which tends to 0 as $\tau_j \to 0$. Consequently,

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega U_2(x,\tau_j)\,dm_{\tau_j} &\geq \int_\Omega U_2(x,\tau_k)\,dm - \int_\Omega\int_0^{\tau_k} h_2\,d\tau dm \\ &= \int_\Omega \tilde{U_2}(x,\tau_k)\,dm - \int_\Omega\int_0^{T} h_2\,d\tau dm\ .\end{aligned}$$

Now, letting $k\to\infty$, by the monotone convergence theorem,

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega U_2(x,\tau_j)\,dm_{\tau_j} &\geq \int_\Omega \lim_{k\to\infty}\tilde{U_2}(x,\tau_k)\,dm - \int_\Omega\int_0^{T} h_2\,d\tau dm \\ &= \int_\Omega \lim_{t\to0^+}\tilde{U_2}(x,t)\,dm - \int_\Omega\int_0^{T} h_2\,d\tau dm \\ &= \int_\Omega \lim_{t\to0^+} U_2(x,t)\,dm\ . \qquad (3.8)\end{aligned}$$

On the other hand, using assumptions (a) and (b) on U_1,

$$\lim_{j\to\infty}\int_\Omega U_1(x,\tau_j)\ dm_{\tau_j} = \int_\Omega U_1(x,0)\ dm\ . \qquad (3.9)$$

The desired result follows by letting $\tau=\tau_j\to 0$ in (3.4) and using (3.8) and (3.9). □

4. Uniqueness of weak solutions

Let u be a continuous very weak solution of (1.1) in Q_T. Suppose that it has a trace triple (μ,λ,ν) where $d\nu = g^m\,dsdt, g\in C(\partial\Omega\times[0,T])$. The Green's potential of $u(\cdot,t)$, $U(\cdot,t)$ has its L^p-norm $(1\le p< n/(n-1))$ bounded by a constant depending only on the trace triple and $\|u\|_{L^1(\rho dx)}$ (see, e.g., Widman [Wi], [4]). By Fatou's lemma, both

$$U^*(x)\equiv\lim_{t\to0^+}U(x,t)\ ,$$

and

$$\tilde{U}^*(x)\equiv U^*(x)+\int_0^T h(x,\tau)\,d\tau\ ,$$

are in $L^p(\Omega)$.

Now we prove the uniqueness part of Theorem 1.1. Let $u_i,\ i=1,2$, be two continuous very weak solutions of (1.1) with the same trace triple and $u_i=g$ along the lateral boundary. For a nonnegative $\varphi\in C_c^\infty(\Omega)$, let θ satisfy $-\Delta\theta=\varphi$ and $\theta=0$ on $\partial\Omega$. We have

$$\int_\Omega U_i(x,t)\varphi(x)\,dx = -\int_\Omega u_i(x,t)\theta(x)\,dx\ .$$

As $t\to0$, the right hand side of this identity tends to

$$\int_\Omega \theta d\mu + \int_{\partial\Omega}\frac{\partial\theta}{\partial\nu}\ d\lambda\ .$$

Therefore,

$$\lim_{t\to0^+}\int_\Omega U_1(x,t)\varphi(x)\ dx = \lim_{t\to0^+}\int_\Omega U_2(x,t)\varphi(x)\ dx.$$

By the monotonicity of $\tilde{U}_i$, we deduce

$$\int_\Omega \tilde{U_1}^*(x)\varphi(x)\,dx = \int_\Omega \tilde{U_2}^*(x)\varphi(x)\,dx\ .$$

We conclude that U_1^* and U_2^* coincide when the two weak solutions share the same trace triple.

Let $T_1 < T$ be fixed. We claim $U_1 \le U_2$ in Q_{T_1}. To this end, we fix $\tau_0 \in (0, T-T_1)$ and let $U(\cdot,t)$ be the Green's potential for $u_1(\cdot, t+\tau_0)$. Both U and U_2 are defined in Q_{T_1}. Applying Proposition 3.4 to U and U_2, we have

$$\begin{aligned}
&\int_\Omega (U(x,t) - U_2(x,t))\,\varphi\,dx \\
\le\ & \int_\Omega (U_1(x,\tau_0) - U_2^*(x))\ dm + \int_{\partial\Omega\times[0,t]} (g^m(x,t+\tau_0) - g^m(x,t))\,d\Sigma \\
=\ & \int_\Omega \left(\tilde{U_1}(x,\tau_0) - \tilde{U_1}^*(x)\right)\ dm - \int_\Omega\int_{\tau_0}^{T_1} h_1(x,\tau)\,d\tau dx + \int_\Omega\int_0^{T_1} h_2(x,\tau)\,d\tau dx \\
& + \int_{\partial\Omega\times[0,t]} (g^m(x,t+\tau_0) - g^m(x,t))\,d\Sigma \\
\le\ & -\int_\Omega\int_{\tau_0}^{T_1} h_1(x,\tau)\,d\tau dx + \int_\Omega\int_0^{T_1} h_2(x,\tau)\,d\tau dx \\
& + \int_{\partial\Omega\times[0,t]} (g^m(x,t+\tau_0) - g^m(x,t))\,d\Sigma \\
\le\ & C_1(m(\Omega) + \Sigma(\partial\Omega\times[0,t])) \sup_{\partial\Omega\times[0,T_1]} |g^m(x,t+\tau_0) - g^m(x,t)| + \tau_0 C_2 m(\Omega)\ ,
\end{aligned}$$

where the constants C_1 and C_2 are independent of τ_0. Using (3.7) and the monotonicity of $\tilde{U_1}$, we can let $\tau_0 \to 0^+$ to conclude

$$\int_\Omega (U_1(x,t) - U_2(x,t))\,\varphi\,dx \le 0\ ,$$

for all nonnegative $\varphi \in C_c^\infty(\Omega)$, hence $U_1(x,t) \le U_2(x,t)$. By reversing the role of U_1 and $U_2, U_2(x,t) \le U_1(x,t)$ also holds. Hence U_1 and U_2 coincide and this implies u_1 and u_2 are the same. The uniqueness part of Theorem 1.1 is proved.

Next we turn to the proof of Theorem 1.2.

A friendly giant with a prescribed Radon measure from $M(\partial\Omega\times(0,T))$ as its lateral trace is constructed in [4]. When the lateral trace is of the form $d\nu = g^m\,dsdt$ where $g \in C(\partial\Omega\times[0,T])$, this friendly giant belongs to $C(\overline{\Omega}\times(0,T])$ and is equal to g on the lateral boundary. Indeed, let us review the construction of the friendly giant. First of all, we may assume that g has been extended as a nonnegative, continuous function on the whole parabolic boundary. Fix a sequence of subdomains, $\{\Omega_k\}, \Omega_k \subset\subset \Omega_{k+1}$, satisfying $\bigcup_k \Omega_k = \Omega$ and an increasing of non-negative, smooth functions $\{\varphi_k\}$ satisfying $\varphi_k \equiv k$ in Ω_k and vanishes outside Ω_{k+1}. We solve the initial-boundary value problem using $g+\varphi_k$ as given data on the parabolic boundary to get an increasing sequence of continuous very weak solutions of (1.1) denoted by $\{u_k\}$. Similarly, let $\{v_k\}$ be the sequence of continuous very weak solutions taking φ_k as initial data and vanishing on the lateral boundary. We have $v_k \le u_k$ for all k. Using (2.1) and (2.2) it can be shown that both sequences converge uniformly in every $\overline{\Omega} \times (\tau, T], \tau > 0$, to continuous very weak solutions u and v which are equal to g and 0 along the lateral boundary respectively. It is known in [6] that v tends to ∞ as t tends to 0 compactly, so does u. We have proved Part (a) of Theorem 1.2.

To prove Part (b) of Theorem 1.2, we note the following result.

Lemma 4.1. *Let u be a continuous very weak solution of (1.1) in Q_T satisfying $u = g$ on $\partial\Omega \times (0, T]$ for some $g \in C(\partial\Omega \times [0, T])$. Then (1.8) holds if and only if*

$$\lim_{t\to 0^+} \int_\Omega u(x,t)\rho(x)\,dx = \infty \ . \tag{4.1}$$

Proof. Let ψ be the function satisfying $-\Delta\psi = 1$ in Ω and vanishes on the boundary. There exists some positive constant C such that $1/C\rho \le \psi \le C\rho$ in Ω. Fix a continuous, piecewise linear function ζ which is equal to 1 on $[0, T']$ for some $T' < T$ and vanishes at $(T'+T)/2$. Using $\psi(x)\zeta(t)$ as a test function in (2.3), for $t \in (0, T')$,

$$\int_\Omega u(x,t)\psi(x)\,dx = \int_t^T \int_\Omega u^m\,dxdt + \int_{T'}^T \int_\Omega (u^m\zeta - u\psi\zeta_t)\,dxdt + \int_t^T \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\zeta g^m\,dsdt\,.$$

Observing that the second and third terms on the right hand side are bounded uniformly for all t, (1.8) and (4.1) are equivalent. □

Now we can prove Part (b) of Theorem 1.2. When a continuous very weak solution u satisfies (1.8), by Lemma 4.1 (4.1) holds. Using the fact that $G(x,y) \ge C_x\rho(y),\ y \in \Omega$, for some $C_x > 0$ [W], its Green's potential U satisfies $\lim_{t\to 0^+} U(x,t) = \infty$ for every $x \in \Omega$. On the other hand, let u_1 be the friendly giant constructed in the proof of Part (a) and U_1 its Green's potential. Similarly, we have $\lim_{t\to 0^+} U_1(x,t) = \infty$. Using Proposition 3.4 and arguing as in the proof of Theorem 1.1, we conclude that U and U_1 coincide, so u is equal to u_1.

The proof of Theorem 1.2 is completed.

Based on the results in [4] and this paper, we point out some further questions.

First of all, in the case of finite total mass (1.6), the prescribed triple trace problem is completely solved. Moreover, the (unique) solution is continuous up to the lateral boundary when the lateral data is a continuous function. However, the uniqueness of the solution in the general case is not known.

In the case of infinite total mass (1.8), we conjecture that this continuous, very weak solution must be a friendly giant. Here we have shown that it is true when the lateral trace ν is given by $d\nu = g^m\,ds$ where g is a continuous function on $\partial\Omega \times [0, T]$. But the general case is open. Conversely, given any Radon measure on the lateral boundary, one would like to show that there is a unique friendly giant whose m-th power takes this measure as its lateral trace.

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K.-S. Chou
Institute of Mathematical Sciences, The Chinese University of Hong Kong, Hong Kong
E-mail address: kschou@math.cuhk.edu.hk

Y. C. Kwong
Department of Mathematics, Northern Illinois University, DeKalb, IL60115, U.S.A.
E-mail address: ykwong@niu.edu