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EXTREMAL SOLUTIONS OF QUASILINEAR ELLIPTIC VARIATIONAL INEQUALITIES IN EXTERIOR DOMAINS

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ABSTRACT. Let $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ be the exterior of the closed unit ball in $\mathbb{R}^N$. In this paper we study variational inequalities of the form

$$u \in K : \langle -\Delta_p u + af(\cdot, u), v - u\rangle \geq 0, \quad \forall v \in K,$$

where Δ_p is the p-Laplacian with $1 < p < \infty$, and the coefficient a is supposed to satisfy a certain decay condition. We are looking for solutions in the closed convex set K of the Beppo-Levi space $D_0^{1,p}(\Omega)$ which is the completion of $C_c^\infty(\Omega)$ with respect to the $\|\nabla \cdot\|_{p,\Omega}$-norm. Our main goal is to establish a sub-supersolution principle by appropriately defined sub-and supersolutions for the above variational inequality in the unbounded domain Ω, and to prove the existence of extremal solutions within an ordered interval of sub-supersolutions. As an application an obstacle problem in the exterior domain Ω is treated.

1. Introduction

Let $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$ be the exterior of the closed unit ball in $\mathbb{R}^N$, and let $X = D_0^{1,p}(\Omega)$, $1 < p < N$, be the Beppo-Levi space, which is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_X = \Big(\int_\Omega |\nabla u|^p \, dx\Big)^{\frac{1}{p}}.$$

In this paper we study quasilinear variational inequalities in the unbounded domain Ω of the form

$$u \in K : \langle -\Delta_p u + af(\cdot, u), v - u\rangle \geq 0, \quad \forall v \in K, \tag{1.1}$$

where K is a closed convex subset of X, and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some growth condition to be specified later, and the coefficient $a : \Omega \to \mathbb{R}$ is a measurable function, which decays like $|x|^{-(N+\alpha)}$ with $\alpha > 0$.

While variational inequalities in bounded domains have been studied quite extensively (see e.g. [3, 5, 8–10, 16] and the references therein), the literature on this topic in unbounded domains is rather limited, see e.g. [1, 4, 12, 13, 15]. In [4] variational inequalities in exterior domains in $\mathbb{R}^2$ have been considered in X with $p = N = 2$, which represents a certain borderline situation regarding the underlying solution

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space, because X for $p = N = 2$ is qualitatively distinct from the present situation where $1 < p < N$, see [4] for more details. The focus of the papers [1, 12, 13, 15] is on existence results for variational inequalities in unbounded domains with nonlinear elliptic operators of Leray-Lions type in a Sobolev space or Sobolev-Orlicz space setting that require certain coercivity-related conditions to meet the assumptions of abstract existence results.

The study of quasilinear elliptic variational inequalities in unbounded domains causes a number of additional difficulties such as e.g. the lack of compact embedding, and therefore cannot be considered as just a straightforward extension of the bounded domain problems. Further the choice of an appropriate solution space which is not too narrow such as the usual Sobolev spaces as used in the above mentioned papers [1, 12, 13, 15], is essential. In our treatment of the variational inequality (1.1), we make use of the Beppo-Levi space X that contains the Sobolev space $W_0^{1,p}(\Omega)$ (note $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$) as a subspace.

The main goal and the novelty of this paper is, first, to establish a sub-supersolution method and to prove the existence of solutions of (1.1) within an ordered interval $[\underline{u}, \overline{u}]$ of appropriately defined sub-supersolutions $\underline{u}$ and $\overline{u}$, respectively, which allows us to deal with variational inequalities that lack coercivity.

Second, we will characterize the solution set topologically and order-theoretically. In particular, we are going to prove the existence of extremal solutions with respect to the underlying natural partial ordering of functions, that is, $u \le v$ iff $u(x) \le v(x)$ for a.e. $x \in \Omega$. Thereby the embedding behavior of X into weighted Lebesgue spaces will play an important role to overcome some of the difficulties that arise in the functional analytic treatment of variational inequalities in unbounded domains.

Moreover, as an application of the obtained results we deal with an obstacle problem, that is, (1.1) along with the unilateral condition

$$K = \{u \in X : u(x) \ge \psi(x) \text{ for a.e. } x \in \Omega\}, \tag{1.2}$$

where the coefficient a is supposed to be positive and satisfies the decay condition above. By imposing a certain growth condition on the nonlinearity $s \mapsto f(x,s)$ in terms of the first eigenvalue λ_1 of the eigenvalue problem (in the distributional sense)

$$-\Delta_p u = \lambda a|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial B(0,1) = \partial\Omega, \tag{1.3}$$

as well as a condition on the obstacle function ψ in terms of the eigenfunction φ_1 associated with λ_1, we are able to prove the existence of extremal solutions of the obstacle problem within the interval $[\varepsilon\varphi_1, M\Gamma]$. Here the function $\Gamma \in X$ denotes the unique solution of the equation (in the distributional sense)

$$-\Delta_p u = a \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial B(0,1), \tag{1.4}$$

where the boundary values on $\partial B(0,1)$ are understood in the sense of traces.

It should be noted that the theory to be developed in this paper holds true if the p-Laplacian Δ_p in (1.1) is replaced by the more general elliptic operator of divergence type

$$\operatorname{div} A(x, \nabla u), \tag{1.5}$$

where $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector field satisfying the following Leray-Lions conditions

(A1) $|A(x,\xi)| \leq k_0(x) + c_0|\xi|^{p-1}$, $\forall \xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, where $c_0 > 0$, $k_0 \in L^{p'}(\Omega)$.

(A2) $(A(x,\xi) - A(x,\hat{\xi}))(\xi - \hat{\xi}) > 0$, $\forall \xi, \hat{\xi} \in \mathbb{R}^N$, $\xi \neq \hat{\xi}$, a.e. $x \in \Omega$.

(A3) $A(x,\xi)\xi \geq \nu|\xi|^p - k_1(x)$, $\forall \xi \in \mathbb{R}^N$, a.e. $x \in \Omega$, where $\nu > 0$ and $k_1 \in L^1(\Omega)$.

Only for the sake of clarity and emphasizing the key ideas and techniques we have confined ourselves to consider the p-Laplacian instead of the more general elliptic operator (1.5).

The paper is organized as follows: In Section 2 we introduce appropriate weighted Lebesgue spaces $L^q(\Omega, w)$ with weight w, and prove a compact embedding result of the underlying solution space X into $L^q(\Omega, w)$. Mapping properties of the Nemytskij operator F generated by f of (1.1) as well as of the operator $\mathcal{F}_a = aF : X \to X^*$ (X^* is the dual of X) are investigated, and a precise definition of solutions of (1.1) is given. In Section 3 we introduce our basic notion of sub-supersolution and establish the method of sub-supersolution. The qualitative characterization of the solution set of (1.1) enclosed by an ordered pair of sub-supersolutions such as compactness, directedness, and extremality of solutions is provided in Section 4. Finally, in the last section, Section 5, we study the obstacle problem (1.1), (1.2) in the exterior domain Ω. To this end we prove various regularity results of the solutions Γ and φ_1 of (1.4) and (1.3), respectively, as well as their qualitative behavior for $x \in \Omega \setminus \overline{B(0,R)}$, $R > 1$.

2. Hypotheses, notations, and preliminaries

Throughout this paper we assume $1 < p < N$ and $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$. Due to the Gagliardo-Nirenberg-Sobolev Inequality, the Beppo-Levi space X is continuously embedded into $L^{p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$ denoting the critical Sobolev exponent, and X can be characterized as

$$X = \Big\{ u \in L^{p*}(\Omega) : \int_\Omega |\nabla u|^p \, dx < \infty, \quad u = 0 \ \text{ on } \partial B(0,1) \Big\}, \tag{2.1}$$

which is a separable and reflexive Banach space with the norm

$$\|u\|_X = \Big(\int_\Omega |\nabla u|^p \, dx \Big)^{\frac{1}{p}}.$$

We use the following notations: for any $\sigma \in (1, \infty)$ the Hölder conjugate is denoted by σ' $(1/\sigma + 1/\sigma' = 1)$, $Y \hookrightarrow Z$ and $Y \hookrightarrow\hookrightarrow Z$ denotes the continuous and compact embedding, respectively, of normed spaces Y, Z.

The assumptions on the coefficient a and the nonlinearity f of the variational inequality (1.1) are as follows:

(Ha) Let $a : \Omega \to \mathbb{R}$ be a measurable function satisfying the decay property

$$|a(x)| \leq c_a \frac{1}{|x|^{N+\alpha}}, \quad \text{for a.e. } x \in \Omega, \tag{2.2}$$

where c_a and α are positive constants.

(Hf) The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, $x \mapsto f(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$ and $s \mapsto f(x,s)$ is continuous for a.e. $x \in \Omega$, and f satisfies the growth condition with $q \in (1, p^*)$ and some positive constant c_f,

$$|f(x,s)| \le k(x) + c_f|s|^{q-1}, \quad \forall\, s \in \mathbb{R}, \text{ and for a.e. } x \in \Omega, \tag{2.3}$$

where $k \in L^{q'}(\Omega, w)$.

The space $L^{q'}(\Omega, w)$ that appears in (Hf) denotes the weighted Lebesgue space with the weight $w : \Omega \to \mathbb{R}_+$ given by

$$w(x) = w(|x|) = \frac{1}{|x|^{N+\alpha}} \quad \text{with } \alpha > 0 \text{ as in (Ha).} \tag{2.4}$$

Clearly, $w \in L^\infty(\Omega)$. By applying spherical coordinates we get

$$\int_\Omega w(x)\,dx \le c \int_1^\infty \frac{1}{\varrho^{N+\alpha}} \varrho^{N-1}\,d\varrho < \infty,$$

which shows that $w \in L^1(\Omega)$, and thus by interpolation we have $w \in L^r(\Omega)$ for $1 \le r \le \infty$.

For any $r \in (1, \infty)$ let us shortly recall the definition of the weighted Lebesgue space $L^r(\Omega, w)$ with weight w defined by

$$L^r(\Omega, w) = \Big\{ u : \Omega \to \mathbb{R} \;\text{ measurable: } \int_\Omega w|u|^r\,dx < \infty \Big\},$$

which is a separable and reflexive Banach space under the norm

$$\|u\|_{r,w} = \Big(\int_\Omega w|u|^r\,dx \Big)^{\frac{1}{r}}.$$

For $r \in [1, \infty)$, let $\|\cdot\|_r$ denote the norm in $L^r(\Omega)$. The following embedding result will play an important role in the functional analytic treatment of the variational inequality (1.1).

Lemma 2.1. *The embedding $X \hookrightarrow\hookrightarrow L^q(\Omega, w)$ is compact for $1 < q < p^*$, that is, the embedding operator $i_w : X \to L^q(\Omega, w)$ given by $u \mapsto i_w u = u$ is linear and compact.*

Proof. Let $u \in X$, then $u \in L^{p^*}(\Omega)$, and thus for some positive constant we get

$$\int_\Omega w|u|^q\,dx \le \|w\|_{\frac{p^*}{p^*-q}} \|u\|_{p^*}^q \le c\|w\|_{\frac{p^*}{p^*-q}} \|u\|_X^q,$$

that is,

$$\|u\|_{q,w} \le c\|w\|_{\frac{p^*}{p^*-q}}^{\frac{1}{q}} \|u\|_X,$$

which shows that $i_w : X \to L^q(\Omega, w)$ is linear and continuous. Since X is reflexive, to prove the compactness of i_w, we only need to prove that i_w is completely continuous. For a sequence $(u_n) \subset X$ such that $u_n \rightharpoonup u$ (weakly) in X, we are going to show that $u_n \to u$ in $L^q(\Omega, w)$. For simplicity of notation and without loss of clarity, we shall use the same notation for a function defined on Ω and its restriction to some subset of Ω.

Let $\varepsilon > 0$ be arbitrarily given. For any $R > 1$, we have

$$\|u_n - u\|_{q,w}^q = \int_{\Omega\setminus B(0,R)} w|u_n - u|^q\,dx + \int_{\Omega\cap B(0,R)} w|u_n - u|^q\,dx. \tag{2.5}$$

Since (u_n) is bounded in X and thus in $L^{p^*}(\Omega)$, with some generic constant c independent of n and R, we can estimate the first integral on the right-hand side of (2.5) as follows:

$$\begin{aligned}\int_{\Omega\setminus B(0,R)} w|u_n - u|^q\,dx &\le c\int_{\Omega\setminus B(0,R)} w\big(|u_n|^q + |u|^q\big)\,dx\\ &\le c\|w\|_{L^{\frac{p^*}{p^*-q}}(\Omega\setminus B(0,R))}\Big(\|u_n\|^q_{L^{p^*}(\Omega\setminus B(0,R))} + \|u\|^q_{L^{p^*}(\Omega\setminus B(0,R))}\Big)\\ &\le c\|w\|_{L^{\frac{p^*}{p^*-q}}(\Omega\setminus B(0,R))}\Big(\|u_n\|^q_{p^*} + \|u\|^q_{p^*}\Big),\end{aligned}$$

which, together with the estimate $\|u_n\|_{p^*} \le c\|u_n\|_X \le c$, yields

$$\int_{\Omega\setminus B(0,R)} w|u_n - u|^q\,dx \le c\,\|w\|_{L^{\frac{p^*}{p^*-q}}(\Omega\setminus B(0,R))}. \tag{2.6}$$

The right-hand side of (2.6) can be further estimated as

$$\|w\|^{\frac{p^*}{p^*-q}}_{L^{\frac{p^*}{p^*-q}}(\Omega\setminus B(0,R))} \le c\int_R^\infty \Big(\frac{1}{\varrho^{N+\alpha}}\Big)^{\frac{p^*}{p^*-q}} \varrho^{N-1}\,d\varrho \le cR^{-(N+\alpha)\frac{p^*}{p^*-q}+N}, \tag{2.7}$$

since $-(N+\alpha)\frac{p^*}{p^*-q} + N < 0$. It follows from (2.6) and (2.7) the existence of $R > 0$ sufficiently large such that

$$\int_{\Omega\setminus B(0,R)} w|u_n - u|^q\,dx < \frac{\varepsilon}{2},\ \ \forall n \in \mathbb{N}. \tag{2.8}$$

Since $X \subset W^{1,p}_{\text{loc}}(\Omega)$, we have $X \subset W^{1,p}(\Omega \cap B(0,R))$. Taking the compact embedding $W^{1,p}(\Omega\cap B(0,R)) \hookrightarrow\hookrightarrow L^q(\Omega\cap B(0,R))$, $1 < q < p^*$, into account, we deduce from the weak convergence of u_n to u in X that

$$u_n \to u \ \ \text{(strongly) in } L^q(\Omega\cap B(0,R)).$$

In view of $u_n \to u$ (strongly) in $L^q(\Omega\cap B(0,R))$ and taking into account that $w \in L^\infty(\Omega)$ one gets

$$\int_{\Omega\cap B(0,R)} w|u_n - u|^q\,dx < \frac{\varepsilon}{2} \ \ \text{for } n \text{ sufficiently large.} \tag{2.9}$$

Thus the estimates (2.8) and (2.9) complete the proof. □

Denote by F the Nemytskij operator associated with f by $F(u)(x) = f(x, u(x))$, then the following lemma holds.

Lemma 2.2. *Under hypothesis (Hf) the Nemytskij operator $F : L^q(\Omega, w) \to L^{q'}(\Omega)$ is continuous and bounded.*

The proof of Lemma 2.2 follows standard arguments and therefore can be omitted. As an immediate consequence of Lemma 2.1 and Lemma 2.2 we get the following result.

Lemma 2.3. *Under hypothesis (Hf) the composed operator* $F \circ i_w : X \to L^{q'}(\Omega, w)$ *is completely continuous.*

By means of the coefficient $a : \Omega \to \mathbb{R}$ let us define the operator $i_a^* : L^{q'}(\Omega, w) \to X^*$ (X^* denoting the dual space of X) through

$$\eta \in L^{q'}(\Omega, w) : \langle i_a^* \eta, \varphi \rangle = \int_\Omega a\eta\varphi \, dx, \quad \forall\, \varphi \in X, \tag{2.10}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of X^* and X.

Lemma 2.4. *Assume hypothesis (Ha). Then* $i_a^* : L^{q'}(\Omega, w) \to X^*$ *is linear and continuous. Analogously, the operators* $i_{|a|}^* : L^{q'}(\Omega, w) \to X^*$ *and* $i_w^* : L^{q'}(\Omega, w) \to X^*$ *defined by*

$$\langle i_{|a|}^* \eta, \varphi \rangle = \int_\Omega |a|\eta\varphi \, dx, \quad \forall\, \varphi \in X,$$

and

$$\langle i_w^* \eta, \varphi \rangle = \int_\Omega w\eta\varphi \, dx, \quad \forall\, \varphi \in X,$$

respectively, are linear and continuous.

Proof. For any $\eta \in L^{q'}(\Omega, w)$, we have the following estimate:

$$\begin{aligned} |\langle i_a^* \eta, \varphi \rangle| &\le \int_\Omega |a|\, |\eta|\, |\varphi| \, dx \le c_a \int_\Omega w\, |\eta|\, |\varphi| \, dx \\ &\le c_a \int_\Omega w^{\frac{1}{q'}}\, |\eta|\, w^{\frac{1}{q}} |\varphi| \, dx \le c_a \|\eta\|_{q',w} \|\varphi\|_{q,w} \\ &\le c\, \|\eta\|_{q',w} \|\varphi\|_X, \quad \forall\, v \in X. \end{aligned}$$

This shows that $i_a^* \eta \in X^*$, since the linearity $\varphi \mapsto \langle i_a^* \eta, \varphi \rangle$ is obvious. Clearly, the proofs for the mappings $i_{|a|}^*$ and i_w^* follow the same line. $\square$

As an immediate consequence of Lemma 2.3 and Lemma 2.4 we obtain the following result.

Lemma 2.5. *Assume hypotheses (Ha) and (Hf). Then the operator* $\mathcal{F}_a := aF = i_a^* \circ F \circ i_w : X \to X^*$ *is completely continuous. Similarly, the operators* $\mathcal{F}_{|a|} := |a|F = i_{|a|}^* \circ F \circ i_w : X \to X^*$ *and* $\mathcal{F}_w := wF = i_w^* \circ F \circ i_w : X \to X^*$ *are completely continuous.*

Bearing in mind the above notations, the precise definition of solutions of the variational inequality (1.1) reads as follows.

Definition 2.6. The function $u \in K \subset X$ is called a solution of (1.1) if there exists a $q \in (1, p^*)$ such that $F(u) \in L^{q'}(\Omega, w)$, and the following inequality holds

$$\langle -\Delta_p u, v - u \rangle + \int_\Omega aF(u)(v - u) \, dx \ge 0, \quad \forall\, v \in K. \tag{2.11}$$

We remark that (2.11) can equivalently be written in the form

$$u \in K : \langle -\Delta_p u + \mathcal{F}_a(u), v - u \rangle \ge 0, \quad \forall\, v \in K. \tag{2.12}$$

3. Sub-supersolution method

In this section we are going to establish a sub-supersolution method for (1.1), which is based on an appropriately generalized notion of sub-supersolutions for variational inequalities. Before we define our basic notion of sub-supersolution, let us introduce some notations. For functions w, z and sets W and Z of functions defined on Ω or subsets of Ω we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. In particular, we denote $w^+ = w \vee 0$, and $w^- = (-w) \vee 0$. For functions $\underline{u} \le \overline{u}$ we denote by

$$[\underline{u}, \overline{u}] = \{u : \ \underline{u}(x) \le u(x) \le \overline{u}(x) \ \text{ for a.e. } x \in \Omega\}$$

the ordered interval formed by $\underline{u}$ and $\overline{u}$ in a corresponding function space where $\underline{u}$ and $\overline{u}$ reside.

Definition 3.1. A function $\underline{u} \in X$ is called a subsolution of the variational inequality (1.1) if $F(\underline{u}) \in L^{q'}(\Omega, w)$ (q' is the Hölder conjugate of $q \in (1, p^*)$) such that

(i) $\underline{u} \vee K \subset K$,
(ii) $\langle -\Delta_p \underline{u}, v - \underline{u}\rangle + \int_\Omega aF(\underline{u})(v - \underline{u})\, dx \ge 0, \quad \forall v \in \underline{u} \wedge K.$

Definition 3.2. A function $\overline{u} \in X$ is called a supersolution of the variational inequality (1.1) if $F(\overline{u}) \in L^{q'}(\Omega, w)$ (q' is the Hölder conjugate of $q \in (1, p^*)$) such that

(i) $\overline{u} \wedge K \subset K$,
(ii) $\langle -\Delta_p \overline{u}, v - \overline{u}\rangle + \int_\Omega aF(\overline{u})(v - \overline{u})\, dx \ge 0, \quad \forall v \in \overline{u} \vee K.$

Remark 3.3. Note that the notion of sub-supersolution defined in Definition 3.1 and Definition 3.2, respectively, have a symmetric structure, that is, one obtains Definition 3.2 for the supersolution $\overline{u}$ from Definition 3.1 by replacing $\underline{u}$ by $\overline{u}$ and interchanging $\vee$ and $\wedge$.

Next, let us recall an existence result for an abstract variational inequality, see e.g. [16, Theorem 4.16, Theorem 4.17].

Theorem 3.4. *Let $A : V \to V^*$ be a bounded pseudomonotone operator from a reflexive Banach space V to its dual V^*, and let $K \ne \emptyset$ be a closed and convex subset of V. If either K is bounded or K is unbounded and $A : V \to V^*$ is coercive (relative to K), then for any $L \in V^*$ there exists at least one solution of the variational inequality*

$$u \in K : \langle Au - L, v - u\rangle \ge 0, \quad \forall v \in K.$$

We remark that a bounded operator $A : V \to V^*$ is called coercive relative to K if there exists $v_0 \in K$ such that

$$\frac{1}{\|u\|_V}\langle Au, u - v_0\rangle \to \infty \quad \text{as} \quad \|u\|_V \to \infty. \tag{3.1}$$

Our sub-supersolution method is established by the following theorem.

Theorem 3.5. *Assume hypotheses (Ha) and (Hf) and let $\underline{u}$ and $\overline{u}$ be sub- and supersolutions of (1.1), respectively, satisfying $\underline{u} \le \overline{u}$. Then there exists at least one solution u of the variational inequality (1.1) such that $\underline{u} \le u \le \overline{u}$ in $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$.*

Proof. First, we introduce the truncated nonlinearity $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$f_0(x,s) = \begin{cases} f(x,\overline{u}(x)) & \text{if} \quad s > \overline{u}(x) \\ f(x,s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x) \\ f(x,\underline{u}(x)) & \text{if} \quad s < \underline{u}(x). \end{cases} \tag{3.2}$$

One readily verifies that $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which in view of (Hf) satisfies the following uniform bound

$$|f_0(x,s)| \le k(x) + c_f\Big(|\overline{u}(x)|^{q-1} + |\underline{u}(x)|^{q-1}\Big) =: k_0(x), \ \forall (x,s) \in \Omega \times \mathbb{R},$$

where $k_0 \in L^{q'}(\Omega, w)$, and thus by Lemma 2.2 the associated Nemytskij operator $F_0 : L^q(\Omega,w) \to L^{q'}(\Omega,w)$ is uniformly bounded and continuous, which due to Lemma 2.3, Lemma 2.4, and Lemma 2.5 implies that the operator

$$aF_0 = \mathcal{F}_{0,a} = i_a^* \circ F_0 \circ i_w : X \to X^* \tag{3.3}$$

is uniformly bounded and completely continuous, and thus, in particular, pseudomonotone.

Next let us introduce the cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$b(x,s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x) \\ -(\underline{u}(x) - s)^{p-1} & \text{if} \quad s < \underline{u}(x), \end{cases} \tag{3.4}$$

and denote by B the associated Nemytskij operator $B(u)(x) = b(x,u(x))$. Then it is clear that b is a Carathéodory function satisfying the growth condition

$$|b(x,s)| \le c_1(x) + c_2|s|^{p-1}, \ \text{with } c_1 \in L^{p'}(\Omega,w), c_2 > 0. \tag{3.5}$$

In view of (3.5), by Lemma 2.2 the Nemytskij operator $B : L^p(\Omega,w) \to L^{p'}(\Omega,w)$ is bounded and continuous. Therefore, by Lemma 2.3, Lemma 2.4, and Lemma 2.5, the operator

$$\mathcal{B}_w = i_w^* \circ B \circ i_w : X \to X^*, \ \langle \mathcal{B}_w(u), \varphi \rangle = \int_\Omega w b(\cdot,u)\varphi \, dx, \ \forall \varphi \in X, \tag{3.6}$$

is completely continuous, and thus, in particular, pseudomonotone. Further, there are positive constants c_3, c_4 such that

$$\langle \mathcal{B}_w(u), u \rangle \ge c_3 \|u\|_{p,w}^p - c_4. \tag{3.7}$$

With the above introduced truncated functions we consider the following auxiliary variational inequality, which is the key tool in our proof.

$$u \in K : \langle -\Delta_p u + \mathcal{F}_{0,a}(u) + \mathcal{B}_w(u), v - u \rangle \ge 0, \quad \forall v \in K. \tag{3.8}$$

The strategy of proof is first, to show the existence of solutions of the auxiliary problem (3.8), and then to prove that any solution u of (3.8) belongs to the interval $[\underline{u}, \overline{u}]$, which completes the proof, because then $\mathcal{B}_w(u) = 0$ and $\mathcal{F}_{0,a}(u) = \mathcal{F}_a(u)$, that is (3.8) reduces to the original variational inequality (1.1) which is equivalent to (2.12).

As for the existence we are going to make use of the abstract existence result given by Theorem 3.4 with $L = 0$, and

$$A = -\Delta_p + \mathcal{F}_{0,a} + \mathcal{B}_w : X \to X^*.$$

The elliptic operator $-\Delta_p : X \to X^*$ is a bounded, continuous, and monotone operator, which implies that A is pseudomonotone. As seen above $\mathcal{F}_{0,a} + \mathcal{B}_w : X \to X^*$ is completely continuous, and thus pseudomonotone, hence it follows that $A : X \to X^*$ is a bounded and pseudomonotone operator. In case $K \subset X$ is bounded the existence of solutions for (3.8) follows from Theorem 3.4. In case that K is unbounded, the existence follows from Theorem 3.4 provided that A is, in addition, coercive (relative to K), which we are going to prove next. From (3.7) and the uniform boundedness of $\mathcal{F}_{0,a}$ we get (with c denoting a generic positive constant)

$$\langle Au, u\rangle \geq \|u\|_X^p - c\|u\|_X + c_3\|u\|_{p,w}^p - c_4. \tag{3.9}$$

Let $v_0 \in K$ be fixed. Then we have

$$|\langle -\Delta_p u, v_0\rangle| \leq \int_\Omega |\nabla u|^{p-1}|\nabla v_0|\, dx \leq c\|u\|_X^{p-1}, \tag{3.10}$$

and by the uniform boundedness of $\mathcal{F}_{0,a} : X \to X^*$ it follows

$$|\langle \mathcal{F}_{0,a}(u), v_0\rangle| \leq \hat{c}\|v_0\|_X =: c, \tag{3.11}$$

and finally by means of (3.5) we obtain the estimate

$$|\langle \mathcal{B}_w(u), v_0\rangle| \leq \|c_1\|_{p',w}\|v_0\|_{p,w} + c_2\|u\|_{p,w}^{p-1}\|v_0\|_{p,w} \leq c\Big(1 + \|u\|_X^{p-1}\Big). \tag{3.12}$$

Thus from (3.9)–(3.12) it follows

$$\langle Au, u - v_0\rangle \geq \|u\|_X^p - c\Big(\|u\|_X^{p-1} + \|u\|_X + 1\Big) \tag{3.13}$$

which proves the coercivity of A (relative to K), since (3.13) implies

$$\frac{1}{\|u\|_X}\langle Au, u - v_0\rangle \to \infty \quad \text{as } \|u\|_X \to \infty.$$

This completes the existence proof for (3.8). To complete the proof of the theorem, it remains to show that any solution u of the auxiliary problem (3.8) satisfies $\underline{u} \leq u \leq \overline{u}$. Let us verify that $\underline{u} \leq u$. By Definition 3.1 of the subsolution we have $\underline{u} \vee K \subset K$ and

$$\langle -\Delta_p \underline{u}, v - \underline{u}\rangle + \int_\Omega aF(\underline{u})(v - \underline{u})\, dx \geq 0, \quad \forall v \in \underline{u} \wedge K.$$

In particular, $v = \underline{u} \wedge u = \underline{u} - (\underline{u} - u)^+$ is an admissible test function in the last inequality, which yields

$$\langle -\Delta_p \underline{u}, -(\underline{u} - u)^+\rangle - \int_\Omega aF(\underline{u})(\underline{u} - u)^+\, dx \geq 0. \tag{3.14}$$

On the other hand for the auxiliary variational inequality (3.8) we can apply the special test function $v = \underline{u} \vee u \in K$, that is $v = u + (\underline{u} - u)^+$ which gives

$$\langle -\Delta_p u + \mathcal{F}_{0,a}(u) + \mathcal{B}_w(u), (\underline{u} - u)^+\rangle \geq 0. \tag{3.15}$$

Adding (3.14) and (3.15) we receive

$$\begin{aligned}(3.16)\qquad &\int_\Omega a\big(F_0(u)-F(\underline{u})\big)(\underline{u}-u)^+dx+\int_\Omega wb(\cdot,u)(\underline{u}-u)^+dx\\ &\geq \langle -\Delta_p\underline{u}-(-\Delta_p u),(\underline{u}-u)^+\rangle.\end{aligned}$$

Denote $\{u<\underline{u}\}:=\{x\in\Omega : u(x)<\underline{u}(x)\}$. By the definition of F_0 we get

$$\int_\Omega a\big(F_0(u)-F(\underline{u})\big)(\underline{u}-u)^+dx=\int_{\{u<\underline{u}\}} a\big(F(\underline{u})-F(\underline{u})\big)(\underline{u}-u)dx=0.$$

Since the right-hand side of (3.16) is nonnegative, we arrive at

$$\int_\Omega wb(\cdot,u)(\underline{u}-u)^+dx\geq 0,$$

which by definition of b yields

$$0\leq \int_{\{u<\underline{u}\}} w\big[-(\underline{u}-u)^{p-1}\big](\underline{u}-u)dx=-\int_{\{u<\underline{u}\}} w(\underline{u}-u)^p\,dx\leq 0,$$

and thus $\|(\underline{u}-u)^+\|_{w,p}=0$, that is, $(\underline{u}-u)^+=0$ which proves $\underline{u}\leq u$. The proof of the inequality $u\leq\overline{u}$ follows by similar arguments and can be omitted, which completes the proof of Theorem 3.5. □

4. Extremal solutions

Let $\mathcal{S}$ denote the solution set of (1.1) (resp. (2.11)) within the ordered interval $[\underline{u},\overline{u}]$ of sub-supersolutions. Then By Theorem 3.5 we have $\mathcal{S}\neq\emptyset$. In this section, first we are going to show that $\mathcal{S}$ is a compact subset of X. Further, if K satisfies, in addition, the lattice condition

$$(4.1)\qquad K\wedge K\subset K \quad\text{and}\quad K\vee K\subset K,$$

then we will show the existence of extremal solutions, that is, the existence of smallest and greatest solutions of $\mathcal{S}$.

Theorem 4.1. *Let the hypotheses of Theorem 3.5 be satisfied. Then the solution set $\mathcal{S}$ is a compact subset of X.*

Proof. Let $(u_n)\subset\mathcal{S}$, which means $\underline{u}\leq u_n\leq\overline{u}$, and u_n solves (1.1) (resp. (2.11)), i.e.,

$$(4.2)\qquad u_n\in K:\langle -\Delta_p u_n, v-u_n\rangle+\int_\Omega aF(u_n)(v-u_n)\,dx\geq 0,\quad \forall v\in K.$$

Clearly, the sequence $(F(u_n))$ is bounded in $L^{q'}(\Omega,w)$, and for $v_0\in K$ fixed we obtain from (4.2) (with c being some positive generic constant)

$$\begin{aligned}\langle -\Delta_p u_n,u_n\rangle &= \|u_n\|_X^p\leq\langle -\Delta_p u_n,v_0\rangle+\int_\Omega aF(u_n)(v_0-u_n)\,dx\\ &\leq \|u_n\|_X^{p-1}\|v_0\|_X+c_a\int_\Omega w|F(u_n)|\big(|v_0|+|u_n|\big)dx\\ &\leq \|u_n\|_X^{p-1}\|v_0\|_X+c\|F(u_n)\|_{q',w}\big(\|v_0\|_{q,w}+\|u_n\|_{q,w}\big)\\ &\leq c\Big(\|u_n\|_X^{p-1}+\|u_n\|_X+1\Big),\end{aligned}$$

which implies that (u_n) is bounded in X. Thus there is a weakly convergent subsequence (u_j) of (u_n) such that $u_j \rightharpoonup u$ (weakly) in X. Since K is weakly closed, we infer that the weak limit $u \in K$. The compact embedding $X \hookrightarrow\hookrightarrow L^q(\Omega, w)$ yields $u_j \to u$ (strongly) in $L^q(\Omega, w)$ and $\underline{u} \le u \le \overline{u}$, and therefore $F(u_j) \to F(u)$ in $L^{q'}(\Omega, w)$. With $v = u$ and u_n replaced by u_j, from (4.2) we obtain

$$\langle -\Delta_p u_j, u_j - u\rangle \le c\|F(u_j)\|_{q',w}\|u - u_j\|_{q,w} \to 0 \text{ as } j \to \infty. \tag{4.3}$$

From (4.3) along with $u_j \rightharpoonup u$ in X and taking into account that $-\Delta_p : X \to X^*$ is monotone we see that

$$\begin{aligned}&\langle -\Delta_p u_j - (-\Delta_p u), u_j - u\rangle \\ &= \textstyle\int_\Omega \Big(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\Big)\Big(\nabla u_j - \nabla u\Big)\, dx \to 0 \text{ as } j \to \infty.\end{aligned} \tag{4.4}$$

Further, by applying Hölder's inequality we can estimate as follows

$$\begin{aligned}&\int_\Omega \Big(|\nabla u_j|^{p-2}\nabla u_j - |\nabla u|^{p-2}\nabla u\Big)\Big(\nabla u_j - \nabla u\Big)\, dx \\ \ge\ & \int_\Omega \Big(|\nabla u_j|^p + |\nabla u|^p\Big)\, dx - \int_\Omega \Big(|\nabla u_j|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla u_j|\Big)\, dx \\ \ge\ & \|u_j\|_X^p + \|u\|_X^p - \|u_j\|_X^{p-1}\|u\|_X - \|u\|_X^{p-1}\|u_j\|_X \\ =\ & (\|u_j\|_X^{p-1} - \|u\|_X^{p-1})(\|u_j\|_X - \|u\|_X) \ge 0.\end{aligned} \tag{4.5}$$

and thus by (4.4) we get

$$\lim_{j\to\infty} (\|u_j\|_X^{p-1} - \|u\|_X^{p-1})(\|u_j\|_X - \|u\|_X) = 0,$$

which implies

$$\lim_{j\to\infty} \|u_j\|_X = \|u\|_X. \tag{4.6}$$

The weak convergence $u_j \rightharpoonup u$ together with (4.6) yields the strong convergence $u_j \to u$ in X, which allows us to pass to the limit in (4.2) with u_n replaced by u_j as $j \to \infty$ showing that the strong limit u of the subsequence (u_j) belongs to $\mathcal{S}$. □

Next we are going to prove that $\mathcal{S}$ is a directed set, which means that for any $u_1, u_2 \in \mathcal{S}$ there exists a $z \in \mathcal{S}$ such that $\max\{u_1, u_2\} \le z$ (directed upward) as well as a $w \in \mathcal{S}$ such that $w \le \min\{u_1, u_2\}$ (directed downward).

Theorem 4.2. *Let the hypotheses of Theorem 3.5 be satisfied, and assume the lattice condition (4.1) on the closed convex set K be fulfilled. Then the solution set $\mathcal{S}$ is a directed set.*

Proof. Let us show that $\mathcal{S}$ is directed upward only, as the proof for directed downward can be carried out in a similar way by obvious modifications.

Given $u_1, u_2 \in \mathcal{S}$, we will prove the existence of a solution $u \in \mathcal{S}$ such that $\max\{u_1, u_2\} \le u$. Our approach is roughly speaking the following: we first construct an appropriate auxiliary variational inequality in terms of the given $u_1, u_2 \in \mathcal{S}$, and show the existence of solutions. Thereby the construction of the auxiliary problem is done in such a way that any solution u of it belongs again to $\mathcal{S}$, and, in addition, satisfies $u_m \le u$ for $m = 1, 2$, which is the desired directed upward of $\mathcal{S}$.

By definition the given $u_m \in \mathcal{S}$, $(m = 1, 2\ )$ satisfy $u_m \in [\underline{u}, \overline{u}]$ and

$$u_m \in K : \langle -\Delta_p u_m, v - u_m\rangle + \int_\Omega aF(u_m)(v - u_m)\, dx \geq 0, \quad \forall v \in K, \tag{4.7}$$

where $F(u_m) \in L^{q'}(\Omega, w)$. Define $u_0 = \max\{u_1, u_2\}$. Then $F(u_0) \in L^{q'}(\Omega, w)$ is given by

$$F(u_0)(x) = f(x, u_0(x)) = \begin{cases} f(x, u_1(x)) & \text{if} \quad x \in \{u_1 \geq u_2\} \\ f(x, u_2(x)) & \text{if} \quad x \in \{u_2 > u_1\}, \end{cases} \tag{4.8}$$

where $\{u_l \geq (>) u_m\}$ stands for $\{x \in \Omega : u_l(x) \geq (>) u_m(x)\}$. Further, we introduce the truncated function $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(x, s) = \begin{cases} f(x, u_0(x)) & \text{if} \quad s < u_0(x) \\ f(x, s) & \text{if} \quad u_0(x) \leq s \leq \overline{u}(x) \\ f(x, \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{4.9}$$

From (Hf) it readily follows that $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is uniformly bounded, i.e.,

$$|\hat{f}(x, s)| \leq \hat{k}(x), \quad \forall (x, s) \in \Omega \times \mathbb{R}, \tag{4.10}$$

where $\hat{k} \in L^{q'}(\Omega, w)$ can be given in terms of k, $\overline{u}$, and u_0. Thus the associated Nemytskij operator $\hat{F}$ is a uniformly bounded and continuous mapping from $L^q(\Omega, w)$ to $L^{q'}(\Omega, w)$, which implies that the operator

$$\hat{\mathcal{F}}_a := i_a^* \circ \hat{F} \circ i_w : X \to X^* \tag{4.11}$$

is uniformly bounded, and completely continuous, thus pseudomonotone. Next define truncated functions $f_m : \Omega \times \mathbb{R} \to \mathbb{R}$ $(m = 1, 2)$ as follows

$$f_m(x, s) = \begin{cases} f(x, u_m(x)) & \text{if} \quad s < u_m(x) \\ f(x, s) & \text{if} \quad u_m(x) \leq s \leq \overline{u}(x) \\ f(x, \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{4.12}$$

By similar arguments as for $\hat{f}$, we have $f_m : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that are uniformly bounded by some $k_m \in L^{q'}(\Omega, w)$, that is

$$|f_m(x, s)| \leq k_m(x), \quad \forall (x, s) \in \Omega \times \mathbb{R}, \tag{4.13}$$

and the associated Nemytskij operators $F_m : L^q(\Omega, w) \to L^{q'}(\Omega, w)$ are uniformly bounded and continuous, which yields uniformly bounded and completely continuous, thus pseudomonotone, operators $\mathcal{F}_{a,m}$ given by

$$\mathcal{F}_{a,m} := i_a^* \circ F_m \circ i_w : X \to X^*. \tag{4.14}$$

Finally, we introduce the cut-off function $\hat{b}$ defined by

$$\hat{b}(x, s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad u_0(x) \leq s \leq \overline{u}(x) \\ -(u_0(x) - s)^{p-1} & \text{if} \quad s < u_0(x), \end{cases} \tag{4.15}$$

which qualitatively behaves like b (see (3.4)) in Section 3, that is, $\hat{b} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and its associated Nemytskij operator $\hat{B}$ is a bounded and continuous mapping from $L^q(\Omega, w)$ to $L^{q'}(\Omega, w)$, and thus the operator

$$\hat{\mathcal{B}}_w := i_a^* \circ \hat{B} \circ i_w : X \to X^*. \tag{4.16}$$

is completely continuous, which implies that $\hat{\mathcal{B}}_w : X \to X^*$ is bounded and pseudomonotone. By means of F_m and $\hat{F}$ we introduce operators T_m $(m = 1, 2)$ by

$$T_m(u) = |F_m(u) - \hat{F}(u)|. \tag{4.17}$$

The properties of F_m and $\hat{F}$ immediately imply that $T_m : L^q(\Omega, w) \to L^{q'}(\Omega, w)$ are uniformly bounded and continuous, and thus

$$\mathcal{T}_{|a|,m} = i_{|a|} \circ T_m \circ i_w = |a| T_m \circ i_w : X \to X^*$$

are uniformly bounded and completely continuous, thus pseudomonotone, operators.

Now we are ready to formulate our basic auxiliary variational inequality, which reads as follows:

$$u \in K : \langle -\Delta_p u + \hat{\mathcal{B}}_w(u) + \hat{\mathcal{F}}_a(u) - \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), v - u \rangle \geq 0, \quad \forall v \in K. \tag{4.18}$$

Since $\hat{\mathcal{F}}_a - \sum_{j=1}^2 \mathcal{T}_{|a|,j} : X \to X^*$ is uniformly bounded and pseudomonotone and $\hat{\mathcal{B}}_w : X \to X^*$ behaves qualitatively like $\mathcal{B}_w$ (see (3.6)), the same arguments as for the operator A in the proof of Theorem 3.5 apply to the operator

$$\hat{A} = -\Delta_p + \hat{\mathcal{B}}_w + \hat{\mathcal{F}}_a - \sum_{j=1}^{2} \mathcal{T}_{|a|,j} : X \to X^*,$$

which ensure that $\hat{A} : X \to X^*$ is a bounded, pseudomonotone and coercive operator, and thus there exist solutions of the auxiliary variational inequality (4.18) by applying Theorem 3.4.

The proof of Theorem 4.2 is finished provided we are able to show that any solution u of the auxiliary variational inequality (4.18) satisfies the inequality

$$u_m \leq u \leq \overline{u}, \quad (m = 1, 2), \tag{4.19}$$

because then we get $\hat{\mathcal{B}}_w(u) = 0$, $\hat{\mathcal{F}}_a(u) = \mathcal{F}_a(u)$, and $\mathcal{T}_{|a|,m}(u) = 0$, that is, (4.18) reduces to the original variational inequality (1.1) (resp. (2.12)) and thus $u \in \mathcal{S}$ and $\max\{u_1, u_2\} \leq u$, which completes the proof for $\mathcal{S}$ being directed upward.

Let us first show that $u_m \leq u$, $(m = 1, 2)$. By assumption $u_m \in \mathcal{S}$, that is $u_m \in [\underline{u}, \overline{u}]$ and

$$u_m \in K : \langle -\Delta_p u_m + \mathcal{F}_a(u_m), v - u_m \rangle \geq 0, \quad \forall v \in K. \tag{4.20}$$

By virtue of the lattice condition (4.1), we may use the special test function $v = u_m \vee u = u + (u_m - u)^+$ in (4.18) which results in

$$\langle -\Delta_p u + \hat{\mathcal{B}}_w(u) + \hat{\mathcal{F}}_a(u) - \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u_m - u)^+ \rangle \geq 0, \tag{4.21}$$

and in (4.20) we may apply the special test function $v = u_m \wedge u = u_m - (u_m - u)^+$ which yields

$$\langle -\Delta_p u_m + \mathcal{F}_a(u_m), -(u_m - u)^+ \rangle \ge 0. \tag{4.22}$$

Adding inequalities (4.21) and (4.22) we arrive at

$$\begin{aligned} &\langle \hat{\mathcal{B}}_w(u) + (\hat{\mathcal{F}}_a(u) - \mathcal{F}_a(u_m)) - \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u_m - u)^+ \rangle \\ &\ge \langle -\Delta_p u_m - (-\Delta_p u), (u_m - u)^+ \rangle. \end{aligned} \tag{4.23}$$

For the right-hand side of (4.23) we get

$$\begin{aligned} &\langle -\Delta_p u_m - (-\Delta_p u), (u_m - u)^+ \rangle \\ &= \int_{\{u<u_m\}} \Big(|\nabla u_m|^{p-2} - \nabla u_m - |\nabla u|^{p-2} - \nabla u \Big)(\nabla u_m - \nabla u) dx \ge 0. \end{aligned} \tag{4.24}$$

Since $u_m \le u_0$ $(m = 1, 2)$, by the definition of $\hat{F}$ one obtains

$$\begin{aligned} \langle \hat{\mathcal{F}}_a(u) - \mathcal{F}_a(u_m), (u_m - u)^+ \rangle &= \int_\Omega a(\hat{F}(u) - F(u_m))(u_m - u)^+ dx \\ &= \int_{\{u<u_m\}} a(F(u_0) - F(u_m))(u_m - u) dx. \end{aligned} \tag{4.25}$$

The third term on the left-hand side of (4.23) is equal to

$$\begin{aligned} \Big\langle \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u_m - u)^+ \Big\rangle &= \int_\Omega \sum_{j=1}^{2} |a||F_j(u) - \hat{F}(u)|(u_m - u)^+ dx \\ &= \int_{\{u<u_m\}} |a||F_j(u) - F(u_0)|(u_m - u) dx \quad (j \ne m) \\ &\quad + \int_{\{u<u_m\}} |a||F(u_m) - F(u_0)|(u_m - u) dx \quad (j = m) \end{aligned} \tag{4.26}$$

Thus by (4.25) and (4.26) we get

$$\begin{aligned} &\langle \hat{\mathcal{F}}_a(u) - \mathcal{F}_a(u_m) - \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u_m - u)^+ \rangle \\ &\le - \int_{\{u<u_m\}} |a||F_j(u) - F(u_0)|(u_m - u) dx \le 0. \end{aligned} \tag{4.27}$$

Taking (4.24) and (4.27) into account from (4.23) we finally obtain

$$\langle \hat{\mathcal{B}}_w(u), (u_m - u)^+ \rangle \ge 0, \tag{4.28}$$

which by definition of $\hat{\mathcal{B}}_w$ and $u_m \le u_0$ results in

$$\begin{aligned} 0 &\le \int_\Omega w\hat{b}(\cdot, u)(u_m - u)^+ dx = - \int_{\{u<u_m\}} w(u_0 - u)^{p-1}(u_m - u) dx \\ &\le - \int_{\{u<u_m\}} w(u_m - u)^p \, dx \le 0, \end{aligned}$$

hence it follows

$$0 = \int_{\{u<u_m\}} w(u_m - u)^p \, dx = \int_\Omega w[(u_m - u)^+]^p \, dx = \|(u_m - u)^+\|_{p,w}^p$$

which yields $(u_m - u)^+ = 0$, that is, $u_m \le u$ $(m = 1, 2)$ proving the first inequality of (4.19).

As for the proof of the second inequality of (4.19), that is, $u \le \overline{u}$ for any solution u of the auxiliary variational inequality (4.18), we recall the definition of the supersolution (see Definition 3.2) according to which $\overline{u} \wedge K \subset K$ and

$$\langle -\Delta_p \overline{u}, v - \overline{u} \rangle + \int_\Omega aF(\overline{u})(v - \overline{u}) \, dx \ge 0, \quad \forall v \in \overline{u} \vee K. \tag{4.29}$$

Taking the special test function $v = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+$ in (4.29) we get

$$\langle -\Delta_p \overline{u}, (u - \overline{u})^+ \rangle + \int_\Omega aF(\overline{u})(u - \overline{u})^+ \, dx \ge 0, \tag{4.30}$$

and using in (4.18) the special test function $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ we obtain

$$\langle -\Delta_p u + \hat{\mathcal{B}}_w(u) + \hat{\mathcal{F}}_a(u) - \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), -(u - \overline{u})^+ \rangle \ge 0. \tag{4.31}$$

Adding (4.30) and (4.31), and taking into account that

$$\langle -\Delta_p u - (-\Delta_p \overline{u}), (u - \overline{u})^+ \rangle \ge 0,$$

one receives

$$\langle -\hat{\mathcal{B}}_w(u) + (\mathcal{F}_a(\overline{u}) - \hat{\mathcal{F}}_a(u)) + \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u - \overline{u})^+ \rangle \ge 0. \tag{4.32}$$

By definition of $\hat{f}$ and f_m and their corresponding Nemytskij operators $\hat{F}$ and F_m, respectively, we obtain for the second and third term on the left-hand side of (4.32)

$$\begin{aligned} \langle \mathcal{F}_a(\overline{u}) - \hat{\mathcal{F}}_a(u), (u - \overline{u})^+ \rangle &= \int_{\{u>\overline{u}\}} a\big(F(\overline{u}) - \hat{F}(u)\big)(u - \overline{u}) \, dx \\ &= \int_{\{u>\overline{u}\}} a\big(F(\overline{u}) - F(\overline{u})\big)(u - \overline{u}) \, dx = 0, \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} \Big\langle \sum_{j=1}^{2} \mathcal{T}_{|a|,j}(u), (u - \overline{u})^+ \Big\rangle &= \sum_{j=1}^{2} \int_{\{u>\overline{u}\}} |a| \big| F_j(u) - \hat{F}(u) \big| (u - \overline{u}) \, dx \\ &= \sum_{j=1}^{2} \int_{\{u>\overline{u}\}} |a| \big| F(\overline{u}) - F(\overline{u}) \big| (u - \overline{u}) \, dx = 0. \end{aligned} \tag{4.34}$$

In view of (4.33) and (4.34), we get from (4.32)

$$\langle \hat{\mathcal{B}}_w(u), (u - \overline{u})^+ \rangle \le 0,$$

which by definition of $\hat{\mathcal{B}}_w$ yields

$$0 \geq \langle \hat{\mathcal{B}}_w(u), (u-\overline{u})^+ \rangle = \int_{\{u>\overline{u}\}} w\hat{b}(\cdot,u)(u-\overline{u})\,dx = \int_{\{u>\overline{u}\}} w(u-\overline{u})^p\,dx \geq 0,$$

and hence

$$0 = \int_{\{u>\overline{u}\}} w(u-\overline{u})^p\,dx = \int_\Omega w[(u-\overline{u})^+]^p\,dx = \|(u-\overline{u})^+\|_{p,w}^p,$$

which implies $(u-\overline{u})^+ = 0$, that is, $u \leq \overline{u}$ in Ω completing the proof of $\mathcal{S}$ being directed upward. The proof for $\mathcal{S}$ being directed downward follows pretty much the same idea by appropriately modifying the auxiliary variational inequality, and therefore can be omitted. This completes the proof of the theorem. □

The compactness of $\mathcal{S}$ by Theorem 4.1 and the directedness of $\mathcal{S}$ by Theorem 4.2 will allow us to show the following existence result of extremal solutions.

Theorem 4.3. *Let the hypotheses of Theorem 4.2 be satisfied. Then the solution set $\mathcal{S}$ has a smallest solution u_* and a greatest solution u^*, that is, $u_*,\ u^* \in \mathcal{S}$ such that $u_* \leq u \leq u^*$ for all $u \in \mathcal{S}$.*

Proof. Let us show the existence of the greatest solution u^* only, since the proof for the existence of u_* follows analogous arguments. Since X is separable with the metric generated by $\|\cdot\|_X$, so is $\mathcal{S}$. Let (w_n) be a dense sequence in $\mathcal{S}$. Using the directedness of $\mathcal{S}$, we can construct inductively a sequence (u_n) in $\mathcal{S}$ such that $w_n \leq u_n \leq u_{n+1}, \forall n \in \mathbb{N}$. Let

$$u^*(x) = \sup\{u_n(x) : n \in \mathbb{N}\} = \lim_{n\to\infty} u_n(x),\ x \in \Omega.$$

As a consequence of the compactness of $\mathcal{S}$, and the pointwise convergence of (u_n), we get $u_n \to u^*$ in X and thus $u^* \in \mathcal{S}$. Since $u^* \geq w_n$ a.e. in Ω for all $n \in \mathbb{N}$, from the density of (w_n) in $\mathcal{S}$, we see that $u^* \geq u$ a.e. in Ω for all $u \in \mathcal{S}$. □

5. Obstacle problem

In this section we apply the theory developed in Section 3 and Section 4 to the following obstacle problem in the exterior domain $\Omega = \mathbb{R}^N \setminus \overline{B(0,1)}$:

$$u \in K : \langle -\Delta_p u, v-u \rangle + \int_\Omega af(\cdot,u)(v-u)\,dx \geq 0, \quad \forall v \in K, \tag{5.1}$$

where

$$K = \{u \in X : u \geq \psi \quad \text{a.e. in } \Omega\} \tag{5.2}$$

and $\psi : \Omega \to \mathbb{R}$ is a measurable function representing the obstacle, which will be specified later. We readily observe that K given by (5.2) satisfies the lattice condition (4.1). The coefficient $a : \Omega \to \mathbb{R}$ in (5.1) is supposed to be positive and fulfills (Ha), that is,

(Ha$_+$) $a : \Omega \to \mathbb{R}$ is measurable and satisfies

$$0 < a(x) \leq c_a \frac{1}{|x|^{N+\alpha}} = c_a w(x), \quad \text{for a.e. } x \in \Omega,$$

where c_a and α are positive constants and w given by (2.4).

We will provide conditions on the nonlinearity f of (5.1) as well as on the obstacle function ψ of (5.2) in terms of the first eigenvalue λ_1 and associated eigenfunction φ_1 of the eigenvalue problem

$$-\Delta_p u = \lambda a|u|^{p-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial B(0,1), \tag{5.3}$$

that will ensure the existence of extremal solutions of (5.1), (5.2) within the order interval $[\varepsilon\varphi_1, M\Gamma]$ for some positive constants ε and M. Here $\Gamma \in X$ denotes the unique solution of the boundary value problem

$$-\Delta_p u = a \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial B(0,1). \tag{5.4}$$

whose existence, uniqueness, regularity, and further qualitative properties will be proved by the next lemma. To this end we use the notation $\Omega_R := \Omega \setminus \overline{B_R}$, where $B_R = B(0,R) \subset \mathbb{R}^N$ denotes the ball with radius $R > 1$.

Lemma 5.1. *Assume hypothesis (Ha$_+$). Then problem (5.4) has a unique solution $\Gamma \in X$ with the following properties:*

(i) $\Gamma \in C^{1,\gamma}(\overline{\Omega \cap B_R})$, *with $\gamma \in (0,1)$ and $R > 1$.*
(ii) $\Gamma(x) > 0$ *for all $x \in \Omega$.*
(iii) *There exist positive constants d_1, d_2 such that*

$$\frac{d_1}{|x|^{\frac{N-p}{p-1}}} \le \Gamma(x) \le \frac{d_2}{|x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \overline{\Omega_R},\ R > 1. \tag{5.5}$$

(iv) $\frac{\partial\Gamma}{\partial n}(x) < 0, \quad \forall x \in \partial B(0,1)$, *where $\partial\Gamma/\partial n$ is the outward normal derivative at $x \in \partial B(0,1)$.*

Proof. $u \in X$ is a solution of (5.4) if

$$\langle -\Delta_p u, \varphi\rangle = \int_\Omega a\varphi\, dx, \quad \forall \varphi \in X. \tag{5.6}$$

Since $w \in L^r(\Omega)$ for all $r \in [1,\infty]$, by hypothesis (Ha$_+$) we have $a \in L^r(\Omega)$ for all $r \in [1,\infty]$. In particular, $a \in L^{p^{*\prime}}(\Omega)$, and thus due to $X \hookrightarrow L^{p^*}(\Omega)$, $\varphi \mapsto \int_\Omega a\varphi\, dx$ belongs to X^*. Since $-\Delta_p : X \to X^*$ is a strictly monotone, bounded, and continuous mapping, the unique solvability of (5.4) follows from standard monotone operator theory.

Ad (i): We note $X \hookrightarrow W^{1,p}_{\text{loc}}(\Omega)$, and thus $X \hookrightarrow W^{1,p}(\Omega \cap B_R)$ for any $R > 1$. Since $a \in L^\infty(\Omega)$, by elliptic regularity results (see e.g. [7, 11]), we get $\Gamma \in C^{1,\gamma}(\overline{\Omega \cap B_R})$, with $\gamma \in (0,1)$. In particular, Γ is in $C^1(\overline{\Omega})$.

Ad (ii): Replacing u by Γ in (5.6) and testing it with the test function $\varphi = \Gamma^- = \max\{-\Gamma, 0\}$ we obtain

$$\langle -\Delta_p\Gamma, \Gamma^-\rangle = -\int_\Omega |\nabla\Gamma^-|^p\, dx = \int_\Omega a\Gamma^-\, dx \ge 0,$$

which yields $\|\Gamma^-\|_X = 0$, i.e., $\Gamma^- = 0$, and thus $\Gamma \ge 0$ in Ω. By (i), Γ is, in particular, continuous in Ω, and hence it follows from Harnack's inequality that $\Gamma(x) > 0$ for all $x \in \Omega$.

Ad (iii): If Φ is given by

$$\Phi(x) = \frac{1}{|x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \overline{\Omega},$$

then

$$-\Delta_p \Phi = 0 \quad \text{in } \Omega.$$

Since $\Gamma(x) > 0$ in Ω, and continuous in $\overline{\Omega}$, we have

$$\delta := \min_{|x|=R} \Gamma(x) > 0, \quad \text{for } R > 1$$

is well defined, and there is some $\varrho > 0$ such that (note: $\Phi(x) = \frac{1}{R^{\frac{N-p}{p-1}}}$, for $|x| = R$)

$$\varrho \frac{1}{R^{\frac{N-p}{p-1}}} = \delta,$$

and thus

$$\varrho\Phi - \Gamma \le 0 \quad \text{on } \partial B_R, \tag{5.7}$$

and the following differential inequality (in the distributional sense) holds

$$-\Delta_p(\varrho\Phi) - (-\Delta_p \Gamma) = -a \le 0 \quad \text{in } \Omega_R. \tag{5.8}$$

Let us denote by X_R the completion of $C_c^\infty(\Omega_R)$ with respect to the Norm

$$\|u\|_{X_R} = \Big(\int_{\Omega_R} |\nabla u|^p \, dx \Big)^{\frac{1}{p}}.$$

Then due to (5.7), we have $0 \le (\varrho\Phi - \Gamma)^+ \in X_R$, which yields when applied as special test function to (5.8)

$$0 \ge \int_{\Omega_R} \Big(|\nabla(\varrho\Phi)|^{p-2}\nabla(\varrho\Phi) - |\nabla\Gamma|^{p-2}\nabla\Gamma \Big) \nabla(\varrho\Phi - \Gamma)^+ \, dx \ge 0.$$

The last inequality implies

$$\int_{\{x\in\Omega_R : \varrho\Phi(x) - \Gamma(x) > 0\}} \Big(|\nabla(\varrho\Phi)|^{p-2}\nabla(\varrho\Phi) - |\nabla\Gamma|^{p-2}\nabla\Gamma \Big) \big(\nabla(\varrho\Phi) - \nabla\Gamma \big) \, dx = 0,$$

and therefore $\operatorname{meas}\{x \in \Omega_R : \varrho\Phi(x) - \Gamma(x) > 0\} = 0$, that is,

$$\varrho\Phi(x) \le \Gamma(x), \quad \forall x \in \overline{\Omega_R}, \ R > 1,$$

which is the first part of inequality (5.5) with $d_1 = \varrho$.

Let us verify the second part of inequality (5.5). To this end we extend the coefficient a to the entire $\mathbb{R}^N$ by a positive constant, such as e.g. by c_a, then the extension $\hat{a} : \mathbb{R}^N \to \mathbb{R}$ given by

$$\hat{a}(x) = \begin{cases} c_a & \text{if} \quad x \in \overline{B(0,1)} \\ a(x) & \text{if} \quad x \in \Omega. \end{cases} \tag{5.9}$$

is a positive, measurable function, which satisfies

$$0 < \hat{a}(x) \le \frac{2c_a}{1 + |x|^{N+\alpha}}, \quad \forall x \in \mathbb{R}^N. \tag{5.10}$$

Consider the following problem in all $\mathbb{R}^N$

$$-\Delta_p u = \hat{a}(x) \quad \text{in } \mathbb{R}^N. \tag{5.11}$$

For solving (5.11) let $\hat{X}$ denote the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\hat{X}} = \Big(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\Big)^{\frac{1}{p}},$$

which can be characterized as

$$\hat{X} = \Big\{u \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p \, dx < \infty\Big\}.$$

In view of (5.10), we may apply [2, Theorem 2.2] to problem (5.11), which results in the existence of a uniquely defined solution $\hat{u} \in \hat{X}$ of (5.11) satisfying $\hat{u} \in \hat{X} \cap C^1(\mathbb{R}^N)$ and the inequality

$$\frac{c_1}{1 + |x|^{\frac{N-p}{p-1}}} \le \hat{u}(x) \le \frac{c_2}{1 + |x|^{\frac{N-p}{p-1}}}, \quad x \in \mathbb{R}^N, \tag{5.12}$$

where c_1, c_2 are some positive constants. In particular, $\hat{u}$ is a distributional solution of

$$-\Delta_p \hat{u} = \hat{a}(x) = a(x) \quad \text{in } \Omega_R,\ R > 1. \tag{5.13}$$

Since $\Gamma(x) > 0$ in Ω and $\hat{u}(x) > 0$ in $\mathbb{R}^N$ and both are continuous, there is some positive constant $\hat{\varrho}$, which may be chosen $\hat{\varrho} > 1$ such that

$$\Gamma(x) - \hat{\varrho}\hat{u}(x) \le 0, \quad \forall x \in \partial B_R. \tag{5.14}$$

Further, by (5.13) in Ω_R we have the equation (in the distributional sense)

$$-\Delta_p \Gamma - (-\Delta_p(\hat{\varrho}\hat{u})) = a - \hat{\varrho}^{p-1} a = \big(1 - \hat{\varrho}^{p-1}\big) a \le 0, \quad \text{in } \Omega_R. \tag{5.15}$$

By (5.14) the restriction of the function $\Gamma - \hat{\varrho}\hat{u}$ to Ω_R satisfies $(\Gamma - \hat{\varrho}\hat{u})^+ \in X_R$, which when taken as test function in (5.15) results in $(\Gamma - \hat{\varrho}\hat{u})^+ = 0$, that is, using (5.12) one gets

$$\Gamma(x) \le \hat{\varrho}\hat{u}(x) \le \frac{\hat{\varrho} c_2}{1 + |x|^{\frac{N-p}{p-1}}} \le \frac{d_2}{|x|^{\frac{N-p}{p-1}}} \quad \text{in } \overline{\Omega_R},$$

where $d_2 = \hat{\varrho} c_2$, which is the second part of (5.5).

Ad (iv): To show $\frac{\partial \Gamma}{\partial n}(x) < 0, \quad \forall x \in \partial B(0,1)$ we note that Γ, in particular, solves the following problem in the domain $\Omega \cap B_R = B_R \setminus \overline{B(0,1)}$

$$-\Delta_p \Gamma = a(x) \quad \text{in } \Omega \cap B_R\ (R > 1), \quad \Gamma = 0 \quad \text{on } \partial B(0,1), \tag{5.16}$$

and by (iii) satisfies $\Gamma \in C^1(\overline{B_R \setminus \overline{B(0,1)}})$, $\Gamma(x) > 0$ in $B_R \setminus \overline{B(0,1)}$. Therefore, we may apply the boundary point lemma [14, Theorem 5.5.1], according to which $\frac{\partial \Gamma}{\partial n} < 0$ on $\partial B(0,1)$, and thus completing the proof. □

The following characterization of the first eigenvalue λ_1 and its associated eigenfunction φ_1 of the eigenvalue problem (5.3) can be deduced from [6, Lemma 1.1] and [6, Theorem 1.1].

Lemma 5.2. *Assume hypothesis (Ha$_+$). Then the eigenvalue problem (5.3) has a first eigenvalue $\lambda_1 > 0$, which is simple and isolated. The associated eigenfunction φ_1 belongs to X and possesses the following properties:*

(i) $\varphi_1 \in C^{1,\gamma}(\overline{\Omega \cap B_R})$, $\gamma \in (0,1)$, *for any* $R > 1$.
(ii) $\varphi_1(x) > 0$ *for all* $x \in \Omega$.
(iii) *There are positive constants* C_1, C_2 *such that*

$$\frac{C_1}{|x|^{\frac{N-p}{p-1}}} \le \varphi_1(x) \le \frac{C_2}{|x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \overline{\Omega_R},\ R > 1. \tag{5.17}$$

Just like in the proof of (iv) of Lemma 5.1, as an immediate consequence of Lemma 5.2 we obtain the following result.

Corollary 5.3. *Under the hypothesis of Lemma 5.2 the eigenfunction φ_1 satisfies*

$$\frac{\partial \varphi_1}{\partial n}(x) < 0, \quad \forall x \in \partial B(0,1),$$

where $\partial\varphi_1/\partial n$ is the outward normal derivative at $x \in \partial B(0,1)$.

By means of Lemma 5.1, Lemma 5.2, and Corollary 5.3 we are able to provide the following order relation between Γ and φ_1.

Corollary 5.4. *Assume hypothesis (Ha$_+$). Then for any $\varepsilon > 0$ there is a constant $M > 0$ such that*

$$\varepsilon\varphi_1(x) \le M\Gamma(x), \quad \forall x \in \overline{\Omega}.$$

Proof. By virtue of Lemma 5.1 (iv) and Corollary 5.3 there are constants $\varrho_1 > 0$, $\varrho_2 > 0$ and constants $\sigma_1 > 0$, $\sigma_2 > 0$ such that

$$-\sigma_2 \le \frac{\partial \Gamma}{\partial n}(x) \le -\sigma_1, \quad -\varrho_2 \le \frac{\partial \varphi_1}{\partial n}(x) \le -\varrho_1, \quad \forall x \in \partial B(0,1). \tag{5.18}$$

For $\varepsilon > 0$ there is a M sufficiently large such that $\varepsilon\varrho_2 < M\sigma_1$, which implies by using (5.18)

$$\frac{\partial (M\Gamma)}{\partial n}(x) \le -M\sigma_1 < -\varepsilon\varrho_2 \le \frac{\partial(\varepsilon\varphi_1)}{\partial n}(x), \quad \forall x \in \partial B(0,1). \tag{5.19}$$

Since $\Gamma = 0$ and $\varphi_1 = 0$ on $\partial B(0,1)$, from (5.19) we obtain $\varepsilon\varphi_1(x) \le M\Gamma(x)$ in an δ-annulus $\overline{B(0, 1+\delta} \setminus B(0,1)$ for $\delta > 0$ small. As both function Γ and φ_1 are continuous and positive in Ω, by choosing M even larger if needed we arrive at

$$\varepsilon\varphi_1(x) \le M\Gamma(x), \quad \forall x \in \overline{B_R} \setminus B(0,1). \tag{5.20}$$

By virtue of (5.5) and (5.17) and choosing M, in addition, large enough such that $\varepsilon C_2 \le M d_1$ we finally obtain

$$\varepsilon\varphi_1(x) \le \frac{\varepsilon C_2}{|x|^{\frac{N-p}{p-1}}} \le \frac{M d_1}{|x|^{\frac{N-p}{p-1}}} \le M\Gamma(x), \quad \forall x \in \mathbb{R}^N \setminus B_R. \tag{5.21}$$

Thus (5.20) and (5.21) complete the proof. □

Now we are ready to prove the following existence result for the obstacle problem (5.1), (5.2).

Theorem 5.5. *Assume hypotheses (Ha$_+$) and (Hf), and suppose the obstacle function ψ satisfies*

$$\text{(5.22)} \qquad \textit{either} \quad \operatorname{ess\,sup}_{x\in\Omega} \frac{\psi(x)}{\Gamma(x)} < \infty \quad \textit{or} \quad \operatorname{ess\,sup}_{x\in\Omega} \frac{\psi(x)}{\varphi_1(x)} < \infty.$$

If f, in addition, satisfies

$$\text{(5.23)} \qquad \limsup_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = \mu < -\lambda_1, \quad \textit{uniformly in } x \in \Omega,$$

and f is bounded below by some constant $\nu < 0$, i.e.,

$$\text{(5.24)} \qquad f(x,s) \geq \nu \quad \forall (x,s) \in \Omega \times \mathbb{R},$$

then there exist extremal solutions of the obstacle problem (5.1), (5.2) within the interval $[\varepsilon\varphi_1, M\Gamma]$ for some positive constants ε small and M large enough.

Proof. We are going to prove that $\underline{u} = \varepsilon\varphi_1$ is a subsolution for $\varepsilon > 0$ small, and $\overline{u} = M\Gamma$ is a supersolution for $M > 0$ large. Let us consider the case

$$\operatorname{ess\,sup}_{x\in\Omega} \frac{\psi(x)}{\Gamma(x)} < \infty.$$

We note that the other case is treated in a similar way.

Clearly, we have $\underline{u} \vee K \subset K$, and $F(\varepsilon\varphi_1) \in L^{q'}(\Omega, w)$. To show that $\underline{u} = \varepsilon\varphi_1$ is a subsolution, by definition the following inequality needs to be shown

$$\text{(5.25)} \qquad \langle -\Delta_p \underline{u}, v - \underline{u}\rangle + \int_\Omega a f(\cdot, \underline{u})(v - \underline{u})\,dx \geq 0, \quad \forall v \in \underline{u} \wedge K.$$

Note, $v \in \underline{u} \wedge K$ is represented by $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in K$, and thus (5.25) is equivalent to

$$\text{(5.26)} \qquad \langle -\Delta_p \underline{u}, -(\underline{u} - \varphi)^+\rangle - \int_\Omega a f(\cdot, \underline{u})(\underline{u} - \varphi)^+\,dx \geq 0, \quad \forall \varphi \in K.$$

Since $(\underline{u} - \varphi)^+ \in \{v \in X : v \geq 0\}$, inequality (5.26) is proved if $\underline{u} = \varepsilon\varphi_1$ is a subsolution of the following equation (in the distributional sense)

$$\text{(5.27)} \qquad -\Delta_p u + a f(\cdot, u) = 0 \quad \text{in } \Omega.$$

For ε small we get from (5.23)

$$\text{(5.28)} \qquad \frac{f(x, \varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}} \leq -\lambda_1 - \delta, \quad \text{with some } \delta > 0 \text{ small.}$$

Using the properties of the eigenfunction φ_1 along with (5.28) we can estimate as follows

$$\begin{aligned} -\Delta_p \underline{u} + a f(\cdot, \underline{u}) &= \lambda_1 a (\varepsilon\varphi_1)^{p-1} + a f(\cdot, \varepsilon\varphi_1) \\ &= a(\varepsilon\varphi_1)^{p-1}\Big[\lambda_1 + \frac{f(\cdot, \varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}\Big] \\ &\leq a(\varepsilon\varphi_1)^{p-1}[\lambda_1 - \lambda_1 - \delta] \leq 0, \end{aligned}$$

which proves that $\underline{u} = \varepsilon\varphi_1$ is a subsolution for the obstacle problem.

To show that $\overline{u} = M\Gamma$ is a supersolution of the obstacle problem we first note that by hypothesis (5.22) we infer that $\psi(x) \leq M\Gamma(x)$ for $M > 0$ sufficiently large, and hence it follows $\overline{u} \wedge K = M\Gamma \wedge K \subset K$. By definition of the supersolution it remains to check the following inequality:

$$\langle -\Delta_p \overline{u}, v - \overline{u} \rangle + \int_\Omega a f(\cdot, \overline{u})(v - \overline{u})\, dx \ge 0, \quad \forall v \in \overline{u} \vee K. \tag{5.29}$$

Making use of $v = \overline{u} \vee \varphi = \overline{u} + (\varphi - \overline{u})^+$ for all $\varphi \in K$, (5.29) is equivalent to

$$\langle -\Delta_p \overline{u}, (\varphi - \overline{u})^+ \rangle + \int_\Omega a f(\cdot, \overline{u})(\varphi - \overline{u})^+\, dx \ge 0, \quad \forall \varphi \in K. \tag{5.30}$$

Since $(\varphi - \overline{u})^+ \in \{v \in X : v \ge 0\}$ for all $\varphi \in K$, a sufficient condition for (5.30) and thus for (5.29) is to show that $\overline{u} = M\Gamma$ is a supersolution of the equation (5.27), which is verified next. Making use of (5.24) and the definition of Γ yields

$$\begin{aligned} -\Delta_p \overline{u} + a f(\cdot, \overline{u}) &= M^{p-1} a + a f(\cdot, M\Gamma) = a\big(M^{p-1} + f(\cdot, M\Gamma)\big) \\ &\ge a\big(M^{p-1} + \nu\big) \ge 0, \end{aligned}$$

for $M > 0$ large enough, which shows that $\overline{u} = M\Gamma$ is supersolution of the obstacle problem. Finally, Corollary 5.4 implies that for even larger M if needed one can always get $\underline{u} = \varepsilon\varphi_1 \le M\Gamma = \overline{u}$. Thus we may apply Theorem 3.5 and Theorem 4.3, which completes the proof. □

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