INVARIANT CONES FOR LINEAR ELLIPTIC SYSTEMS WITH GRADIENT COUPLING

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ABSTRACT. We discuss counterexamples to the validity of the weak Maximum Principle for linear elliptic systems with zero and first order couplings and prove, through a suitable reduction to a nonlinear scalar equation, a quite general result showing that some algebraic condition on the structure of gradient couplings and a cooperativity condition on the matrix of zero order couplings guarantee the existence of invariant cones in the sense of Weinberger [22].

1. INTRODUCTION

We consider smooth vector-valued functions $u = (u_1, \ldots, u_m)$ of the variable x in a bounded open subset $\Omega \subset \mathbb{R}^n$ satisfying linear systems of partial differential inequalities of the following form

$$(1.1) \qquad Au + \sum_{i=1}^{n} B^{(i)} D_i u + Cu \geq 0 \ \text{ in } \ \Omega$$

where A is the second order operator

$$(1.2) \qquad Au = \begin{pmatrix} \Delta u_1 \\ . \\ . \\ . \\ \Delta u_m \end{pmatrix}$$

$B^{(i)}$ and C are $m \times m$ real matrices and with constant coefficients, and for $i = 1, \ldots, n$,

$$(1.3) \qquad D_i u = \begin{pmatrix} \frac{\partial u_1}{\partial x_i} \\ . \\ . \\ . \\ \frac{\partial u_m}{\partial x_i} \end{pmatrix}$$

denotes the $i - th$ column of the Jacobian matrix of the vector function u.
Note that the above defined structure of the systems allows coupling between the u_j and their gradients but not at the level of second derivatives.

Specific assumptions on the $B^{(i)}$ and C will be made later on.

Systems of this kind naturally arise in several different contexts such as modeling of simultaneous diffusions of m substances which decay spontaneously or in the case of systems describing switching diffusion processes in probability theory. In the latter case the homogeneous Dirichlet problem for system (1.1) describes discounted exit times from Ω, see for example [12].

We are interested here in investigating the validity of the weak Maximum Principle, **wMP** in short, that is the sign propagation property from the boundary to the interior for solutions $u = (u_1, \ldots, u_m)$ of the differential inequalities (1.1), i.e.,

(1.4)   **wMP** : $u \leq 0$ on $\partial\Omega$ $\implies$ $u \leq 0$ in Ω.

The vector function u will be always assumed to belong to $[C^2(\Omega)]^m \cap [C^0(\overline{\Omega})]^m$ and we will adopt the standard notation $u \leq 0$ if $u_j \leq 0$ for each $j = 1, \ldots, m$. We adopt the same notation for real-valued matrices, namely for a matrix A, $A \geq 0$ means that all its entries are nonnegative.

The validity of **wMP** is well-understood in the scalar case $m = 1$ even for general degenerate elliptic fully nonlinear partial differential inequalities such as

$$F(x, u, \nabla u, \nabla^2 u) \geq 0$$

in a bounded Ω and also in some unbounded domain of $\mathbb{R}^n$, see [9], [10] for recent results in this direction. Let us point out that the **wMP** property in the scalar case is related, and in fact equivalent, to the positivity of the principal eigenvalue (may be a pseudo one, if degeneracy occurs in the dependence of F with respect to Hessian matrix $\nabla^2 u$) of the Dirichlet problem for F, see [3], [4].

The case $m > 1$ has been the object of several papers mainly in the case of diagonal weakly coupled systems, that is when the matrices $B^{(i)}$ are diagonal and couplings between the functions u_j only occur at the level of zero-order terms, described by a matrix $C = (c_{jk})_{j,k}$ satisfying the cooperativity condition

(1.5) $c_{jk} \geq 0$ for $j \neq k$, $\sum_{k=1}^{m} c_{jk} \leq 0$ for $j = 1, \ldots, m$

Referring to the aforementioned exit time model, condition (1.5) requires the discount factor for the j-th process to dominate the sum of the interactions coefficients with all the other processes.

In the framework of purely weak cooperative couplings, let us mention the results in Section 8 of the book by Protter and Weinberger [19] and the references therein. For generalizations of those results in some semilinear cases see [20], [6], [1], while [7] contains results in the same direction concerning fully nonlinear uniformly elliptic operators $F = F(x, u, \nabla u, \nabla^2 u)$.

The recent paper [11] extends the validity of some of the results in [7] concerning **wMP** to a large class of fully nonlinear degenerate elliptic operators.

In particular, for the case of linear systems as (1.1) with no coupling in first derivatives (i.e. when each $B^{(i)}$ is diagonal), the main result in [11] is that **wMP** holds

true for system (1.1) provided C is cooperative.

Let us also point out that the main result in [11] holds even in the more general case where the Laplace operator is replaced by more general expressions $\mathrm{Tr}(A_j \nabla^2 u_j)$ satisfying $\mathrm{Tr}(A_j) > 0$ for $j = 1, \ldots, m$.

When coupling in first order terms occurs in (1.1), simple examples as the following one taken from [7] show that the **wMP** property (1.4) may indeed fail:

Example 1.1. The vector $u(x_1, x_2) = (1 - x_1^2 - x_2^2, \frac{1}{3}x_1^3 + 4x_2 - 20)$ is a solution of

$$\begin{cases} \Delta u_1 + \dfrac{\partial u_2}{\partial x_2} = 0 \\ \Delta u_2 + \dfrac{\partial u_1}{\partial x_1} = 0 \end{cases}$$

in the unit ball $\Omega \subset \mathbb{R}^2$, $u_1 = 0$, $u_2 < 0$ on $\partial\Omega$ but $u_1 > 0$ in Ω. Observe that the zero-order matrix is $C \equiv 0$ in this example, so that (1.5) is fulfilled.

As a matter of fact, even a first-order coupling of arbitrarily small size in the system can be responsible of the loss of **wMP**, as the following example shows:

Example 1.2. The system

$$\begin{cases} \Delta u + \varepsilon \dfrac{\partial v}{\partial x_1} \geq 0 \\ \Delta v + \varepsilon' \dfrac{\partial u}{\partial x_1} \geq 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, fulfills **wMP** if and only if $\varepsilon = \varepsilon' = 0$.

Indeed the validity of **wMP** when $\varepsilon = \varepsilon' = 0$ is classical. Conversely, if, say, $\varepsilon \neq 0$, then **wMP** is violated by the pair

$$u(x) = \delta - |x - \bar{x}|^2, \qquad v(x) = v(x_1) = e^{\sigma x_1} - H,$$

by choosing $\bar{x} \in \Omega$ and $\delta > 0$ small enough to have $u < 0$ on $\partial\Omega$, then σ with the same sign as ε and with $|\sigma|$ large enough to have the two inequalities in the system, and finally H large enough to have $v < 0$ on $\partial\Omega$.

Example 1.2 enlightens an instability property of **wMP** for cooperative systems with respect to first order perturbations. This is in striking contrast with the scalar case: indeed, for a uniformly elliptic scalar inequality, not only the presence of a first order term does not affect the validity of **wMP** when the zero-order term is nonpositive, but in addition **wMP** is stable with respect to perturbations of the coefficients, in the L^∞ norm. This can be seen as a consequence of the fact that **wMP** is characterized by the positivity of the associated principal eigenvalue, and the latter depends continuously on the coefficients of the operator, see e.g. [19], [4] and also [6], [7] where such characterization in terms of the same notion of principal eigenvalue as in [4] is extended to cooperative systems without first-order coupling. Example 1.2 reveals either that such notion does not exist when there is a first-order coupling, or that it is not continuous with respect to the coefficients.

According to the above considerations, two perspectives can be adopted in order to investigate the sign-propagation properties for coupled systems such as (1.1).

The first one consists in strengthening the hypotheses on the coefficients of the operators, namely the cooperativity condition (1.5). The second one is to replace **wMP** by some different kind of propagation property which reflects in some way the geometry of the coupling terms.
We will explore both directions.

Observe that the systems in Examples 1.1 and 1.2 fulfill the cooperativity condition (1.5) in the "border case", that is, when all inequalities are replaced by equalities. A natural question is then whether it is possible, for **wMP** to hold, to allow some coupling in first-order terms in the system, at least when the cooperativity conditions (1.5) hold with strict inequalities, i.e.,

$$(1.6) \qquad c_{jk} \geq K \ \text{ for } j \neq k \ , \qquad \sum_{k=1}^{m} c_{jk} \leq -K \ \text{ for } \ j = 1, \ldots, m,$$

with K possibly very large. The next result shows that this is not possible.

Proposition 1.3. *Let $\varepsilon > 0$, $\alpha, \tilde{c} \geq 0$. Then the following system with $m = 2$ and $n = 1$*

$$(1.7) \qquad \begin{cases} u'' \pm \varepsilon v' - cu + \alpha v \geq 0 \\ v'' - \tilde{c}v \geq 0 \end{cases} \qquad x \in I_\rho = (0, \rho)$$

*where u,v are scalar functions of $x \in \mathbb{R}$ does not satisfy **wMP**, provided that*

$$(1.8) \qquad \zeta(\rho\sqrt{c})\sqrt{c} > \frac{\alpha}{\varepsilon} \quad \text{ where } \ \zeta(\tau) := \frac{\cosh \tau - 1}{\sinh \tau - \tau} \ .$$

Remark 1.4. Since $\zeta(0^+) = +\infty$ and $\zeta(+\infty) = 1$, this proposition entails that, for every $\varepsilon, K > 0$, there exists a system of the type (1.1), with $B^{(i)}$ satisfying $|B_{jk}^{(i)}| \leq \varepsilon$ and $C = (c_{jk})_{j,k}$ satisfying (1.6), for which **wMP** fails. Namely, even an arbitrary small amount of coupling at the level of first derivatives can prevent the validity of **wMP** although the zero order matrix is, so to say, "very strongly cooperative". It also shows that, for any $\varepsilon, c > 0$ and $\alpha, \tilde{c} \geq 0$, **wMP** fails for (1.7) in a small enough interval I_ρ.
The fact that **wMP** fails when the diagonal zero-order term c is sufficiently large or when the size ρ of the interval is sufficiently small can be surprising if one has in mind the picture for the scalar equation where both having a large –negative– zero-order term and a small domain help the validity of the Maximum Principle. This phenomenon could be related to a non-monotonic structure of the system when a first-order coupling is in force.

Remark 1.5. A few more comments are in order here. We are considering a system with coupled gradients ($\varepsilon > 0$). The first part of Proposition 1.3 says that **wMP** cannot be satisfied in all bounded domains as soon as $\varepsilon > 0$, whatever the amount of cooperativity ($\alpha > 0$) is. The second part means that in a fixed interval **wMP** fails for c large enough. In cooperative systems under consideration ($0 \leq \alpha \leq c$) an excess of coercivity with respect to the coupling (c large compared with α/ε) seems to be responsible for invalidating **wMP**. In particular this is the case in any interval I_ρ when $\alpha = 0$.

We exhibit in Proposition 1.6 below that the same qualitative phenomenon occurs for a larger class of systems. The proofs of Propositions 1.3 and 1.6 are detailed in Section 2.

Proposition 1.6. *For every $\varepsilon \neq 0$, $\tilde{c} > 0$ and $\tilde{\varepsilon}, \alpha, \beta \in \mathbb{R}$, there exists $c > 0$ large enough such that the system*

$$(1.9) \qquad \begin{cases} u'' - \varepsilon v' - cu + \alpha v \geq 0 \\ v'' - \tilde{\varepsilon} u' - \tilde{c} v + \beta u \geq 0 \end{cases} \quad in \ (0,1)$$

violates the **wMP**.

If, instead, $c > 0$ is also fixed in the system (1.9), there exists an interval $I \subset (0,1)$ in which such system does not fulfill **wMP**.

Let us turn now to the positive results. The study of sufficient conditions for the validity of the weak Maximum Principle in the form **wMP** in the case where coupling occurs also at the level of first or second order derivatives is apparently less explored in literature, see however [14], [15], [16] and also [18], [17] for the related issue of maximum norm estimates of the form $\sup_{x \in \Omega} |u(x)| \leq C \sup_{x \in \Omega} |f(x)|$ for solutions u of non-homogeneous systems of equations involving higher order couplings.

The **wMP** property (1.4) can be understood in the framework of the general theory of invariant sets introduced by H. F. Weinberger in [22] in the context of elliptic and parabolic weakly coupled systems. We refer to the recent paper by G. Kresin and V. Mazya [16] where the notion of invariance is thoroughly developed for general systems with couplings at the first and the second order in the case $C \equiv 0$. According to the notion introduced in [22], a set $S \subseteq \mathbb{R}^m$ is invariant for system (1.1) if the following property holds

$$(1.10) \qquad \mathbf{INV}: \qquad u(x) \in S \text{ for all } x \in \partial\Omega \implies u(x) \in S \text{ for all } x \in \Omega$$

The sign propagation property (1.4) can then be rephrased as the property of the negative orthant $\mathbb{R}^m_- = \{u = (u_1, \dots u_m) : u_j \leq 0 \ , \ j = 1, \dots, m\}$ being an invariant set for system (1.1) of partial differential inequalities.

In [22] it is proved in particular that **wMP** holds for weakly coupled uniformly elliptic systems such as

$$(1.11) \qquad \mathrm{Tr}(A_j \nabla^2 u_j) + b_j \cdot \nabla u_j + f(u) = 0 \ , \ j = 1, \dots, m$$

under the condition that the vector field f satisfies the property that for any p belonging to the outward normal cone to $\mathbb{R}^m_-$ at a point u on the boundary of $\mathbb{R}^m_-$ the inequality

$$(1.12) \qquad p \cdot f(u) \leq 0$$

holds. For $f(u) = Cu$, this geometric condition turns out to be the cooperativity property (1.5) of matrix C. Note also that this condition implies that $\mathbb{R}^m_-$ is invariant under the flow $du/dt = Cu \, , t > 0$.

Let us observe that Example 1.2 and Proposition 1.3 show that $\mathbb{R}^m_-$ fails to be an invariant set even when the coupling of the first order terms is very small. As a

matter of fact, the matrix of the first order couplings in those cases is

$$\begin{pmatrix} 0 & -\varepsilon \\ 0 & 0 \end{pmatrix}$$

which is not diagonalizable. This is indeed consistent with results in [16]. It is in fact shown in that paper, see in particular results in Section 3, that the sufficient conditions involving the relations between the geometry of a closed convex set S and the matrices $B^{(i)}$ which imply the invariance of S, necessarily require, in the case $S = \mathbb{R}_-^m$, the diagonal structure of the first order couplings.

On the account of the example (1.7) exhibited in Proposition 1.3 we are forced to investigate the validity of a weaker form of the sign propagation property or, in other words, to single out an appropriate invariant set for system (1.1) when first order couplings occur.

It turns out that under some algebraic conditions, including notably the simultaneous diagonalizability of the matrices $B^{(i)}$, a cone propagation type result holds:

Theorem 1.7. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Assume that there exists an invertible $m \times m$ matrix Q such that, for all $i = 1, \dots, n$,*

$$(1.13) \quad Q^{-1}B^{(i)}Q = \mathrm{Diag}\big(\beta_1^{(i)}, \dots, \beta_m^{(i)}\big) \quad \text{for some } \beta_j^{(i)} \in \mathbb{R}, \quad (j = 1, \dots, m)$$

$$(1.14) \qquad\qquad\qquad\qquad Q^{-1} \geq 0$$

and, moreover,

$$(1.15) \qquad\qquad\qquad\qquad Q^{-1}CQ \quad \text{is cooperative}$$

Then the convex cone $S = \{u \in \mathbb{R}^m : Q^{-1}u \leq 0\}$ is invariant for (1.1).

Remark 1.8. Concerning the linear algebraic conditions of Theorem 1.7, observe first that a matrix Q simultaneously satisfying (1.13) for $i = 1, \dots, n$ exists if the $B^{(i)}$'s have a common basis of eigenvectors. This is the case when the matrices $B^{(i)}$ commute each other for all $i = 1, \dots, n$.

Observe also that if Q is an invertible M-matrix, that is $Q = sI - X$ where $X \geq 0$ and s is strictly greater than the spectral radius of X, then Q fulfills condition (1.14), see [5], [2].

Next, it is perhaps worth to point out that conditions (1.14) and (1.15) are compatible.

For example, $Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is an invertible M-matrix, $C = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$ is cooperative and $Q^{-1}CQ = \begin{pmatrix} -4 & 3 \\ 0 & -1 \end{pmatrix}$ is cooperative as well.

If no coupling occurs in first derivatives, so that $Q = Q^{-1} = I$, the above result reproduces the one in [11].

Remark 1.9. A related remark is that permutation matrices satisfies both $Q^{-1} \geq 0$ and $Q \geq 0$, so that in this case the conclusion of Theorem 1.7 is in fact that the negative orthant R_-^m is invariant. However, it is easy to check that in this situation condition (1.13) implies that each $B^{(i)}$ is diagonal and the results of [11] apply.

A further remark is that one cannot expect in general the invariance of the negative orthant R^m_-. This is indeed coherent with results in [16]: Lemma 2 there states in fact that the geometric sufficient condition on the matrices $B^{(i)}$ guaranteeing the invariance of R^m_- implies their diagonal structure.

Remark 1.10. Theorem 1.7 can in fact be extended (with a completely analogous proof) to a second order matrix operator

$$Au = \begin{pmatrix} \mathrm{Tr}(A\nabla^2 u_1) \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Tr}(A\nabla^2 u_m) \end{pmatrix}$$

where A is a positive semidefinite matrix such that $A\nu\cdot\nu \geq \lambda > 0$ for some direction $\nu \in \mathbb{R}^n$. For some applications of this notion of directional uniform ellipticity condition see [8], [9], [10], [21].

A key role in the proof of this result, which is postponed to the next section, is based on a reduction to a suitable fully nonlinear scalar differential inequality governed by the elliptic convex Bellman-type operator F defined, on scalar functions $\psi : \Omega \to \mathbb{R}$, as

$$(1.16) \qquad F[\psi] = \Delta\psi + \max_{j=1,\ldots,m} \sum_{i=1}^n \beta_j^{(i)} \frac{\partial\psi}{\partial x_i} = \Delta\psi + \max_{j=1,\ldots,m} b_j \cdot \nabla\psi$$

where $\beta_j^{(i)}$ are as in (1.13) and $b_j := (\beta_j^{(1)}, \ldots, \beta_j^{(m)})$.

The main ingredients in the proof are results in [11], see in particular Theorems 1.1 and 1.3, and the notion of generalized principal eigenvalue for scalar fully nonlinear degenerate elliptic operators and its relations with the validity of **wMP**, see [3].

Let us give a simple example illustrating the result of Theorem 1.7.

Example 1.11. Let $u = (u_1, u_2)$ be a solution of

$$\begin{cases} \Delta u_1 + 6\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_1} - u_1 \geq 0 \\ \Delta u_2 - 8\dfrac{\partial u_1}{\partial x_1} - u_2 \geq 0 \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. In this case $B^{(1)} = \begin{pmatrix} 6 & 1 \\ -8 & 0 \end{pmatrix}$, $B^{(2)} = 0$, $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and Theorem 1.7 applies with

$$Q = \begin{pmatrix} -1 & 1/2 \\ 4 & -1 \end{pmatrix} \qquad Q^{-1} = \begin{pmatrix} 1 & 1/2 \\ 4 & 1 \end{pmatrix}$$

yielding that inequality $u_2 \leq \min(-2u_1; -4u_1)$ propagates from $\partial\Omega$ to the whole Ω.

The result of Theorem 1.7 can be somewhat refined by a suitable weakening of the assumptions there. Firstly, observe that $B^{(i)}$ is not necessarily diagonalizable.

A suitable change of basis leads to an upper triangular matrix, which yields a real Jordan canonical form of $B^{(i)}$.

Suppose that the $B^{(i)}$'s have a common eigenspace of dimension $k \leq m$ and consider a basis of $\mathbb{R}^m$ where the first k vectors are linearly independent (common) eigenvectors of $B^{(i)}$, then we can find an $m \times m$ invertible real matrix $\hat{Q}$ that produces a real Jordan canonical form $J^{(i)} = \hat{Q}^{-1} B^{(i)} \hat{Q}$, $i = 1, \ldots, n$.

In this setting we have the following:

Theorem 1.12. *Let Ω be a bounded open subset of $\mathbb{R}^n$. Suppose that the $B^{(i)}$'s have a common eigenspace of dimension $k \leq m$, and let $\hat{Q}$ be an $m \times m$ invertible real matrix yielding the Jordan canonical form*

$$J^{(i)} = \hat{Q}^{-1} B^{(i)} \hat{Q} \ \ i = 1, \ldots, n$$

Let the $k \times m$ submatrix of the first k rows of $J^{(i)}$ be made up by a $k \times k$ diagonal block Λ plus the $k \times (m - k)$ zero matrix.

Assume in addition that the $k \times m$ sub-matrix containing the first k rows of $\hat{C} = \hat{Q}^{-1} C \hat{Q}$ is made by a cooperative $k \times k$ block plus the $k \times (m - k)$ zero matrix. If the $k \times m$ sub-matrix formed with the first k rows of $\hat{P}$ is positive, then the closed convex set

$$\hat{S} = \left\{ u \in \mathbb{R}^m : \sum_{j=1}^{m} p_{1j} u_j \leq 0, \ldots, \sum_{j=1}^{m} p_{kj} u_j \leq 0 \right\}$$

where p_{ij} are the entries of the matrix $\hat{P} := \hat{Q}^{-1}$ is invariant for system (1.1).

An illustrative example is provided next:

Example 1.13. Consider the 2×2 system

$$\begin{cases} \Delta u_1 + \dfrac{\partial u_2}{\partial x_1} - u_1 \geq 0 \\ \Delta u_2 + \dfrac{\partial u_1}{\partial x_1} - u_2 \geq 0 \end{cases}$$

in a domain Ω of $\mathbb{R}^2$. In this case $B^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B^{(2)} = 0$, $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and Theorem 1.12 applies with

$$\hat{Q} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \hat{Q}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

Since the first row of $\hat{P} = \hat{Q}^{-1}$ is nonnegative the above result yields the invariance of the convex set $S = \{u = (u_1, u_2) : u_1 + u_2 \leq 0\}$.

Note that **wMP**, that is the invariance of $\mathbb{R}^2_-$, does not hold true in this example. Indeed, the vector $u = \left((x_1 - x_1^2)(x_2 - x_2^2)^3, (x_1^2 + 2x_1 - 4)(x_2 - x_2^2) \right)$ is a solution in the square $\Omega = [0, 1] \times [0, 1]$ taking non positive values on $\partial\Omega$ with $u_2 \leq 0$, $u_1 \geq 0$ in Ω.

2. Proofs of the results

The first part of this section is dedicated to the proofs of Proposition 1.3 and 1.6.

Proof of Proposition 1.3. We restrict to the case $-\varepsilon$, with $\varepsilon > 0$. In fact, we can reduce to it by using the change of coordinate $x \to \rho - x$. We also observe that the argument is not affected by $\tilde{c} \geq 0$, so that we will omit to mention it when discussing on the parameters.

Firstly, we observe that $v(x) = -x$ obviously satisfies the second equation with $v \leq 0$ in $[0, \rho]$.

Next, we introduce the sequence of functions

$$u_k(x) = -\frac{\varepsilon}{c} \left\{ \frac{1 - e^{-\sqrt{c}/k}}{2 \sinh(\sqrt{c}/k)} e^{\sqrt{c}x} + \frac{e^{\sqrt{c}/k} - 1}{2 \sinh(\sqrt{c}/k)} e^{-\sqrt{c}x} \right.$$
$$\left. -1 - \frac{\alpha}{\varepsilon} \left(\frac{\sinh(\sqrt{c}x)}{k \sinh(\sqrt{c}/k)} - x \right) \right\}.$$

Then, a direct computation, shows that for all $k \in \mathbb{N}$,

$$u_k'' - \varepsilon v' - c u_k + \alpha v = 0 \quad \text{in } I_\rho \equiv (0, \rho)$$

and $u_k(0) = 0$.

The case $c = 0$ is ruled out either by taking the limit as $k \to +\infty$ or directly by putting $v = -x$ and $c = 0$ in the above equation.

Note also that as $k \to \infty$

(2.1) $$u_k(x) \to u_0(x) = -\frac{\varepsilon}{c} \left\{ \cosh \sqrt{c}x - 1 - \frac{\alpha}{\varepsilon} \left(\frac{\sinh \sqrt{c}x}{\sqrt{c}} - x \right) \right\}.$$

Let the parameters ε, c, α and ρ be fixed. For large $k \in \mathbb{N}$:

$$u_k'(0) = -\frac{\varepsilon}{c} \left\{ \frac{1 - \cosh(\sqrt{c}/k)}{\sinh(\sqrt{c}/k)/\sqrt{c}} - \frac{\alpha}{\varepsilon} \left(\frac{\sqrt{c}/k}{\sinh(\sqrt{c}/k)} - 1 \right) \right\}$$
$$= \frac{\varepsilon}{2k} + o(1/k)$$

so that $u_k'(0) > 0$ for k large enough. Since $u_k(0) = 0$, we also have $u_k(x) > 0$ for some $x \in I_\rho$ for such $k \in \mathbb{N}$. So **wMP** will be violated if $u_k(\rho) \leq 0$.

Next, rearranging (2.1) for $x = \rho$,

$$u_k(\rho) = -\frac{\varepsilon}{c} \left\{ \cosh \sqrt{c}\rho - 1 - \frac{\alpha}{\varepsilon} \left(\frac{\sinh \sqrt{c}\rho}{\sqrt{c}} - \rho \right) \right\} + u_k(\rho) - u_0(\rho)$$
$$= -\frac{\varepsilon}{c} \frac{\sinh \sqrt{c}\rho - \sqrt{c}\rho}{\sqrt{c}} \left\{ \zeta(\sqrt{c}\rho)\sqrt{c} - \frac{\alpha}{\varepsilon} \right\} + u_k(\rho) - u_0(\rho),$$

where the function ζ is defined in (1.8). Therefore condition $\zeta(\sqrt{c}\rho)\sqrt{c} > \frac{\alpha}{\varepsilon}$ in (1.8) yields $u_k(\rho) \leq 0$, for large $k \in \mathbb{N}$, so that **wMP** is not satisfied. Once established this fact, we search for condition (1.8) to prove that **wMP** fails.

A straightforward calculation shows that $\zeta(\tau) \to \infty$ as $\tau \to 0^+$ and $\zeta(\tau) \to 1$ as $\tau \to \infty$. Therefore there exists $\rho_0 = \rho_0(c; \frac{\alpha}{\varepsilon})$ such that condition (1.8) holds for

$\rho < \rho_0$, and **wMP** is not satisfied, thereby proving that as soon as $\varepsilon > 0$ there are intervals I_ρ, small enough, where **wMP** fails, whatever c and α are.

On the other hand, let $\rho > 0$ be fixed. The function $\zeta(\sqrt{c}\rho)\sqrt{c}$ is increasing with respect to c, see the Appendix, and $\zeta(\sqrt{c}\rho) \to 1$ as $c \to \infty$, so that

$$\lim_{c\to\infty} \zeta(\sqrt{c}\rho)\sqrt{c} = \infty.$$

Hence there exists $c_0 = c_0(\rho; \frac{\alpha}{\varepsilon})$ such that condition (1.8) holds for $c > c_0$, and **wMP** is not satisfied, thereby proving that as soon as $\varepsilon > 0$ then **wMP** fails in any interval I_ρ and for any $\alpha \geq 0$ when a sufficiently large c is taken.

By the increasing monotonicity of the function $c \to \zeta(\sqrt{c}\rho)\sqrt{c}$ we get

$$\inf_{c>0} \zeta(\sqrt{c}\rho)\sqrt{c} = \lim_{c\to 0^+} \zeta(\sqrt{c}\rho)\sqrt{c} = \frac{3}{\rho}.$$

It follows that, if $\frac{\alpha}{\varepsilon} < \frac{3}{\rho}$, then condition (1.8) is satisfied for all $c > 0$. This means that in this case we can choose $c_0(\rho; \frac{\alpha}{\varepsilon}) = 0$.

Finally, recalling that $\zeta(\sqrt{c}\rho) \to 1$ as $c \to \infty$, then $\zeta(\sqrt{c}\rho)\sqrt{c} \cong \sqrt{c}$ for $c \geq c_1(\rho)$. Hence condition (1.8) is equivalent to

$$\sqrt{c} > \frac{\alpha}{\varepsilon}.$$

It follows that, if $\frac{\alpha}{\varepsilon} > c_1$, then we can choose $c_0(\rho; \frac{\alpha}{\varepsilon}) = \left(\frac{\alpha}{\varepsilon}\right)^2$. $\square$

Proof of Proposition 1.6. Up to replacing $u(x), v(x)$ with $u(-x), v(-x)$, it is not restrictive to assume that $\varepsilon > 0$. We claim that, for c sufficiently large, there exists a pair (u, v) satisfying (1.9) in a strict sense, namely

$$\inf_{(0,1)} \left(u'' - \varepsilon v' - cu + \alpha v \right) > 0 \quad , \quad \inf_{(0,1)} \left(v'' - \tilde\varepsilon u' - \tilde c v + \beta u \right) > 0$$

such that $u(0) = 0 > u(1)$, $v(0) \leq 0$, $v(1) \leq 0$ and $u'(0) = 0$.
Then, for $\delta > 0$ sufficiently small, the pair of functions $(u(x) + \delta x, v(x))$ still satisfies the system (1.9) and both functions are ≤ 0 on the boundary of $(0, 1)$, but $u(x) > 0$ for $x > 0$ small, hence **wMP** is violated.

Let us construct the pair of strict subsolutions (u, v) by setting:

$$v(x) = x^2 - x, \qquad u(x) = \sigma\chi(x),$$

where χ is a smooth, non-increasing function satisfying

$$\chi(0) = \chi'(0) = 0, \qquad \chi(x) = -1 \text{ for } x \geq \min\left\{ \frac{1}{4}, \frac{\varepsilon}{4|\alpha| + 1} \right\}$$

Let $\sigma := \min(\sigma_1, \sigma_2)$, with $\sigma_1 := 1/(|\beta| + |\tilde\varepsilon| \|\chi'\|_{L^\infty((0,1))} + 1)$ and $\sigma_2 := \varepsilon/(8\|\chi''\|_{L^\infty((0,1))})$. We compute, for $x \in (0, 1)$,

$$v'' - \tilde\varepsilon u' - \tilde c v + \beta u \geq 2 - \sigma\big(|\beta| + \tilde\varepsilon\|\chi'\|_{L^\infty((0,1))}\big) \geq 1,$$

where we have used that $\sigma \leq \sigma_1$. Next, for $x \in (0, 1)$, we have

$$u'' - \varepsilon v' - cu + \alpha v \geq \varepsilon(1 - 2x) - |\alpha|x - \sigma|\chi''| - c\sigma\chi.$$

We estimate the above right-hand side considering first the case $0 < x \leq \min\{1/4, \varepsilon/(4|\alpha|+1)\}$, where we have

$$u'' - \varepsilon v' - cu + \alpha v \geq \frac{\varepsilon}{4} - \sigma\|\chi''\|_{L^\infty((0,1))} \geq \frac{\varepsilon}{8}.$$

Next, for $\min(1/4, |\varepsilon|/(4|\alpha|+1)) < x < 1$, we find

$$u'' - \varepsilon v' - cu + \alpha v \geq -\varepsilon - |\alpha| + c\sigma.$$

As a consequence, if $c \geq (2\varepsilon + |\alpha|)/\sigma$, there holds

$$\begin{cases} u'' - \varepsilon v' - cu + \alpha v \geq \dfrac{\varepsilon}{8} \\ v'' - \tilde{\varepsilon}u' - \tilde{c}v + \beta u \geq 1 \,. \end{cases}$$

The first statement of the proposition is thereby proved.

Let us turn to the second statement. We have seen above that **wMP** fails for (1.9) provided c is larger than some $\bar{c} > 0$, and more precisely that it is violated by a pair (u, v) with $v < 0$ on $(0, 1)$ and $u > 0$ somewhere. Consider the pair (u, v) associated with $c = \bar{c}$ and let I be a connected component of the set where $u > 0$ in $(0, 1)$, hence $u = 0$ on ∂I. For $c \leq \bar{c}$ there holds in I,

$$u'' - \varepsilon v' - cu + \alpha v \geq u'' - \varepsilon v' - \bar{c}u + \alpha v \geq 0.$$

This means that **wMP** fails in I and then concludes the proof. $\qquad\square$

Let us go now to the proof of Theorem 1.7.

Proof of Theorem 1.7. Assume that $u \in [C^2(\Omega)]^m \cap [C^0(\overline{\Omega})]^m$ satisfies (1.1) and that $Q^{-1}u \leq 0$ on $\partial\Omega$. Set

$$\hat{B}^{(i)} := Q^{-1}B^{(i)}Q \quad \text{and} \quad \hat{C} := Q^{-1}CQ.$$

Observe that the change of unknown $u = Qv$ gives, on the account of assumptions (1.13), (1.14), that v satisfies

$$(2.2) \qquad Av + \sum_{i=1}^{n} \hat{B}^{(i)}D_i v + \hat{C}v \geq 0 \ \text{ in } \ \Omega \ \text{ and } \ v \leq 0 \text{ on } \ \partial\Omega$$

that is, componentwise,

$$\begin{cases} \Delta v_1 + b_1 \cdot \nabla v_1 + \hat{C}_1 v \quad \geq 0 \\ \qquad\qquad \dots \\ \Delta v_m + b_m \cdot \nabla v_m + \hat{C}_m v \geq 0 \end{cases}$$

where $b_j = (\beta_j^{(1)}, \dots, \beta_j^{(m)})$ and $\hat{C}_j$ is the j-th row of $\hat{C}$, for $j = 1, \dots, m$.
We now employ the argument of the proof of Theorem 1 in [11] which reduces the above system to a scalar inequality governed by the uniformly elliptic (nonlinear) Bellman operator F in (1.16).
By viscosity calculus results based on the cooperativity condition (1.15), see [11], [7], since $v = (v_1, \dots, v_m)$ is a classical solution of (2.2) then the scalar function

$$v^*(x) := \max_{j=1,\dots,m} (v_j)^+(x),$$

where $+$ denotes the positive part of the various functions, is a continuous weak solution in the viscosity sense, see [13], of

$$(2.3) \qquad\qquad F[v^*] \geq 0 \ \text{ in } \ \Omega \ \text{ and } \ v^* = 0 \text{ on } \ \partial\Omega.$$

Suppose indeed that a smooth function φ touches from above v^* at some point in Ω. If at that point $v^* = 0$ then clearly $F[\varphi] \geq 0$ there. Otherwise φ touches from above the component v_{j_0} realizing the positive maximum v^* at that point and thus there holds

$$\Delta\varphi + b_{j_0} \cdot \nabla\varphi + \hat{C}_{j_0}v \geq 0.$$

But then recalling that $\hat{C}$ fulfills the cooperativity condition (1.5), one infers that

$$\hat{C}_{j_0}v \leq v_{j_0} \sum_k \hat{C}_{j_0 k} \leq 0$$

whence again $F[\varphi] \geq 0$.

In order to apply the general results of [3] we need to show that the generalized principal eigenvalue of F is positive. This amounts, by its very definition, to find a strict supersolution which is strictly positive in $\overline{\Omega}$. The latter is simply provided by $\psi(x) = \psi(x_1, \ldots, x_m) = 1 - \delta e^{\gamma x_1}$.
Indeed, this function satisfies

$$F[\psi] = -\delta\gamma e^{\gamma x_1} \left(\gamma + \min_{j=1,\ldots,m} \beta_j^{(1)} \right)$$

which is strictly negative in $\mathbb{R}^n$ provided $\gamma > |\min_{j=1,\ldots,m} \beta_j^{(1)}|$. We then choose δ small enough, depending on γ and Ω, so that $\psi > 0$ in $\overline{\Omega}$.
Summing up, ψ is positive in $\overline{\Omega}$ and satisfies there $F[\psi] < 0$, hence also $F[\psi] + \lambda\psi < 0$ for $\lambda > 0$ suitably small. This implies that the numerical index $\mu_1(F, \Omega)$ defined by

$$\mu_1(F, \Omega) = \sup\{\lambda \in \mathbb{R} : \psi \in C(\overline{\Omega}), \ \psi > 0, \ F[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\}$$

is strictly positive. Therefore, according to [3], the weak Maximum Principle for the scalar problem (2.3) holds, that is $v^* \leq 0$ in Ω.

This means that $Q^{-1}u = v \leq 0$ in Ω and the proof is complete. $\qquad\square$

We conclude the section with the proof of Theorem 1.12

Proof of Theorem 1.12. Following the same lines of the proof of Theorem 1.7, we set $u = \hat{Q}v$. When multiplying by $\hat{P} = \hat{Q}^{-1}$, this time we keep, by assumption, the positivity for the first k equations, which again by the assumptions made are decoupled in the gradient variables. So, letting $\tilde{C}$ be the diagonal part of $\hat{C}$ and $\tilde{v} = (v_1, \ldots, v_k)$, we get

$$\begin{cases} \Delta v_1 + b_1 \cdot \nabla v_k + \tilde{C}_1 \tilde{v} \geq 0 \\ \qquad\qquad \cdots \\ \Delta v_k + b_m \cdot \nabla v_k + \tilde{C}_k \tilde{v} \geq 0 \end{cases}$$

where $\tilde{C}_j$ is the j-th row of $\hat{C}$. The conclusion follows as in the proof of Theorem 2 with k instead of m. $\qquad\square$

3. Appendix

In this short section we provide the proof of the increasing monotonicity of the function $c \to \zeta(\tau\sqrt{c})\sqrt{c}$, where

$$\zeta(\tau) = \frac{\cosh\tau - 1}{\sinh\tau - \tau}.$$

We recall that this fact has been used in the proof of Proposition 1.3. At this purpose it is sufficient to show that the function

$$\varphi(t) := \frac{t(\cosh t - 1)}{\sinh t - t} = \frac{\cosh t - 1}{\frac{\sinh t}{t} - 1}$$

is increasing for $t > 0$. Since

$$\varphi'(t) = \frac{\sinh t \left(\frac{\sinh t}{t} - 1\right) - (\cosh t - 1)\frac{t\cosh t - \sinh t}{t^2}}{\left(\frac{\sinh t}{t} - 1\right)^2}$$

it is enough to prove that

$$\sinh t \left(\frac{\sinh t}{t} - 1\right) - (\cosh t - 1)\frac{t\cosh t - \sinh t}{t^2} > 0,$$

that is

$$(3.1) \qquad \frac{1}{t^2}(t + \sinh t)(\cosh t - 1) > \sinh t \qquad \text{for } t > 0.$$

In fact, let $(a_n)_{n\in\mathbb{N}}$ be the coefficients of the Taylor expansion

$$\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots + \frac{t^{2k+1}}{(2k+1)!} + \cdots$$

and $(b_n)_{n\in\mathbb{N}}$ be coefficients of the Taylor expansion of the left-hand side in (3.1):

$$\frac{1}{t^2}(t + \sinh t)(\cosh t - 1) = \frac{1}{t^2}\left(2t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right)\left(\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right)$$

$$= t + \left(\frac{2}{4!} + \frac{1}{3!}\frac{1}{2!}\right)t^3 + \left(\frac{2}{6!} + \frac{1}{3!}\frac{1}{4!} + \frac{1}{5!}\frac{1}{2!}\right)t^5 + \cdots$$

$$+ \left(\frac{2}{(2k+2)!} + \frac{1}{3!}\frac{1}{(2k)!} + \cdots + \frac{1}{(2k+1)!}\frac{1}{2!}\right)t^{2k+1} + \cdots$$

Both expansions only contain odd powers. We compare their coefficients:

$$b_1 = 1 = a_1; \quad b_3 = \frac{1}{6} = a_3,$$

and, for $k \geq 2$,

$$b_{2k+1} = \frac{2}{(2k+2)!} + \cdots + \frac{1}{(2k-1)!} \frac{1}{4!} + \frac{1}{(2k+1)!} \frac{1}{2!}$$

$$\geq \frac{1}{(2k-1)!} \frac{1}{4!} + \frac{1}{(2k+1)!} \frac{1}{2!}$$

$$= \frac{1}{2} \left\{ \frac{k(2k+1)}{6} + 1 \right\} \frac{1}{(2k+1)!}$$

$$> \frac{2}{(2k+1)!} = 2a_{2k+1}.$$

As a consequence $a_n \leq b_n$ for every $n \in \mathbb{N}$, and therefore (3.1) holds, as it was to be proved.

A picture of the function $\zeta(\sqrt{c}\rho)\sqrt{c}$ for different values of $\rho > 0$ is shown in Figure 1 below, where condition (1.8) with the threshold c_0 is graphically exhibited on the track $\rho = \frac{1}{2}$ for different values of $\frac{\alpha}{\varepsilon}$.
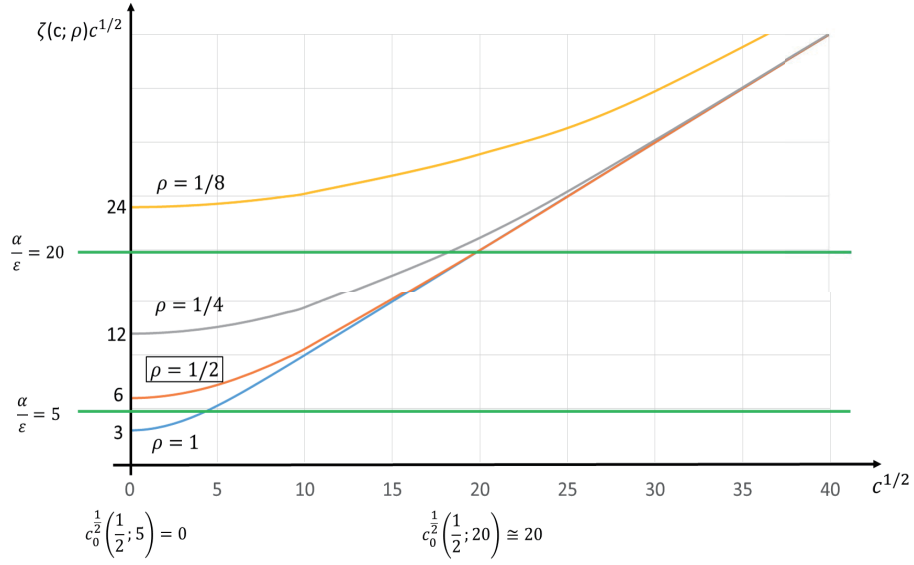

FIGURE 1. The function $\zeta(\sqrt{c}\rho)\sqrt{c}$

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