Pure and Applied Functional Analysis
Volume 6, Number 6, 2021, 1533–1543

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON DIRECTIONAL CONTINUITY, LIPSCHITZIAN PROPERTIES, AND DIFFERENTIABILITY

BINGWU WANG

Abstract. In this paper we give a survey on various concepts related to directional differentiabilities, calmness/Lipschitzian properties, and continuity. Some basic results are presented with some basic techniques involving directional constructions exhibited.

1. Introduction

It has been a very active field of research recently on directional differentiabilities and corresponding nonsmooth constructions in variational analysis and its applications; we refer the readers to [2–5, 8–14, 18, 19] for directional constructions with applications, and [1, 6, 7, 15, 16, 20] for the general framework of variational analysis and related discussions. In this survey paper we explore some basic notions of directional differentiabilities. We shall present some basic results and exhibit some basic techniques special to directional constructions. Most results have parallel non-directional versions while some proofs need slightly more delicate arguments.

By X, Y we mean Banach spaces with norm $\|\cdot\|$; $\mathbb{B}_X$ denotes the closed unit ball of X, and $\mathbb{B}_X(\bar{x};\delta)$ denotes the closed ball in X centered at $\bar{x}$ with radius δ (we often omit the subscript X when there is no ambiguity). The dual spaces of X and Y are denoted by X^*, Y^*. We use $\mathfrak{B}(X,Y)$ to denote the space of all continuous linear operators from X to Y.

By a *bornology* β on X we mean a family of bounded subsets of X that satisfies the following conditions:

(i) Λ is centrally symmetric for all $\Lambda \in \beta$;
(ii) The union of all sets in the family is the whole space X;
(iii) $\lambda\Lambda \in \beta$ if $\Lambda \in \beta$ and $\lambda > 0$;
(iv) For any $\Lambda_1, \Lambda_2 \in \beta$, there is $\Lambda \in \beta$ such that $\Lambda_1 \cup \Lambda_2 \subseteq \Lambda$.

Note that the results in this sequel still hold if we drop conditions (i), (iii), and (iv) in the above definition. The following bornologies are commonly used:

(i) the *Gâteaux bornology* ($\beta = G$), which contains all centrally symmetric finite sets;

2020 *Mathematics Subject Classification.* 49J53, 49J50, 49J52 .

Key words and phrases. Differentiability, Directional β-differentiability, weak differentiability, uniform weak differentiability, variational analysis, generalized differentiation.

(ii) the *Hadamard bornology* ($\beta = H$), which contains all centrally symmetric compact sets;
(iii) the *weak Hadamard bornology* ($\beta = WH$), which contains all centrally symmetric weakly compact sets;
(iv) the *Fréchet bornology* ($\beta = F$), which contains all centrally symmetric bounded sets.

In what follows, by $\beta_1 \leq \beta_2$ or $\beta_2 \geq \beta_1$, we mean $\beta_1 \subseteq \beta_2$; and we also assume that $G \leq \beta$ for all β in consideration.

Recall that cone(Ω) denotes the cone generated by the set $\Omega \subset X$, i.e.,

$$\operatorname{cone}(\Omega) = \cup_{\lambda \geq 0} \lambda\Omega.$$

A directional neighborhood ([10]) $D(\bar{x}, \bar{u}; \delta, \eta)$ at $\bar{x} \in X$ in direction $\bar{u} \in X$ is defined as

$$D(\bar{x}, \bar{u}; \delta, \eta) := \mathbb{B}(\bar{x}; \delta) \cap (\bar{x} + \operatorname{cone}(\bar{u} + \eta\mathbb{B}))$$

for given $\delta, \eta > 0$, and $D^\circ(\bar{x}, u; \delta, \eta)$ is the set of interior points of $D(\bar{x}, \bar{u}; \delta, \eta)$. Note that $D(\bar{x}, \bar{u}; \delta, \eta)$ reduces to the usual neighborhood $\mathbb{B}(\bar{x}; \delta)$ when $\bar{u} = 0$.

2. Directional continuity and Lipschitzian roperties

In this section, we present the notion of directional continuity, calmness, and Lipschitzian properties.

Definition 2.1. Let $f \colon X \to Y$ and $\bar{x}, \bar{u} \in X$. We say that f is *directionally continuous* at $\bar{x}$ in the direction $\bar{u}$ if for any $\varepsilon > 0$, there are $\delta, \eta > 0$ such that $\|f(x) - f(\bar{x})\| < \varepsilon$ for all $x \in D(\bar{x}, \bar{u}; \delta, \eta)$.

When $\bar{u} = 0$, the directional continuity reduces to the continuity in normal sense. When $\bar{u} \neq 0$, it is a rather different concept; see example 2.3 below. Before we see the example, we define the directional calmness/Lipschitzian property.

Definition 2.2. Let $f \colon X \to Y$, $\bar{x}, \bar{u} \in X$, and let β be a bornology on X. We say that

(i) f is *directionally β-Lipschitzian* around $\bar{x}$ in direction $\bar{u}$ if for each $\Lambda \in \beta$, there are $\delta, \eta > 0$ such that

$$\ell_{f,\Lambda}(\bar{x}, \bar{u}; \delta, \eta) := \sup_{x, x+th \in D(\bar{x}, \bar{u}; \delta, \eta),\, h \in \Lambda, t > 0} \frac{\|f(x + th) - f(x)\|}{t} < \infty.$$

(ii) f is *directionally β-calm* at $\bar{x}$ in direction $\bar{u}$ if for each $\Lambda \in \beta$, there are $\delta, \eta > 0$ such that

$$\ell_{f,\Lambda}(\bar{x}, \bar{u}; \delta, \eta) := \sup_{\bar{x}+th \in D(\bar{x}, \bar{u}; \delta, \eta),\, h \in \Lambda, t > 0} \frac{\|f(\bar{x} + th) - f(\bar{x})\|}{t} < \infty.$$

The constant $\ell_{f,\Lambda}(\bar{x}, \bar{u}; \delta, \eta)$ is called a *Lipschitzian (resp. calmness) modulus* of f around $\bar{x}$ in direction $\bar{u}$ with respect to Λ. When $\bar{u} = 0$, the directional β-calmness/Lipschitzian property reduces to the β-calmness/Lipschitzian property introduced in [21]. When $\bar{u} = 0$, and $\beta = F$, the directional Lipschitzian property reduces to the usual Lipschitzian property, and the directional calmness reduces to the usual calmness. When $\beta = F$, the directional F-Lipschitzian property reduces to the directional Lipschitzian property introduced in [13].

Example 2.3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(u,v) = -\sqrt{1-(u-1)^2-v^2}$ for all $(u,v) \in \mathbb{R}^2$; then f is convex on the disk $\{(u,v) \mid (u-1)^2+v^2 \le 1\}$, and is directionally continuous at $\bar{x} = (0,0)$ in direction $\bar{u} = (1,0)$. However, we can check that f is not directionally G-calm at $\bar{x}$ in direction $\bar{u}$, and consequently, not directionally calm at or Lipschitzian around $\bar{x}$ in direction $\bar{u}$.

This examples shows, *the directional continuity of convex functions does not imply the directional calmness/Lipschitzian property.* This is very different to the classical result that the continuity (actually boundedness is sufficient) of convex functions implies the local Lipschitzian property.

Next we present a relation of the directional calmness/Lipschitzian property among different bornologies. The case $\bar{u} = 0$ is established in [21].

Proposition 2.4. *If f is directionally β-Lipschitzian around (resp. directionally β-calm at) $\bar{x}$ in direction $\bar{u}$ with $\beta \ge H$, then f is directionally F-Lipschitzian around (resp. F-calm at) $\bar{x}$ in direction $\bar{u}$.*

Proof. The proof is a slight modification of the proof of Proposition 2.2 in [21]); for completeness, we reproduce it here (cf. the proof of Proposition 2.2.1 in [6] or the proof of Proposition 2.3 in [17]). For simplicity, we only present the proof to the case of directional β-Lipschitzian property, the proof to the case of directional β-calmness is similar. Suppose, to the contrary, that f is not directionally F-Lipschitzian around $\bar{x}$ at the direction $\bar{u}$. Then for each $k \in \mathbb{N}$, there are $u_k, v_k \in D(\bar{x}, \bar{u}; 1/k, 1/k)$ with $u_k \ne v_k$ and $\|f(v_k) - f(u_k)\| \ge k\|v_k - u_k\|$. Let $t_k = \sqrt{k}\|v_k - u_k\|$, $h_k = (v_k - u_k)/t_k$. Then

$$t_k \le \sqrt{k}(\|v_k - \bar{x}\| + \|u_k - \bar{x}\|) \le \sqrt{k}(2/k) = 2/\sqrt{k} \to 0$$

and $\|h_k\| = 1/\sqrt{k} \to 0$ (as $k \to \infty$) with $v_k = u_k + t_k h_k$. Let $\Lambda = \{h_1, h_2 \ldots\} \cup \{0\}$; then Λ is a compact set in X, and so is $\Lambda \cup (-\Lambda)$ which is centrally symmetric. It follows that $\Lambda \cup (-\Lambda) \in \beta$ since $\beta \ge H$. By the directional β-Lipschitzian property of f around $\bar{x}$, there exist $\delta, \eta > 0$ such that

$$\|f(u_k + t_k h_k) - f(u_k)\|/t_k \le \ell_{f,\Lambda}(\bar{x}, \bar{u}; \delta, \eta) < \infty$$

for all large $k \in \mathbb{N}$. On the other hand,

$$\begin{aligned} & \|f(u_k + t_k h_k) - f(u_k)\|/t_k = \|f(v_k) - f(u_k)\|/t_k \\ \ge \quad & k\|v_k - u_k\|/(\sqrt{k}\|v_k - u_k\|) = \sqrt{k} \to \infty \quad (\text{as } k \to \infty), \end{aligned}$$

which is a contradiction. □

By the result above, we can derive a relation between the directional continuity and the directional calmness/Lipschitzian property.

Proposition 2.5. *If f is directionally β-Lipschitzian around (resp. directionally β-calm at) $\bar{x}$ in direction $\bar{u}$ with $\beta \ge H$, then f is directionally continuous at $\bar{x}$ in direction $\bar{u}$.*

Proof. It is easy to check that the F-calmness/Lipschitzian property implies the directional continuity; and then the result follows from Proposition 2.4. $\square$

In the case when X is finite-dimensional, next result shows that *there is only one directional Lipschitzian property among different bornologies.*

Proposition 2.6. *Let $f\colon X \to Y$. If $\dim X < \infty$ and f is directionally G-Lipschitzian around $\bar{x} \in X$ in direction $\bar{u} \in X$, then f is directionally F-Lipschitzian around $\bar{x}$ in direction $\bar{u}$.*

Proof. (cf. the proofs of Theorem 2.4 in [21] and Theorem 2.5 in [13]) Let $\dim X = n$ and $e_1, e_2, \ldots, e_n$ are unit vectors in X that form a basis. We use the ℓ^1 norm on X, i.e., $\|\sum_{i=1}^n \lambda_i e_k\| = \sum_{i=1}^n |\lambda_i|$ for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. By the directional Gâteaux Lipschitzian property of f, we can find some $\bar{\delta}, \bar{\eta} > 0$ such that

$$\|f(x+th) - f(x)\| \leq Mt \tag{2.1}$$

for some $M > 0$ and for all $x \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$, $t > 0$ with $x + th \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$, $h \in \Lambda := \{\pm\bar{u}, \pm e_1, \pm e_2, \ldots, \pm e_n\}$. Fix $x \in D^\circ(\bar{x}; \bar{u}; \bar{\delta}, \bar{\eta})$ and $\mathbb{B}(x; \delta_1) \subseteq D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$ for some $\delta_1 > 0$. For any $x' \in \mathbb{B}(x; \delta_1)$, let $x' - x = \sum_{i=1}^n \lambda_i e_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$; then $\|x' - x\| = \sum_{i=1}^n |\lambda_i| \leq \delta_1$; consequently $\|\sum_{i=1}^k \lambda_i e_i\| = \sum_{i=1}^k |\lambda_i| \leq \delta_1$ and $v_k := x + \sum_{i=1}^k \lambda_i e_i \in \mathbb{B}(x; \delta_1)$ for all $k = 1, \ldots, n$. By (2.1) we have

$$\|f(x) - f(x')\| \leq \sum_{i=1}^n \|f(v_i) - f(v_{i-1})\| \leq \sum_{i=1}^n M|\lambda_i| = M\|x - x'\|, \tag{2.2}$$

for all $x' \in \mathbb{B}(x; \delta_1)$, where $v_0 := x$. Now for any $x_1, x_2 \in D^\circ(\bar{x}; \bar{u}; \bar{\delta}, \bar{\eta})$, we can find $\delta_2 > 0$ such that $\mathbb{B}(x_1; \delta_2), \mathbb{B}(x_2; \delta_2) \subseteq D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$. Choose $l \in \mathbb{N}$ such that $\|x_2 - x_1\|/l < \delta_2$, and let $w_0 := x_1$, $w_k := x_1 + \frac{k}{l}(x_2 - x_1)$ for $k = 1, \ldots, l$. Then $w_l = x_2$, $\mathbb{B}(w_k; \delta_2) \subseteq D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$, and $w_k \in \mathbb{B}(w_{k-1}; \delta_2)$ for all $k = 1, \ldots, l$. By (2.2), we have

$$\|f(x_2) - f(x_1)\| \leq \sum_{i=1}^l \|f(w_i) - f(w_{i-1})\| \leq \sum_{i=1}^l M\frac{\|x_2 - x_1\|}{l} = M\|x_2 - x_1\| \tag{2.3}$$

for all $x_1, x_2 \in D^\circ(\bar{x}; \bar{u}; \bar{\delta}, \bar{\eta})$.

Now we consider the case $x_1 = \bar{x}$, $x_2 \neq \bar{x}$ with $x_2 \in D^\circ(\bar{x}; \bar{u}; \bar{\delta}, \bar{\eta})$. By (2.1) again, we have $\|f(\bar{x} + t\bar{u}) - f(\bar{x})\| \leq Mt$ for all $t \in (0, \delta_3)$ for some $\delta_3 > 0$. Now choose $\bar{t} \in (0, \delta_3)$ so small such that $\bar{t}, \bar{t}\|\bar{u}\| < \|x_2 - \bar{x}|$ and $\bar{x} + \bar{t}\bar{u} \in D^\circ(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$. Then by (2.3) we have

$$\begin{aligned}
& \|f(x_2) - f(x_1)\| \\
\leq\ & \|f(x_2) - f(\bar{x} + \bar{t}\bar{u})\| + \|f(\bar{x} + \bar{t}\bar{u}) - f(\bar{x})\| \\
\leq\ & M\|x_2 - \bar{x} - \bar{t}\bar{u}\| + M\bar{t} \\
\leq\ & M(\|x_2 - \bar{x}\| + \bar{t}\|\bar{u}\|) + M\bar{t} \\
\leq\ & M(\|x_2 - \bar{x}\| + \|x_2 - \bar{x}\|) + M\|x_2 - \bar{x}\| \\
\leq\ & 3M\|x_2 - \bar{x}\|.
\end{aligned}$$

Combining this with (2.3), we obtain

$$\|f(x_2) - f(x_1)\| \leq 3M\|x_2 - x_1\|$$

for all $x_1, x_2 \in D^\circ(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta}) \cup \{\bar{x}\}$. It derives that $\ell_{f,\mathbb{B}}(\bar{x}, \bar{u}; \frac{\delta}{2}, \frac{\eta}{2}) \leq 3M$ and the proof is complete. $\square$

We present a corollary of Proposition 2.5 and Proposition 2.6 to end this section.

Proposition 2.7. *Let $f\colon X \to Y$. If $\dim X < \infty$ and f is directionally Gâteaux Lipschitzian around $\bar{x} \in X$ in direction $\bar{u} \in X$, then f is directionally continuous at $\bar{x}$ in direction $\bar{u}$.*

3. Directional differentiabilities

We explore various kinds of directional differentiabilities in this section.

Definition 3.1. Let $f\colon X \to Y$, $\bar{x}, \bar{u} \in X$, and let β be a bornology on X. We say that

(i) f is *directionally strictly β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if there exists a continuous linear operator $A\colon X \to Y$ such that for any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ with

$$\left\| \frac{f(x+th) - f(x)}{t} - A(h) \right\| \leq \varepsilon \tag{3.1}$$

for all $x \in D(\bar{x}, \bar{u}; \delta, \eta)$, $t > 0$ with $x + th \in D(\bar{x}, \bar{u}; \delta, \eta)$, and $h \in \Lambda$.

(ii) f is *directionally β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if there exists a continuous linear operator $A\colon X \to Y$ such that for any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ with (3.1) holds for all $x = \bar{x}$, $t > 0$ with $\bar{x} + th \in D(\bar{x}, \bar{u}; \delta, \eta)$ and $h \in \Lambda$.

When $\bar{u} = 0$, these reduce to the β-/strict β-differentiabilities in the classical sense. The case of directional strict G/F-differentiabilities were introduced in [13]. The continuous Linear operator A in the above definions is called a directional β-derivative. Let us proceed to examine the uniqueness of such derivatives. We start with the directional β-differentiability. The result below shows that the derivative is never unique in this case unless X is of dimension one.

Proposition 3.2. *If f is directionally β-differentiable at $\bar{x}$ in direction $\bar{u} \neq 0$, then the set of directional β-derivatives is*

$$\Big\{ T \in \mathfrak{B}(X, Y) \mid T(\bar{u}) = \lim_{t \to 0^+} \frac{f(\bar{x} + t\bar{u}) - f(\bar{x})}{t} \Big\}. \tag{3.2}$$

Proof. If f is directionally β-differentiable at $\bar{x}$ in direction $\bar{u} \neq 0$, and T is a directional β-derivative, then clearly $T(\bar{u}) = \bar{y} := \lim_{t \to 0^+} \frac{f(\bar{x}+t\bar{u}) - f(\bar{x})}{t}$. Next we show the converse, i.e., if f is β-differentiable at $\bar{x}$ in direction $\bar{u}$, and $T \in \mathfrak{B}(X, Y)$ with $T(\bar{u}) = \bar{y}$, then T is a directional β-derivative. Fix $\varepsilon > 0$ and $\Lambda \in \beta$. Then there is a constant $M > 0$ such that $\|h\| \leq M$ for all $h \in \Lambda$. Let $A \in \mathfrak{B}(X, Y)$ be a directional β-derivative of f at $\bar{x}$ in the direction $\bar{u}$, then $A(\bar{u}) = T(\bar{u}) = \bar{y}$, so there exists $0 < \eta_1 < \|\bar{u}\|/2$ such that

$$\|A(x) - T(x)\| < \frac{\|\bar{u}\|}{4M}\varepsilon \quad \forall x \in \bar{u} + \eta_1 \mathbb{B}. \tag{3.3}$$

On the other hand, we can find $\delta, \eta > 0$ such that $\eta < \eta_1$ and

$$\left\| \frac{f(\bar{x}+th)-f(\bar{x})}{t} - A(h) \right\| < \frac{\varepsilon}{2} \tag{3.4}$$

for all $h \in \Lambda$ and $t > 0$ with $\bar{x}+th \in D(\bar{x}, \bar{u}; \delta, \eta)$. It follows that $h = t'(\bar{u}+\eta h')$ for some $t' > 0$ and $h' \in \mathbb{B}$, and then $t' = \frac{\|h\|}{\|\bar{u}+\eta h'\|} \leq \frac{M}{\|\bar{u}\|-\eta} < \frac{M}{\|\bar{u}\|-\frac{\|\bar{u}\|}{2}} = \frac{2M}{\|\bar{u}\|}$. Consequently, $\|A(h)-T(h)\| = t'\|A(\bar{u}+\eta h') - T(\bar{u}+\eta h')\| \leq \frac{2M}{\|\bar{u}\|} \cdot \frac{\|\bar{u}\|}{4M}\varepsilon = \frac{\varepsilon}{2}$ due to (3.3). Together with (3.4), we derive

$$\begin{aligned} & \left\| \frac{f(\bar{x}+th)-f(\bar{x})}{t} - T(h) \right\| \\ \leq \ & \left\| \frac{f(\bar{x}+th)-f(\bar{x})}{t} - A(h) \right\| + \|A(h)-T(h)\| \\ < \ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for all $h \in \Lambda$ and $t > 0$ with $\bar{x}+th \in D(\bar{x}, \bar{u}; \delta, \eta)$. Therefore T is a directional β-derivative of f at $\bar{x}$ in direction $\bar{u}$, and the conclusion follows. $\square$

Now let us exam some other variants of the directional differentiability.

Definition 3.3. Let $f\colon X \to Y$, $\bar{x}, \bar{u} \in X$, and let β be a bornology on X.

(i) We say that f is *directionally β-differentiable of type B* at $\bar{x}$ in direction $\bar{u}$ if there exists a continuous linear operator $A\colon X \to Y$ such that for any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ with (3.1) holds for all $x \in D(\bar{x}, \bar{u}; \delta, \eta) \cap \{\bar{x}+\lambda\bar{u} \mid \lambda \geq 0\}$, $t > 0$ with $x+th \in D(\bar{x}, \bar{u}; \delta, \eta)$, and $h \in \Lambda$.

(ii) We say that f is *directionally β-differentiable of type C* at $\bar{x}$ in direction $\bar{u}$ if there exists a continuous linear operator $A\colon X \to Y$ such that for any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ with (3.1) holds for all $x \in D(\bar{x}, \bar{u}; \delta, \eta)$, $t > 0$ with $x+th \in D(x, \bar{u}; \delta, \eta)$, and $h \in \Lambda$.

Note that the type C directional differentiability in the case $Y = \mathbb{R}$, $\beta = F$ was given in [10]. It does not guarantee the uniqueness of the corresponding derivative, either; in fact, we can show by a similar argument like in the proof of Proposition 3.2 that,

Proposition 3.4. *If a function is directionally β-differentiable of type C at $\bar{x}$ in direction $\bar{u}$, then the set of derivatives is also (3.2).*

On the other hand, type B directional differentiability in Definition 3.3 can guarantee the uniqueness of the β-derivative, as below:

Proposition 3.5. *If a function $f\colon X \to Y$ is strictly directionally β-differentiable or directionally β-differentiable of type B at $\bar{x}$ in direction $\bar{u}$, then the β-derivative is unique.*

Proof. It is clear that the directional strict β-differentiability implies the directional β-differentiability of type B; so it suffices prove the case of type B differentiability. Let T, S be two directional β-derivatives of f at $\bar{x}$ in direction $\bar{u}$ and fix $h \in \mathbb{B}$. Then for any $\varepsilon > 0$, we can find $\delta, \eta > 0$, $t > 0$ with (3.1) holds for $A = T$ and $A = S$ for all $x \in D(\bar{x}, \bar{u}; \delta, \eta) \cap \{\bar{x}+\lambda\bar{u} \mid \lambda \geq 0\}$, $t > 0$ with $x+th \in D(\bar{x}, \bar{u}; \delta, \eta)$.

Now we choose $\lambda > 0$ such that $x := \bar{x} + \lambda\bar{u} \in D^\circ(\bar{x}, \bar{u}; \delta, \eta)$, and then we can choose $t > 0$ such that $x + th = \bar{x} + \lambda\bar{u} + th \in D(\bar{x}, \bar{u}; \delta, \eta)$. Now applying (3.1) for $A = T$ and $A = S$ for x and $x + th$, we have $\|(T - S)(h)\| \leq 2\varepsilon$. Since this is true for any $h \in \mathbb{B}$, we derive that $\|T - S\| \leq 2\varepsilon$ for all $\varepsilon > 0$. Therefore $T = S$. ☐

Next we study the relation between directional differentiabilities and directional calmness/Lipschitzian property.

Proposition 3.6. *Let $f\colon X \to Y$ with $\bar{x}, \bar{u} \in X$, and let β be a bornology on X.*

(i) *If f is directionally β-differentiable at $\bar{x}$ in direction $\bar{u}$, then it is directionally β-calm at $\bar{x}$; in particular, f is directionally calm at $\bar{x}$ in direction $\bar{u}$ if $\beta \geq H$.*

(ii) *If f is directionally strictly β-differentiable at $\bar{x}$ in direction $\bar{u}$, then it is directionally β-Lipschitzian around $\bar{x}$ in direction $\bar{u}$; in particular, f is directionally Lipschitzian around $\bar{x}$ in direction $\bar{u}$ if $\beta \geq H$.*

Proof. Both the first assertion in (i) and that in (ii) can be derived directly from the definition of the directional differentiabilities. The corresponding second assertions can then be derived taking into account of Proposition 2.4 ☐

We then have the following corollary of the above result which gives a relation of directional differentiabilities and continuity.

Proposition 3.7. *Let $f\colon X \to Y$ with $\bar{x}, \bar{u} \in X$, and let β be a bornology on X. If f is directionally β-differentiable or directionally strictly β-differentiable at $\bar{x}$ in direction $\bar{u}$ with $\beta \geq H$, then f is directionally continuous at $\bar{x}$ in direction $\bar{u}$.*

The following result generalizes the classical result (when $\bar{u} = 0$); the general scheme of the proof is standard while a much more involved analysis involving the directional neighborhood is used.

Proposition 3.8. *Let $f\colon X \to Y$ with $\bar{x}, \bar{u} \in X$. Suppose that f is directionally Gâteaux-differentiable (resp. directionally strictly Gâteaux-differentiable) at $\bar{x}$ in direction $\bar{u}$, and that f is directionally H-Lipschitzian around $\bar{x}$ at direction $\bar{u}$. Then f is directionally H-differentiable (resp. directionally strictly H-differentiable) at $\bar{x}$ in direction $\bar{u}$.*

Proof. By Theorem 2.4, f is directionally F-Lipschitzian around $\bar{x}$ in direction $\bar{u}$ under the assumptions made; then we can find $\bar{\delta}, \bar{\eta} > 0$ such that $\ell := \ell_{f,\mathbb{B}}(x, \bar{u}; \tilde{\delta}, \tilde{\eta}) < \infty$. We only consider the case of the directional strict differentiability (the proof of the case of the directional differentiability corresponds to the situation $x_k = \bar{x}$, $\bar{h} = \lambda\bar{u}$ in case (i) below). Suppose that f is not directionally H-differentiable at $\bar{x}$ in direction $\bar{u}$. Then there are sequences $\delta_k, \eta_k, t_k \downarrow 0$, $\varepsilon > 0$, a compact set $\Lambda \in H$, and $h_k \in \Lambda$, $x_k, x_k + t_k h_k \in D(\bar{x}, \bar{u}; \delta_k, \eta_k)$ such that

$$\left\|\frac{f(x_k + t_k h_k) - f(x_k)}{t_k} - A(h_k)\right\| > \varepsilon \tag{3.5}$$

for all $k \in \mathbb{N}$, where A is a directional G-derivative of f at $\bar{x}$. Since Λ is compact, without loss of generality, assume that $\lim_{k\to\infty} h_k = \bar{h}$ for some $\bar{h} \in X$. Then we have

$$(\ell + \|A\|)\|h_k - \bar{h}\| < \frac{\varepsilon}{2} \tag{3.6}$$

when k is sufficiently large. By the directional strict G-differentiability of f at $\bar{x}$ in direction $\bar{u}$ we can find $\bar{\delta} \in (0, \tilde{\delta})$, $\bar{\eta} \in (0, \tilde{\eta})$ such that

$$\left\| \frac{f(x+tw) - f(x)}{t} - A(w) \right\| < \frac{\varepsilon}{2} \tag{3.7}$$

for all $x \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$, $t > 0$ with $x + tw \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$, and $w \in \{\bar{h}, -\bar{h}\}$. Let $x_k = \bar{x} + \delta'_k(\bar{u} + \eta_k h'_k)$, $x_k + t_k h_k = \bar{x} + \delta''_k(\bar{u} + \eta_k h''_k)$ with $\delta'_k, \delta''_k \in [0, \delta_k)$, $h'_k, h''_k \in \mathbb{B}$, and we consider two cases:

Case 1. $\bar{u} = 0$ or $\bar{h} = \lambda \bar{u}$ for some $\lambda \geq 0$. If $\bar{u} = 0$, then clearly $x_k + t_k \bar{h} \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$ when k is large. If $\bar{h} = \lambda \bar{u}$ for some $\lambda > 0$, then

$$\begin{aligned} x_k + t_k \bar{h} &= \bar{x} + \delta'_k(\bar{u} + \eta_k h'_k) + t_k \lambda \bar{u} \\ &= \bar{x} + (\delta'_k + t_k \lambda)(\bar{u} + \frac{\delta'_k}{\delta'_k + t_k \lambda} \eta_k h'_k) \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta}) \end{aligned}$$

when k is sufficiently large. Taking into account (3.6) and (3.7), we have

$$\begin{aligned} & \left\| \frac{f(x_k + t_k h_k) - f(x_k)}{t_k} - A(h_k) \right\| \\ \leq & \left\| \frac{f(x_k + t_k \bar{h}) - f(x_k)}{t_k} - A(\bar{h}) \right\| \\ & \qquad + \left\| \frac{f(x_k + t_k h_k) - f(x_k + t_k \bar{h})}{t_k} \right\| + \|A(\bar{h} - h_k)\| \\ \leq & \left\| \frac{f(x_k + t_k \bar{h}) - f(x_k)}{t_k} - A(\bar{h}) \right\| \\ & \qquad + \ell \|h_k - \bar{h}\| + \|A\| \cdot \|\bar{h} - h_k\| \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

which contradics to (3.5).

Case 2. Now assume that $\bar{u} \neq 0$ and $\bar{h} \neq \lambda u$ for all $\lambda \geq 0$. We proceed to show that

$$\frac{\delta'_k}{t_k} \geq M \qquad \text{for some } M > 0 \text{ and all } k \in \mathbb{N}. \tag{3.8}$$

If this is not true, then there is a subsequence of $\{\frac{\delta'_k}{t_k}\}$ that converges to 0. Without loss of generality, assume that the whole sequence converges to 0. Because

$$x_k + t_k h_k = \bar{x} + \delta'_k(\bar{u} + \eta_k h'_k) + t_k h_k = \bar{x} + \delta''_k(\bar{u} + \eta_k h''_k),$$

we have

$$\frac{\delta'_k}{t_k}(\bar{u} + \eta_k h'_k) + h_k = \frac{\delta''_k}{t_k}(\bar{u} + \eta_k h''_k),$$

and then

$$\frac{\delta''_k}{t_k} = \frac{\|\frac{\delta'_k}{t_k}(\bar{u} + \eta_k h'_k) + h_k\|}{\|\bar{u} + \eta_k h''_k\|} \to \frac{\|\bar{h}\|}{\|\bar{u}\|} \quad (\text{as } k \to \infty).$$

It follows that

$$h_k = \frac{\delta''_k}{t_k}(\bar{u} + \eta_k h''_k) - \frac{\delta'_k}{t_k}(\bar{u} + \eta_k h'_k) \to \frac{\|\bar{h}\|}{\|\bar{u}\|} \bar{u}$$

as $k \to \infty$, which is a contradiction. So (3.8) holds. Now we have

$$x_k + t_k h_k - t_k \bar{h} = \bar{x} + \delta_k'(\bar{u} + \eta_k h_k') + t_k(h_k - \bar{h}) = \bar{x} + \delta_k'\Big(\bar{u} + \eta_k h_k' + \frac{t_k}{\delta_k'}(h_k - \bar{h})\Big)$$

and

$$\Big\|\eta_k h_k' + \frac{t_k}{\delta_k'}(h_k - \bar{h})\Big\| \quad \le \quad \eta_k + \frac{1}{M}\|h_k - \bar{h}\| < \bar{\eta}$$

when k is sufficiently large; so $x_k + t_k h_k - t_k \bar{h} \in D(\bar{x}, \bar{u}; \bar{\delta}, \bar{\eta})$ for such k. and consequently, by (3.6) and (3.7),

$$\begin{aligned}
&\Big\|\frac{f(x_k + t_k h_k) - f(x_k)}{t_k} - A(h_k)\Big\| \\
\le\ &\Big\|\frac{f(x_k + t_k h_k) - f(x_k + t_k h_k - t_k \bar{h})}{t_k} - A(\bar{h})\Big\| \\
&\qquad + \Big\|\frac{f(x_k + t_k h_k - t_k \bar{h}) - f(x_k)}{t_k}\Big\| + \|A(\bar{h} - h_k)\| \\
\le\ &\Big\|\frac{f(x_k + t_k h_k + t_k(-\bar{h})) - f(x_k + t_k h_k)}{t_k} - A(-\bar{h})\Big\| \\
&\qquad + \ell\|h_k - \bar{h}\| + \|A\| \cdot \|\bar{h} - h_k\| \\
<\ &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which contradics to (3.5). □

Combining Proposition 2.6, 3.6, and 3.8, we have the following corollary in the case $\dim X < \infty$.

Proposition 3.9. *Let $f\colon X \to Y$ with $\bar{x} \in X$, $\bar{u} \in X$ and $\dim X < \infty$. Then the directional strict G-differentiability of f at $\bar{x}$ in direction $\bar{u}$ is equivalent to the directional strict F-differentiability of f at $\bar{x}$ in direction $\bar{u}$.*

4. Weak directional differentiabilities

Following [17,21], we can extend the study of directional differentiabilities to their *weak* variants. Here we only present the definitions and omit the further discussions. In the definition below, $\langle y^*, f\rangle\colon X \to \mathbb{R}$ is defined as $\langle y^*, f\rangle(x) = \langle y^*, f(x)\rangle$, where $y^* \in Y^*$. By τ we mean a linear topology on Y^*; a sequence $\{y_k^*\}$ is said τ-convergent if it is convergent under the topology τ.

Definition 4.1. Let $f\colon X \to Y$ and $\bar{x}, \bar{u} \in X$, and β be a bornology.

(i) f is said *weakly directionally β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if $\langle y^*, f\rangle$ is directionally β-differentiable at $\bar{x}$ in direction $\bar{u}$ for all $y^* \in Y^*$.
(ii) f is said *weakly directionally strictly β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if $\langle y^*, f\rangle$ is directionally strictly β-differentiable at $\bar{x}$ in direction $\bar{u}$ for all $y^* \in Y^*$.

(iii) f is said *τ-uniformly weakly directionally β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if there is a function $A^*\colon Y^* \to X^*$ such that for any τ-convergent sequence $\{y_k^*\} \subset Y^*$, any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ such that

$$|\langle y_k^*, [f(\bar{x}+th) - f(\bar{x})]/t\rangle - \langle A^*(y_k^*), h\rangle| < \varepsilon$$
$$\forall k \in \mathbb{N},\ h \in \Lambda,\ t > 0 \text{ with } \bar{x}+th \in D(\bar{x}, \bar{u}; \delta, \eta).$$

(iv) f is said *τ-uniformly weakly directionally strictly β-differentiable* at $\bar{x}$ in direction $\bar{u}$ if there is a function $A^*\colon Y^* \to X^*$ such that for any τ-convergent sequence $\{y_k^*\} \subset Y^*$, any $\varepsilon > 0$, $\Lambda \in \beta$, there are $\delta, \eta > 0$ such that

$$|\langle y_k^*, [f(x+th) - f(x)]/t\rangle - \langle A^*(y_k^*), h\rangle| < \varepsilon$$
$$\forall k \in \mathbb{N},\ h \in \Lambda,\ x \in D(\bar{x}, \bar{u}; \delta, \eta), t > 0 \text{ with } x, x+th \in D(\bar{x}, \bar{u}; \delta, \eta).$$

The non-directional versions of the weak differentiabilities were introduced in [17] (the case $\bar{u} = 0$), and the non-directional uniform weak differentiabilities first appeared in [21]. Extensive discussions on the theoretical issues as well as applications to variational analysis can be found in these two papers. We will explore the directional versions of these results in separate papers.

References

[1] J. M. Borwein and Q. J. Zhu, *Techniques of Variational Analysis*, Springer, 2005.

[2] R. W. Chaney, *On second derivatives for nonsmooth functions*, Nonlinear Anal. **9** (1985), 1189–1209.

[3] R. W. Chaney, *Second-order directional derivatives for nonsmooth functions*, J. Math. Anal. Appl. **128** (1987), 495–511.

[4] R. W. Chaney, *Second-order necessary conditions in constrained semismooth optimization*, SIAM J. Control Optim. **25** (1987), 1072–1081.

[5] C. C. Chou, X. Li, K. F. Ng and S. Shi, *Optimality conditions for lower semicontinuous functions*, J. Math. Anal. Appl. **202** (1996), 686–701.

[6] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[7] F. Facchinei and J. S. Pang, *Finite-dimensional Variational Inequalities and Complementary Problems*, Vol. I and II. Springer, New York (2003).

[8] H. Gfrerer, *On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs*, Set-Valued Var. Anal. **21** (2013), 151–176.

[9] I. Ginchev and B. S. Mordukhovich, *On directionally dependent subdifferentials*, C. R. Bulg. Acad. Sci. **64** (2011), 497–508.

[10] I. Ginchev and B. S. Mordukhovich, *Directional subdifferentials and optimality conditions*, Positivity **16** (2012), 707–737.

[11] L. R. Huang and K. F. Ng, *Second-order necessary and sufficient conditions in nonsmooth optimization*, Math. Program. **66** (1994), 379–402.

[12] L. R. Huang and K. F.Ng, *On some relations between Chaney's generalized second order directional derivative and that of Ben-Tal and Zowe*, SIAM J. Control Optim. **34** (1996), 1220–1235.

[13] P. Long, B. Wang and X. Yang, *Calculus of directional subdifferentials and coderivatives in Banach spaces*, Positivity **21** (2017), 223–254.

[14] P. Long, B. Wang and X. Yang, *Calculus of directional coderivatives and normal cones in Asplund spaces*, Positivity **21** (2017), 1115–1142.

[15] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I: Basic Theory*, Springer, Berlin, 2006.

[16] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation II: Applications*, Springer, Berlin (2006).

[17] B. S. Mordukhovich and B. Wang, *Differentiability and regularity of Lipschitzian mappings*, Proc. Amer. Math. Soc. **131** (2002), 389–399.
[18] J. P. Penot, *The directional subdifferential of the difference of two convex functions*, J. Glob. Optim. **49** (2010), 505–519.
[19] J. P. Penot, *Directionally limiting subdifferentials and second-order optimality conditions*, Optim. Lett. **8** (2013), 1191–1200.
[20] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.
[21] B. Wang and X. Yang, *Weak differentiability with applications to variational analysis*, Set-Valued Var. Anal. **24** (2016), 299–321.

Manuscript received August 27 2020
revised September 16 2020

Bingwu Wang
Department of Mathematics & Statistics, Eastern Michigan University, Ypsilanti, MI 48197, USA
E-mail address: bwang@emich.edu