



ON DIFFERENTIAL PROPERTIES OF MULTIFUNCTIONS DEFINED IMPLICITLY BY SET-VALUED INCLUSIONS

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ABSTRACT. In the present paper, several properties concerning generalized derivatives of multifunctions implicitly defined by set-valued inclusions are studied by techniques of variational analysis. Set-valued inclusions are problems formalizing the robust fulfilment of cone constraint systems, whose data are affected by a “crude knowledge” of uncertain elements, so they can not be cast in traditional generalized equations.

The focus of this study is on the first-order behaviour of the solution mapping associated with a parameterized set-valued inclusion, starting with Lipschitzian properties and then considering its graphical derivative. In particular, a condition for the Aubin continuity of the solution mapping is established in terms of outer prederivative of the set-valued mapping defining the inclusion. A large class of parameterized set-valued inclusions is singled out, whose solution mapping turns out to be convex. Some relevant consequences on the graphical derivative are explored. In the absence of that property, formulae for the inner and outer approximation of the graphical derivative are provided by means of prederivatives of the problem data. A representation useful to calculate the coderivative of the solution mapping is also obtained via the subdifferential of a merit function.

1. INTRODUCTION AND PROBLEM STATEMENT

The concept of implicit function has been devised to enable calculations and, more generally, to deal with solutions of parameterized problems that can not be explicitly solved. Historically, the study of conditions under which a smooth equation system determines its variables as a function of parameters, as well as the continuity and differentiability properties of the function so defined, was the theme of fruitful speculations in classic analysis. While the first implicit function theorem, as modernly meant, seems to be due to Cauchy, as a matter of fact functions defined implicitly by equations can be traced back to earlier works authored by the founding fathers of differential calculus (for detailed historical remarks, see [7, Commentary to Chapter 1] and references therein). When specific features of modern variational analysis, with the acceptance of set-valued mappings as basic mathematical objects, led to address more general class of problems, such as inequality and cone constraint systems, variational inequalities and equilibrium problems, similar questions have been posed with reference to the multifunction counterpart of the

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original concept of implicit function. As a result, a comprehensive theory of multifunctions implicitly defined by generalized equations flourished in the last decades or so, which has been brought now to a high level of development (see, among others, [5, 7–9, 12, 15–21, 23–26, 28]). Although generalized equations are a problem format able to subsume the vast majority of mathematical conditions encountered in optimization and variational analysis, the treatment of constraint systems arising in robust optimization seems to be left out by such a formalism. After the seminal paper [4], robust optimization considers constraint systems of the form

$$f(x, \omega) \in C,$$

for given $f : \mathbb{R}^n \times \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ and $C \subset \mathbb{R}^m$, where $x \in \mathbb{R}^n$ represents the decision vector, whereas $\omega \in \mathbb{R}^k$ the data element of the problem. In many decision environments, described and discussed by concrete examples in [4], while the knowledge of the data may be partly or fully uncertain, reducing to the crude fact that ω belongs to a given uncertain set $\Omega \subseteq \mathbb{R}^k$, on the other hand the constraint system $f(x, \omega) \in C$ must be satisfied independently of the actual realization of $\omega \in \Omega$. This feature of the problem leads to the concept of robust feasibility, formalized by the set-valued inclusion

$$(1.1) \quad \Phi_f(x) = f(x, \Omega) = \{f(x, \omega) \mid \omega \in \Omega\} \subseteq C$$

and to the related notion of robust optimal solution to uncertain optimization problems. It is worth noting that the same problem format arises when considering vector optimization problems, which are characterized by a criterion function affected by uncertain data elements (see [13]).

In spite of the clear motivation and the urgent demand for skills on the aforementioned issue, to the best of the author's knowledge the solution analysis of set-valued inclusions is still very little explored. In fact, an error bound estimate was achieved in [6], under a C -concavity assumption, by techniques of convex analysis. A different approach to error bounds and to solution existence has been proposed more recently in [29], which is based on the C -increase behaviour, a sort of set-valued counterpart of the decrease principle (see [5]). Conditions for solution existence, global error bounds and characterizations of the contingent cone to the solution set are also investigated in [30], following the convex analysis approach initiated in [6]. Besides, a perturbation analysis of the solution set to parameterized set-valued inclusions has been started in [31]. More precisely, given a set-valued mapping $F : P \times X \rightrightarrows Y$ and a nonempty closed set $C \subset Y$, the following parameterized set-valued inclusion problem is considered there: find $x \in X$ such that

$$(SVI_p) \quad F(p, x) \subseteq C,$$

where p plays the role of a parameter. The above class of parameterized set-valued inclusions implicitly defines the solution mapping $\mathcal{S} : P \rightrightarrows X$ as being

$$\mathcal{S}(p) = \{x \in X \mid F(p, x) \subseteq C\}.$$

The paper [31] contains sufficient conditions for several quantitative forms of semicontinuity of \mathcal{S} , including those known as Lipschitz lower semicontinuity and calmness in the variational analysis literature.

The present paper carries on this research line, focussing instead on the Aubin and Lipschitz continuity of \mathcal{S} , as well as on its first-order behaviour. In particular, a first attempt of studying the graphical derivative of \mathcal{S} is undertaken. In consideration of the fact that, as recognized in [26, Chapter 9], “*the notion of Lipschitz continuity [...] singles out a class of functions which, although not necessarily differentiable, have a property akin to differentiability in furnishing estimates of magnitudes, if not the directions, of changes*” (and the same could be repeated for multifunctions, to a certain extent), the subject of the investigations here reported can be regarded as an introduction to the sensitivity analysis of problems (SVI_p).

The contents of the paper are organized in the subsequent sections according to the following outline. In Section 2 basic notions and tools needed for implementing the study of the subject by a variational technique are recalled. Since a part of the analysis refers to concepts that find in merely metric spaces their natural setting, this section is arranged in two subsections, presenting material in the absence or in the presence of a vector structure. The two results furnished with a full proof in this section capture the main ideas behind the approach of study proposed in the paper: namely, a proper error bound estimate and a behaviour of outer prederivative, which is adequate to supporting it. In Section 3 the metric space formulation of results about the behaviour of the solution mapping to parameterized set-valued inclusion is presented. In particular, a condition for the Aubin continuity of this set-valued mapping is established in terms of nondegeneracy of the strong slope of a merit function. It is clear that error bound estimates play a fundamental role here. Section 4 contains the main findings of the paper and some discussion of them. A condition for the Aubin continuity of implicit multifunctions defined by (SVI_p) in normed vector spaces is expressed in terms of problem data, by means of the outer prederivatives, and some consequence for its graphical derivative are discussed. The class of C -concave parameterized set-valued inclusions is shown to exhibit a convex solution mapping, which is thereby protodifferentiable, with a convex process as a graphical derivative. In the absence of C -concavity, some formulae for the inner and outer approximation of the graphical derivative are presented. Elements for a representation of the coderivative of the solution mapping conclude that section.

2. PRELIMINARIES

2.1. Variational analysis tools in metric spaces. Throughout the pre-sent subsection (P, d) , (X, d) and (Y, d) denote metric spaces. Given a function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, define $[\varphi \leq \alpha] = \varphi^{-1}([-\infty, \alpha])$, $[\varphi > \alpha] = \varphi^{-1}((\alpha, +\infty])$ and $[\varphi = \alpha] = \varphi^{-1}(\alpha)$. The symbol $\text{dom } \varphi = \varphi^{-1}(\mathbb{R})$ stands for the domain of φ . Given $x \in X$ and $r \geq 0$, the closed ball centered at x with radius r is denoted by $B(x, r) = [d(\cdot, x) \leq r]$. If $S \subseteq X$, define $\text{dist}(x, S) = \inf_{z \in S} d(x, z)$ and $B(S, r) = [\text{dist}(\cdot, S) \leq r]$. The symbol $\iota(\cdot; S) : X \rightarrow \{0, +\infty\}$ denotes the indicator function of the set S . Given two subsets $A, B \subseteq X$, the excess of A over B is indicated by $\text{exc}(A, B) = \sup_{a \in A} \text{dist}(a, B)$, whereas the Pompeiu-Hausdorff distance between A and B by $\text{haus}(A, B) = \max\{\text{exc}(A, B), \text{exc}(B, A)\}$. To deal with the case of the empty set, throughout the paper the usual conventions $\sup_{x \in \emptyset} = -\infty$ and $\inf_{x \in \emptyset} = +\infty$ are accepted and employed, whenever needed. The topological closure, the interior and the boundary of a set $S \subseteq X$ are denoted

by $\text{cl } S$, $\text{int } S$, and $\text{bd } S$, respectively. Given a set-valued mapping $\Phi : X \rightrightarrows Y$, $\text{dom } \Phi = \{x \in X \mid \Phi(x) \neq \emptyset\}$ and $\text{graph } \Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}$ stand for the domain and the graph of Φ , respectively. Given $C \subseteq Y$, the upper inverse image of C through Φ is indicated by $\Phi^{+1}(C) = \{x \in X \mid \Phi(x) \subseteq C\}$. The acronyms l.s.c. and u.s.c. stand for lower and upper semicontinuous, respectively.

Given a set-valued mapping $\Phi : X \rightrightarrows Y$ and a closed set $C \subset Y$, the solution set of the set-valued inclusion

$$(SVI) \quad \Phi(x) \subseteq C,$$

namely the set $\Phi^{+1}(C)$, can be conveniently reformulated via level/sublevel sets of the merit function $\nu_{\Phi, C} : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$, defined through the excess as being

$$(2.1) \quad \nu_{\Phi, C}(x) = \text{exc}(\Phi(x), C) = \sup_{y \in \Phi(x)} \text{dist}(y, C).$$

To this aim, observe that, if $\text{dom } \Phi = X$, then it is $[\nu_{\Phi, C} \geq 0] = X$. Thus, in this case the following equality holds

$$\Phi^{+1}(C) = [\nu_{\Phi, C} = 0].$$

More in general, in the case $X \setminus \text{dom } \Phi \neq \emptyset$, under the convention about $\sup_{x \in \emptyset}$, one has $X \setminus \text{dom } \Phi = [\nu_{\Phi, C} = -\infty]$ and hence

$$\Phi^{+1}(C) = [\nu_{\Phi, C} \leq 0].$$

Since the elements of $X \setminus \text{dom } \Phi$ are trivial solutions of (SVI), the equality $\text{dom } \Phi = X$ will be maintained as a standing assumption in the rest of the paper.

Besides, it is useful to note that, whenever Φ takes bounded values, one has $[\nu_{\Phi, C} < +\infty] = X$.

Remark 2.1. As one expects, the function $\nu_{\Phi, C}$ defined through (2.1) inherits various properties from Φ . For the purposes of the present investigations, it is useful to recall that if $\Phi : P \rightrightarrows X$ is l.s.c. at $x_0 \in X$, then $\nu_{\Phi, C}$ is l.s.c. at the same point. If Φ is Hausdorff C -u.s.c. (in particular, u.s.c.) at x_0 , then $\nu_{\Phi, C}$ is u.s.c. at the same point (see [29, Lemma 2.3]). Notice that, whenever $\Phi : P \rightrightarrows X$ is l.s.c. on X , then $\Phi^{+1}(C)$ is a closed (possibly empty) subset of X .

The concept of Lipschitz continuity can be adapted in different ways if referred to set-valued mappings. In the context of mappings with bounded values, it seems to be natural to extend immediately the notion valid for functions via the Pompeiu-Hausdorff distance. Accordingly, a set-valued mapping $\Phi : P \rightrightarrows X$ between metric spaces is said to be Lipschitz continuous with rate $\kappa > 0$ in a subset $S \subseteq P$ if

$$\text{haus}(\Phi(p_1), \Phi(p_2)) \leq \kappa d(p_1, p_2), \quad \forall p_1, p_2 \in S.$$

In more general contexts of interest to variational analysis, a more general notion gained a wide attention, inasmuch as it revealed to be intertwined with profound phenomena of regularity. This notion¹, playing a crucial role in the present paper, is recalled below.

¹Introduced under the name of ‘‘pseudo-Lipschitz’’ property in [1], later on it became popular as ‘‘Lipschitz-likeness’’ or ‘‘Aubin property/continuity’’.

Definition 2.2 (Aubin continuity). A set-valued mapping $\Phi : P \rightrightarrows X$ between metric spaces is said to be *Aubin continuous* at $(\bar{p}, \bar{x}) \in \text{graph } \Phi$ with rate $\kappa > 0$ if there exist positive δ and r such that

$$(2.2) \quad \text{dist}(x, \Phi(p_1)) \leq \kappa d(p_1, p_2), \quad \forall p_1, p_2 \in B(\bar{p}, \delta), \quad \forall x \in \Phi(p_2) \cap B(\bar{x}, r).$$

The value

$$(2.3) \quad \text{lip } \Phi(\bar{p}, \bar{x}) = \inf\{\kappa > 0 \mid \exists \delta, r > 0 \text{ such that inequality (2.2) holds}\}$$

is called *modulus of Aubin continuity* of Φ at (\bar{p}, \bar{x}) .

Another Lipschitzian property for set-valued mappings, which is worth being mentioned in connection with the subject of the present investigations, is calmness. The behaviour that it postulates can be obtained from condition (2.2), by fixing $p_1 = \bar{p}$, so it results in a property weaker than Aubin continuity.

From inclusion (2.2), by taking $p_2 = \bar{p}$ and $p_1 = p \in B(\bar{p}, \delta)$, one gets

$$\text{dist}(x, \Phi(p)) \leq \kappa d(p, \bar{p}), \quad \forall p \in B(\bar{p}, \delta), \quad \forall x \in \Phi(\bar{p}) \cap B(\bar{x}, r),$$

which implies, in particular, the existence of $\ell > 0$ such that

$$(2.4) \quad \Phi(p) \cap B(\bar{x}, \ell d(p, \bar{p})) \neq \emptyset, \quad \forall p \in B(\bar{p}, \delta).$$

The behaviour of Φ derived in (2.4) as a further consequence of the Aubin continuity of Φ , which can be regarded as a local version of inner semicontinuity, is called Lipschitz lower semicontinuity in [14].

A standard technique for establishing solution existence (solvability) and estimates of the distance from the solution set (error bounds) to inequalities in metric spaces relies on a quantitative employment of metric completeness via the Ekeland variational principle, which enables to replace iteration schemes. Such a technique can be fruitfully implemented by means of the notion of strong slope of a function $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ at a point x_0 in a metric space X , defined as

$$|\nabla\varphi|(x_0) = \begin{cases} 0, & \text{if } x_0 \text{ is a local minimizer of } \varphi, \\ \limsup_{x \rightarrow x_0} \frac{\varphi(x_0) - \varphi(x)}{d(x, x_0)}, & \text{otherwise.} \end{cases}$$

After [3, 11], the usage of this tool has become standard in variational analysis. It is well known that, whenever X is a normed vector space $(\mathbb{X}, \|\cdot\|)$ and φ is Fréchet differentiable at $x_0 \in \text{dom } \varphi$, with derivative $\widehat{D}\varphi(x_0) \in \mathbb{X}^*$, then it holds $|\nabla\varphi|(x_0) = \|\widehat{D}\varphi(x_0)\|$, while if φ is convex on \mathbb{X} and continuous at x_0 , then $|\nabla\varphi|(x_0) = \text{dist}(\mathbf{0}^*, \partial\varphi(x_0))$, where $\partial\varphi(x_0)$ denotes the subdifferential of φ at x_0 in the sense of convex analysis.

In what follows, to deal with set-valued mappings depending on a parameter, a partial variant of the strong slope with respect to the variable x of a function $\varphi : P \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$, defined on the product of metric spaces, at $(p_0, x_0) \in P \times X$, will be considered, which is defined as

$$|\nabla_x\varphi|(p_0, x_0) = \begin{cases} 0, & \text{if } (p_0, x_0) \text{ is a local} \\ & \text{minimizer of } \varphi, \\ \limsup_{x \rightarrow x_0} \frac{\varphi(p_0, x_0) - \varphi(p_0, x)}{d(x, x_0)}, & \text{otherwise.} \end{cases}$$

A sufficient condition for the behaviour of the solution set to parameterized inequalities, which is able to trigger the forthcoming analysis, can be expressed in terms of partial strong slope. The next lemma, whose role is fundamental according to the approach here followed, extends to a metric space setting an analogous tool of analysis valid in more structured contexts (see [5, Theorem 3.6.3]).

Lemma 2.3 (Parametric basic lemma). *Let P , X and Y be metric spaces and let $(\bar{p}, \bar{x}) \in P \times X$. Suppose that X and a function $\nu : P \times X \rightarrow [0, +\infty]$ satisfy the following conditions:*

- (i) (X, d) is metrically complete;
- (ii) $\nu(\bar{p}, \bar{x}) = 0$;
- (iii) the function $p \mapsto \nu(p, \bar{x})$ is u.s.c. at \bar{p} ;
- (iv) there exists $\delta_1 > 0$ such that, for every $p \in B(\bar{p}, \delta_1)$, each function $x \mapsto \nu(p, x)$ is l.s.c. on X ;
- (v) there exists $\delta_2 > 0$ such that

$$\sigma = \inf\{|\nabla_x \nu|(p, x) \mid (p, x) \in [B(\bar{p}, \delta_2) \times B(\bar{x}, \delta_2)] \cap [\nu > 0]\} > 0.$$

Then, there exist positive η and ζ such that

- (t) $[\nu(p, \cdot) = 0] \cap B(\bar{x}, \eta) \neq \emptyset$, for every $p \in B(\bar{p}, \zeta)$;
- (tt) the following estimate holds

$$(2.5) \quad \text{dist}(x, [\nu(p, \cdot) = 0]) \leq \frac{\nu(p, x)}{\sigma}, \quad \forall (p, x) \in B(\bar{p}, \zeta) \times B(\bar{x}, \eta).$$

Proof. (t) Take an arbitrary $\tilde{\sigma} \in (0, \sigma)$. As it is $\nu(\bar{p}, \bar{x}) = 0$, then according to hypothesis (iii) there exists $0 < \delta_3 < \min\{\delta_1, \delta_2\}$ such that

$$(2.6) \quad \nu(p, \bar{x}) < \frac{\tilde{\sigma}\delta_2}{3}, \quad \forall p \in B(\bar{p}, \delta_3).$$

Set $\zeta = \delta_3$ and fix an arbitrary $p \in B(\bar{p}, \zeta)$. Then consider the corresponding function $\nu(p, \cdot) : X \rightarrow [0, +\infty]$. Since it is $\zeta < \delta_1$, by hypothesis (iv) $\nu(p, \cdot)$ is l.s.c. on X (and bounded from below). Moreover, because of inequality (2.6), clearly it is

$$\nu(p, \bar{x}) \leq \inf_{x \in X} \nu(p, x) + \frac{\tilde{\sigma}\delta_2}{3}.$$

Thus, by the Ekeland variational principle, which can be invoked owing to hypothesis (i), there exists $x_p \in X$ such that

$$\nu(p, x_p) \leq \nu(p, \bar{x}) < \frac{\tilde{\sigma}\delta_2}{3},$$

$$(2.7) \quad d(x_p, \bar{x}) \leq \frac{\delta_2}{3},$$

and

$$\nu(p, x_p) < \nu(p, x) + \tilde{\sigma}d(x, x_p), \quad \forall x \in X \setminus \{x_p\},$$

whence one readily obtains

$$|\nabla_x \nu|(p, x_p) = \limsup_{x \rightarrow x_p} \frac{\nu(p, x_p) - \nu(p, x)}{d(x, x_p)} \leq \tilde{\sigma} < \sigma.$$

Notice that, as it is $\zeta < \delta_2$, it is true that $(p, x_p) \in [\mathbb{B}(\bar{p}, \delta_2) \times \mathbb{B}(\bar{x}, \delta_2)]$. This fact entails that $\nu(p, x_p) = 0$ for, if it were $\nu(p, x_p) > 0$, one would find contradicted the hypothesis (v). So, one is forced to admit that $\nu(p, x_p) = 0$. Therefore, taking into account inequality (2.7), it suffices to set $\eta = \delta_2/3$ in order to get

$$(2.8) \quad x_p \in [\nu(p, \cdot) = 0] \cap \mathbb{B}(\bar{x}, \eta) \neq \emptyset.$$

By the arbitrariness of $p \in \mathbb{B}(\bar{p}, \zeta)$, the above argument proves the assertion (t) in the thesis.

(tt) Fix $(p, x) \in \mathbb{B}(\bar{p}, \zeta) \times \mathbb{B}(\bar{x}, \eta) \cap [\nu > 0]$, where ζ and η are as in the proof of (t), and set $r_{p,x} = \nu(p, x)/\tilde{\sigma}$, with $\tilde{\sigma} \in (0, \sigma)$.

Let us consider first the case $r_{p,x} \geq 2\eta$. In such an event, since as a consequence of (2.8) it holds

$$\text{dist}(x, [\nu(p, \cdot) = 0]) \leq d(x, \bar{x}) + \text{dist}(\bar{x}, [\nu(p, \cdot) = 0]) \leq 2\eta,$$

then inequality (2.5) is immediately proved.

Let us consider now the case $r_{p,x} < 2\eta$. Take a positive \tilde{r} in such a way that $r_{p,x} < \tilde{r} < 2\eta$. Since it is $\nu(p, x) < \tilde{r}\tilde{\sigma}$ and $\nu(p, x) \leq \inf_{z \in X} \nu(p, z) + \tilde{r}\tilde{\sigma}$, one can employ the same argument as for the proof of the assertion (t), thus getting $\tilde{x}_p \in X$ such that

$$(2.9) \quad d(\tilde{x}_p, x) \leq \tilde{r}$$

and

$$(2.10) \quad |\nabla_x \nu|(p, \tilde{x}_p) \leq \tilde{\sigma} < \sigma.$$

Since on account of inequality (2.9) it holds

$$d(\tilde{x}_p, \bar{x}) \leq d(\tilde{x}_p, x) + d(x, \bar{x}) \leq \tilde{r} + \eta \leq 3\eta = \delta_2,$$

the only way to avoid a contradiction following from inequality (2.10) is to admit that $\tilde{x}_p \in [\nu(p, \cdot) = 0]$. Consequently, it results in

$$\text{dist}(x, [\nu(p, \cdot) = 0]) \leq d(x, \tilde{x}_p) \leq \tilde{r}.$$

As the argument leading to the last inequality works for every $\tilde{r} \in (r_{p,x}, 2\eta)$, one can deduce that

$$\text{dist}(x, [\nu(p, \cdot) = 0]) \leq r_{p,x}$$

and hence, by arbitrariness of $\tilde{\sigma} \in (0, \sigma)$, one can achieve the inequality (2.5). This completes the proof. \square

2.2. Variational analysis tools in normed vector spaces. Throughout the current subsection, $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ denote normed vector spaces. The null vector in a normed vector space is indicated by $\mathbf{0}$. Define $\mathbb{B} = \mathbb{B}(\mathbf{0}, 1)$ and $\mathbb{S} = \text{bd } \mathbb{B}$. Given a set $S \subseteq \mathbb{X}$, cone S stands for the conic hull of S . The (topological) dual space of \mathbb{X} is denoted by \mathbb{X}^* and its null element by $\mathbf{0}^*$, while the bilinear form defining the duality pairing between normed vector spaces is indicated by $\langle \cdot, \cdot \rangle$. The acronym p.h. stands for positively homogeneous.

The next remark collects some properties of the excess over a cone, which may occur in a vector space setting, in view of a subsequent employment through the function $\nu_{\Phi, C}$.

Remark 2.4 (Excess over a cone). Let $C \subseteq \mathbb{Y}$ be a closed, convex cone.

- (i) For any $A, B \subseteq \mathbb{Y}$ and $t \in (0, +\infty)$, as a straightforward consequence of the sublinearity of the function $y \mapsto \text{dist}(y, C)$, one has $\text{exc}(A + B, C) \leq \text{exc}(A, C) + \text{exc}(B, C)$ and $\text{exc}(tA, C) = t\text{exc}(A, C)$.
- (ii) For any $A \subseteq \mathbb{Y}$ it holds $\text{exc}(A + C, C) = \text{exc}(A, C)$ (see [29, Remark 2.1(iv)]).
- (iii) Since it is $\text{dist}(y, C) \leq \|y\|$ for every $y \in \mathbb{Y}$, then given any $r > 0$, the inequality $\text{exc}(r\mathbb{B}, C) \leq r$ holds.
- (iv) For any $y \in \mathbb{Y} \setminus C$ and $r > 0$, it holds $\text{exc}(y + r\mathbb{B}, C) = \text{dist}(y, C) + r$ (see [29, Lemma 2.1]).
- (v) Let $S \subseteq \mathbb{Y}$ be a set such that $S \not\subseteq C$. Then, for every $r > 0$, it holds

$$\begin{aligned} \text{exc}(S + r\mathbb{B}, C) &= \sup_{y \in S} \text{dist}(y + r\mathbb{B}, C) = \sup_{y \in S \setminus C} [\text{dist}(y, C) + r] \\ &= \text{exc}(S, C) + r. \end{aligned}$$

- (vi) It is easy to see that for any $A, B \subseteq \mathbb{Y}$, it holds $\text{exc}(A, C) \leq \text{exc}(A, B) + \text{exc}(B, C)$.

Given two nonempty subsets $K, S \subseteq \mathbb{Y}$, their $*$ -difference (a.k.a. Pontryagin difference) is defined as

$$K^*S = \{y \in \mathbb{Y} \mid y + S \subseteq K\}.$$

It is readily seen that $\mathbf{0} \in K^*S$ iff $S \subseteq K$. In what follows, several conditions will be expressed in terms of the following quantity

$$(2.11) \quad |K^*S| = \sup\{r > 0 \mid r\mathbb{B} \subseteq K^*S\},$$

which can be regarded as a measure of how much the set S is inner to K (for more details on the $*$ -difference, see for instance [27]).

Given a set-valued mapping $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ between normed vector spaces, several notions of first-order approximations of Φ can be found in variational analysis, which reveal to be suitable in connection with the present approach of study. Let $x_0 \in \text{dom } \Phi$. After [10], a p.h. set-valued mapping $H_\Phi(x_0; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be an outer prederivative of Φ at x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Phi(x) \subseteq \Phi(x_0) + H_\Phi(x_0; x - x_0) + \epsilon\|x - x_0\|\mathbb{B}, \quad \forall x \in \text{B}(x_0, \delta).$$

In contrast with [22], a p.h. set-valued mapping $H_\Phi(x_0; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ is said to be an inner prederivative of Φ at x_0 if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\Phi(x_0) + H_\Phi(x_0; x - x_0) \subseteq \Phi(x) + \epsilon\|x - x_0\|\mathbb{B}, \quad \forall x \in \text{B}(x_0, \delta).$$

For expanding the discussion about outer and inner prederivatives, the reader may refer to [10, 22].

Graphical differentiation represents a different way of approximating set-valued mappings. It is based on the notion of conical approximation of sets. Given a nonempty set $S \subseteq \mathbb{Y}$ and $y \in S$, let $\text{T}(S; y)$ denote, in particular, the contingent cone to S at y . Recall that $\text{T}(S; y)$ is always a closed cone and, whenever S is convex, $\text{T}(S; y)$ too is convex and can be represented as

$$(2.12) \quad \text{T}(S; y) = \text{cl}[\text{cone}(S - y)]$$

(see, for instance, [28, Proposition 11.1.2(d)]). Besides, in view of the technique of proof employed in a subsequent section, the following variational characterization of the contingent cone to any set S at y will be helpful

$$(2.13) \quad \mathbb{T}(S; y) = \left\{ v \in \mathbb{Y} \mid \liminf_{t \downarrow 0} \frac{\text{dist}(y + tv, S)}{t} = 0 \right\}$$

(see [28, Proposition 11.1.5]).

The graphical (contingent) derivative of Φ at $(x_0, y_0) \in \text{graph } \Phi$ is the set-valued mapping $\mathbb{D}\Phi(x_0, y_0) : \mathbb{X} \rightrightarrows \mathbb{Y}$ defined via the graphical relation

$$\text{graph } \mathbb{D}\Phi(x_0, y_0) = \mathbb{T}(\text{graph } \Phi; (x_0, y_0)).$$

Namely, the fact that $v \in \mathbb{D}\Phi(x_0, y_0)(z)$ means that there exist sequences $(z_n)_n$ in \mathbb{X} , with $z_n \rightarrow z$, $(v_n)_n$ in \mathbb{Y} , with $v_n \rightarrow v$, and $(t_n)_n$ in $(0, +\infty)$, with $t_n \downarrow 0$, as $n \rightarrow \infty$, such that

$$(2.14) \quad y_0 + t_n v_n \in \Phi(x_0 + t_n z_n), \quad \forall n \in \mathbb{N}.$$

From the very definition, one readily sees that $\mathbb{D}\Phi(x_0, y_0)$ is a p.h. set-valued mapping.

If for each $v \in \mathbb{D}\Phi(x_0, y_0)(z)$ and choice of the sequence $(t_n)_n$ in $(0, +\infty)$, with $t_n \downarrow 0$, there exist sequences $(z_n)_n$ in \mathbb{X} , with $z_n \rightarrow z$, and $(v_n)_n$ in \mathbb{Y} , with $v_n \rightarrow v$, such that inclusion (2.14) holds, then Φ is said to be protodifferentiable at $(x_0, y_0) \in \text{graph } \Phi$. Detailed accounts on graphical differentiation can be found in [2, 7, 26, 28].

An aspect which should be the subject of meditation is that, while the concept of approximation provided by outer/inner prederivatives refer to an element $x_0 \in \text{dom } \Phi$ and consider the whole set $\Phi(x_0)$, graphical derivatives refer to an element $(x_0, y_0) \in \text{graph } \Phi$ and are affected only by the local geometry of Φ near (x_0, y_0) .

Other convenient derivative-like objects for set-valued mappings are coderivatives. They act between dual spaces and can be introduced via normal cones to the graph of set-valued mappings. Accordingly, the Fréchet coderivative of $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ at $(x_0, y_0) \in \text{graph } \Phi$ is the set-valued mapping $\widehat{\mathbb{D}}^*\Phi(x_0, y_0) : \mathbb{Y}^* \rightrightarrows \mathbb{X}^*$ defined by

$$\widehat{\mathbb{D}}^*\Phi(x_0, y_0)(y^*) = \{x^* \mid (x^*, -y^*) \in \widehat{\mathbb{N}}(\text{graph } \Phi; (x_0, y_0))\},$$

where, if $S \subseteq \mathbb{X} \times \mathbb{Y}$ and $w_0 \in S$, the subset

$$\widehat{\mathbb{N}}(S; w_0) = \left\{ w^* \in \mathbb{X}^* \times \mathbb{Y}^* \mid \limsup_{S \ni w \rightarrow w_0} \frac{\langle w^*, w - w_0 \rangle}{\|w - w_0\|} \leq 0 \right\}$$

denotes the Fréchet normal cone to S at w_0 . For more material on coderivative, see [5, 7, 19, 26, 28]. In view of a subsequent employment, let us recall the following equality linking the Fréchet normal cone (and hence the coderivative) with the Fréchet subdifferential via the indicator and the distance function:

$$(2.15) \quad \widehat{\mathbb{N}}(S; w_0) = \widehat{\partial}\iota(\cdot; S)(w_0) = \bigcup_{\kappa > 0} \kappa \widehat{\partial}\text{dist}(\cdot, S)(w_0),$$

where

$$\widehat{\partial}\varphi(x_0) = \left\{ x^* \in \mathbb{X}^* \mid \liminf_{x \rightarrow x_0} \frac{\varphi(x) - \varphi(x_0) - \langle x^*, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}$$

denotes the Fréchet subdifferential of a function $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ at $x_0 \in \text{dom } \varphi$ (see, for instance, [19, Corollary 1.96]).

The next proposition explains how outer prederivatives of a set-valued mapping Φ can be exploited for estimating the strong slope of the function $\nu_{\Phi,C}$, at points which fail to be a solution of the set-valued inclusion defined by Φ and C : roughly speaking, such first-order approximations of Φ must admit a direction, along which their values are strictly inner to C .

Proposition 2.5. *Let $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ be a set-valued mapping between Banach spaces, let $C \subseteq \mathbb{Y}$ be a closed, convex cone and let $x_0 \in \mathbb{X} \setminus \Phi^{+1}(C)$. Suppose that*

- (i) Φ is l.s.c. on x_0 ;
- (ii) Φ admits $H_{\Phi}(x_0; \cdot)$ as an outer prederivative at x_0 ;
- (iii) it holds

$$(2.16) \quad \sigma_H(x_0) = \sup_{u \in \mathbb{S}} |C^*H_{\Phi}(x_0; u)| > 0.$$

Then, the following estimate holds

$$(2.17) \quad |\nabla \nu_{\Phi,C}|(x_0) \geq \sigma_H(x_0).$$

Proof. In the light of Remark 2.1, by virtue of hypothesis (i), the function $\nu_{\Phi,C}$ turns out to be l.s.c. on x_0 . Since it is $x_0 \in \mathbb{X} \setminus \Phi^{+1}(C)$, one has $\nu_{\Phi,C}(x_0) > 0$. Then, there exists $\delta > 0$ such that $\nu_{\Phi,C}(x) > 0$ for every $x \in B(x_0, \delta)$. According to hypothesis (ii), fixed any $\epsilon \in (0, \sigma_H(x_0))$ there exists $\delta_\epsilon \in (0, \delta)$ such that

$$(2.18) \quad \Phi(x_0 + v) \subseteq \Phi(x_0) + H_{\Phi}(x_0; v) + \epsilon \|v\| \mathbb{B}, \quad \forall v \in \delta_\epsilon \mathbb{B}.$$

By virtue of hypothesis (iii), there exists $u_\epsilon \in \mathbb{S}$ such that

$$|C^*H_{\Phi}(x_0; u_\epsilon)| > \sigma_H(x_0) - \epsilon,$$

and hence, recalling definition (2.11), there exists $r_\epsilon > \sigma_H(x_0) - \epsilon$ such that

$$H_{\Phi}(x_0; u_\epsilon) + r_\epsilon \mathbb{B} \subseteq C.$$

Since the set-valued mapping $H_{\Phi}(x_0; \cdot)$ is positively homogeneous and C is a cone, the last inclusion entails

$$(2.19) \quad H_{\Phi}(x_0; tu_\epsilon) + tr_\epsilon \mathbb{B} \subseteq C, \quad \forall t > 0.$$

By combining inclusions (2.18) and (2.19), one finds

$$\begin{aligned} \Phi(x_0 + tu_\epsilon) + tr_\epsilon \mathbb{B} &\subseteq \Phi(x_0) + [H_{\Phi}(x_0; tu_\epsilon) + tr_\epsilon \mathbb{B}] + t\epsilon \mathbb{B} \\ &\subseteq \Phi(x_0) + C + t\epsilon \mathbb{B}, \quad \forall t \in (0, \delta_\epsilon). \end{aligned}$$

Consequently, it results in

$$(2.20) \quad \text{exc}(\Phi(x_0 + tu_\epsilon) + tr_\epsilon \mathbb{B}, C) \leq \text{exc}(\Phi(x_0) + C + t\epsilon \mathbb{B}, C), \quad \forall t \in (0, \delta_\epsilon).$$

On the other hand, by Remark 2.4(ii) and (v), recalling that $\Phi(x_0) \not\subseteq C$ as well as $\Phi(x_0 + tu_\epsilon) \not\subseteq C$ for every $t \in (0, \delta_\epsilon)$, because $\delta_\epsilon < \delta$, so $x_0 + tu_\epsilon \in [\nu_{\Phi,C} > 0]$, one obtains

$$\text{exc}(\Phi(x_0 + tu_\epsilon) + tr_\epsilon \mathbb{B}, C) = \nu_{\Phi,C}(x_0 + tu_\epsilon) + tr_\epsilon$$

and

$$\text{exc}(\Phi(x_0) + C + t\epsilon \mathbb{B}, C) = \nu_{\Phi,C}(x_0) + t\epsilon.$$

In the light of inequality (2.20), the above equalities yield

$$\frac{\nu_{\Phi,C}(x_0) - \nu_{\Phi,C}(x_0 + tu_\epsilon)}{t} \geq r_\epsilon - \epsilon, \quad \forall t \in (0, \delta_\epsilon),$$

whence one gets

$$\sup_{x \in B(x_0, t) \setminus \{x_0\}} \frac{\nu_{\Phi,C}(x_0) - \nu_{\Phi,C}(x)}{\|x - x_0\|} \geq r_\epsilon - \epsilon > \sigma_H(x_0) - 2\epsilon, \quad \forall t \in (0, \delta_\epsilon).$$

Thus, one obtains

$$|\nabla \nu_{\Phi,C}|(x_0) = \lim_{t \downarrow 0} \sup_{x \in B(x_0, t) \setminus \{x_0\}} \frac{\nu_{\Phi,C}(x_0) - \nu_{\Phi,C}(x)}{\|x - x_0\|} \geq \sigma_H(x_0) - 2\epsilon.$$

By arbitrariness of ϵ the estimate in (2.17) follows from the last inequality. \square

The proof of Proposition 2.5 should help to understand a possible reading of the crucial condition (2.16): it prescribes a behaviour of Φ near x_0 , which results in the existence of a descent direction for $\nu_{\Phi,C}$, with a rate controlled by $\sigma_H(x_0)$.

Remark 2.6. As a caveat regarding condition (2.16), it must be noticed that such a requirement can be satisfied only if $\text{int } C \neq \emptyset$.

3. LIPSCHITZIAN BEHAVIOUR IN METRIC SPACES

Pursuing the research line presented in [31], the study of properties of the solution mapping \mathcal{S} associated to (SVI_p) will be carried out by means of the merit function $\nu_{F,C} : P \times X \rightarrow \mathbb{R} \cup \{\pm\infty\}$, given by

$$(3.1) \quad \nu_{F,C}(p, x) = \text{exc}(F(p, x), C) = \sup_{y \in F(p, x)} \text{dist}(y, C).$$

Such an approach allows one to embed the analysis of the quantitative stability properties of \mathcal{S} into a framework, which is suitable for applying the parametric basic lemma.

In what follows, consistently with the material exposed in Section 2, $\text{dom } F = P \times X$ will be kept as a standing assumption, so it is true that $[\nu_{F,C}(p, x) \geq 0] = P \times X$.

Proposition 3.1 (Parametric solvability and error bound). *Given a parameterized problem (SVI_p) , let $(\bar{p}, \bar{x}) \in P \times X$. Suppose that:*

- (i) (X, d) is metrically complete;
- (ii) $\bar{x} \in \mathcal{S}(\bar{p})$;
- (iii) the set-valued mapping $p \rightsquigarrow F(p, \bar{x})$ is Hausdorff C -u.s.c. at \bar{p} ;
- (iv) there exists $\delta_1 > 0$ such that for every $p \in B(\bar{p}, \delta_1)$ each set-valued mapping $x \rightsquigarrow F(p, x)$ is l.s.c. on X ;
- (v) there exists $\delta_2 > 0$ such that

$$\sigma_\nabla = \inf\{|\nabla_x \nu_{F,C}|(p, x) \mid (p, x) \in [B(\bar{p}, \delta_2) \times B(\bar{x}, \delta_2)] \setminus \text{graph } \mathcal{S}\} > 0.$$

Then, there exist positive η and ζ such that

- (t) $\mathcal{S}(p) \cap B(\bar{x}, \eta) \neq \emptyset$, for every $p \in B(\bar{p}, \zeta)$;
- (tt) the following estimate holds

$$(3.2) \quad \text{dist}(x, \mathcal{S}(p)) \leq \frac{\nu_{F,C}(p, x)}{\sigma_\nabla}, \quad \forall (p, x) \in B(\bar{p}, \zeta) \times B(\bar{x}, \eta).$$

Proof. It suffices to apply the parametric basic lemma (Lemma 2.3) with $\nu = \nu_{F,C}$, after having noted that, by Remark 2.1, under the current hypotheses $\nu_{F,C}(\cdot, \bar{x})$ is u.s.c. at \bar{p} and each function $\nu_{F,C}(p, \cdot)$ is l.s.c. on X , for every p near \bar{p} . Then, it remains to remember that $\mathcal{S}(p) = [\nu_{F,C}(p, \cdot) = 0]$. \square

It is worth remarking that, as a consequence of assertion (t), one gets that each problem (SVI_p) , for every p near \bar{p} , does admit a solution. In other words, it is $\bar{p} \in \text{int dom } \mathcal{S}$. The error bound inequality (3.2) says that $\nu_{F,C}$ works as a residual in estimating the distance from the solution set to (SVI_p) . While to compute the term in the left side of (3.2) one needs to find explicitly the solutions to (SVI_p) , what might be considerably difficult, the residual in the right-side is expressed in terms of problem data, so is expected to be more easily computed.

An important consequence of the above error bound can be established upon an additional hypothesis on F . This leads to the next result about the Lipschitzian behaviour of \mathcal{S} .

Theorem 3.2 (Aubin continuity of \mathcal{S}). *Given a parameterized problem (SVI_p) , let $(\bar{p}, \bar{x}) \in P \times X$. Suppose that all the hypotheses of Proposition 3.1 are in force and suppose, in addition, that*

- (vi) *there exist positive τ and s such that for every $x \in B(\bar{x}, s)$ each set-valued mapping $p \rightsquigarrow F(p, x)$ is Lipschitz with rate ℓ in $B(\bar{p}, \tau)$.*

Then, $\bar{p} \in \text{int dom } \mathcal{S}$, \mathcal{S} is Aubin continuous at (\bar{p}, \bar{x}) and the following estimate holds

$$(3.3) \quad \text{lip } \mathcal{S}(\bar{p}, \bar{x}) \leq \frac{\ell}{\sigma_{\nabla}}.$$

Proof. From assertion (t) in Proposition 3.1 it follows that $\bar{p} \in \text{int dom } \mathcal{S}$ and that there exist positive ζ, η such that the estimate (3.2) holds true. So, setting $\delta = \min\{\zeta, \tau\}$ and $r = \min\{\eta, s\}$, by remembering the inequality in Remark 2.4(vi), one obtains

$$\begin{aligned} \nu_{F,C}(p_1, x) &\leq \text{exc}(F(p_1, x), F(p_2, x)) + \text{exc}(F(p_2, x), C) \\ &\leq \ell d(p_1, p_2), \quad \forall p_1, p_2 \in B(\bar{p}, \delta), \quad \forall x \in B(\bar{x}, r) \cap \mathcal{S}(p_2). \end{aligned}$$

Thus, on account of inequality (3.2), it results in

$$(3.4) \quad \text{dist}(x, \mathcal{S}(p_1)) \leq \frac{\ell}{\sigma_{\nabla}} d(p_1, p_2), \quad \forall p_1, p_2 \in B(\bar{p}, \delta), \quad \forall x \in B(\bar{x}, r) \cap \mathcal{S}(p_2),$$

which shows that condition (2.2) is satisfied with rate $\kappa = \ell/\sigma_{\nabla}$. The estimate in the assertion comes as a direct consequence of the definition of modulus of Aubin continuity, in the light of inequality (3.4). \square

A comparison of Theorem 3.2 with [31, Theorem 3.3] should be useful in order to evaluate its impact. The latter result provides a sufficient condition for the calmness of \mathcal{S} , under a milder set of hypotheses. Nonetheless, inasmuch as Aubin continuity implies calmness, Theorem 3.2 establishes an enhanced Lipschitzian property of \mathcal{S} .

In the same vein, it is worth noting that, since Aubin continuity implies Lipschitz lower semicontinuity as seen in Section 2, Theorem 3.2 contains a sufficient condition also for the Lipschitz lower semicontinuity of \mathcal{S} . Of course, in establishing a

stronger Lipschitz behaviour, the invoked hypotheses are stronger than those appearing in [31, Theorem 3.1], which is a condition specifically tailored for Lipschitz lower semicontinuity.

The next example aims at illustrating the crucial role played by the condition in hypothesis (v) of Proposition 3.1.

Example 3.3. Let $P = X = \mathbb{R}$ and $Y = \mathbb{R}^m$ be endowed with their usual (Euclidean) metric structure. Consider the parameterized set-valued inclusion (SVI_p) with data $F : \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^m$ and C defined by

$$F(p, x) = \{y = (y_1, \dots, y_m) \in \mathbb{R}^m \mid \min_{i=1, \dots, m} y_i \geq x^2 - p\} \text{ and } C = \mathbb{R}_+^m.$$

Fixed $\bar{p} = 0$, it is clear that $\bar{x} = 0 \in \mathcal{S}(0)$. More generally, since it is readily seen that $F(p, x) \subseteq \mathbb{R}_+^m$ iff $x^2 - p \geq 0$, for the problem under consideration the solution set-valued mapping $\mathcal{S} : \mathbb{R} \rightrightarrows \mathbb{R}$ can be computed explicitly, resulting in

$$\mathcal{S}(p) = \begin{cases} \mathbb{R}, & \forall p \in (-\infty, 0], \\ (-\infty, -\sqrt{p}] \cup [\sqrt{p}, +\infty), & \forall p \in (0, +\infty). \end{cases}$$

The set-valued mapping $p \rightsquigarrow F(p, 0)$ is evidently Hausdorff \mathbb{R}_+^m -u.s.c. (though failing to be u.s.c.) at $\bar{p} = 0$. Moreover, as a consequence of the continuity of the function $x \mapsto x^2 - p$, each set-valued mapping $x \rightsquigarrow F(p, x) = (x^2 - p)\{(1, \dots, 1)\} + \mathbb{R}_+^m$ is l.s.c. on \mathbb{R} . From the definition of F , one deduces

$$\nu_{F,C}(p, x) = \begin{cases} 0, & \forall (p, x) \in \text{graph } \mathcal{S}, \\ \sqrt{m}|x^2 - p|, & \forall (p, x) \in (\mathbb{R} \times \mathbb{R}) \setminus \text{graph } \mathcal{S}. \end{cases}$$

Consequently, fixed any $(p, x) \in (\mathbb{R} \times \mathbb{R}) \setminus \text{graph } \mathcal{S}$, one obtains

$$|\nabla_x \nu_{F,C}|(p, x) = \left| \frac{\partial}{\partial x} \sqrt{m}(p - x^2) \right| = 2\sqrt{m}|x|.$$

Thus, if considering the graph \mathcal{S} near its point $(0, 0)$, one sees that for any fixed $\delta > 0$ there exists $(p, 0) \in [B(0, \delta) \times B(0, \delta)] \setminus \text{graph } \mathcal{S}$ such that $|\nabla_x \nu_{F,C}|(p, 0) = 0$. This leads to conclude that $\sigma_{\nabla} = 0$, so hypothesis (v) of Proposition 3.1 in this case is not satisfied. One can check that, whereas the nonemptiness in assertion (t) actually takes place, the function $\nu_{F,C}$ fails to work as a residual for $\text{dist}(x, \mathcal{S}(p))$. Indeed, taking $x = 0$, one finds

$$(3.5) \quad \text{dist}(0, \mathcal{S}(p)) = \begin{cases} 0, & \forall p \in (-\infty, 0], \\ \sqrt{p}, & \forall p \in (0, +\infty). \end{cases}$$

Clearly, the inequality

$$\text{dist}(0, \mathcal{S}(p)) = \sqrt{p} \leq \kappa p = \kappa \nu_{F,C}(p, 0), \quad \forall p \in (0, \zeta),$$

can not be true for any choice of positive κ and ζ . For a similar reason, it is worth noting that the expression (3.5) reveals that \mathcal{S} fails to be Aubin continuous at $(0, 0)$. Nevertheless, one can check by using its definition that the set-valued mapping $p \rightsquigarrow F(p, x)$ is Lipschitz continuous with rate $\ell = 1$ in \mathbb{R} , for every $x \in \mathbb{R}$.

4. FIRST-ORDER ANALYSIS IN NORMED VECTOR SPACES

Unless otherwise stated, throughout the present section, $(\mathbb{P}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|)$ will be normed vector spaces, whereas $(\mathbb{X}, \|\cdot\|)$ will be assumed to be a Banach space. Whenever considered, the product space $\mathbb{P} \times \mathbb{X}$ will be assumed to be equipped with the max-norm $\|(p, x)\| = \max\{\|p\|, \|x\|\}$. Moreover, in view of the employment of condition (2.16), it will be assumed $\text{int } C \neq \emptyset$.

In a normed vector space setting, a sufficient condition for error bounds and the Aubin continuity of a multifunction, defined implicitly by (SVI_p) , can be established in terms of outer prederivative as follows.

Proposition 4.1. *Given a parameterized set-valued inclusion (SVI_p) , let $(\bar{p}, \bar{x}) \in \mathbb{P} \times \mathbb{X}$. Suppose that:*

- (i) $\bar{x} \in \mathcal{S}(\bar{p})$;
- (ii) the set-valued mapping $p \rightsquigarrow F(p, \bar{x})$ is Hausdorff C -u.s.c. at \bar{p} ;
- (iii) there exists $\delta_1 > 0$ such that for every $p \in B(\bar{p}, \delta_1)$ each set-valued mapping $x \rightsquigarrow F(p, x)$ is l.s.c. on \mathbb{X} ;
- (iv) there exists $\delta_2 > 0$ such that, for every $p \in B(\bar{p}, \delta_2)$, each set-valued mapping $x \rightsquigarrow F(p, x)$ admits an outer prederivative $H_{F(p, \cdot)}(x; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at each point $x \in B(\bar{p}, \delta_2)$;
- (v) it holds

$$\sigma_H(\bar{p}, \bar{x}) = \inf\{\sigma_{H_{F(p, \cdot)}}(x) \mid (p, x) \in [B(\bar{p}, \delta_2) \times B(\bar{x}, \delta_2)] \setminus \text{graph } \mathcal{S}\} > 0,$$

where $\sigma_{H_{F(p, \cdot)}}(x)$ is defined as in (2.16).

Then, there exist positive η and ζ such that the estimate (3.2) holds true with σ_∇ replaced by $\sigma_H(\bar{p}, \bar{x})$.

If, in addition,

- (vi) there exist positive τ and s such that for every $x \in B(\bar{x}, s)$ each set-valued mapping $p \rightsquigarrow F(p, x)$ is Lipschitz with rate ℓ in $B(\bar{p}, \tau)$,

then \mathcal{S} is Aubin continuous at (\bar{p}, \bar{x}) and the following estimate holds

$$(4.1) \quad \text{lip } \mathcal{S}(\bar{p}, \bar{x}) \leq \frac{\ell}{\sigma_H(\bar{p}, \bar{x})}.$$

Proof. If $(p, x) \notin \text{graph } \mathcal{S}$, then it is $F(p, x) \not\subseteq C$, or, equivalently, it holds $x \in \mathbb{X} \setminus F^{+1}(p, \cdot)(C)$. Thus, under the above hypotheses it is possible to apply Proposition 2.5. Consequently, for every $(p, x) \in [B(\bar{p}, \delta_2) \times B(\bar{x}, \delta_2)] \setminus \text{graph } \mathcal{S}$ one obtains

$$|\nabla_x \nu_{F, C}|(p, x) \geq \sigma_{H_{F(p, \cdot)}}(x),$$

which implies

$$\sigma_\nabla \geq \sigma_H(\bar{p}, \bar{x}).$$

The last inequality, on account of hypothesis (v), enables one to apply Proposition 3.1 and, under the hypothesis (vi), Theorem 3.2. This completes the proof. \square

Proposition 4.1 has the following consequence on the graphical derivative of \mathcal{S} , which is relevant to the sensitivity analysis of (SVI_p) .

Corollary 4.2 (Lipschitz continuity of $\text{DS}(\bar{p}, \bar{x})$). *Given a parameterized set-valued inclusion (SVI_p) , let $(\bar{p}, \bar{x}) \in \mathbb{P} \times \mathbb{X}$. Under the hypotheses (i) – (vi) of Proposition 4.1, $\text{dom DS}(\bar{p}, \bar{x}) = \mathbb{P}$ and $\text{DS}(\bar{p}, \bar{x}) : \mathbb{P} \rightrightarrows \mathbb{X}$ is Lipschitz continuous on \mathbb{P} with rate $\kappa \leq \ell/\sigma_H(\bar{p}, \bar{x})$.*

Proof. The thesis follows from the Aubin continuity of \mathcal{S} in the light of [26, Exercise 9.49]. Indeed, a perusal of the argument suggested there certifies that the finite-dimensional setting does not affect the reasoning. \square

Other useful properties of $\text{DS}(\bar{p}, \bar{x})$ can be established in the presence of a specific geometric property of F , which appeared in connection with the study of set-valued inclusions already in [6].

Definition 4.3 (C -concavity). A set-valued mapping $\Phi : \mathbb{X} \rightrightarrows \mathbb{Y}$ between normed vector spaces is called C -concave in \mathbb{X} , where C is a convex cone in \mathbb{Y} , if it holds

$$\Phi(tx_1 + (1 - t)x_2) \subseteq t\Phi(x_1) + (1 - t)\Phi(x_2) + C, \quad \forall x_1, x_2 \in \mathbb{X}, \forall t \in [0, 1].$$

A remarkable class of C -concave set-valued mappings emerging in the context of robust convex optimization is singled out below.

Example 4.4. Let $f : \mathbb{X} \times \Omega \rightarrow \mathbb{Y}$ be a given mapping, with $\Omega \neq \emptyset$, and let $C \subseteq \mathbb{Y}$ be a convex cone. If each single-valued mapping $f(\cdot, \omega) : \mathbb{X} \rightarrow \mathbb{Y}$ is C -concave in \mathbb{X} , i.e. for every $x_1, x_2 \in \mathbb{X}$ and $t \in [0, 1]$ it holds

$$f(tx_1 + (1 - t)x_2, \omega) - tf(x_1, \omega) - (1 - t)f(x_2, \omega) \in C,$$

for every $\omega \in \Omega$, then the set-valued mapping $\Phi_f : \mathbb{X} \rightrightarrows \mathbb{Y}$ defined by

$$\Phi_f(x) = \{y = f(x, \omega) \mid \omega \in \Omega\} = f(x, \Omega)$$

turns out to be C -concave in \mathbb{X} . Indeed, taken any pair $x_1, x_2 \in \mathbb{X}$ and $t \in [0, 1]$, if y is an arbitrary element of $\Phi_f(tx_1 + (1 - t)x_2)$, then there exists $\omega \in \Omega$ such that

$$\begin{aligned} y &= f(tx_1 + (1 - t)x_2, \omega) \in tf(x_1, \omega) + (1 - t)f(x_2, \omega) + C \\ &\subseteq t\Phi_f(x_1) + (1 - t)\Phi_f(x_2) + C, \end{aligned}$$

which shows that

$$\Phi_f(tx_1 + (1 - t)x_2) \subseteq t\Phi_f(x_1) + (1 - t)\Phi_f(x_2) + C.$$

It is worth noting that if, in particular, $f : \mathbb{X} \times \Omega \rightarrow \mathbb{R}^m$ is given by $f = (f_1, \dots, f_m)$, where $f_i(\cdot, \omega) : \mathbb{X} \rightarrow \mathbb{R}$ is concave for every $i = 1, \dots, m$, and $\omega \in \Omega$, then f turns out to be \mathbb{R}_+^m -concave. Thus, the set-valued inclusion $\Phi_f(x) \subseteq \mathbb{R}_+^m$ expresses in this case the robust fulfilment of the convex inequality system

$$\begin{cases} -f_1(x, \omega) \leq 0 \\ \vdots \\ -f_m(x, \omega) \leq 0, \end{cases}$$

which typically defines the feasible region in robust convex optimization (see [4]).

Remark 4.5. The notion of C -concavity for set-valued mappings is evidently a generalization of that of concavity, as presented in [30, Definition 2.3]. Consequently, several further examples of C -concave set-valued mappings, including among others the class of fans introduced by Ioffe (see [10]), can be found therein.

The C -concavity of F in $\mathbb{P} \times \mathbb{X}$ yields the following important geometric property of \mathcal{S} .

Proposition 4.6 (Convexity of \mathcal{S}). *With reference to a parameterized set-valued inclusion (SVI_p), suppose that $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is C -concave on $\mathbb{P} \times \mathbb{X}$. Then, $\mathcal{S} : \mathbb{P} \rightrightarrows \mathbb{X}$ is a convex set-valued mapping.*

Proof. Taken arbitrary $p_1, p_2 \in \text{dom } \mathcal{S}$ and $x_1, x_2 \in \mathbb{X}$, with $x_i \in \mathcal{S}(p_i)$, $i = 1, 2$, by virtue of the C -concavity of F one has

$$\begin{aligned} F(t(p_1, x_1) + (1-t)(p_2, x_2)) &\subseteq tF(p_1, x_1) + (1-t)F(p_2, x_2) + C \\ &\subseteq tC + (1-t)C + C \subseteq C, \quad \forall t \in [0, 1]. \end{aligned}$$

This inclusion shows that

$$tx_1 + (1-t)x_2 \in \mathcal{S}(tp_1 + (1-t)p_2), \quad \forall t \in [0, 1].$$

By arbitrariness of $x_i \in \mathcal{S}(p_i)$, from the last inclusion one can deduce that

$$t\mathcal{S}(p_1) + (1-t)\mathcal{S}(p_2) \subseteq \mathcal{S}(tp_1 + (1-t)p_2), \quad \forall t \in [0, 1],$$

thereby completing the proof. \square

As one expects, the convexity of \mathcal{S} , that is the convexity of its graph, induces a similar geometric property in its graphical approximation, as stated next.

Corollary 4.7 (Sublinearity of $\text{DS}(\bar{p}, \bar{x})$). *With reference to a parameterized set-valued inclusion (SVI_p), let $(\bar{p}, \bar{x}) \in \text{graph } \mathcal{S}$. If $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is C -concave on $\mathbb{P} \times \mathbb{X}$, then $\text{DS}(\bar{p}, \bar{x}) : \mathbb{P} \rightrightarrows \mathbb{X}$ is a closed sublinear set-valued mapping (a.k.a. convex process).*

If, in addition, $\text{graph } \mathcal{S}$ is closed, then \mathcal{S} is protodifferentiable at (\bar{p}, \bar{x}) .

Proof. Since by Proposition 4.6 $\text{graph } \mathcal{S}$ is convex, so is $\text{T}(\text{graph } \mathcal{S}; (\bar{p}, \bar{x}))$. Having a closed, convex cone as a graph, the set-valued mapping $\text{DS}(\bar{p}, \bar{x}) : \mathbb{P} \rightrightarrows \mathbb{X}$ must be closed and sublinear.

As for the second assertion, it is useful to recall that a sufficient condition for protodifferentiability is graph regularity (see [26, Proposition 8.41]), which in turn comes here as a consequence of the convexity of the graph along with the outer semicontinuity (see [26, Example 8.39]), the latter property being guaranteed by the fact that the graph of \mathcal{S} is closed (see [7, Theorem 3B.2(c)]). \square

Convexity interacts also with Aubin continuity yielding an enhanced Lipschitzian behaviour of \mathcal{S} , according to the assertion below.

Corollary 4.8 (Lipschitz continuity of \mathcal{S} under truncation). *With reference to a parameterized set-valued inclusion (SVI_p), let $(\bar{p}, \bar{x}) \in \text{graph } \mathcal{S}$. Suppose that all the hypotheses (i) – (vi) of Proposition 4.1 are fulfilled. If $F : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ is C -concave on $\mathbb{P} \times \mathbb{X}$, then \mathcal{S} has a Lipschitz continuous graphical localization (not necessarily single-valued) around $(\bar{p}, \bar{x}) \in \text{graph } \mathcal{S}$, i.e. there exist neighbourhoods V of \bar{p} and U of \bar{x} such that the truncated mapping $p \rightsquigarrow \mathcal{S}(p) \cap U$ is Lipschitz continuous on V .*

Proof. Observe that, in the light of Proposition 4.6, \mathcal{S} is convex. Consequently, it takes convex values. Since \mathcal{S} is also Aubin continuous at (\bar{p}, \bar{x}) by virtue of the hypotheses taken, the convexity of its images allows one to apply [7, Theorem 3E.3],

after having noticed that this result can be extended to normed vector spaces with the same proof. \square

In the absence of convexity of \mathcal{S} , the following formulae provide inner and outer approximations of $\text{DS}(\bar{p}, \bar{x})$ in terms of proper prederivatives.

Theorem 4.9 (Inner approximation of $\text{DS}(\bar{p}, \bar{x})$). *Given a parameterized set-valued inclusion (SVI_p) , let $(\bar{p}, \bar{x}) \in \mathbb{P} \times \mathbb{X}$. Suppose that:*

- (i) $\bar{x} \in \mathcal{S}(\bar{p})$;
- (ii) the set-valued mapping $p \rightsquigarrow F(p, \bar{x})$ is Hausdorff C -u.s.c. at \bar{p} ;
- (iii) there exists $\delta_1 > 0$ such that for every $p \in \text{B}(\bar{p}, \delta_1)$ each set-valued mapping $x \rightsquigarrow F(p, x)$ is l.s.c. on \mathbb{X} ;
- (iv) there exists $\delta_2 > 0$ such that, for every $p \in \text{B}(\bar{p}, \delta_2)$, each set-valued mapping $x \rightsquigarrow F(p, x)$ admits an outer prederivative $H_{F(p, \cdot)}(x; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at each point $x \in \text{B}(\bar{p}, \delta_2)$, such that $\sigma_H(\bar{p}, \bar{x}) > 0$;
- (v) F admits an outer prederivative $H_F((\bar{p}, \bar{x}); \cdot) : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ at (\bar{p}, \bar{x}) .

Then, the following approximation holds

$$(4.2) \quad \text{DS}(\bar{p}, \bar{x})(p) \supseteq H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(C), \quad \forall p \in \mathbb{P}.$$

Proof. Take an arbitrary $v \in H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(C)$. Observe first of all that, since $\text{DS}(\bar{p}, \bar{x})$ and $H_F((\bar{p}, \bar{x}); \cdot)$ are both p.h. set-valued mappings, it suffices to prove the validity of inclusion (4.2) in the case $(p, v) \in \mathbb{B} \times \mathbb{B}$. That said, in order to prove that $v \in \text{DS}(\bar{p}, \bar{x})(p)$, one has to show that $(p, v) \in \text{graph } \text{DS}(\bar{p}, \bar{x}) = \text{T}(\text{graph } \mathcal{S}; (\bar{p}, \bar{x}))$. On account of the characterization recalled in (2.13), this can be done by showing that

$$\liminf_{t \downarrow 0} \frac{\text{dist}((\bar{p}, \bar{x}) + t(p, v), \text{graph } \mathcal{S})}{t} = 0.$$

The last equality means that for every $\tau > 0$ and $\epsilon > 0$ there must exist $t \in (0, \tau)$ such that

$$(4.3) \quad \frac{\text{dist}((\bar{p}, \bar{x}) + t(p, v), \text{graph } \mathcal{S})}{t} \leq \epsilon.$$

Fix positive ϵ and τ . Since under the current hypotheses it is possible to apply Proposition 4.1, one gets the existence of ζ and η such that the estimate

$$(4.4) \quad \text{dist}(\bar{x} + tv, \mathcal{S}(\bar{p} + tp)) \leq \frac{\nu_{F,C}(\bar{p} + tp, \bar{x} + tv)}{\sigma_H(\bar{p}, \bar{x})}, \quad \forall t \in (0, \min\{\zeta, \eta\})$$

holds true. By hypothesis (v), there exists $0 < \delta_\epsilon < \min\{\zeta, \eta, \tau\}$ such that

$$F((\bar{p}, \bar{x}) + t(p, v)) \subseteq F(\bar{p}, \bar{x}) + tH_F((\bar{p}, \bar{x}); (p, v)) + \epsilon t \sigma_H(\bar{p}, \bar{x}) \mathbb{B}, \quad \forall t \in (0, \delta_\epsilon).$$

As it is $v \in H_F((\bar{p}, \bar{x}); (p, v)) \subseteq C$, from the above inclusion one obtains

$$F((\bar{p}, \bar{x}) + t(p, v)) \subseteq F(\bar{p}, \bar{x}) + C + \epsilon t \sigma_H(\bar{p}, \bar{x}) \mathbb{B}, \quad \forall t \in (0, \delta_\epsilon).$$

Thus, when passing to the excess, by virtue of what observed in Remark 2.4(i), (ii) and (iii), one finds

$$(4.5) \quad \begin{aligned} \nu_{F,C}((\bar{p}, \bar{x}) + t(p, v)) &\leq \text{exc}(F(\bar{p}, \bar{x}) + C + \epsilon t \sigma_H(\bar{p}, \bar{x}) \mathbb{B}, C) \\ &\leq \nu_{F,C}(\bar{p}, \bar{x}) + \epsilon t \sigma_H(\bar{p}, \bar{x}), \\ &= \epsilon t \sigma_H(\bar{p}, \bar{x}), \quad \forall t \in (0, \delta_\epsilon). \end{aligned}$$

Now, it is proper to observe that

$$\begin{aligned} \text{dist}((\bar{p}, \bar{x}) + t(p, v), \text{graph } \mathcal{S}) &= \inf_{(q,w) \in \text{graph } \mathcal{S}} \|(\bar{p}, \bar{x}) + t(p, v) - (q, w)\| \\ &\leq \inf_{w \in \mathcal{S}(\bar{p}+tp)} \|(\bar{p}, \bar{x}) + t(p, v) - (\bar{p} + tp, w)\| \\ &= \inf_{w \in \mathcal{S}(\bar{p}+tp)} \|\bar{x} + tv - w\| \\ &= \text{dist}(\bar{x} + tv, \mathcal{S}(\bar{p} + tp)). \end{aligned}$$

Therefore, by combining the last inequality chain with inequalities (4.4) and (4.5), one obtains

$$\text{dist}((\bar{p}, \bar{x}) + t(p, v), \text{graph } \mathcal{S}) \leq \frac{\nu_{F,C}(\bar{p} + tp, \bar{x} + tv)}{\sigma_H(\bar{p}, \bar{x})} \leq \epsilon t, \quad \forall t \in (0, \delta_\epsilon).$$

As it is $\delta_\epsilon < \tau$, from the last inequality one can deduce the existence of $t \in (0, \tau)$, for which inequality (4.3) is fulfilled, thereby completing the proof. \square

Remark 4.10. (i) The reader should notice that hypotheses (iv) and (v) of Theorem 4.9 speak about different mathematical objects. Indeed, the outer prederivatives mentioned in hypothesis (iv) provide partial first-order approximations, each for the mapping $F(p, \cdot)$ (depending only on x) near \bar{x} , with $p \in B(\bar{p}, \delta_2)$. In contrast, $H_F((\bar{p}, \bar{x}); \cdot)$ provides a first-order joint approximation of F as a multifunction of both the variables p and x .

(ii) It is readily seen that each set $H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(C)$ is a cone in \mathbb{X} , being the upper inverse image of a cone through a p.h. set-valued mapping. In particular, whenever $H_F((\bar{p}, \bar{x}); \cdot)$ is C -concave, it can be shown to be a convex cone. By virtue of these features, such a set is expected to be more easily computable than the set $DS(\bar{p}, \bar{x})(p)$, in the spirit of implicit function theorems.

Theorem 4.11 (Outer approximation of $DS(\bar{p}, \bar{x})$). *Given a parameterized set-valued inclusion (SVI_p) , let $(\bar{p}, \bar{x}) \in \mathbb{P} \times \mathbb{X}$, with $\bar{x} \in \mathcal{S}(\bar{p})$. Suppose that F admits an inner prederivative $H_F((\bar{p}, \bar{x}); \cdot) : \mathbb{P} \times \mathbb{X} \rightrightarrows \mathbb{Y}$ at (\bar{p}, \bar{x}) , which is l.s.c. on $\mathbb{P} \times \mathbb{X}$. Then, the following approximation holds*

$$(4.6) \quad DS(\bar{p}, \bar{x})(p) \subseteq \bigcap_{y \in F(\bar{p}, \bar{x})} H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(T(C; y)), \quad \forall p \in \mathbb{P}.$$

Proof. Fix $p \in \mathbb{P}$ and take an arbitrary $v \in DS(\bar{p}, \bar{x})(p)$. According to the definition of graphical (contingent) derivative, there exist sequences $(p_n)_n$ in \mathbb{P} , with $p_n \rightarrow p$, $(v_n)_n$ in \mathbb{X} , with $v_n \rightarrow v$, and $(t_n)_n$ in $(0, +\infty)$, with $t_n \downarrow 0$, such that $\bar{x} + t_n v_n \in \mathcal{S}(\bar{p} + t_n p_n)$ for every $n \in \mathbb{N}$, which means

$$(4.7) \quad F((\bar{p}, \bar{x}) + t_n(p_n, v_n)) \subseteq C, \quad \forall n \in \mathbb{N}.$$

Fix $\epsilon > 0$. Since $H_F((\bar{p}, \bar{x}); \cdot)$ is an inner prederivative of F at (\bar{p}, \bar{x}) , there exists $\delta_\epsilon > 0$ such that

$$(4.8) \quad F(\bar{p}, \bar{x}) + tH_F((\bar{p}, \bar{x}); (q, w)) \subseteq F(p, x) + \epsilon t\mathbb{B}, \quad \forall (q, w) \in \mathbb{B} \times \mathbb{B}, \quad \forall t \in (0, \delta_\epsilon).$$

Since, without loss of generality, it is possible to assume that $(p_n, v_n) \in \mathbb{B} \times \mathbb{B}$ and $t_n \in (0, \delta_\epsilon)$, by combining inclusions (4.8) and (4.7), one obtains

$$F(\bar{p}, \bar{x}) + t_n H_F((\bar{p}, \bar{x}); (p_n, v_n)) \subseteq C + \epsilon t_n \mathbb{B}, \quad \forall n \in \mathbb{N}.$$

Let y be an arbitrary element of $F(\bar{p}, \bar{x})$. As it is $F(\bar{p}, \bar{x}) \subseteq C$ by hypothesis, the last inclusion implies

$$H_F((\bar{p}, \bar{x}); (p_n, v_n)) \subseteq \frac{C - y}{t_n} + \epsilon \mathbb{B} \subseteq \text{cl}[\text{cone}(C - y)] + \epsilon \mathbb{B}, \quad \forall n \in \mathbb{N}.$$

Since C is convex, by virtue of the representation (2.12) valid for its contingent cone, the last inclusion gives

$$(4.9) \quad H_F((\bar{p}, \bar{x}); (p_n, v_n)) \subseteq T(C; y) + \epsilon \mathbb{B} \subseteq B(T(C; y), \epsilon), \quad \forall n \in \mathbb{N}.$$

As the set-valued mapping $H_F((\bar{p}, \bar{x}); \cdot)$ is l.s.c. at (p, v) and $B(T(C; y), \epsilon)$ is a closed set, from the last inclusion it follows

$$(4.10) \quad H_F((\bar{p}, \bar{x}); (p, v)) \subseteq B(T(C; y), \epsilon).$$

Indeed, if one assumes that

$$H_F((\bar{p}, \bar{x}); (p, v)) \cap [\mathbb{Y} \setminus B(T(C; y), \epsilon)] \neq \emptyset,$$

there must exist δ_C such that

$$H_F((\bar{p}, \bar{x}); (q, w)) \cap [\mathbb{Y} \setminus B(T(C; y), \epsilon)] \neq \emptyset, \quad \forall (q, w) \in B(p, \delta_C) \times B(v, \delta_C),$$

which contradicts inclusion (4.9), because $(p_n, v_n) \rightarrow (p, v)$ as $n \rightarrow \infty$, so that $(p_n, v_n) \in B(p, \delta_C) \times B(v, \delta_C)$. By arbitrariness of ϵ , as $T(C; y)$ is a closed set, inclusion (4.10) entails

$$H_F((\bar{p}, \bar{x}); (p, v)) \subseteq T(C; y),$$

what amounts to say

$$v \in H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(T(C; y)).$$

Since this reasoning is valid for every $y \in F(\bar{p}, \bar{x})$, one achieves the inclusion (4.6) in the thesis. \square

For the set $\bigcap_{y \in F(\bar{p}, \bar{x})} H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(T(C; y))$ what has been said in Remark 4.10(ii) can be repeated.

Remark 4.12. Since it holds $T(C; y) = \mathbb{Y}$ whenever $y \in \text{int} C$ and one has $H_F^{+1}((\bar{p}, \bar{x}); (p, \cdot))(\mathbb{Y}) = \mathbb{X}$, it should be clear that formula (4.6) yields an useful outer approximation of $\text{DS}(\bar{p}, \bar{x})$, provided that $F(\bar{p}, \bar{x}) \not\subseteq \text{int} C$. On the other hand, in the case $F(\bar{p}, \bar{x}) \subseteq \text{int} C$, if F is u.s.c. at (\bar{p}, \bar{x}) , there exists $\delta_0 > 0$ such that

$$F(p, x) \subseteq C, \quad \forall (p, x) \in B(\bar{p}, \delta_0) \times B(\bar{x}, \delta_0).$$

This means that

$$B(\bar{x}, \delta_0) \subseteq \mathcal{S}(p), \quad \forall p \in B(\bar{p}, \delta_0)$$

and hence $B(\bar{p}, \delta_0) \times B(\bar{x}, \delta_0) \subseteq \text{graph } \mathcal{S}$. In other terms, $(\bar{p}, \bar{x}) \in \text{int graph } \mathcal{S}$. Consequently, it results in $T(\text{graph } \mathcal{S}; (\bar{p}, \bar{x})) = \mathbb{P} \times \mathbb{X}$. According to the definition of graphical (contingent) derivative, this fact yields

$$\text{DS}(\bar{p}, \bar{x})(p) = \mathbb{X}, \quad \forall p \in \mathbb{P}.$$

So, the circumstance $F(\bar{p}, \bar{x}) \subseteq \text{int } C$, under an upper semicontinuity assumption, turns out to be of less interest.

Following a similar technique as in [5, Theorem 5.5.1], the following estimate of the coderivative of \mathcal{S} can be established.

Proposition 4.13 (Coderivative of \mathcal{S} via merit function). *Let (SVI_p) be a parameterized set-valued inclusion and let $(\bar{p}, \bar{x}) \in \mathbb{P} \times \mathbb{X}$. Suppose that:*

- (i) $\bar{x} \in \mathcal{S}(\bar{p})$;
- (ii) the set-valued mapping $p \rightsquigarrow F(p, \bar{x})$ is Hausdorff C -u.s.c. at \bar{p} ;
- (iii) there exists $\delta_1 > 0$ such that for every $p \in B(\bar{p}, \delta_1)$ each set-valued mapping $x \rightsquigarrow F(p, x)$ is l.s.c. on \mathbb{X} ;
- (iv) there exists $\delta_2 > 0$ such that, for every $p \in B(\bar{p}, \delta_2)$, each set-valued mapping $x \rightsquigarrow F(p, x)$ admits an outer prederivative $H_{F(p, \cdot)}(x; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Y}$ at each point $x \in B(\bar{p}, \delta_2)$, such that $\sigma_H(\bar{p}, \bar{x}) > 0$.

Then, there exist positive ζ and η such that the following representation holds

$$(4.11) \quad \widehat{D}^* \mathcal{S}(p, x)(x^*) = \{p^* \in \mathbb{P}^* \mid (p^*, -x^*) \in \text{cone } \widehat{\nu}_{F, C}(p, x)\}, \\ \forall (p, x) \in [\text{int } B(\bar{p}, \zeta) \times \text{int } B(\bar{x}, \eta)] \cap \text{graph } \mathcal{S}.$$

Proof. The above hypotheses ensure the validity of the first assertion in Proposition 4.1. Thus, there exists positive ζ and η such that

$$(4.12) \quad \text{dist}(x, \mathcal{S}(p)) \leq \frac{\nu_{F, C}(p, x)}{\sigma_H(\bar{p}, \bar{x})}, \quad \forall (p, x) \in B(\bar{p}, \zeta) \times B(\bar{x}, \eta).$$

Now, take an arbitrary pair $(p, x) \in [\text{int } B(\bar{p}, \zeta) \times \text{int } B(\bar{x}, \eta)] \cap \text{graph } \mathcal{S}$, $x^* \in \mathbb{X}^*$ and $p^* \in \widehat{D}^* \mathcal{S}(p, x)(x^*)$. According to the representation of a Fréchet normal cone in terms of Fréchet subdifferential in (2.15), one has

$$(p^*, -x^*) \in \widehat{N}(\text{graph } \mathcal{S}; (p, x)) = \bigcup_{\kappa > 0} \kappa \widehat{\text{dist}}(\cdot, \text{graph } \mathcal{S})(p, x).$$

By applying a well-known variational description of Fréchet subgradients (see, for instance, [19, Theorem 1.88(i)]), the above inclusion means that, for some $\kappa > 0$, there must exist a function $\varphi \in C^1(\mathbb{P} \times \mathbb{X})$ with $\widehat{D}\varphi(p, x) = (p^*, -x^*)$ and $\varphi(p, x) = \kappa \text{dist}((p, x), \text{graph } \mathcal{S}) = 0$, such that

$$\varphi(q, z) \leq \kappa \text{dist}((q, z), \text{graph } \mathcal{S}), \quad \forall (q, z) \in \mathbb{P} \times \mathbb{X}.$$

By combining the last inequality with inequality (4.12) and taking $r > 0$ in such a way that $B(p, r) \times B(x, r) \subseteq B(\bar{p}, \zeta) \times B(\bar{x}, \eta)$, one obtains

$$\begin{aligned} \varphi(q, z) &\leq \varphi(p, x) + \kappa \text{dist}((q, z), \text{graph } \mathcal{S}) \\ &\leq \varphi(p, x) + \kappa \text{dist}(z, \mathcal{S}(q)) \leq \varphi(p, x) + \frac{\kappa}{\sigma_H(\bar{p}, \bar{x})} \nu_{F, C}(q, z), \\ &\forall (q, z) \in B(p, r) \times B(x, r). \end{aligned}$$

This inequality says that (p, x) is a local minimizer of the function $(q, z) \mapsto -\varphi(q, z) + (\kappa/\sigma_H(\bar{p}, \bar{x}))\nu_{F, C}(q, z)$. Consequently, by well-known calculus rules of the Fréchet

subdifferential (see, for instance, [19, Proposition 1.107(i) and Proposition 1.114]), it must be

$$(p^*, -x^*) = \widehat{D}\varphi(p, x) \in \frac{\kappa}{\sigma_H(\bar{p}, \bar{x})} \widehat{\partial}\nu_{F,C}(p, x).$$

Thus, the above argument proves the inclusion

$$\widehat{D}^*\mathcal{S}(p, x)(x^*) \subseteq \{p^* \in \mathbb{P}^* \mid (p^*, -x^*) \in \text{cone } \widehat{\partial}\nu_{F,C}(p, x)\}.$$

For proving the reverse inclusion it suffices to observe that for every $\kappa > 0$ it holds

$$\kappa\nu_{F,C}(q, z) \leq \iota((q, z); \text{graph } \mathcal{S}), \quad \forall (q, z) \in \mathbb{P} \times \mathbb{X},$$

which, by known properties of the Fréchet subdifferential, implies

$$\kappa\widehat{\partial}\nu_{F,C}(p, x) \subseteq \widehat{\partial}\iota(\cdot; \text{graph } \mathcal{S})(p, x) = \widehat{N}(\text{graph } \mathcal{S}; (p, x)),$$

thereby completing the proof. □

Remark 4.14. Whenever F is, in particular, C -concave in $\mathbb{P} \times \mathbb{X}$, function $\nu_{F,C}$ turns out to be convex in $\mathbb{P} \times \mathbb{X}$. Indeed, for every $(p_1, x_1), (p_2, x_2) \in \mathbb{P} \times \mathbb{X}$ and $t \in [0, 1]$, by virtue of the relations recalled in Remark 2.4(i) and (ii), it holds

$$\begin{aligned} \nu_{F,C}(t(p_1, x_1) + (1-t)(p_2, x_2)) &= \text{exc}(F(t(p_1, x_1) + (1-t)(p_2, x_2)), C) \\ &\leq \text{exc}(tF(t(p_1, x_1) + (1-t)(p_2, x_2)) + C, C) \\ &= \text{exc}(tF(p_1, x_1) + (1-t)F(p_2, x_2), C) \\ &\leq t\text{exc}(F(p_1, x_1), C) + (1-t)\text{exc}(F(p_2, x_2), C) \\ &= t\nu_{F,C}(p_1, x_1) + (1-t)\nu_{F,C}(p_2, x_2). \end{aligned}$$

Thus, in such an event the Fréchet subdifferential in formula (4.11) can be replaced with the subdifferential in the sense of convex analysis.

5. CONCLUSIONS

The findings exposed in Section 3 and 4 provide some elements for a first-order analysis of multifunctions implicitly defined by parameterized set-valued inclusions. These elements are formulated in the language of modern variational analysis, speaking of Lipschitzian properties and generalized derivatives. The methodology employed for the main achievements is based on an error bound estimate, which describes the metric behaviour of the solution mapping near points of its graph, under proper infinitesimal conditions. Such an approach leaves open some technical questions that could be matter for a future deepening of the present research line. Among them, the following ones are to be mentioned:

1. Several results presented in Section 4 invoke the condition $\sigma_H(\bar{p}, \bar{x}) > 0$. It would be useful to work out this condition in relation to specific forms taken by the outer prederivative (e.g. fans and, in particular, those fans generated by bundles of linear operators).
2. A general assumption on the set-valued inclusions considered in Section 4 is that $\text{int } C \neq \emptyset$ (remember the caveat in Remark 2.6). The author is aware of the fact that this condition might be severe in the context of infinite-dimensional spaces. It would be helpful therefore to devise surrogates of the condition (2.16), which allow one to avoid involving the topological interior of C .

3. In order to complete the analysis of $\widehat{D}^*\mathcal{S}(p, x)$ with a representation in terms of problem data, it would be proper to find how to express $\widehat{\partial}\nu_{F,C}(p, x)$ via the coderivative of F .

As a final remark, it should be added that an impact evaluation of the main achievements on the treatment of robust optimization problems would deserve a specifically devoted analysis.

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REFERENCES

- [1] J.-P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res. **9**(1) (1984), 87–111.
- [2] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, Birkhäuser Boston, Boston, MA, 2009.
- [3] D. Azé, J.-N. Corvellec, and R.E. Lucchetti, *Variational pairs and applications to stability in nonsmooth analysis*, Nonlinear Anal. **49**(5) (2002), 643–670.
- [4] A. Ben-Tal and A. Nemirovski, *Robust convex optimization*, Math. Oper. Res. **23**(4) (1998), 769–805.
- [5] J. M. Borwein and Q. J. Zhu, *Techniques of variational analysis*, Springer-Verlag, New York, 2005.
- [6] M. Castellani, *Error bounds for set-valued maps*, Generalized convexity and optimization for economic and financial decisions, 121–135, Pitagora, Bologna, 1999.
- [7] A.L. Dontchev and R.T. Rockafellar, *Implicit functions and solution mappings. A view from variational analysis*, Springer, New York, 2014.
- [8] M. Durea and R. Strugariu, *Openness stability and implicit multifunction theorems: applications to variational systems*, Nonlinear Anal. **75**(3) (2012), 1246–1259.
- [9] H. Gfrerer and J.V. Outrata, *On Lipschitzian properties of implicit multifunctions* SIAM J. Optim. **26**(4) (2016), 2160–2189.
- [10] A.D. Ioffe, *Nonsmooth analysis: Differential calculus of nondifferentiable mappings*, Trans. Amer. Math. Soc. **266**(1) (1981), 1–56.
- [11] A.D. Ioffe, *Metric regularity and subdifferential calculus*, Uspekhi Mat. Nauk **55**(3) (2000), 103–162.
- [12] A.D. Ioffe, *Implicit functions: a metric theory*, Set-Valued Var. Anal. **25**(4) (2017), 679–699.
- [13] A.A. Khan, C. Tammer and C. Zălinescu, *Set-valued optimization. An introduction with applications*, Springer, Heidelberg, 2015.
- [14] D. Klattke and B. Kummer, *Nonsmooth equations in optimizations. Regularity, calculus, methods and applications*, Kluwer Academic Publishers, Dordrecht, 2002.
- [15] Y.S. Ledyaev and Q.J. Zhu, *Implicit multifunction theorems*, Set-Valued Anal. **7**(3) (1999), 209–238.
- [16] G.M. Lee, N.N. Tam, and N.D. Yen, *Normal coderivative for multifunctions and implicit function theorems*, J. Math. Anal. Appl. **338**(1) (2008), 11–22.
- [17] B. S. Mordukhovich, *Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis*, Trans. Amer. Math. Soc. **343**(2), (1994), 609–657.
- [18] B. S. Mordukhovich, *Lipschitzian stability of constraint systems and generalized equations*, Nonlinear Anal. **22**(2) (1994), 173–206.
- [19] B. S. Mordukhovich, *Variational analysis and generalized differentiation. I. Basic theory*, Springer-Verlag, Berlin, 2006.
- [20] H. Van Ngai and M. Théra, *Error bounds and implicit multifunction theorem in smooth Banach spaces and applications to optimization*, Set-Valued Anal. **12**(1–2) (2004), 195–223.

- [21] H. Van Ngai, N.H. Tron and M. Théra, *Implicit multifunction theorems in complete metric spaces*, Math. Program. **139**(1–2) (2013), Ser. B, 301–326.
- [22] C.H.J. Pang, *Generalized Differentiation with Positively Homogeneous Maps: Applications in Set-Valued Analysis and Metric Regularity*, Math. Oper. Res. **36**(3) (2011), 377–397.
- [23] S.M. Robinson, *Stability theory for systems of inequalities. II. Differentiable nonlinear systems*, SIAM J. Numer. Anal. **13**(4) (1976), 497–513.
- [24] S.M. Robinson, *Generalized equations and their solutions. I. Basic theory. Point-to-set maps and mathematical programming*, Math. Programming Stud. No. **10** (1979), 128–141.
- [25] S.M. Robinson, *An implicit-function theorem for a class of nonsmooth functions*, Math. Oper. Res. **16**(2) (1991), 292–309.
- [26] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.
- [27] A.M. Rubinov and A.A. Vladimirov, *Differences of convex compacta and metric spaces of convex compacta with applications: a survey*, in Quasidifferentiability and related topics, 263–296, Nonconvex Optim. Appl., 43, Kluwer Acad. Publ., Dordrecht, 2000.
- [28] W. Schirotzek, *Nonsmooth analysis*, Springer, Berlin, 2007.
- [29] A. Uderzo, *On some generalized equations with metrically C -increasing mappings: solvability and error bounds with applications to optimization*, Optimization **68** (2019), 227–253.
- [30] A. Uderzo, *Solution analysis for a class of set-inclusive generalized equations: a convex analysis approach*, Pure Appl. Funct. Anal. **5**(3) (2020), 769–790.
- [31] A. Uderzo, *On the quantitative solution stability of parameterized set-valued inclusions*, Set-valued Var. Anal. **29** (2021), 425–451.

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