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SPLIT COMMON NULL POINT PROBLEMS AND NEW HYBRID METHODS FOR MAXIMAL MONOTONE OPERATORS IN TWO BANACH SPACES

NARIYUKI MINAMI AND WATARU TAKAHASHI

ABSTRACT. In this paper, we deal with split common null point problems under new hybrid methods for maximal monotone operators in two Banach spaces. Using metric resolvents and generalized resolvents of maximal monotone operators in Banach spaces, we prove strong convergence theorems under hybrid methods for finding solutions of split common null point problems in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem and the split common null point problem in two Banach spaces.

1. INTRODUCTION

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [6] also considered the following problem: Given maximal monotone mappings $G : H_1 \to 2^{H_1}$, and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the split common null point problem [6] is to find a point $z \in H_1$ such that

$$z \in G^{-1}0 \cap A^{-1}(B^{-1}0),$$

where $G^{-1}0$ and $B^{-1}0$ are the null point sets of G and B, respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [3], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

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where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Furthermore, if $G^{-1}0 \cap A^{-1}(B^{-1}0)$ is nonempty, then for $\gamma > 0$, $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to

$$z = J_\lambda(I - \gamma A^*(I - Q_\mu)A)z, \tag{1.2}$$

where J_λ and Q_μ are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem, the split common null point problem and the split common fixed point problem; see, for instance, [3, 6, 16, 17, 27, 28]. However, it is difficult to have such results outside Hilbert spaces. Takahashi [23, 24] and Hojo and Takahashi [10] extended the results of (1.1) and (1.2) in Hilbert spaces to Banach spaces. Furthermore, by using the methods of [13, 14, 20], Takahashi [25] proved a strong convergence theorem for two metric resolvents of maximal monotone operators in two Banach spaces. Furtheremore Takahashi [26] proved a strong convergence theorem for two generalized resolvents of maximal monotone operators in two Banach spaces; These theorems solved the split common null point problems in two Banach spaces.

In this paper, we consider split common null point problems in two Banach spaces. We first prove a strong convergence theorem under a new hybrid method for metric resolvents and generalized resolvents of maximal monotone operators with generalized projections in two Banach spaces. Furthermore, we prove another strong convergence theorem under the hybrid method for generalized resolvents and metric resolvents of maximal monotone operators with metric projections in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem and the split common null point problem in two Banach spaces.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [8, 15].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [21,22].

Lemma 2.1 ([21])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi_E : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.4}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.5}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.6}$$

The following lemma was proved by Kamimura and Takahashi [11].

Lemma 2.2 ([11])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C. We know the following result.

Lemma 2.3 ([9,21])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) *$z = P_C x$;*

(2) $\langle z - y, J(x - z)\rangle \geq 0, \quad \forall y \in C.$

For any $x \in E$, we also know that there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. The following results are well-known. For example, see [1, 2, 11].

Lemma 2.4 ([1, 2, 11])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$*;*
(2) $\langle z - y, Jx - Jz\rangle \geq 0, \quad \forall y \in C.$

Lemma 2.5 ([1, 2, 11])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$$

for all $y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [5, 19]; see also [22, Theorem 3.5.4].

Theorem 2.6 ([5, 19])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r; see [22]. We define J_r by $x_r = J_r x$. Such a J_r is denoted by

$$J_r = (I + rJ^{-1}B)^{-1}$$

and is called the metric resolvent of B. For $r > 0$, the Yosida approximation $A_r : E \to E^*$ is defined by

$$A_r x = \frac{J(x - J_r x)}{r}, \quad \forall x \in E.$$

Lemma 2.7 ([22]). *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let J_r and A_r be the metric resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\langle J_r x - u, J(x - J_r x)\rangle \geq 0, \quad \forall x \in E,\ u \in B^{-1}0$;
(2) $(J_r x, A_r x) \in B, \quad \forall x \in E$;
(3) $F(J_r) = B^{-1}0$.

For all $x \in E$ and $r > 0$, we also consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r; see [12]. We define Q_r by $x_r = Q_r x$. Such a Q_r is called the generalized resolvent of B. For $r > 0$, the Yosida approximation $B_r : E \to E^*$ is defined by

$$B_r x = \frac{Jx - JQ_r x}{r}, \quad \forall x \in E.$$

The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [22]. In case a Banach space is a Hilbert space, we have that $J_r = Q_r$ for all $r > 0$. Such a J_r is simply called the resolvent of B.

Lemma 2.8 ([12]). *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let Q_r and B_r be the generalized resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x), \quad \forall x \in E,\ u \in B^{-1}0$;
(2) $(Q_r x, B_r x) \in B, \quad \forall x \in E$;
(3) $F(Q_r) = B^{-1}0$.

3. Main results

In this section, using a new hybrid method, we first prove a strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces; see also [13, 14, 20].

Theorem 3.1. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $J_\mu^A = (I + \mu J_E^{-1} A)^{-1}$ be the metric resolvent of A for all $\mu > 0$, let $Q_\lambda^B = (J_E + \lambda B)^{-1} J_E$ be the generalized resolvent of B for all $\lambda > 0$ and let $Q_\eta^G = (J_F + \eta G)^{-1} J$ be the generalized resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_{\eta_n}^G T x_n)\big), \\ y_n = J_{\mu_n}^A z_n, \\ u_n = Q_{\lambda_n}^B y_n, \\ B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \ge \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \ge r_n \phi_F(T x_n, Q_{\eta_n}^G T x_n)\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \ge \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \le r_n \le \frac{1}{\|T\|^2} \text{ and } b \le \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Proof. It is obvious that $B_n \cap C_n \cap D_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $\Omega \subset B_n \cap C_n \cap D_n \cap Q_n$ for all $n \in \mathbb{N}$, we first show that, for $z \in \Omega \subset B^{-1}0$,

$$2\langle y_n - z, J_E y_n - J_E u_n\rangle \ge \phi_E(y_n, u_n) + \phi_E(u_n, y_n).$$

In fact, since $Q_{\lambda_n}^B$ is the generalized resolvent, we have from [4] that

$$\langle Q_{\lambda_n}^B y_n - z, J_E y_n - J_E Q_{\lambda_n}^B y_n\rangle \ge 0$$

for all $z \in \Omega \subset B^{-1}0$. Thus, we get that

$$\langle Q_{\lambda_n}^B y_n - y_n + y_n - z, J_E y_n - J_E Q_{\lambda_n}^B y_n\rangle \ge 0$$

and hence

$$2\langle y_n - z, J_E y_n - J_E Q_{\lambda_n}^B y_n\rangle \ge 2\langle y_n - Q_{\lambda_n}^B y_n, J_E y_n - J_E Q_{\lambda_n}^B y_n\rangle.$$

We have from (2.5) that

$$2\langle y_n - z, J_E y_n - J_E Q_{\lambda_n}^B y_n\rangle \ge \phi_E(y_n, Q_{\lambda_n}^B y_n) + \phi_E(Q_{\lambda_n}^B y_n, y_n).$$

This implies that

$$2\langle y_n - z, J_E y_n - J_E u_n\rangle \ge \phi_E(y_n, u_n) + \phi_E(u_n, y_n).$$

Next, let us show that, for $z \in \Omega \subset T^{-1}(G^{-1}0)$,

$$2\langle x_n - z, J_E x_n - J_E z_n\rangle \ge r_n \phi_F(T x_n, Q_{\eta_n}^G T x_n).$$

In fact, we have that

$$\begin{aligned} 2\langle x_n - z,\ J_E x_n - J_E z_n\rangle &= 2\langle x_n - z, r_n T^*(J_F T x_n - J_F Q_{\eta_n}^G T x_n)\rangle \\ &= 2 r_n \langle T x_n - T z, J_F T x_n - J_F Q_{\eta_n}^G T x_n\rangle \\ &\ge r_n \phi_F(T x_n, Q_{\eta_n}^G T x_n). \end{aligned}$$

We can also show that, for $z \in \Omega \subset A^{-1}0$,

$$\begin{aligned} &\langle z_n - z, J_E(z_n - y_n)\rangle - \|z_n - y_n\|^2 \\ &\qquad = \langle z_n - z, J_E(z_n - J_{\mu_n}^A z_n)\rangle - \|z_n - J_{\mu_n}^A z_n\|^2 \end{aligned}$$

$$\geq \|z_n - J^A_{\mu_n} z_n\|^2 - \|z_n - J^A_{\mu_n} z_n\|^2$$
$$= 0.$$

We finally show that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$. From

$$Q_1 = \{z \in E : \langle x_1 - z, J_E x_1 - J_E x_1\rangle \geq 0\} = E,$$

it is obvious that $\Omega \subset Q_1$. Suppose that $\Omega \subset Q_k$ for some $k \in \mathbb{N}$. Then we have $\Omega \subset B_k \cap C_k \cap D_k \cap Q_k$. From $x_{k+1} = \Pi_{B_k \cap C_k \cap D_k \cap Q_k} x_1$, we get that

$$\langle x_{k+1} - z, J_E x_1 - J_E x_{k+1}\rangle \geq 0, \quad \forall z \in B_k \cap C_k \cap D_k \cap Q_k$$

and hence

$$\langle x_{k+1} - z, J_E x_1 - J_E x_{k+1}\rangle \geq 0, \quad \forall z \in \Omega.$$

Then $\Omega \subset Q_{k+1}$. We have by induction that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$. Thus we have that $\Omega \subset B_n \cap C_n \cap D_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since Ω is a nonempty, closed and convex subset of E, there exists $z_0 \in \Omega$ such that $z_0 = \Pi_\Omega x_1$. From $x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1$, we have that

$$\phi_E(x_{n+1}, x_1) \leq \phi_E(y, x_1)$$

for all $y \in B_n \cap C_n \cap D_n \cap Q_n$. Since $z_0 \in \Omega \subset B_n \cap C_n \cap D_n \cap Q_n$, we have that

$$\phi_E(x_{n+1}, x_1) \leq \phi_E(z_0, x_1), \quad \forall n \in \mathbb{N}. \tag{3.1}$$

This means that $\{x_n\}$ is bounded. We show that $\lim_{n\to\infty} \phi_E(x_{n+1}, x_n) = 0$. From the definition of Q_n, we have that $x_n = \Pi_{Q_n} x_1$. From $x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1$ we have that $x_{n+1} \in Q_n$. Thuen we have that

$$\phi_E(x_n, x_1) \leq \phi_E(x_{n+1}, x_1)$$

for all $n \in \mathbb{N}$. This implies that $\{\phi_E(x_n, x_1)\}$ is bounded and nondecreasing. Then there exists the limit of $\{\phi_E(x_n, x_1)\}$. From Lemma 2.5, we have that

$$\begin{aligned}\phi_E(x_{n+1}, x_n) = \phi_E(x_{n+1}, \Pi_{Q_n} x_1) &\leq \phi_E(x_{n+1}, x_1) - \phi_E(\Pi_{Q_n} x_1, x_1)\\ &= \phi_E(x_{n+1}, x_1) - \phi_E(x_n, x_1)\end{aligned}$$

for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \phi_E(x_{n+1}, x_n) = 0$. From Lemma 2.2, we get that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.2}$$

From $x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1$, we have $x_{n+1} \in C_n$. This implies that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, Q^G_{\eta_n} Tx_n). \tag{3.3}$$

Furthermore, we claim that $\{J_E x_n - J_E z_n\}$ is bounded. That $\{J_E x_n - J_E z_n\}$ is bounded is proved as follows. We first have that

$$\|J_E x_n - J_E z_n\| = \|r_n T^*(J_F T x_n - J_F Q^G_{\eta_n} T x_n)\|.$$

Furthermore, we have that

$$\|J_F T x_n\| = \|T x_n\| \leq \|T\| \|x_n\|.$$

We also have that, for $z \in T^{-1}(G^{-1}0)$,

$$\begin{aligned}(\|Tz\| - \|Q^G_{\eta_n} T x_n\|)^2 &\leq \phi_F(Tz, Q^G_{\eta_n} T x_n)\\ &\leq \phi_F(Tz, Tx_n) \leq (\|Tz\| + \|Tx_n\|)^2\end{aligned}$$

$$\leq \|T\|^2(\|z\| + \|x_n\|)^2.$$

Using this, we have that

$$\|Q^G_{\eta_n} Tx_n\| \leq \|T\|(\|z\| + \|x_n\|) + \|Tz\| \leq \|T\|(\|z\| + \|x_n\|) + \|T\|\|z\|.$$

Then, we have that

$$\|J_F Q^G_{\eta_n} Tx_n\| = \|Q^G_{\eta_n} Tx_n\| \leq \|T\|(2\|z\| + \|x_n\|).$$

Hence, we have that

$$\begin{aligned}\|J_E x_n - J_E z_n\| &= \|r_n T^*(J_F Tx_n - J_F Q^G_{\eta_n} Tx_n)\| \\ &\leq \frac{1}{\|T\|^2}\|T\| \left(\|J_F Tx_n\| + \|J_F Q^G_{\eta_n} Tx_n)\|\right) \\ &\leq \frac{1}{\|T\|^2}\|T\| \left(\|T\|\|x_n\| + \|T\|(2\|z\| + \|x_n\|)\right) \\ &\leq 2(\|x_n\| + \|z\|).\end{aligned}$$

This implies that $\{J_E x_n - J_E z_n\}$ is bounded. Since $r_n \geq a > 0$ for all $n \in \mathbb{N}$, we have from (3.3) that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \geq a\phi_F(Tx_n, Q^G_{\eta_n} Tx_n). \tag{3.4}$$

Since $\|x_n - x_{n+1}\| \to 0$ from (3.2) and $\{J_E x_n - J_E z_n\}$ is bounded, we get that

$$\lim_{n\to\infty} \phi_F(Tx_n, Q^G_{\eta_n} Tx_n) = 0. \tag{3.5}$$

Therefore, we get from Lemma 2.2 that

$$\lim_{n\to\infty} \|Tx_n - Q^G_{\eta_n} Tx_n\| = 0. \tag{3.6}$$

Furthermore, since F is uniformly smooth, we have from (3.6) that

$$\lim_{n\to\infty} \|J_F Tx_n - J_F Q^G_{\eta_n} Tx_n\| = 0. \tag{3.7}$$

Since $\|J_E x_n - J_E z_n\| = \|r_n T^*(J_F Tx_n - J_F Q^G_{\eta_n} Tx_n)\|$ and $\{r_n\}$ is bounded, we get from (3.7) that

$$\lim_{n\to\infty} \|J_E x_n - J_E z_n\| = 0. \tag{3.8}$$

Since E^* is uniformly smooth, we have from (3.8) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.9}$$

We also have from $x_{n+1} \in D_n$ that

$$\langle z_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2$$

and hence

$$\|z_n - x_{n+1}\| \geq \|z_n - y_n\|.$$

From $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - z_n\| \to 0$, we have that $\lim_{n\to\infty} \|z_n - y_n\| = 0$. Using $y_n = J^A_{\mu_n} z_n$, we have that

$$\lim_{n\to\infty} \|z_n - J^A_{\mu_n} z_n\| = 0. \tag{3.10}$$

Furthermore,we have from $x_{n+1} \in B_n$ that

$$2\langle y_n - x_{n+1}, Jy_n - Ju_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n).$$

From $y_n - z_n \to 0$, $z_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|y_n - x_{n+1}\| \to 0$. Then we get that $\lim_{n\to\infty} \phi_E(y_n, u_n) = 0$ and hence

$$\lim_{n\to\infty} \|y_n - Q^B_{\lambda_n} y_n\| = 0. \tag{3.11}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From $\|z_n - x_n\| \to 0$, we have that $\{z_{n_i}\}$ converges weakly to w. Since $\lim_{n\to\infty} \|J^A_{\mu_n} z_n - z_n\| = 0$ from (3.10), $\{J^A_{\mu_n} z_n\}$ converges weakly to w. Since $J^A_{\mu_n}$ is the metric resolvent of A, we have that

$$\frac{J_E(z_n - J^A_{\mu_n} z_n)}{\mu_n} \in AJ^A_{\mu_n} z_n$$

for all $n \in \mathbb{N}$. From the monotonicity of A we have that

$$0 \leq \Big\langle s - J^A_{\mu_n} z_n, t^* - \frac{J_E(z_n - J^A_{\mu_n} z_n)}{\mu_n}\Big\rangle$$

for all $(s, t^*) \in A$. Since $\|J_E(z_n - J^A_{\mu_n} z_n)\| \to 0$ and $0 < b \leq \mu_n$ for all $n \in \mathbb{N}$, we have that $0 \leq \langle s - w, t^* - 0\rangle$ for all $(s, t^*) \in A$. Since A is maximal monotone, we have that $w \in A^{-1}0$. Furthermore, since T is bounded and linear, we also have that $\{Tx_{n_i}\}$ converges weakly to Tw. Using this and $\lim_{n\to\infty} \|Tx_n - Q^G_{\eta_n} Tx_n\| = 0$, we have that $\{Q^G_{\eta_n} Tx_n\}$ converges weakly to Tw. Since $Q^G_{\eta_n}$ is the generalized resolvent of G, we have that

$$\frac{J_F Tx_n - J_F Q^G_{\eta_n} Tx_n}{\eta_n} \in GQ^G_{\eta_n} Tx_n$$

for all $n \in \mathbb{N}$. From the monotonicity of G we have that

$$0 \leq \Big\langle u - Q^G_{\eta_n} Tx_n, v^* - \frac{J_F Tx_n - J_F Q^G_{\eta_n} Tx_n}{\eta_n}\Big\rangle$$

for all $(u, v^*) \in B$. Since $\|J_F Tx_n - J_F Q^G_{\eta_n} Tx_n\| \to 0$ from (3.7) and $0 < b \leq \eta_n$ for all $n \in \mathbb{N}$, we have that $0 \leq \langle u - Tw, v^* - 0\rangle$ for all $(u, v^*) \in G$. Since G is maximal monotone, we have that $Tw \in G^{-1}0$. We show $w \in B^{-1}0$. Since E is uniformly smooth, from $u_n = Q^B_{\lambda_n} y_n$ and (3.11) we have that

$$\lim_{n\to\infty} \|Jy_n - Ju_n\| = 0.$$

From $\lambda_n \geq b$, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \|Jy_n - Ju_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \|B^B_{\lambda_n} y_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n} \|Jy_n - Ju_n\| = 0.$$

For $(p, p^*) \in B$, from the monotonicity of B and $B^B_{\lambda_n} y_n \in BQ^B_{\lambda_n} y_n$, we have

$$\langle p - u_n, p^* - B^B_{\lambda_n} y_n\rangle \geq 0$$

for all $n \in \mathbb{N}$. From $u_n \rightharpoonup w$ and $B^B_{\lambda_n} y_n \to 0$, we get $\langle p - w, p^* \rangle \geq 0$. From the maximality of B, we have $w \in B^{-1}0$. Therefore, we have $w \in \Omega$.

From $z_0 = \Pi_\Omega x_1$, $w \in \Omega$ and (3.14), we have that

$$\begin{aligned}\phi_E(z_0, x_1) \leq \phi_E(w, x_1) &\leq \liminf_{i\to\infty} \phi_E(x_{n_i}, x_1)\\ &\leq \limsup_{i\to\infty} \phi_E(x_{n_i}, x_1) \leq \phi_E(z_0, x_1).\end{aligned}$$

From $z_0 = \Pi_\Omega x_1$, we have $w = z_0$. Furthermore, we get that

$$\lim_{i\to\infty} \phi_E(x_{n_i}, x_1) = \phi_E(w, x_1) = \phi_E(z_0, x_1).$$

This implies that

$$\lim_{i\to\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, x_1 \rangle + \|x_1\|^2) = \|w\|^2 - 2\langle w, x_1 \rangle + \|x_1\|^2).$$

Thus we get $\lim_{i\to\infty} \|x_{n_i}\| = \|w\|$. From the Kadec-Klee property of E, we have that $x_{n_i} \to w = z_0$. Therefore, we have $x_n \to z_0$. This completes the proof. $\square$

Next, using the hybrid method, we prove another strong convergence theorem for finding a solution of the split common null point problem in two Banach spaces.

Theorem 3.2. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $Q^A_\mu = (J_E + \mu A)^{-1} J_E$ be the generalized resolvent of A for all $\mu > 0$, let $J^B_\lambda = (I + \lambda J_E^{-1} B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$ and let $J^G_\eta = (I + \eta J_F^{-1} G)^{-1}$ be the metric resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F (T x_n - J^G_{\eta_n} T x_n), \\ y_n = Q^A_{\mu_n} z_n, \\ u_n = J^B_{\lambda_n} y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n) \rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n) \rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\}, \{\mu_n\}, \{\eta_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2} \text{ and } b \leq \lambda_n, \mu_n, \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

Proof. It is obvious that $B_n \cap C_n \cap D_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. Let us show $\Omega \subset B_n \cap C_n \cap D_n \cap Q_n$ for all $n \in \mathbb{N}$. Since $J^B_{\lambda_n}$ is the metric resolvent, we have that, for $z \in \Omega \subset B^{-1}0$,

$$\langle J^B_{\lambda_n} y_n - z, J_E(y_n - J^B_{\lambda_n} y_n)\rangle \geq 0.$$

From this, we get that $\langle J^B_{\lambda_n} y_n - y_n + y_n - z, J_E(y_n - J^B_{\lambda_n} y_n)\rangle \geq 0$ and hence

$$\langle y_n - z, J_E(y_n - J^B_{\lambda_n} y_n)\rangle \geq \|y_n - J^B_{\lambda_n} y_n\|^2.$$

This implies that

$$\langle y_n - z, J_E(y_n - u_n)\rangle \geq \|y_n - u_n\|^2.$$

Then we have that $\Omega \subset B_n$. To show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$, let us show that $\langle z_n - z, J_E(x_n - z_n)\rangle \geq 0$ for all $z \in \Omega \subset T^{-1}(G^{-1}0)$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in \Omega$,

$$\begin{aligned}
\langle z_n - z, &J_E(x_n - z_n)\rangle = \langle z_n - x_n + x_n - z, J_E(x_n - z_n)\rangle \\
&= \langle -r_n J_E^{-1} T^* J_F(Tx_n - J^G_{\eta_n} Tx_n) \\
&\qquad + x_n - z, J_E(r_n J_E^{-1} T^* J_F(Tx_n - J^G_{\eta_n} Tx_n))\rangle \\
&= \langle -r_n J_E^{-1} T^* J_F(Tx_n - J^G_{\eta_n} Tx_n) + x_n - z, r_n T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\rangle \\
&= -r_n^2 \langle J_E^{-1} T^* J_F(Tx_n - J^G_{\eta_n} Tx_n), T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\rangle \\
&\qquad + \langle x_n - z, r_n T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\rangle \\
&= -r_n^2 \|T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\|^2 + \langle x_n - z, r_n T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\rangle \\
&\geq -r_n^2 \|T^* J_F(Tx_n - J^G_{\eta_n} Tx_n)\|^2 + r_n \|Tx_n - J^G_{\eta_n} Tx_n\|^2 \\
&\geq -r_n^2 \|T\|^2 \|Tx_n - J^G_{\eta_n} Tx_n\|^2 + r_n \|Tx_n - J^G_{\eta_n} Tx_n\|^2 \\
&= r_n\big(1 - r_n \|T\|^2\big) \|Tx_n - J^G_{\eta_n} Tx_n\|^2 \\
&\geq 0.
\end{aligned} \tag{3.12}$$

Then we have that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Next, to show that $\Omega \subset D_n$, let us show that

$$2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n)$$

for all $z \in \Omega$. In fact, we have that

$$\begin{aligned}
2\langle z_n - z, &J_E z_n - J_E y_n\rangle - \phi_E(z_n, y_n) \\
&= 2\langle z_n - z, J_E z_n - J_E Q^A_{\mu_n} z_n\rangle - \phi_E(z_n, Q^A_{\mu_n} z_n) \\
&\geq \phi(z_n, Q^A_{\mu_n} z_n) - \phi_E(z_n, Q^A_{\mu_n} z_n) \\
&= 0.
\end{aligned} \tag{3.13}$$

Then we have that $\Omega \subset D_n$ for all $n \in \mathbb{N}$. We show that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$. Since $Q_1 = \{z \in E : \langle x_1 - z, J_E(x_1 - x_1)\rangle \geq 0\} = E$, it is obvious that $\Omega \subset Q_1$. Suppose that $\Omega \subset Q_k$ for some $k \in \mathbb{N}$. Then $\Omega \subset B_k \cap C_k \cap D_k \cap Q_k$. From $x_{k+1} = P_{B_k \cap C_k \cap D_k \cap Q_k} x_1$, we have that

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in B_k \cap C_k \cap D_k \cap Q_k.$$

From $\Omega \subset B_k \cap C_k \cap D_k \cap Q_k$, we have that

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in \Omega.$$

Then $\Omega \subset Q_{k+1}$. We have by induction that $\Omega \subset Q_n$ for all $n \in \mathbb{N}$. Thus we have that $\Omega \subset B_n \cap C_n \cap D_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since Ω is nonempty, closed and convex, there exists $w_1 \in \Omega$ such that $w_1 = P_\Omega x_1$. From $x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in B_n \cap C_n \cap D_n \cap Q_n$. Since $w_1 \in \Omega \subset B_n \cap C_n \cap D_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - w_1\|. \tag{3.14}$$

This means that $\{x_n\}$ is bounded. We show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of Q_n, we have that $x_n = P_{Q_n} x_1$. From $x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1$ we have that $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_n - x_1\| = c$. If $c = 0$, then $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Assume that $c > 0$. Since $x_n = P_{Q_n} x_1$, $x_{n+1} \in Q_n$ and $\frac{x_n + x_{n+1}}{2} \in Q_n$, we have that

$$\|x_1 - x_n\| \leq \left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| \leq \frac{1}{2}\Big(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|\Big)$$

and hence

$$\lim_{n\to\infty} \left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| = c.$$

Since E is uniformly convex, we get that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

From $x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1$, we have $x_{n+1} \in C_n$. This implies that

$$\langle z_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0$$

and hence

$$\langle z_n - x_n + x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0.$$

Then we have that

$$\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq \|x_n - z_n\|^2$$

and hence

$$\|x_n - z_n\| \leq \|x_n - x_{n+1}\|.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ we have that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.15}$$

On the other hand, we have that

$$\begin{aligned}\|x_n - z_n\| &= \|J_E(x_n - z_n)\| \\ &= \|r_n T^* J_F(Tx_n - J_{\eta_k}^G Tx_n)\| \\ &= r_n \|T^* J_F(Tx_n - J_{\eta_k}^G Tx_n)\| \\ &\geq a\|T^* J_F(Tx_n - J_{\eta_k}^G Tx_n)\|.\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we have that

$$\lim_{n\to\infty}\|T^*J_F(Tx_n - J^G_{\eta_k}Tx_n)\| = 0.$$

Since $J^G_{\eta_n}$ is the metric resolvent, we have that, for $Tz \in G^{-1}0$,

$$\begin{aligned}\langle x_n - z, T^*J_F(Tx_n - J^G_{\eta_n}Tx_n)\rangle &= \langle Tx_n - Tz, J_F(Tx_n - J^G_{\eta_n}Tx_n)\rangle \\ &\geq \|Tx_n - J^G_{\eta_n}Tx_n\|^2.\end{aligned}$$

Then we get that

$$\lim_{n\to\infty}\|Tx_n - J^G_{\eta_n}Tx_n\| = 0. \tag{3.16}$$

Furthermore, from $x_{n+1} = P_{B_n\cap C_n\cap D_n\cap Q_n}x_1$ and $x_{n+1}\in D_n$, we have that

$$2\langle z_n - x_{n+1}, J_Ez_n - J_Ey_n\rangle \geq \phi_E(z_n, y_n)$$

and hence

$$2\langle z_n - x_n + x_n - x_{n+1}, J_Ez_n - J_Ey_n\rangle \geq \phi_E(z_n, y_n). \tag{3.17}$$

Let us show that $\{J_Ez_n - J_Ey_n\}$ is bounded. Since $Q^A_{\mu_n}$ is the generalized resolvent, we have that, for $z\in A^{-1}0$,

$$2\langle z_n - z, J_Ez_n - J_Ey_n\rangle \geq \phi_E(z_n, y_n)$$

and hence

$$\phi_E(z_n, y_n) + \phi_E(z, z_n) - \phi_E(z, y_n) \geq \phi_E(z_n, y_n).$$

This implies that

$$\phi_E(z, z_n) \geq \phi_E(z, y_n).$$

Thus we have that, for $z\in A^{-1}0$,

$$(\|z\| - \|y_n\|)^2 \leq \phi_E(z, y_n) \leq \phi_E(z, z_n) \leq (\|z\| + \|z_n\|)^2.$$

Using this, we have that

$$|\|z\| - \|y_n\|| \leq \|z\| + \|z_n\|$$

and hence

$$\|y_n\| \leq 2\|z\| + \|z_n\|.$$

Hence, we have that

$$\begin{aligned}\|J_Ez_n - J_Ey_n\| &\leq \|J_Ez_n\| + \|J_Ey_n\| \\ &= \|z_n\| + \|y_n\| \\ &\leq 2\|z\| + 2\|z_n\|.\end{aligned}$$

This implies that $\{J_Ez_n - J_Ey_n\}$ is bounded. From (3.17), $\|x_n - x_{n+1}\|\to 0$ and $\|x_n - z_n\|\to 0$, we have that $\lim_{n\to\infty}\phi_E(z_n, y_n) = 0$. Then we get from Lemma 2.2 that $\|z_n - y_n\|\to 0$ and hence

$$\lim_{n\to\infty}\|z_n - Q^A_{\mu_n}z_n\| = 0. \tag{3.18}$$

Since $x_{n+1}\in B_n$, we have that

$$\langle y_n - x_{n+1}, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2$$

and hence

$$\|y_n - x_{n+1}\| \geq \|y_n - u_n\|.$$

From $y_n - z_n \to 0$, $z_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|y_n - x_{n+1}\| \to 0$. Then we get that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0 \tag{3.19}$$

and hence

$$\lim_{n\to\infty} \|y_n - J^B_{\lambda_n} y_n\| = 0. \tag{3.20}$$

Since $\{x_n\}$ converges weakly to w, we have from $\lim_{n\to\infty} \|x_n - z_n\| = 0$ that $\{z_n\}$ converges weakly to w. We also have from (3.18) that $\{Q^A_{\mu_n} z_n\}$ converges weakly to w. Since $Q^A_{\mu_n}$ is the generalized resolvent of A, we have that

$$\frac{J_E z_n - J_E Q^A_{\mu_n} z_n}{\mu_n} \in A Q^A_{\mu_n} z_n$$

for all $n \in \mathbb{N}$. From the monotonicity of A we have that

$$0 \leq \left\langle s - Q^A_{\mu_n} z_n, t^* - \frac{J_E z_n - J_E Q^A_{\mu_n} z_n}{\mu_n} \right\rangle$$

for all $(s, t^*) \in A$. Since E is uniformly smooth, from (3.18) we have that

$$\|J_E z_n - J_E Q^A_{\mu_n} z_n\| \to 0.$$

Using $0 < b \leq \mu_n$ for all $n \in \mathbb{N}$, we have that $0 \leq \langle s - w, t^* - 0 \rangle$ for all $(s, t^*) \in A$. Since A is maximal monotone, we have that $w \in A^{-1}0$. Furthermore, since T is bounded and linear, we also have that $\{Tx_n\}$ converges weakly to Tw. From (3.16) we have that $\{J^G_{\eta_n} Tx_n\}$ converges weakly to Tw. Since $J^G_{\eta_n}$ is the metric resolvent of G, we have that

$$\frac{J_F(Tx_n - J^G_{\eta_n} Tx_n)}{\eta_n} \in G J^G_{\eta_n} Tx_n$$

for all $n \in \mathbb{N}$. From the monotonicity of G we have that

$$0 \leq \left\langle u - J^G_{\eta_n} Tx_n, v^* - \frac{J_F(Tx_n - J^G_{\eta_n} Tx_n)}{\eta_n} \right\rangle$$

for all $(u, v^*) \in G$. Since $\|J_F(Tx_n - J^G_{\eta_n} Tx_n)\| \to 0$ and $0 < b \leq \eta_n$ for all $n \in \mathbb{N}$, we have that $0 \leq \langle u - Tw, v^* - 0 \rangle$ for all $(u, v^*) \in G$. Since G is maximal monotone, we have that $Tw \in G^{-1}0$. We show $w \in B^{-1}0$. From $\lambda_n \geq b$ and (3.20), we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \|J_E(y_n - J^B_{\lambda_n} y_n)\| = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \|A^B_{\lambda_n} y_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n} \|J_E(y_n - J^B_{\lambda_n} y_n)\| = 0.$$

For $(p, p^*) \in B$, from the monotonicity of B, we have

$$\langle p - J^B_{\lambda_n} y_n, p^* - A^B_{\lambda_n} y_n \rangle \geq 0$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get from $J^B_{\lambda_n} y_n \rightharpoonup w$ that $\langle p - w, p^* \rangle \geq 0$. By the maximality of B, we have $w \in B^{-1}0$. Therefore, we have $w \in \Omega$.

From $w_1 = P_\Omega x_1$, $w \in \Omega$ and (3.14), we have that

$$\begin{aligned}\|x_1 - w_1\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - w_1\|.\end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - w_1\|$$

and hence $w = w_1$. Furthermore, from the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = w_1.$$

Therefore, we have $x_n \to w = w_1$. This completes the proof. □

4. Applications

In this section, using Theorems 3.1 and 3.2, we get new strong convergence theorems which are connected with the split feasibility problem and the split common null point problem in Banach spaces. Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^* \rangle + f(x),\ \forall y \in E\}$$

for all $x \in E$. Then we know that ∂f is a maximal monotone operator; see [18] for more details. Let C be a nonempty, closed and convex subset of E and let i_C be the indicator function, that is,

$$i_C = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then we have that ∂i_C is a maximal monotone operator and the generalized resolvent $Q_r = \Pi_C$ for all $r > 0$, where Π_C is the generalized projection of E onto C. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.4 that

$$\begin{aligned} z = Q_r x &\Leftrightarrow Jz + r\partial i_C(z) \ni Jx \\ &\Leftrightarrow Jx - Jz \in r\partial i_C(z) \\ &\Leftrightarrow i_C(y) \geq \left\langle y - z, \frac{Jx - Jz}{r} \right\rangle + i_C(z),\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, Jx - Jz \rangle,\ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y\in C} \phi(y, x) \\ &\Leftrightarrow z = \Pi_C. \end{aligned}$$

Furthermore, the metric resolvent $J_r = P_C$ for all $r > 0$, where P_C is the metric projection of E onto C. In fact, for any $x \in E$ and $r > 0$, we have that

$$\begin{aligned} z = J_r x &\Leftrightarrow J(z - x) + r\partial i_C(z) \ni 0 \\ &\Leftrightarrow J(x - z) \in r\partial i_C(z) \end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow i_C(y) \geq \langle y - z, \frac{J(x-z)}{r} \rangle + i_C(z),\ \forall y \in E \\
&\Leftrightarrow 0 \geq \langle y - z, J(x-z) \rangle,\ \forall y \in C \\
&\Leftrightarrow z = P_C x.
\end{aligned}
$$

As a direct consequence of Theorem 3.1, we have the following theorem for finding a solution of the split common null point problem in two Banach spaces.

Theorem 4.1. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $J_\mu^A = (I + \mu J_E^{-1} A)^{-1}$ be the metric resolvent of A for all $\mu > 0$, let $Q_\lambda^B = (J_E + \lambda B)^{-1} J_E$ be the generalized resolvent of B for all $\lambda > 0$ and let $Q_\eta^G = (J_F + \eta G)^{-1} J$ be the generalized resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$
\begin{cases}
z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F Q_\eta^G T x_n)\big), \\
y_n = J_\mu^A z_n, \\
u_n = Q_\lambda^B y_n, \\
B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n \rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\
C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n \rangle \geq r_n \phi_F(T x_n, Q_\eta^G T x_n)\}, \\
D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n) \rangle \geq \|z_n - y_n\|^2\}, \\
Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n \rangle \geq 0\}, \\
x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N},
\end{cases}
$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Next, using Theorem 3.1, we have the following theorem for finding a solution of the split feasibility problem in two Banach spaces.

Theorem 4.2. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and let H be a nonempty, closed and convex subset of F. Let P_C be the metric projection of E onto C, let Π_C be the generalized projection of E onto C. and let Π_H be the generalized projection of F onto H. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = C \cap D \cap T^{-1}H \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n T^*(J_F T x_n - J_F \Pi_H T x_n)\big), \\ y_n = P_C z_n, \\ u_n = \Pi_D y_n, \\ B_n = \{z \in E : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n)\}, \\ C_n = \{z \in E : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Tx_n, \Pi_H T x_n)\}, \\ D_n = \{z \in E : \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E x_1 - J_E x_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Proof. We have that $Q^G_{\eta_n} = \Pi_H$, $J^A_{\mu_n} = P_C$ and $Q^B_{\lambda_n} y_n = Pi_D$ in Theorem 3.1. Therefore, we have the desired result from Theorem 3.1. □

Similarly, using Theorem 3.2 and the proofs in Theorems 4.1 and 4.2, we have the following strong convergence theorems for the split common null point problem and the split feasibility problem in two Banach spaces.

Theorem 4.3. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $A, B \subset E \times E^*$ be maximal monotone operators and let $G \subset F \times F^*$ be a maximal monotone operator. Let $Q^A_\mu = (J_E + \mu A)^{-1} J_E$ be the generalized resolvent of A for all $\mu > 0$, let $J^B_\lambda = (I + \lambda J_E^{-1} B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$ and let $J^G_\eta = (I + \eta J_F^{-1} G)^{-1}$ be the metric resolvent of G for all $\eta > 0$. Let $T : E \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap T^{-1}(G^{-1}0) \neq \emptyset.$$

Let $x_1 \in E$ and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F(Tx_n - J^G_\eta T x_n), \\ y_n = Q^A_\mu z_n, \\ u_n = J^B_\lambda y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following:

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $w_1 \in \Omega$*, where* $w_1 = P_\Omega x_1$*.*

Theorem 4.4. *Let* E *and* F *be uniformly convex and uniformly smooth Banach spaces and let* J_E *and* J_F *be the duality mappings on* E *and* F*, respectively. Let* C *and* D *be nonempty, closed and convex subsets of* E *and let* H *be a nonempty, closed and convex subset of* F*. Let* Π_C *be the generalized projection of* E *onto* C*, let* P_D *be the metric projection of* E *onto* D *and let* P_H *be the metric projection of* F *onto* H*. Let* $T : E \to F$ *be a bounded linear operator such that* $T \neq 0$ *and let* T^* *be the adjoint operator of* T*. Suppose that*

$$\Omega = C \cap D \cap T^{-1}H \neq \emptyset.$$

Let $x_1 \in E$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} T^* J_F(Tx_n - P_H T x_n), \\ y_n = \Pi_C z_n, \\ u_n = P_D y_n, \\ B_n = \{z \in E : \langle y_n - z, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2\}, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ D_n = \{z \in E : 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n)\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{B_n \cap C_n \cap D_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ *and* $a \in \mathbb{R}$ *satisfy the following:*

$$0 < a \leq r_n \leq \frac{1}{\|T\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $w_1 \in \Omega$*, where* $w_1 = P_\Omega x_1$*.*

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Nariyuki Minami
Keio University School of Medicine, 4-1-1 Hiyoshi, Kohoku-ku, Yokohama 223-8521, Japan
E-mail address: minami@a5.keio.jp

Wataru Takahashi
Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net