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A NECESSARY OPTIMALITY CONDITION INVOLVING MEASURES OF NONCOMPACTNESS

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Abstract. A "large" approximating cone to a closed set is introduced and a sufficient condition for tangential transversality is proved. Moreover, a nonseparation result and a Lagrange multiplier theorem are also obtained.

1. Introduction

The calculus of variations is the oldest branch of optimization, dating from over three hundred years ago. The development of functional analysis made it possible to consider this problem as a constrained optimization problem in an infinite-dimensional space. As early as in 1965, Dubovickiĭ and Miljutin applied systematically this approach and, moreover, realized the importance of convex approximations of closed sets for obtaining necessary conditions for nonlinear problems in optimization. This approach proves its efficiency during the years in the seminal works of, e.g., P. D. Loewen and R.-T. Rockafellar ([18], and [19]), R.-T. Rockafellar (cf. [20] and [22]), A. Ioffe and V. Tihomirov (cf. [13]), F. Clarke ([6]) and etc.

It is our understanding that to obtain a necessary optimality condition for the basic problem of calculus of variations considered as a constrained optimization problem in an infinite-dimensional space one needs:

- to find a Lagrange multiplier, that is a nontrivial linear functional separating the tangent cones to the epigraph of the integral functional and the constraint, respectively;
- to prove regularity of the obtained Lagrange multiplier;
- to find reasonable conditions connecting the obtained Lagrange multiplier with some selections of normals to the epigraph of the integrand.

In this paper we apply methods of non-smooth analysis in order to prove an abstract Lagrange multiplier rule which would be suitable to apply to the basic problem of calculus of variations even in the case when the integral functional is not Lipschitz with respect to the state variable. But in contrast to the aforementioned authors, we concentrate our attention on the study of the tangential approximating cones rather then the normal ones. Another difference in the techniques is the

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fact that we don't use any variational principles. The main idea of our approach is to find a large (but not convex) approximating cone to the epigraph of the integral functional for which there is some small, but still positive, "inaccuracy" of the approximation allowed. Then we build a Lagrange multiplier separating the tangent cone to the constraint from the convex hull of the Clarke tangent cone to the epigraph and a suitable convex subcone of the "large" one. This method is a continuation of the variational approach we together with Apostolov began to develop in [1].

The paper is organized as follows. Some preliminaries as well as part of the ideas of the proposed method are presented in the next section. The third section contains the definition of the "large" approximating cone and a sufficient condition for tangential transversality (whose proof is rather technical). In section 4 a nonseparation result and a Lagrange multiplier theorem are proved.

2. Preliminaries

Throughout the paper if Y is a Banach space, we will denote by $\mathbf{B}_Y$ $[\overline{\mathbf{B}}_Y]$ its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space. If S is a closed subset of Y and $y \in S$, we will denote by $\widehat{T}_S(y)$ the Clarke tangent cone to S at y, i.e.

$$\widehat{T}_S(y) := \left\{ v \in Y : \begin{array}{c} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \\ \text{such that for every } t \in [0,\delta] \text{ it holds true that} \\ S \cap (y + \delta\overline{\mathbf{B}}) + tv \subset S + t\varepsilon\overline{\mathbf{B}} \end{array} \right\}.$$

In this paper we continue the research in [1] whose starting point was the famous Aubin condition from [7] for the basic problem of the calculus of variations. We formulated there an abstract (infinite-dimensional) version of this condition and established its relation to the original idea of "perturbation" and subsequent "correction" of the epigraph of the considered functional (introduced by J.M. Borwein and H.M. Strojwas in 1985 in [5]):

Definition 2.1. Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which has finite value at $(\overline{x}, \overline{y}) \in X \times Y$. It is said that f satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ iff there exist positive reals $\overline{\delta} > 0$ and $K > 0$ such that for every $t \in [0, \overline{\delta}]$ the following inclusion holds true:

$$\begin{aligned} \operatorname{epi} f \cap \big((\overline{x}, \overline{y}, f(\overline{x}, \overline{y})) + \overline{\delta} \cdot \overline{\mathbf{B}}_{X\times Y\times R}\big) + t\left(\overline{\mathbf{B}}_X, \mathbf{0}, 0\right) \subset \\ \subset \operatorname{epi} f + t\left(\mathbf{0}, K \cdot \overline{\mathbf{B}}_Y, K[-1,1]\right). \end{aligned}$$

The Aubin condition enables one to prove an abstract Lagrange multiplier rule (cf. [1]) for a natural optimization problem:

Corollary 2.2. *Let X and Y be Banach spaces. We consider the optimization problem*

$$f(x,y) \to \min \quad \textit{subject to} \quad (x,y) \in S \ ,$$

where $f : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper and satisfies the Aubin condition at $(\overline{x}, \overline{y}, f(\overline{x}, \overline{y}))$ and $S := \{(Ly, y) : y \in Y\}$, where $L : Y \longrightarrow X$ is *a compact linear operator. Let* $(\overline{x},\overline{y})$ *be a solution of the above problem. Then there exists a triple* $(\xi,\eta,\zeta)\in X^*\times Y^*\times\mathbb{R}$ *such that*

(i) $(\xi,\eta,\zeta)\neq(\mathbf{0},\mathbf{0},0)$*;*
(ii) $\zeta\in\{0,1\}$*;*
(iii) $\langle\xi,Ly\rangle+\langle\eta,y\rangle=0$ *for every* $y\in Y$*;*
(iv) $\langle\xi,u\rangle+\langle\eta,v\rangle+\zeta w\geq 0$ *for every* $(u,v,w)\in\widehat{T}_{epif}(\overline{x},\overline{y},f(\overline{x},\overline{y}))$*.*

The proof of this assertion is based on the concept of *subtransversality* and a general sufficient condition for it.

Transversality is a classical concept of mathematical analysis and differential topology. Recently, it has proven to be useful in variational analysis as well (cf. [11]). In the literature there exist many notions generalizing the classical transversality as well as transversality of cones (cf., for example, [23]). Some of them are introduced under different names by different authors, but actually coincide. We refer to [16] for a survey of terminology and comparison of the available concepts. The central ones among them are *transversality* and *subtransversality*. They are also objects of study in the recent book [12].

The term subtransversality is recently introduced in [8] in relation to proving linear convergence of the alternating projections algorithm. However, this property has been around for more than 20 years, but under different names – see Remark 4 in [16] and the references therein. It is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cone and a sum rule for the limiting subdifferential. The precise definition is

Definition 2.3. Let A and B be closed subsets of the Banach space X. A and B are said to be subtransversal at $x_0\in A\cap B$, if there exists $K>0$ such that

$$d(x,A\cap B)\leq K(d(x,A)+d(x,B))$$

for all x in a fixed neighborhood of x_0.

From our point of view the most remarkable thing about subtransversality is the fact that it implies a rather general nonseparation result which is crucial for obtaining necessary optimality conditions of Pontryagin maximum principle type (including optimal control problems with infinite-dimensional state space). Moreover, subtransversality is a natural assumption for proving abstract Lagrange multiplier rule. But the intriguing thing is to verify the subtransversality assumption in nontrivial cases. Our approach to verification of subtransversality is proving the following local property:

Definition 2.4. Let A and B be closed subsets of the Banach space X. We say that A and B are tangentially transversal at $x_0\in A\cap B$, if there exist $M>0$, $\delta>0$ and $\eta>0$ such that for any two different points $x^A\in(x_0+\delta\overline{\mathbf{B}})\cap A$ and $x^B\in(x_0+\delta\overline{\mathbf{B}})\cap B$, there exists a sequence $\{t_m\}$, $t_m\searrow 0$, such that for every $m\in\mathbb{N}$ there exist $w_m^A\in X$ with $\|w_m^A\|\leq M$ and $x^A+t_mw_m^A\in A$, and $w_m^B\in X$ with $\|w_m^B\|\leq M$, $x^B+t_mw_m^B\in B$, and the following inequality holds true

$$\|x^A-x^B+t_m(w_m^A-w_m^B)\|\leq\|x^A-x^B\|-t_m\eta\,.$$

The above condition is a stronger condition than subtransversality (c.f. [2], Proposition 2.8). For a discussion on this notion the reader is referred to [2]. It happens that usually tangential transversality is easier to verify than subtransversality when the information known concerns the tangential structure of the sets. We presented in [1] a general sufficient condition for tangential transversality:

Theorem 2.5. *Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$, $q_1 > 0$, $q_2 > 0$, such that $q_1 + q_2 < 1$ and:*

(i) there exist bounded "ball covering" sets M_A and M_B such that $M_A - M_B$ is εq_1-dense in $\varepsilon\overline{\boldsymbol{B}}$ and "correcting" sets U_A, U_B such that

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;

(ii) there exist two bounded sets D_A and D_B such that $D_A - D_B$ is εq_2-dense in $U_A - U_B$ and they are "η-uniform" with $\eta := (1 - q_1 - q_2)/3$, i.e. for each $t \in [0, \delta]$

$$A \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_A \subset A + t\eta\overline{\boldsymbol{B}} \text{ and } B \cap (x_0 + \delta\overline{\boldsymbol{B}}) + tD_B \subset B + t\eta\overline{\boldsymbol{B}}.$$

Then A and B are tangentially transversal at x_0.

The underlying idea is that in many cases the uniformness of the local approximation of a closed set can be used instead of some suitable compactness assumption. This is especially important in the infinite-dimensional case. If the property of a set being "η-uniform" (from condition *(ii)* written above) holds true for every $\eta > 0$, we arrive to the concept of uniform tangent set which was introduced in [15] and its study was continued in [4]. It happened to be very useful for obtaining necessary conditions for optimal control problems in infinite dimensional state space, because the diffuse variations (which are naturally defined and easy to calculate for local approximation of the reachable set) are uniform. Examples 2.5 and 4.11 in [14] further motivate the importance of "uniformity of approximation".

Definition 2.6. Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$.

The next theorem is the main result from [4]. It establishes one of the important properties of the uniform tangent sets whose direct corollary is the fact that every uniform tangent set $D_S(x_0)$ to the set S at the point x_0 is contained in the Clarke tangent cone $\widehat{T}_S(x_0)$.

Theorem 2.7. *Let S be a closed subset of X and x_0 belong to S. The following are equivalent*

(1) *$D_S(x_0)$ is a uniform tangent set to S at the point x_0*
(2) *for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta\overline{\mathbf{B}})$ the set $S \cap (x + t(v + \varepsilon\overline{\mathbf{B}}))$ is non empty for each $t \in [0, \lambda]$.*

The basic properties of uniform tangent sets are gathered in the next proposition taken from [3]:

Proposition 2.8. *Let S be a closed subset of X and let $x_0 \in S$. Let $D_S(x_0)$ be a uniform tangent set to S at the point x_0. Then, the following hold true:*

(1) *the set $c\,D_S(x_0)$ is a uniform tangent set to S at x_0 for each fixed constant $c > 0$;*
(2) *if $D'_S(x_0) \subset D_S(x_0)$, then $D'_S(x_0)$ is a uniform tangent set to S at x_0;*
(3) *if $D'_S(x_0)$ is another uniform tangent set to S at x_0, then $D_S(x_0) \cup D'_S(x_0)$ is a uniform tangent set to S at x_0;*
(4) *the convex closed closure $\overline{co}\,D_S(x_0)$ of $D_S(x_0)$ is a uniform tangent set to S at x_0*
(5) *if S is convex, then $(S - x_0) \cap M\overline{\mathbf{B}}$ is a uniform tangent set to S at x_0 for every $M > 0$.*

Throughout the paper we consider the following optimization problem

$$\varphi(x,y) \to \min \quad \text{subject to} \quad (x,y) \in A, \tag{2.1}$$

where X and Y are Banach spaces, $\varphi : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, $L : Y \longrightarrow X$ a bounded linear operator,

$$S := \{(Ly, y) \in X \times Y \ : \ y \in Y\}$$

is a closed linear subspace of $X \times Y$ and $A := S + \{(\xi, \eta)\}$, where (ξ, η) is a fixed point of $X \times Y$.

In this paper we prove an analogue of Corollary 2.2 under much weaker assumptions. We have to note that the "correcting set" in the Definition 2.1 depends only on the second and the third variables. We generalize the Aubin condition imposed on the epigraph of the considered function by assuming "a variational condition (A1)" and "a measure of noncompactness condition (A2)". The "variational condition" allows one to use a "correcting set" which contains an open ball in the first variable and thus to go away from the Lipshitz continuous case. Regarding the "measure of noncompactness condition", it replaces the compactness of the linear operator L in Corollary 2.2. Moreover, these two conditions are closely intertwined - the "measure of noncompactness condition" poses some restrictions on the size of the "correcting set". In fact, the case when L is the integration operator from $Y = L^1([a,b],\mathbb{R}^n)$ to $X = L^\infty([a,b],\mathbb{R}^n)$ could be important for future applications of our results. Clearly, this operator is bounded but not compact, and it maps weakly compact sets in Y to totally bounded sets in X, thus allowing to use weakly compact sets as "correcting sets". To deal with the "measure of noncompactness condition" is not a trivial task. We solved it by combining ideas from the proofs of Theorem 2.5 (cf. [1]) and the main result of [17]. The obtained Lagrange multiplier separates the constraint from an arbitrary convex subcone of the "large" approximating cone D_η introduced in Section 3.

3. A general sufficient condition for tangential transversality

Definition 3.1. Let us fix $\eta > 0$. It is said that $(u,v,w) \in D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ if there exists $\delta(u,v,w) > 0$ such that for each $t \in [0, \delta(u,v,w)]$ the following inclusion holds true:

$$epi\ \varphi \cap \big((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \delta(u,v,w)\overline{\mathbf{B}}\big) + t(u,v,w) \subset epi\ \varphi + t\eta\|u - Lv\|\overline{\mathbf{B}}\ .$$

Note that in the above definition the constant $\delta(u,v,w)$ may depend on the vector (u,v,w).

Theorem 3.2. *Let in the above setting the following assumptions hold true:*

(a) *"variational condition" : there exist a positive real $\delta > 0$, an uniform tangent set D to epi φ at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ and a "correcting set" $U \subset X\times Y\times\mathbb{R}$ (having the appearance $U=(U_X,U_Y,U_{\mathbb{R}})$) such that for each $t\in[0,\delta]$ we have*

$$epi\ \varphi \cap \left((\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))+\delta\bar{\mathbf{B}}\right)+t\left(\bar{\mathbf{B}}_{\mathbf{X}},\mathbf{0},0\right)\subset epi\ \varphi + tU + tD\ .$$

(b) *"measure of noncompactness condition" : The set U is bounded and*

$$\mu\left(U_X - L(U_Y)\right) < 1$$

where μ denotes the ball measure of noncompactness (in X).

(c) *"density condition": there exists $R>0$ such that $D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))+\hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))\cap\ R\ \bar{\mathbf{B}} - S\times(-\infty,0]$ is dense in the closed unit ball of $X\times Y\times\mathbb{R}$ centered at the origin for some $\eta>0$ satisfying*

$$\eta < \frac{1}{(1-q)^{-1}(1+\|L\|)(1+M+\|L\|M+q)(1+(1+\|L\|)R)},$$

where $M := 2\sup\{\|z\|:\ z\in U\cup D\}+1\ .$

Then the sets epi φ and $(S+(\bar{x},\bar{y}))\times(-\infty,\varphi(\bar{x},\bar{y})]$ are tangentially transversal at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$.

Proof. Let us first fix the constants involved. The reason for defining them in this particular way is to be clarified in the proof.

$$M := 2\sup\{\|z\|:\ z\in U\cup D\}+1,$$

$$\tilde{q} \text{ and } q \text{ are such that } \mu\left(U_X - L(U_Y)\right) < \tilde{q} < q < 1,$$

$$\sigma>0 \text{ is such that } q=\tilde{q}+\sigma(1+\|L\|).$$

Since $\tilde{q} > \mu\left(U_X - L(U_Y)\right)$, there exist a finite set $F\subset X$ and a finite set $G\subset\mathbb{R}$ such that

$$(U_X - L(U_Y),\mathbf{0},U_{\mathbb{R}})\subset\left(F+\tilde{q}\bar{\mathbf{B}}_{\mathbf{X}},\mathbf{0},G+\tilde{q}[-1,1]\right).$$

Let us denote

$$T := \frac{1+\|L\|}{1-q}\left(\overline{\mathrm{conv}}\left(\{\mathbf{0}\}\cup F\right),\mathbf{0},\overline{\mathrm{conv}}\left(\{\mathbf{0}\}\cup G\right)+[-1,1]\right)\ .$$

Note that for every $z\in T$ it is true that

$$\|z\| \le \frac{1+\|L\|}{1-q}\max\left\{(1+\|L\|)M+q,\ M+1+q\right\}\le$$

$$\le \frac{(1+\|L\|)(1+M+\|L\|M+q)}{1-q} =: N\ .$$

Because D is a uniform tangent set to *epi* φ at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$, there exists $\hat{\delta}>0$, so that for each $t\in(0,\hat{\delta})$

$$epi\ \varphi\cap((\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))+\hat{\delta}\bar{\mathbf{B}})+tD\subseteq epi\ \varphi+t\sigma\bar{\mathbf{B}}. \tag{3.1}$$

We continue with the definition of the basic constants:

$$\eta > 0 \text{ is such that } \eta < \frac{1}{N(1+(1+\|L\|)R)},$$

$$\rho > 0 \text{ is such that } \rho < \frac{1-\eta(N(1+(1+\|L\|)R)}{\eta(1+\|L\|)+2}.$$

Since T is compact, there exists a finite ρ-net

$$H_\rho := \{p_1, p_2, \ldots, p_k\}$$

for T. Without loss of generality, we may think that $H_\rho \subset T$. The "density condition" implies that

$$D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R\,\bar{\mathbf{B}} - S \times (-\infty, 0]$$

is dense in the closed unit ball of $X \times Y \times \mathbb{R}$ centered at the origin. Therefore

$$D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ RN\bar{\mathbf{B}} - S \times (-\infty, 0]$$

is dense in $T \subset RN\bar{\mathbf{B}}$. Let

$$\{(u_i, v_i, w_i)\}_{i=1}^k \subset D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ RN\ \bar{\mathbf{B}}$$

and $\{(L(q_i), q_i, r_i)\}_{i=1}^k \subset S \times (-\infty, 0]$ be such that

$$\|p_i - ((u_i, v_i, w_i) - (L(q_i), q_i, r_i))\| < \rho \quad \text{for every} \quad i \in \{1, 2, \ldots, k\}.$$

Since $(u_i, v_i, w_i) \in D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ RN\ \bar{\mathbf{B}}$, it has the appearance

$$(u_i, v_i, w_i) = (u_i^1, v_i^1, w_i^1) + (u_i^2, v_i^2, w_i^2), (u_i^1, v_i^1, w_i^1) \in D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$$

and

$$(u_i^2, v_i^2, w_i^2) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ RN\ \bar{\mathbf{B}}$$

for every $i \in \{1, 2, \ldots, k\}$.

As $(u_i^2, v_i^2, w_i^2) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, there exists $\delta(u_i^2, v_i^2, w_i^2) > 0$ such that that for each $t \in [0, \delta(u_i^2, v_i^2, w_i^2)]$ the following inclusion holds true:

$$epi\ \varphi \cap \big((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \delta(u_i^2, v_i^2, w_i^2)\bar{\mathbf{B}}\big) + t(u_i^2, v_i^2, w_i^2) \subset epi\ \varphi + t\eta\bar{\mathbf{B}}$$

for each $i = 1, \ldots, k$.

We put

$$0 < \tilde{\delta} := \min\left\{\delta, \hat{\delta}, \delta(u_i^1, v_i^1, w_i^1), \delta(u_i^2, v_i^2, w_i^2), i = 1, \ldots, k\right\},$$

(where $\delta(u_1^1, v_1^1, w_1^1), \delta(u_2^1, v_2^1, w_2^1), \ldots, \delta(u_k^1, v_k^1, w_k^1)$ come from the definition of $D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$) and

$$\bar{\delta} := \frac{\tilde{\delta}}{2\left(\dfrac{q}{1-q}(1+\|L\|)^2(1+M) + (1+\|L\|)(1+M+\|L\|M) + 1\right)}.$$

Let us denote by A the epigraph $epi\ \varphi$ and by B the set $(S + (\bar{x}, \bar{y})) \times (-\infty, \varphi(\bar{x}, \bar{y})]$. The reference point $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ belongs to the intersection $A \cap B$. We consider the space $X \times Y \times \mathbb{R}$ endowed with the usual uniform norm.

Let $z^A \in A \cap \bar{\mathbf{B}}_{\bar\delta}(\bar x, \bar y, \varphi(\bar x, \bar y))$ and $z^B \in B \cap \bar{\mathbf{B}}_{\bar\delta}(\bar x, \bar y, \varphi(\bar x, \bar y))$ be different. We fix $t \in [0, \bar\delta]$ and we denote

$$w := -\frac{z^A - z^B}{\|z^A - z^B\|} \in \bar{\mathbf{B}}_{X\times Y\times\mathbb{R}} .$$

1st step: Reducing to a variational condition with a smaller constant. We put $M_1^A := (\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0)$ and $M_1^B := (\mathbf{0}, \bar{\mathbf{B}}_{\mathbf{Y}}, [-1,1])$. Then $\bar{\mathbf{B}}_{X\times Y\times\mathbb{R}} = M_1^A - M_1^B$ and therefore $w = m_1^A - m_1^B$ for some $m_1^A \in M_1^A$, $m_1^B \in M_1^B$. It is straightforward to check that

$$B + tM_1^B \subset B + tU_1^B \quad \text{for every} \quad t > 0 , \quad \text{where} \quad U_1^B := \big(L(\bar{\mathbf{B}}_{\mathbf{Y}}), \mathbf{0}, [-1,1]\big) .$$

Hence there exists $u_1^B \in U_1^B$ such that $z_1^B := z^B + tm_1^B - tu_1^B \in B$. Putting $U_1^A := M_1^A$, $u_1^A := m_1^A$, we have $z_1^A := z^A + tm_1^A - tu_1^A = z^A \in A$. Thus

$$z_1^A - z_1^B = (z^A + tm_1^A - tu_1^A) - (z^B + tm_1^B - tu_1^B) = z^A - z^B + tw - t(u_1^A - u_1^B) .$$

Moreover,

$$\|z_1^A - z^A\| = 0, \|z_1^A - (\bar x, \bar y, \varphi(\bar x, \bar y))\| \le \bar\delta \text{ and}$$
$$\|z_1^B - z^B\| \le t(1 + \|L\|), \|z_1^B - (\bar x, \bar y, \varphi(\bar x, \bar y))\| \le \bar\delta(2 + \|L\|).$$

Now we are to replace the "correcting term" $u_1^A - u_1^B$ by another difference, more suitable to be corrected. As

$$U_1^A - U_1^B = \big(\bar{\mathbf{B}}_{\mathbf{X}} - L(\bar{\mathbf{B}}_{\mathbf{Y}}), \mathbf{0}, [-1,1]\big) \subset \big((1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, [-1,1]\big) \subset M_2^A - M_2^B$$

where $M_2^A := \big((1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\big)$, $M_2^B := (\mathbf{0}, \mathbf{0}, [-1,1])$, there exist $m_2^A \in M_2^A$, $m_2^B \in M_2^B$ such that $u_1^A - u_1^B = m_2^A - m_2^B$.

Now (having in mind that $\bar\delta$ is less than $\frac{\delta}{1+\|L\|}$) we can apply the "variational condition" scaling the parameter t and obtain

$$A \cap \bar{\mathbf{B}}_{\bar\delta}(\bar x, \bar y, \varphi(\bar x, \bar y)) + t\big((1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\big) \subset A + t(1 + \|L\|)(U + D) .$$

Therefore there exists $u_2^A \in U_2^A := (1 + \|L\|)U$ and $d_2^A \in D_2^A := (1 + \|L\|)D$ such that $z_2^A := z_1^A + tm_2^A - tu_2^A - td_2^A \in A$. Putting $U_2^B := M_2^B$, $u_2^B := m_2^B$, we have $z_2^B := z_1^B + tm_2^B - tu_2^B = z_1^B \in B$. Hence

$$\begin{aligned} z_2^A - z_2^B &= (z_1^A + tm_2^A - tu_2^A - td_2^A) - (z_1^B + tm_2^B - tu_2^B) \\ &= z_1^A - z_1^B + t(m_2^A - m_2^B) - t(u_2^A - u_2^B) - td_2^A \\ &= z^A - z^B + tw - t(u_1^A - u_1^B) + t(m_2^A - m_2^B) - t(u_2^A - u_2^B) - td_2^A \\ &= z^A - z^B + tw - t(u_2^A - u_2^B) - td_2^A . \end{aligned}$$

Moreover,

$$\|z_2^A - z^A\| = t(1 + \|L\|)M, \|z_2^A - (\bar x, \bar y, \varphi(\bar x, \bar y))\| \le \bar\delta(1 + (1 + \|L\|)M) \text{ and}$$
$$\|z_2^B - z^B\| \le t(1 + \|L\|), \|z_2^B - (\bar x, \bar y, \varphi(\bar x, \bar y))\| \le \bar\delta(2 + \|L\|).$$

We continue again by calculating

$$U_2^A - U_2^B = ((1 + \|L\|)U_X, (1 + \|L\|)U_Y, (1 + \|L\|)U_{\mathbb{R}} + [-1,1]) = M_3^A - M_3^B$$

where

$$M_3^A := ((1 + \|L\|)U_X, \mathbf{0}, 0)$$

and

$$M_3^B := (\mathbf{0}, -(1+\|L\|)U_Y, -(1+\|L\|)U_{\mathbb{R}} + [-1,1]) \ .$$

Then there exist $m_3^A \in M_3^A$, $m_3^B \in M_3^B$ such that $u_2^A - u_2^B = m_3^A - m_3^B$. Again it is easy to check that $B + tM_3^B \subset B + tU_3^B$ for every $t > 0$ where

$$U_3^B := ((1+\|L\|)L(U_Y), \mathbf{0}, -(1+\|L\|)U_{\mathbb{R}} + [-1,1]) \ .$$

Hence there exists $u_3^B \in U_3^B$ such that $\hat{z}_3^B := z_2^B + tm_3^B - tu_3^B \in B$. Not touching A, we put $U_3^A := M_3^A$, $u_3^A := m_3^A$, and we have $\hat{z}_3^A := z_2^A + tm_3^A - tu_3^A = z_2^A \in A$. Thus

$$\begin{aligned} \hat{z}_3^A - \hat{z}_3^B &= (z_2^A + tm_3^A - tu_3^A) - (z_2^B + tm_3^B - tu_3^B) \\ &= z_2^A - z_2^B + t(m_3^A - m_3^B) - t(u_3^A - u_3^B) \\ &= z^A - z^B + tw - t(u_2^A - u_2^B) - td_2^A + t(m_3^A - m_3^B) - t(u_3^A - u_3^B) \\ &= z^A - z^B + tw - t(u_3^A - u_3^B) - td_2^A \ . \end{aligned}$$

We set $d_3^A := d_2^A$ and write this equality as follows

$$\hat{z}_3^A - \hat{z}_3^B = z^A - z^B + tw - t(u_3^A - u_3^B) - td_3^A.$$

Now the last "correcting set" has the appearance

$$\begin{aligned} U_3^A - U_3^B &= ((1+\|L\|)(U_X - L(U_Y)), \mathbf{0}, (1+\|L\|)U_{\mathbb{R}} + [-1,1]) \\ &= (1+\|L\|)K + (\mathbf{0}, \mathbf{0}, [-1,1]) \ , \end{aligned}$$

where by K we have denoted the set $(U_X - L(U_Y), \mathbf{0}, U_{\mathbb{R}})$.

Let us remind that D is a uniform tangent set and z^A is an arbitrary point from $A \cap \bar{\mathbf{B}}_{\bar{\delta}}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$. Because $\tilde{\delta} < \hat{\delta}$, we have that

$$\begin{aligned} \|\hat{z}_3^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| &= \|z_2^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| \\ &\le \bar{\delta}(1 + (1+\|L\|)M) < \hat{\delta}, \end{aligned}$$

according to our choice of M and $\bar{\delta}$.

Therefore, we can apply (3.1) starting from $\hat{z}_3^A$ and obtain that there exists $o_3^A := (o_X^3, o_Y^3, o_R^3)$ with $\|o_3^A\| \le \sigma$ such that

$$z_3^A := \hat{z}_3^A + td_3^A - to_3^A \in A.$$

By setting $z_3^B := \hat{z}_3^B - t(L(o_Y^3), o_Y^3, 0) \in B$, we obtain that

$$\begin{aligned} z_3^A - z_3^B &= \hat{z}_3^A + td_3^A - to_3^A - \hat{z}_3^B + t(L(o_Y^3), o_Y^3, 0) \\ &= z^A - z^B + tw - t(u_3^A - u_3^B) - td_3^A + td_3^A - to_3^A + t(L(o_Y^3), o_Y^3, 0) \\ &= z^A - z^B + tw - t(u_3^A - u_3^B) - t(o_X^3, o_Y^3, o_R^3) + t(L(o_Y^3), o_Y^3, 0) \\ &= z^A - z^B + tw - t(u_3^A - u_3^B) - t(o_X^3 - L(o_Y^3), 0, o_R^3). \end{aligned}$$

Let us remind that

$$u_3^A - u_3^B \in U_3^A - U_3^B = ((1+\|L\|)(U_X - L(U_Y)), \mathbf{0}, (1+\|L\|)U_{\mathbb{R}} + [-1,1])$$

and $(o_X^3 - Lo_Y^3, 0, o_R^3) \in \ (1+\|L\|)\sigma(\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, [-1,1])$. Since

$$0 < q = \tilde{q} + (1+\|L\|)\sigma < 1,$$

we obtain that

$$u_3^A - u_3^B + (o_X^3 - Lo_Y^3, 0, o_R^3) \in (1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) + (\mathbf{0}, \mathbf{0}, [-1,1]) .$$

Hence,

$$\begin{aligned} z_3^A - z_3^B &\in z^A - z^B + tw \\ &\quad -t\left((1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) + (\mathbf{0}, \mathbf{0}, [-1,1])\right) . \end{aligned} \tag{3.2}$$

Moreover, the following estimates hold true

$$\begin{aligned} \|z_3^A - z^A\| &\le \|z_3^A - \hat{z}_3^A\| + \|\hat{z}_3^A - z^A\| \\ &\le t(1+\|L\|)M + t\sigma \le t(1 + M + \|L\|M) \\ \|z_3^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| &\le \|z_3^A - z^A\| + \|z^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| \\ &\le \bar{\delta}(1 + M + \|L\|M) + \bar{\delta} = \bar{\delta}(2 + M + \|L\|M). \end{aligned}$$

Also,

$$\begin{aligned} \|z_3^B - z^B\| &\le \left\|z_3^B - \hat{z}_3^B\right\| + \left\|\hat{z}_3^B - z_2^B\right\| + \left\|z_2^B - z^B\right\| \\ &\le t\|(L(o_Y^3), o_Y^3, 0)\| + t\left\|m_3^B - u_3^B\right\| + t(1+\|L\|) \\ &\le t\sigma \max\{\|L\|, 1\} + t(1+\|L\|)(1 + M + \|L\|M) + t(1+\|L\|) \\ &\le t(1+\|L\|)(1 + (1 + M + \|L\|M) + 1) \\ &= t(1+\|L\|)(3 + M + \|L\|M). \end{aligned}$$

$$\begin{aligned} \|z_3^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| &\le \|z_3^B - z^B\| + \|z^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\| \\ &\le t(1+\|L\|)(3 + M + \|L\|M) + \bar{\delta} \\ &\le \bar{\delta}((1+\|L\|)(3 + M + \|L\|M) + 1). \end{aligned}$$

We are ready for the next step.

2nd step: Reducing to a compact correcting set.
We are going to use repeatedly the following induction step:

Induction step. For each $\bar{\zeta} \in (0, \tilde{\delta})$, $\zeta \in (0,1)$ and for each $\tilde{z}^A \in A \cap \bar{\mathbf{B}}_{\bar{\zeta}}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, $\tilde{z}^B \in B \cap \bar{\mathbf{B}}_{\bar{\zeta}}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$, $m \in \left(\zeta\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right)$ and for the fixed $t \in [0, \bar{\delta}]$ there exist $\tilde{z}^{A,*} \in A$, $\tilde{z}^{B,*} \in B$ such that

$$\tilde{z}^{A,*} - \tilde{z}^{B,*} = \tilde{z}^A - \tilde{z}^B + tm - tu\ , \quad \text{where} \ \ u \in \zeta\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right)$$

$$\text{and} \ \left\|\tilde{z}^{A,*} - \tilde{z}^A\right\| \le t\zeta(1+M)\ , \ \left\|\tilde{z}^{B,*} - \tilde{z}^B\right\| \le t\zeta(1+\|L\|)M\ .$$

Proof of the induction step. In fact we repeat the second half of the construction in the previous step. Since $\bar{\zeta} < \delta$ and $t\zeta < \delta$, we apply the "variational condition" with the parameter $t\zeta$ and obtain

$$A \cap \bar{\mathbf{B}}_{\bar{\zeta}}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + t\left(\zeta\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right) \subset A + t\zeta U + t\zeta D\ .$$

Therefore there exists $u^A = (u_X^A, u_Y^A, u_{\mathbb{R}}^A) \in \zeta U$ and $d^A \in \zeta D$ such that

$$z^{A,*} := \tilde{z}^A + tm - tu^A - td^A \in A\ .$$

Since $t\zeta < \hat{\delta}$ and $\bar{\zeta} < \hat{\delta}$, we obtain from (3.1) there exists $o^A := (o^A_X, o^A_Y, o^A_{\mathbb{R}})$ with $\|o^A\| \le \zeta\sigma$ such that

$$\tilde{z}^{A,*} := z^{A,*} + td^A - to^A \in A.$$

By setting $\tilde{z}^{B,*} := \tilde{z}^B - t\left(L(o^A_Y + u^A_Y), o^A_Y + u^A_Y, 0\right) \in B$, we obtain that

$$\begin{aligned}\tilde{z}^{A,*} - \tilde{z}^{B,*} &= z^{A,*} + td^A - to^A - \tilde{z}^B + t\left(L(o^A_Y + u^A_Y), o^A_Y + u^A_Y, 0\right)\\ &= \tilde{z}^A - \tilde{z}^B + tm - tu^A - td^A + td^A - to^A\\ &\quad + t\left(L(o^A_Y + u^A_Y), o^A_Y + u^A_Y, 0\right)\\ &= \tilde{z}^A - \tilde{z}^B + tm - t(u^A_X, u^A_Y, u^A_{\mathbb{R}}) - t(o^A_X, o^A_Y, o^A_{\mathbb{R}})\\ &\quad + t\left(L(o^A_Y + u^A_Y), o^A_Y + u^A_Y, 0\right)\\ &= \tilde{z}^A - \tilde{z}^B + tm - t(u^A_X + o^A_X - L(o^A_Y + u^A_Y), 0, u^A_{\mathbb{R}} + o^A_{\mathbb{R}})\\ &\subseteq \tilde{z}^A - \tilde{z}^B + tm - t\zeta(U_X + \sigma\bar{\mathbf{B}}_{\mathbf{X}} - L(U_Y + \sigma\bar{\mathbf{B}}_{\mathbf{Y}}), \mathbf{0}, U_{\mathbb{R}} + \sigma[-1,1]).\end{aligned}$$

Note that

$$\begin{aligned}&(U_X + \sigma\bar{\mathbf{B}}_{\mathbf{X}} - L(U_Y + \sigma\bar{\mathbf{B}}_{\mathbf{Y}}), \mathbf{0}, U_{\mathbb{R}} + \sigma[-1,1])\\ &\qquad\qquad \subset (U_X - L(U_Y) + \sigma(1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, U_{\mathbb{R}} + \sigma[-1,1]).\end{aligned}$$

Having in mind that $0 < q = \tilde{q} + (1 + \|L\|)\sigma < 1$ and

$$(U_X - L(U_Y), \mathbf{0}, U_{\mathbb{R}}) = K \subset \left(F + \tilde{q}\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + \tilde{q}[-1,1]\right),$$

we obtain that

$$(U_X - L(U_Y) + \sigma(1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, U_{\mathbb{R}} + \sigma[-1,1]) \subset \left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right).$$

Therefore, the formula for $\tilde{z}^{A,*} - \tilde{z}^{B,*}$ in the induction step is proved. It remains to estimate the distance from $\tilde{z}^{A,*}$ and $\tilde{z}^{B,*}$ to the starting points. Indeed, we have (according to our choice of M and σ)

$$\left\|\tilde{z}^{A,*} - \tilde{z}^A\right\| = t\|m - u^A - o^A\| \le t(\|m\| + \|u^A\| + \|o^A\|) < t\zeta(M + \sigma) < t\zeta(1 + M)$$

and

$$\begin{aligned}\left\|\tilde{z}^{B,*} - \tilde{z}^B\right\| &= t\|(-L(u^A_Y + o^A_Y), -u^A_Y - o^A_Y, 0)\|\\ &\le t\max\{\|L\|(\|u^A_Y\| + \zeta\sigma), \|u^A_Y\| + \zeta\sigma\}\\ &\le t\zeta(1 + \|L\|)M\ .\end{aligned}$$

End of the proof of the induction step.

We proceed by constructing z^A_4 and z^B_4. According to (3.2) we have that

$$\begin{aligned}z^A_3 - z^B_3 \in\ & z^A - z^B + tw - tm\\ & -t\left((1 + \|L\|)\left(F, \mathbf{0}, G + q[-1,1]\right) + (\mathbf{0}, \mathbf{0}, [-1,1])\right),\end{aligned} \tag{3.3}$$

where $m \in (1 + \|L\|)q\bar{\mathbf{B}}_{\mathbf{X}}$.

We apply the induction step with $\bar{\zeta} := \bar{\delta}((1 + \|L\|)(1 + M + \|L\|M) + 1)$, $\zeta := q(1 + \|L\|)$, $\tilde{z}^A := z^A_3$, $\tilde{z}^B := z^B_3$, m and t. We denote the resulting $\tilde{z}^{A,*}$ by z^A_4 and

$\tilde{z}^{B,*}$ by z_4^B. Thus we have $z_4^A \in A$, $z_4^B \in B$ and

$$\begin{aligned} z_4^A - z_4^B &\in z_3^A - z_3^B + tm - tq(1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) \\ &\subset z^A - z^B + tw - tm - t(1+\|L\|)\,(F, \mathbf{0}, G + q[-1,1]) \\ &\quad - t\,(\mathbf{0}, \mathbf{0}, [-1,1]) + tm - tq(1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) \\ &= z^A - z^B + tw - t(1+\|L\|)\left(F + qF, \mathbf{0}, G + qG + (q+q^2)[-1,1]\right) \\ &\quad - t\,(\mathbf{0}, \mathbf{0}, [-1,1]) - t\left(q^2(1+\|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right) \ . \end{aligned}$$

This could be written as

$$z_4^A - z_4^B \in z^A - z^B + tw - tV_4 - t\left(q^2(1+\|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right)$$

if we put

$$V_4 := (1+\|L\|)\left(F + qF, \mathbf{0}, G + qG + (q+q^2)[-1,1]\right) + (\mathbf{0}, \mathbf{0}, [-1,1]) \ .$$

Moreover, we can estimate the distance

$$\begin{aligned} \left\|z_4^A - z^A\right\| &\le \left\|z_4^A - z_3^A\right\| + \left\|z_3^A - z^A\right\| \\ &\le tq(1+\|L\|)(1+M) + t(1+M+M\|L\|) \\ &\le t(1+q)(1+\|L\|)(1+M)\,\left\|z_4^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le \left\|z_4^A - z^A\right\| + \left\|z^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le t(1+q)(1+\|L\|)(1+M) + \bar{\delta} \\ &\le \bar{\delta}(1 + (1+q)(1+\|L\|)(1+M)). \end{aligned}$$

Analogously,

$$\begin{aligned} \left\|z_4^B - z^B\right\| &\le \left\|z_4^B - z_3^B\right\| + \left\|z_3^B - z^B\right\| \\ &\le tq(1+\|L\|)^2 M + t(1+\|L\|)(3+M+M\|L\|) \\ &\le t(1+\|L\|)(q(1+\|L\|)M + 3 + M + M\|L\|)\,\left\|z_4^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le \left\|z_4^B - z^B\right\| + \left\|z^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le t(1+\|L\|)(q(1+\|L\|)M + 3 + M + M\|L\|) + \bar{\delta} \\ &\le \bar{\delta}(1 + (1+\|L\|)(q(1+\|L\|)M + 3 + M + M\|L\|)). \end{aligned}$$

Let for $n \ge 4$ the points $z_n^A \in A$, $z_n^B \in B$ have been constructed satisfying

$$z_n^A - z_n^B \in z^A - z^B + tw - tm_n - tV_n$$

where $m_n \in \left(q^{n-2}(1+\|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right)$ and V_n has the appearance

$$\begin{aligned} V_n := (1+\|L\|)\,\big(F + qF + \cdots + q^{n-3}F, \mathbf{0}, G + qG \\ + \cdots + q^{n-3}G + (q + \cdots + q^{n-2})[-1,1]\big) + (\mathbf{0}, \mathbf{0}, [-1,1]) \ . \end{aligned}$$

Moreover, let

$$\left\|z_n^A - z^A\right\| \le t(1 + q + \cdots + q^{n-3})(1+\|L\|)(1+M) \quad \text{and}$$

$$\left\|z_n^B - z^B\right\| \le t(q + \cdots + q^{n-3})(1+\|L\|)^2 M + t(1+\|L\|)(3+M+M\|L\|).$$

Next we apply the induction step with $\zeta := q^{n-2}(1+\|L\|)$, $\tilde{z}^A := z_n^A$, $\tilde{z}^B := z_n^B$, $m := m_n$ and t. We denote the resulting $\tilde{z}^{A,*}$ by z_{n+1}^A and $\tilde{z}^{B,*}$ by z_{n+1}^B. Thus we have $z_{n+1}^A \in A$, $z_{n+1}^B \in B$ and

$$\begin{aligned} z_{n+1}^A - z_{n+1}^B &\in z_n^A - z_n^B + tm_n - tq^{n-2}(1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) \\ &\subset z^A - z^B + tw - tm_n - tV_n + tm_n \\ &\quad - tq^{n-2}(1+\|L\|)\left(F + q\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, G + q[-1,1]\right) \\ &= z^A - z^B + tw - tV_{n+1} - t\left(q^{n-1}(1+\|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right). \end{aligned}$$

where

$$\begin{aligned} V_{n+1} := (1+\|L\|)\big(&F + qF + \cdots + q^{n-2}F, \mathbf{0}, G + qG \\ &+ \cdots + q^{n-2}G + (q + \cdots + q^{n-1})[-1,1]\big) + (\mathbf{0}, \mathbf{0}, [-1,1]) \ . \end{aligned}$$

One can directly check that

$$\begin{aligned} \left\|z_{n+1}^A - z^A\right\| &\le \left\|z_{n+1}^A - z_n^A\right\| + \left\|z_n^A - z^A\right\| \\ &\le tq^{n-2}(1+\|L\|)(1+M) + t(1+q+\cdots+q^{n-3})(1+\|L\|)(1+M) \\ &\le t(1+q+\cdots+q^{n-2})(1+\|L\|)(1+M) \end{aligned}$$

and

$$\begin{aligned} \left\|z_{n+1}^B - z^B\right\| &\le \left\|z_{n+1}^B - z_n^B\right\| + \left\|z_n^B - z^B\right\| \\ &\le tq^{n-2}(1+\|L\|)^2 M + t(q+\cdots+q^{n-3})(1+\|L\|)^2 M \\ &\quad + t(1+\|L\|)(3+M+M\|L\|)t(q+\cdots+q^{n-2})(1+\|L\|)^2 M \\ &\quad + t(1+\|L\|)(3+M+M\|L\|). \end{aligned}$$

Estimating the distance to the reference point, we have

$$\begin{aligned} \left\|z_{n+1}^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| &\le \left\|z_{n+1}^A - z^A\right\| + \left\|z^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le \bar{\delta}\left(1 + (1+q+\cdots+q^{n-2})(1+\|L\|)(1+M)\right) \end{aligned}$$

and

$$\begin{aligned} \left\|z_{n+1}^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| &\le \left\|z_{n+1}^B - z^B\right\| + \left\|z^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \\ &\le \bar{\delta}\left(1 + (q+\cdots+q^{n-2})(1+\|L\|)^2 M \right. \\ &\quad \left. + (1+\|L\|)(3+M+M\|L\|)\right). \end{aligned}$$

Let us note that for every $n \ge 4$ we have

$$\begin{aligned} &F + qF + q^2F + \ldots q^{n-2}F \\ &\quad = \frac{1-q^{n-1}}{1-q}\left(\frac{1-q}{1-q^{n-1}} \cdot F + q \cdot \frac{1-q}{1-q^{n-1}} \cdot F + \cdots + q^{n-2} \cdot \frac{1-q}{1-q^{n-1}} \cdot F\right) \\ &\qquad \subset \frac{1-q^{n-1}}{1-q} \cdot \overline{\operatorname{conv}}\ F \subset \frac{1}{1-q}\ \overline{\operatorname{conv}}\left(\{\mathbf{0}\} \cup F\right) =: P \end{aligned}$$

and similarly,

$$G + qG + q^2G + \dots q^{n-2}G \subset \frac{1 - q^{n-1}}{1 - q} \cdot \overline{\text{conv}}\ G \subset \frac{1}{1 - q}\ \overline{\text{conv}}\,(\{\mathbf{0}\} \cup G) =: Q\ .$$

Thus for every $n \geq 4$ it is true that

$$V_n \subset (1 + \|L\|) \left(P, \mathbf{0}, Q + \frac{1}{1 - q}[-1, 1] \right) =: T\ .$$

The set T is a polytope (closed convex hull of finitely many points) contained in $X \times \{\mathbf{0}\} \times \mathbb{R}$. Note that stopping at stage n for n sufficiently large we can assume that the term $m_n \in \left(q^{n-2}(1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right)$ is as small as desired. It is sufficient for our purposes. Nevertheless we prefer to work with the infinite version of this construction. Indeed, the sequences $\{z_n^A\}_{n=1}^\infty \subset A$ and $\{z_n^B\}_{n=1}^\infty \subset B$ are fundamental as the distances between two subsequent terms are majorized by a geometric progression with ratio $q \in (0, 1)$. Let

$$z_n^A \longrightarrow_{n\to\infty} \tilde{z}^A \ \text{ and } \ z_n^B \longrightarrow_{n\to\infty} \tilde{z}^B\ .$$

Apparently $\tilde{z}^A \in A$ and $\tilde{z}^B \in B$ because A and B are closed. Keeping in mind that $V_n \subset T$ for every $n \in \mathbb{N}$, we can let n tend to infinity in the inclusion

$$z_n^A - z_n^B \in z^A - z^B + tw - tT - t\left(q^{n-2}(1 + \|L\|)\bar{\mathbf{B}}_{\mathbf{X}}, \mathbf{0}, 0\right) \ \text{ and obtain that}$$

$$\tilde{z}^A - \tilde{z}^B \in \overline{z^A - z^B + tw - tT} = z^A - z^B + tw - tT$$

because of the compactness of T. Letting n tend to infinity in the distance estimates, we obtain the inequalities

$$\left\|\tilde{z}^A - z^A\right\| \leq t\ \frac{1}{1 - q}(1 + \|L\|)(1 + M) \ \text{ and}$$

$$\left\|\tilde{z}^B - z^B\right\| \leq t\ \frac{q}{1 - q}(1 + \|L\|)^2 M + t(1 + \|L\|)(3 + M + M\|L\|).$$

Therefore

$$\left\|\tilde{z}^A - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \leq \bar{\delta} \left(1 + \frac{1}{1 - q}(1 + \|L\|)(1 + M) \right) < \tilde{\delta},$$

$$\left\|\tilde{z}^B - (\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))\right\| \leq \bar{\delta} \left(1 + \frac{q}{1 - q}(1 + \|L\|)^2 M + (1 + \|L\|)(3 + M + M\|L\|) \right) < \tilde{\delta}.$$

3rd step: Using the density assumption.

Let us remind that $H_\rho := \{p_1, p_2, \dots, p_k\}$ is a finite ρ-net for T,

$$\{(u_i, v_i, w_i)\}_{i=1}^k \subset D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ NR\ \bar{\mathbf{B}}$$

and $\{(L(q_i), q_i, r_i)\}_{i=1}^k \subset S \times (-\infty, 0]$ are such that

$$\|p_i - ((u_i, v_i, w_i) - (L(q_i), q_i, r_i))\| < \rho \ \text{ for every } \ i \in \{1, 2, \dots, k\},$$

$$(u_i, v_i, w_i) = (u_i^1, v_i^1, w_i^1) + (u_i^2, v_i^2, w_i^2), (u_i^1, v_i^1, w_i^1) \in D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$$

and

$$(u_i^2, v_i^2, w_i^2) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ NR\ \bar{\mathbf{B}}$$

for every $i \in \{1, 2, \ldots, k\}$, where N is an upper bound for the norms of the elements of T.

Now $\tilde{z}^A$ and $\tilde{z}^B$ are in $\tilde{\delta}$ neighbourhood of $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$. As $\tilde{z}^A - \tilde{z}^B \in z^A - z^B + tw - tT$, there exists $i \in \{1, 2, \ldots, k\}$ such that

$$\tilde{z}^A - \tilde{z}^B \in z^A - z^B + tw - tp_i + t\rho\bar{\mathbf{B}} \ .$$

Therefore

$$\tilde{z}^A - \tilde{z}^B \in z^A - z^B + tw - t(u_i, v_i, w_i) + t(L(q_i), q_i, r_i) + 2t\rho\bar{\mathbf{B}} \ .$$

Now the definition of $D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ yields the existence of a point $\bar{z}_1^A \in A$ with

$$\bar{z}_1^A \in \tilde{z}^A + t(u_i^1, v_i^1, w_i^1) - t\eta\|u_i^1 - L(v_i^1)\|\bar{\mathbf{B}} \ .$$

Analogously, the definition of the Clarke tangent cone $\hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ yields the existence of a point $\bar{z}^A \in A$ with

$$\bar{z}^A \in \bar{z}_1^A + t(u_i^2, v_i^2, w_i^2) - t\eta\bar{\mathbf{B}} \ .$$

The last two inclusions give us

$$\bar{z}^A \in \tilde{z}^A + t(u_i, v_i, w_i) - t\eta(1 + \|u_i^1 - L(v_i^1)\|)\bar{\mathbf{B}} \ .$$

Apparently we have

$$\bar{z}^B := \tilde{z}^B + t(L(q_i), q_i, r_i) \in B \ .$$

Then we have

$$\begin{aligned} \bar{z}^A - \bar{z}^B &\in \tilde{z}^A - \tilde{z}^B + t(u_i, v_i, w_i) - t(L(q_i), q_i, r_i) - t\eta(1 + \|u_i^1 - L(v_i^1)\|)\bar{\mathbf{B}} \\ &\subset z^A - z^B + tw - t(u_i, v_i, w_i) + t(L(q_i), q_i, r_i) + 2t\rho\bar{\mathbf{B}} \\ &\qquad + t(u_i, v_i, w_i) - t(L(q_i), q_i, r_i) - t\eta(1 + \|u_i^1 - L(v_i^1)\|)\bar{\mathbf{B}} \\ &= z^A - z^B + tw - t\eta(1 + \|u_i^1 - L(v_i^1)\|)\bar{\mathbf{B}} + 2t\rho\bar{\mathbf{B}} \ . \end{aligned}$$

Therefore

$$\begin{aligned} \left\|\bar{z}^A - \bar{z}^B\right\| &\le \left\|z^A - z^B + tw\right\| + t\eta(1 + \|u_i^1 - L(v_i^1)\|) + 2t\rho \\ &= \left\|z^A - z^B\right\| - t\left(1 - \eta(1 + \|u_i^1 - L(v_i^1)\|) - 2\rho\right) \ . \end{aligned}$$

Let $p_i = (p_i^X, \mathbf{0}, p_i^{\mathbb{R}})$. Because

$$\|p_i - ((u_i, v_i, w_i) - (L(q_i), q_i, r_i))\| < \rho,$$

we have $\|q_i - v_i\| < \rho$, and hence

$$\left\|p_i^X - (u_i - L(v_i))\right\| \le \left\|p_i^X - (u_i - L(q_i))\right\| + \|L(v_i) - L(q_i)\| \le (1 + \|L\|)\rho \ .$$

Hence

$$\begin{aligned} \|u_i^1 - L(v_i^1)\| &\le \|u_i - L(v_i) - u_i^1 - L(v_i^1)\| + \|u_i - L(v_i) - p_i^X\| + \|p_i^X\| \\ &= \|u_i^2 - L(v_i^2)\| + \|u_i - L(v_i) - p_i^X\| + \|p_i^X\| \\ &\le (1 + \|L\|)NR + (1 + \|L\|)\rho + \max\{\|p\| : \ p \in T\} \\ &\le (1 + \|L\|)(\rho + NR) + N. \end{aligned}$$

Thus we obtain that

$$\left\|\bar{z}^A - \bar{z}^B\right\| \le \left\|z^A - z^B\right\| - t\left(1 - \eta\left(N + (1 + \|L\|)(\rho + NR)\right) - 2\rho\right) \ .$$

Then the coefficient $\alpha := 1-\eta\,(N+(1+\|L\|)\,(\rho+NR))-2\rho$ is positive because of the choice of the constants η and ρ. As the set $\{(u_i,v_i,w_i)\}_{i=1}^k \cup \{(L(q_i),q_i,r_i)\}_{i=1}^k$ is bounded, this proves tangential transversality of A and B at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$. $\square$

4. A Lagrange multiplier rule

Lemma 4.1. *Let* $D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) \cap R\ \bar{\mathbf{B}} - S\times(-\infty,0]$ *be dense in the closed unit ball* $\bar{\mathbf{B}}$ *of* $X\times Y\times\mathbb{R}$. *Then, for each* $\varepsilon>0$ *there exist* $\tilde{p}=(\tilde{u},\tilde{v},\tilde{w}) \in D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) \cap 2R\ \bar{\mathbf{B}}$ *and* $\tilde{s}=(L(\tilde{q}),\tilde{q},\tilde{r}) \in S\times(-\infty,0]$ *with unit norm such that* $\|\tilde{p}-\tilde{s}\|<\varepsilon$ *and* $\|\tilde{u}-L(\tilde{v})\|<\varepsilon$.

Proof. Let us fix an arbitrary $\varepsilon\in(0,1)$. We consider the vector $z:=(\mathbf{0},\mathbf{0},-1)\in X\times Y\times\mathbb{R}$. Now the density of

$$D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) \cap R\ \bar{\mathbf{B}} - S\times(-\infty,0]$$

yields the existence of two vectors

$$p=(u,v,w)\in D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) \cap R\ \bar{\mathbf{B}}$$

and $s=(L(q),q,r)\in S\times(-\infty,0]$, such that

$$\|z-(p-s)\| < \frac{\varepsilon}{4\max\{1,\|L\|\}}\,,$$

hence

$$\|u-L(q)\|<\frac{\varepsilon}{4}\,,\quad \|v-q\|<\frac{\varepsilon}{4\max\{1,\|L\|\}}\,,\quad |-1-(w-r)|<\frac{\varepsilon}{2}\,.$$

As $s=(L(q),q,r)\in S\times(-\infty,0]$, we have $(L(q),q,r-1)\in S\times(-\infty,0]$. Also,

$$\|(L(q),q,r-1)\|\geq |r-1|\geq 1$$

since $r\leq 0$. Moreover, $\|(u,v,w)\|\geq|w|\geq|r-1|-\varepsilon/2>1/2$, and hence

$$\tilde{p}:=\frac{(u,v,w)}{\|(u,v,w)\|}\in D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y})) \cap 2R\ \bar{\mathbf{B}}$$

$$\text{and}\quad \tilde{s}:=\frac{(L(q),q,r-1)}{\|(L(q),q,r-1)\|}\in S\times(-\infty,0]\,.$$

Therefore, if $\tilde{p}:=(\tilde{u},\tilde{v},\tilde{w})$, we have

$$\begin{aligned}\|\tilde{u}-L(\tilde{v})\| &= \left\|\frac{u}{\|(u,v,w)\|}-L\left(\frac{v}{\|(u,v,w)\|}\right)\right\| \\ &= \frac{\|u-L(v)\|}{\|(u,v,w)\|}\leq 2\,(\|u-L(q)\|+\|L(q)-L(v)\|) \\ &< 2\left(\frac{\varepsilon}{4}+\|L\|\ \|v-q\|\right)\leq\frac{\varepsilon}{2}\,.\end{aligned}$$

Apparently, $\|\tilde{p}\| = 1$ and $\|\tilde{s}\| = 1$. We estimate

$$\begin{aligned}
\|\tilde{p} - \tilde{s}\| &= \left\| \frac{(u,v,w)}{\|(u,v,w)\|} - \frac{(L(q),q,r-1)}{\|(L(q),q,r-1)\|} \right\| \\
&\leq \left\| \frac{(u,v,w)}{\|(u,v,w)\|} - \frac{(u,v,w)}{\|(L(q),q,r-1)\|} \right\| \\
&\quad + \left\| \frac{(u,v,w)}{\|(L(q),q,r-1)\|} - \frac{(L(q),q,r-1)}{\|(L(q),q,r-1)\|} \right\| \\
&= \|(u,v,w)\| \left| \frac{1}{\|(u,v,w)\|} - \frac{1}{\|(L(q),q,r-1)\|} \right| \\
&\quad + \frac{\|(u,v,w) - (L(q),q,r-1)\|}{\|(L(q),q,r-1)\|} \\
&= \frac{|\|(L(q),q,r-1)\| - \|(u,v,w)\||}{\|(L(q),q,r-1)\|} + \frac{\|(u,v,w) - (L(q),q,r-1)\|}{\|(L(q),q,r-1)\|} \\
&\leq 2\frac{\|(u,v,w) - (L(q),q,r-1)\|}{\|(L(q),q,r-1)\|} \\
&\leq 2\max\{\|u - L(q)\|, \|v-q\|, |w-(r-1)|\} < \varepsilon\,.
\end{aligned}$$

The proof is complete. □

Lemma 4.2. *(Nonseparation result) Let A be the epigraph epi φ and B be the set $(S + (\bar{x},\bar{y}))\times(-\infty,\varphi(\bar{x},\bar{y})]$. Let A and B be tangentially transversal at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ with constants $\delta > 0$, $\alpha > 0$ and $\Omega > 0$. Let there exist $v^A := (u,v,w)$ with unit norm that belongs to the sum of the cone $D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ and $\hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))\cap 2R\ \bar{\mathbf{B}}$, $v^B := (L(q),q,r)$ with unit norm that belongs to $S\times(-\infty,0]$ such that*

$$\eta\left(1 + \frac{\alpha}{2\Omega} + (1+\|L\|)2R\right) < \frac{\alpha}{2\Omega}, \|v^A - v^B\| < \frac{\alpha}{2\Omega} \quad \text{and} \quad \|u - L(v)\| < \frac{\alpha}{2\Omega}\ .$$

Then A and B can not be locally separated at $(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$.

Proof. We set $z^0 := (\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$. Since $v^A := (u,v,w)$ belongs to $D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))+ \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))\cap 2R\ \bar{\mathbf{B}}$ there exist $v_1^A := (u_1,v_1,w_1) \in D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ and $v_2^A := (u_2,v_2,w_2) \in \hat{T}_{epi\ \varphi}(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))\cap 2R\ \bar{\mathbf{B}}$ such that

$$(u,v,w) = (u_1,v_1,w_1) + (u_2,v_2,w_2).$$

The definitions of the cone $D_\eta(\bar{x},\bar{y},\varphi(\bar{x},\bar{y}))$ and the Clarke tangent cone imply the existence $\tilde{\delta} > 0$ such that for every $t \in [0,\tilde{\delta}]$ it holds true that

$$z_1^A(t) := z^0 + tv_1^A - t\eta\|u_1 - L(v_1)\|z_t^1 \in A \quad \text{for some} \quad z_t^1 \in \bar{\mathbf{B}}$$

and

$$z^A(t) := z_1^A(t) + tv_2^A - t\eta z_t^2 \in A \quad \text{for some} \quad z_t^2 \in \bar{\mathbf{B}}.$$

From these two inclusions we obtain that

$$z^A(t) = z^0 + tv^A - t\eta(1 + \|u_1 - L(v_1)\|)z_t \in A \quad \text{for some} \quad z_t \in \bar{\mathbf{B}}.$$

Also,

$$\begin{aligned}\|u_1 - L(v_1)\| &\le \|u_1 + u_2 - L(v_1 + v_2)\| + \|u_2 - L(v_2)\| \\ &\le \|u - L(v)\| + (1 + \|L\|)2R \le \frac{\alpha}{2\Omega} + (1 + \|L\|)2R.\end{aligned}$$

Apparently

$$z^B(t) := z^0 + tv^B \in B \text{ for every } t > 0 .$$

Clearly,

$$\|z^B(t) - z^0\| = t .$$

Using Theorem 2.3 from [2], we obtain the existence of $z^{AB}(t) \in A \cap B$ with $\|z^{AB}(t) - z^A(t)\| \le \frac{\Omega}{\alpha}\|z^A(t) - z^B(t)\|$ and $\|z^{AB}(t) - z^B(t)\| \le \frac{\Omega}{\alpha}\|z^A(t) - z^B(t)\|$. Therefore

$$\begin{aligned}\|z^{AB}(t) - z^B(t)\| &\le \frac{\Omega}{\alpha}\|z^A(t) - z^B(t)\| \\ &= \frac{\Omega}{\alpha}\, t\, \left\|v^A - v^B - \eta(1 + \|u_1 - L(v_1)\|)z_t\right\| \\ &\le \frac{\Omega}{\alpha} \cdot \|z^B(t) - z^0\| \cdot \left(\|v^A - v^B\| + \eta(1 + \|u_1 - L(v_1)\|\right) \\ &< \frac{\Omega}{\alpha} \cdot \|z^B(t) - z^0\| \cdot \left(\frac{\alpha}{2\Omega} + \eta\left(1 + \frac{\alpha}{2\Omega} + (1 + \|L\|)2R\right)\right) \\ &\le \frac{\Omega}{\alpha} \cdot \|z^B(t) - z^0\| \cdot \left(\frac{\alpha}{2\Omega} + \frac{\alpha}{2\Omega}\right) \\ &= \|z^B(t) - z^0\|.\end{aligned}$$

Therefore $z^{AB}(t) \ne z^0$. Moreover,

$$\|z^{AB}(t) - z^0\| \le \|z^B(t) - z^0\| + \|z^{AB}(t) - z^B(t)\| < 2\|z^B(t) - z^0\| \le 2t$$

which tends to zero as t tends to zero. Thus $z^{AB}(t) \to z^0$ as t tends to zero, and the sets A and B can not be locally separated at z^0. □

Recall that X and Y are Banach spaces and $\varphi : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous function, $L : Y \longrightarrow X$ a bounded linear operator and

$$S := \{(Ly, y) \in X \times Y \ : \ y \in Y\}$$

be a closed linear subspace of $X \times Y$. Let us consider the optimization problem

$$\varphi(x, y) \to \min \quad \text{subject to} \quad (x, y) \in S \tag{4.1}$$

Let the following assumptions hold true:

(A1) "variational condition": there exist a positive real $\delta > 0$, an uniform tangent set D that is intersection of a cone and a closed ball centered at the origin, and a "correcting set" $U \subset X \times Y \times \mathbb{R}$ (having the appearance $U = (U_X, U_Y, U_\mathbb{R})$) such that for each $t \in [0, \delta]$ we have

$$epi\ \varphi \cap \left((\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \delta\bar{\mathbf{B}}\right) + t\left(\bar{\mathbf{B}}_\mathbf{X}, \mathbf{0}, 0\right) \subset epi\ \varphi + tU + tD .$$

(A2) "measure of noncompactness condition": The set $U_\mathbb{R} \subset \mathbb{R}$ is bounded and

$$\mu\left(U_X - L(U_Y)\right) < 1$$

where μ denotes the ball measure of noncompactness (in X).

Theorem 4.3 (Lagrange multiplier rule). *Let $(\bar{x}, \bar{y})$ be a solution of the problem (4.1). Let the Assumptions (A1) and (A2) hold true. Then for each $\varepsilon > 0$ there exists $\eta > 0$ such that for every closed convex cone $C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$ which is contained in the cone $D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$ there exists a triple $(\xi_1, \xi_2, \xi_3) \in X^* \times Y^* \times \mathbb{R}$ such that*

(i) $\|(\xi_1, \xi_2, \xi_3)\| = 1$, *hence it is not trivial;*
(ii) $\xi_3 \geq 0$*;*
(iii) $\langle \xi_1, u\rangle + \langle \xi_2, v\rangle = 0$ *for every* $(u, v) \in S$*;*
(iv) $\langle \xi_1, u\rangle + \langle \xi_2, v\rangle + \xi_3 w \geq 0$ *for every* $(u, v, w) \in C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$*;*
(v) $\langle \xi_1, u\rangle + \langle \xi_2, v\rangle + \xi_3 w \geq -\varepsilon$ *for every* $(u, v, w) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap \bar{\mathbf{B}}$.

Proof. Let us assume that there exists $R_1 > 0$ and $\eta_1 > 0$ such that the set

$$D_{\eta_1}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R_1\ \bar{\mathbf{B}} - S \times (-\infty, 0]$$

is dense in the closed unit ball $\bar{\mathbf{B}}$ and $\eta_1 > 0$ is so small that

$$\eta_1 < \frac{1}{(1-q)^{-1}(1 + \|L\|)(1 + M + \|L\|M + q)(1 + (1 + \|L\|)R_1)},$$

where $M := 2\sup\{\|z\| :\ z \in U \cup D\} + 1$. Applying Theorem 3.2, we obtain that the sets *epi* φ and $S \times (-\infty, \varphi(\bar{x}, \bar{y})]$ are tangentially transversal at $(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$ with constants $\delta > 0$, $\alpha > 0$ and $\Omega > 0$.

If there exist $R_2 > 0$ and $\eta_2 > 0$ such that $\eta_2 > 0$ is so small that

$$\eta_2 \left(1 + \frac{\alpha}{2\Omega} + (1 + \|L\|)2R_2\right) < \frac{\alpha}{2\Omega}$$

and the set

$$D_{\eta_2}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R_2\ \bar{\mathbf{B}} - S \times (-\infty, 0]$$

is dense in the closed unit ball $\bar{\mathbf{B}}$, then according to Lemma 4.1, there exist

$$\tilde{p} = (\tilde{u}, \tilde{v}, \tilde{w}) \in D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ 2R\ \bar{\mathbf{B}}$$

and $\tilde{s} = (L(\tilde{q}), \tilde{q}, \tilde{r}) \in S \times (-\infty, 0]$ with unit norm such that

$$\|\tilde{p} - \tilde{s}\| < \frac{\alpha}{2\Omega} \text{ and } \|\tilde{u} - L(\tilde{v})\| < \frac{\alpha}{2\Omega}.$$

Applying Lemma 4.2, we obtain that the sets epi φ and $S \times (-\infty, \varphi(\bar{x}, \bar{y})]$ can not be locally separated at $(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$. But this contradicts the assumption that $(\bar{x}, \bar{y})$ is a solution of the problem (4.1).

The obtained contradiction shows that at least one of the above written assumptions is wrong. Since the nature of these assumptions is very similar, we derive the conclusion: For each $R > 0$ there exists $\eta > 0$, such that the set

$$D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R\ \bar{\mathbf{B}} - S \times (-\infty, 0]$$

is NOT dense in the closed unit ball $\bar{\mathbf{B}}$.

We fix $R := \frac{1}{\varepsilon}$ and $\eta > 0$ with the above property. Let $C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$ be an arbitrary closed convex cone contained in $D_\eta(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y}))$. It is straightforward that

$$C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R\ \bar{\mathbf{B}} - S \times (-\infty, 0]$$

is not dense in the closed unit ball $\bar{\mathbf{B}}$. Then there exist $(x_0, y_0, z_0) \in X \times Y \times \mathbb{R}$ and $\varepsilon > 0$ such that

$$Q := (x_0, y_0, z_0) + \varepsilon\bar{\mathbf{B}} \subset \bar{\mathbf{B}}$$

and

$$\Big(C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y})) + \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R\bar{\mathbf{B}} - S \times (-\infty, 0]\Big) \cap Q = \emptyset\,.$$

Clearly, P and Q can be strongly separated because Q has nonempty interior. Hence, there exists a non-zero triple $(\xi_1, \xi_2, \xi_3) \in X^* \times Y^* \times \mathbb{R}$ and a real α such that

$$\langle \xi_1, u_1 \rangle\ + \langle \xi_2, v_1 \rangle\ + \xi_3 w_1 \geq \alpha > \langle \xi_1, u_2 \rangle\ + \langle \xi_2, v_2 \rangle\ + \xi_3 w_2$$

for all $(u_1, v_1, z_1) \in P$ and $(u_2, v_2, w_2) \in Q$. Clearly, without loss of generality we may think that $\|(\xi_1, \xi_2, \xi_3)\| = 1$. Since $(\mathbf{0}, \mathbf{0}, 0)$ lies in P, we have that $\alpha \leq 0$. Hence,

$$\langle \xi_1, u' + d_1 - u'' \rangle + \langle \xi_2, v' + d_2 - v'' \rangle + \xi_3(w' + d_3 - w'') \geq \alpha \tag{4.2}$$

for all $(u', v', w') \in\ C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$, $(d_1, d_2, d_3) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ \bar{\mathbf{B}}$ and $(u'', v'', w'') \in S \times (-\infty, 0]$. By taking $d_1 = u' = u'' = \mathbf{0}$, $d_2 = v' = v'' = \mathbf{0}$, $d_3 = w' = 0$ and $w'' < 0$ we obtain that $\xi_3 \geq 0$. By taking $(u, v, w) \in C_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi((\bar{x}, \bar{y}))$ and $(d_1, d_2, d_3) = (u'', v'', w'') = \mathbf{0}_{X \times Y \times \mathbb{R}}$, we obtain (iv). We take an arbitrary element $(u, v, w) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ \bar{\mathbf{B}}$ and $(u', v', w') = (u'', v'', w'') = \mathbf{0}_{X \times Y \times \mathbb{R}}$. Then $R(u, v, w) \in \hat{T}_{epi\ \varphi}(\bar{x}, \bar{y}, \varphi(\bar{x}, \bar{y})) \cap\ R\ \bar{\mathbf{B}}$ and we obtain

$$\langle \xi_1, Ru \rangle\ + \langle \xi_2, Rv \rangle\ + \xi_3 Rw \geq \alpha \text{ hence } \langle \xi_1, u \rangle\ + \langle \xi_2, v \rangle\ + \xi_3 w \geq \alpha\varepsilon$$

which implies (v) because

$$\alpha \geq \langle \xi_1, x_0 \rangle\ + \langle \xi_2, y_0 \rangle\ + \xi_3 z_0 \geq -\|(\xi_1, \xi_2, \xi_3)\| \cdot \|(x_0, y_0, z_0)\| > -1.$$

By taking $(u, v, 0) \in S \times (-\infty, 0]$ and $(u', v', w') = (d_1, d_2, d_3) = \mathbf{0}_{X \times Y \times \mathbb{R}}$, we obtain

$$\langle \xi_1, u \rangle\ + \langle \xi_2, v \rangle\ \geq \alpha.$$

Because S is a linear space, the last inequality implies (iii). This completes the proof. $\square$

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