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OPTIMALITY CONDITIONS IN NONLINEAR VECTOR OPTIMIZATION WITH VARIABLE ORDERING STRUCTURES

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Abstract. This paper investigates nonlinear vector optimization problems with variable ordering structures used for the objectives and the constraints. For these problems new necessary and sufficient optimality conditions are presented without differentiability assumptions. This theory works with non-constant Lagrange multipliers, it does not need any constraint qualification, and the functions are not assumed to be convex. These optimality conditions are applied to so-called conditional vector optimization problems being characterized by constraints and/or objective functions, which are only valid under a certain condition.

1. Introduction

In vector optimization one minimizes or maximizes several objective functions subject to certain constraints. These problems are difficult to treat, if one uses variable ordering structures for the objectives and constraints. This approach utilizes variable ordering structures as introduced by Eichfelder [3, 4]. For these nonlinear problems we present new optimality conditions without differentiability assumptions. The presented necessary optimality conditions turn out to be even sufficient. This theory does not require convexity of the objective and constraint functions.

Optimality conditions in variable ordering structures using subdifferentials, derivatives, and coderivatives can already be found in [2, 4–6, 9, 13]. In contrast to the existing theory we formulate necessary optimality conditions in vector optimization without any constraint qualification. This point has already been made in a general duality theory [10, Section 5]. The reason for this fact is that necessary optimality conditions are obtained using a strict separation theorem, which opens up more topological possibilities than the standard separation theorem. The price one has to pay for this advantage is that the Lagrange multipliers are not constants (or linear functionals in an infinite dimensional setting) but functions of the variables. Therefore, these optimality conditions are much more complicated in practice.

The presented optimality conditions are applied to so-called conditional vector optimization problems. For these special problems constraints and/or objectives depend on a given condition. For instance, an inequality constraint is taken into account, if the preimage of the constraint function fulfills a certain condition. If this condition is not fulfilled, this constraint vanishes. In analogy, an objective function is taken into consideration, if the preimage of the objective function satisfies a given

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condition. Therefore, such an optimization problem is called a conditional vector optimization problem in this paper.

The class of conditional vector optimization problems is not large, but there are important applications leading to conditional nonlinear vector optimization problems. For instance, conditional vector optimization problems occur in structural optimization [1, Example 1] and image registration of medical data [12], [4, Subsection 1.3.1]; and Achtziger and Kanzow speak of vanishing constraints [1].

This paper is organized as follows: Section 2 gives the problem formulation and basic definitions. Optimality conditions for nonlinear vector optimization problems with variable orderings are presented in Section 3 as necessary and sufficient conditions. These conditions are also specialized to problems with only one objective function. In Section 4 these optimality conditions are applied to conditional vector optimization problems.

2. Problem formulation

For the problem investigated in this paper we have the following standard assumption.

Assumption 2.1. Let X, Y and Z be real linear spaces, and let $\hat{S}$ be a nonempty subset of X. Let $f : \hat{S} \to Y$ and $g : \hat{S} \to Z$ be given vector functions. Let $C_f : \hat{S} \rightrightarrows Y$ and $C_g : \hat{S} \rightrightarrows Z$ be set-valued maps so that for every $x \in \hat{S}$ the image sets $C_f(x)$ and $C_g(x)$ are convex cones in Y and Z, respectively. Let the constraint set

$$S := \{x \in \hat{S} \mid g(x) \in -C_g(x)\}$$

be nonempty.

The set-valued maps C_f and C_g define a variable ordering structure in the real linear spaces Y and Z, respectively. We use these variable ordering structures as introduced by Eichfelder [3, 4]. In the following we define the order relations $\leq_f$ and $\leq_g$, i.e. for arbitrary $x_1, x_2 \in \hat{S}$ we define

$$f(x_1) \leq_{C_f} f(x_2) \;:\Longleftrightarrow\; f(x_2) - f(x_1) \in C_f(x_1)$$

and

$$g(x_1) \leq_{C_g} g(x_2) \;:\Longleftrightarrow\; g(x_2) - g(x_1) \in C_g(x_1)$$

(for simplicity we only use this way for the definition of the variable ordering structure [4, p. 5]). For an arbitrary $x \in \hat{S}$ the constraint $g(x) \in -C_g(x)$ is then equivalent to the inequality constraint $g(x) \leq_{C_g} 0_Z$.

The use of the superset $\hat{S}$ has the advantage that we can also treat functions, which are not defined on the whole space X. Since $\hat{S}$ may also be a discrete set, discrete vector optimization problems are also covered by this problem class.

Under Assumption 2.1 we investigate the *vector optimization problem with variable orderings*

$$\begin{array}{c} \min_{C_f(x)} \; f(x) \\ \text{subject to} \\ g(x) \in -C_g(x) \\ x \in \hat{S}. \end{array} \tag{2.1}$$

Minimal solutions of this vector optimization problem with variable orderings are understood in the following sense.

Definition 2.2. Let Assumption 2.1 be satisfied. A feasible element $\bar{x} \in S$ is called a *minimal solution* of the vector optimization problem with variable orderings (2.1), if the image $f(\bar{x})$ is a nondominated element of the image set $f(S)$, i.e.

$$\nexists\, x \in S: \quad f(\bar{x}) \in \{f(x)\} + \big(C_f(x) \backslash \{0_Y\}\big). \tag{2.2}$$

For a better understanding of the condition (2.2) notice the equivalences (for an arbitrary $x \in S$)

$$\begin{aligned} f(\bar{x}) \in \{f(x)\} + \big(C_f(x) \backslash \{0_Y\}\big) \quad &\Leftrightarrow \quad f(\bar{x}) - f(x) \in C_f(x) \text{ and } f(x) \neq f(\bar{x}) \\ &\Leftrightarrow \quad f(x) \leq_{C_f} f(\bar{x}) \text{ and } f(x) \neq f(\bar{x}). \end{aligned}$$

Hence, the condition (2.2) is equivalent to the condition

$$\nexists\, x \in S: \quad f(x) \leq_{C_f} f(\bar{x}) \text{ and } f(x) \neq f(\bar{x}).$$

This condition is an extension of the well-known Edgeworth-Pareto optimality concept ([8, Definition 11.3]) to variable ordering structures ([4, Definition 2.7, (a)]). In order to have a unified approach to constraints and objectives we restrict ourselves to the concept of nondominated elements (formally, the notion of minimal elements with respect to an ordering map (see [4, Definition 2.7, (b)]) could also be used).

It is well-known that there are various applications which lead to a vector optimization problem with variable orderings (e.g. compare [4]). For instance, such problems may appear in image registration of medical data [4, Subsection 1.3.1].

3. Optimality conditions

In this section we present necessary and sufficient optimality conditions for the vector optimization problem with variable orderings (2.1). In general, the known standard approach of optimality conditions cannot be extended to variable orderings but a new theory is needed for this general case. We begin with a simple multiplier-free necessary optimality condition.

Lemma 3.1. *Let Assumption 2.1 be satisfied. If $\bar{x}$ is a minimal solution of the vector optimization problem with variable orderings (2.1), then we have*

$$(0_Y, 0_Z) \notin \bigcup_{\substack{x \in \hat{S} \\ f(x) \neq f(\bar{x})}} \left\{ \begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix} \right\} + C_f(x) \times C_g(x). \tag{3.1}$$

Proof. We prove this result by contraposition. Assume that the condition (3.1) is not true. Then there exist some $x \in \hat{S}$, $f(x) \neq f(\bar{x})$, and vectors $y(x) \in C_f(x)$ and $z(x) \in C_g(x)$ with

$$f(x) - f(\bar{x}) + y(x) = 0_Y \tag{3.2}$$

and

$$g(x) + z(x) = 0_Z. \tag{3.3}$$

The equation (3.3) implies

$$g(x) = -z(x) \in -C_g(x),$$

which means that x is feasible. The equation (3.2) implies

$$f(\bar{x}) = f(x) + y(x) \in \{f(x)\} + C_f(x)$$

and with $f(x) \neq f(\bar{x})$ we then get

$$f(\bar{x}) \in \{f(x)\} + \big(C_f(x) \backslash \{0_Y\}\big).$$

Hence, $\bar{x}$ is not a minimal solution of problem (2.1). □

Now we formulate a necessary optimality condition with an extension of the known theory of Lagrange multipliers.

Theorem 3.2. *Let Assumption 2.1 be satisfied, and in addition, let Y and Z be locally convex spaces. If $\bar{x}$ is a minimal solution of the vector optimization problem with variable orderings (2.1), then for every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$, $C_f(x) \neq Y$ and closed sets $C_f(x)$ and $C_g(x)$ there are continuous linear functionals $\ell_x^f \in C_f^*(x) \backslash \{0_{Y^*}\}$ and $\ell_x^g \in C_g^*(x)$ ($C_f^*(x)$ and $C_g^*(x)$ denote the dual cone in the topological dual space Y^* and Z^*, respectively) with*

$$\ell_x^f\big(f(x) - f(\bar{x})\big) + \ell_x^g\big(g(x)\big) > 0. \tag{3.4}$$

Proof. Let $\bar{x}$ be a minimal solution of the vector optimization problem with variable orderings (2.1), and choose an arbitrary $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$, $C_f(x) \neq Y$ and closed sets $C_f(x)$ and $C_g(x)$. By Lemma 3.1 we then get

$$(0_Y, 0_Z) \notin \left\{ \begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix} \right\} + C_f(x) \times C_g(x).$$

The product cone $C_f(x) \times C_g(x)$ is closed and convex. By a strict separation theorem (compare [8, Theorem 3.18]) there exist continuous linear functionals $\ell_x^f \in Y^*$ and $\ell_x^g \in Z^*$ with $\big(\ell_x^f, \ell_x^g\big) \neq \big(0_{Y^*}, 0_{Z^*}\big)$ and

$$\begin{aligned} \ell_x^f\big(f(x) - f(\bar{x})\big) + \ell_x^f\big(y(x)\big) + \ell_x^g\big(g(x)\big) + \ell_x^g\big(z(x)\big) > 0 \\ \text{for all } y(x) \in C_f(x) \text{ and } z(x) \in C_g(x). \end{aligned} \tag{3.5}$$

We now show that $\ell_x^f \in C_Y^*(x)$. If we choose $z(x) = 0_Z$, we obtain from the inequality (3.5)

$$\ell_x^f\big(y(x)\big) > -\ell_x^f\big(f(x) - f(\bar{x})\big) - \ell_x^g\big(g(x)\big) \text{ for all } y(x) \in C_f(x),$$

which means that $\ell_x^f\big(y(x)\big)$ has a lower bound for all $y(x) \in C_f(x)$. Since $C_f(x)$ is a cone, this implies

$$\ell_x^f\big(y(x)\big) \geq 0 \text{ for all } y(x) \in C_f(x),$$

i.e. $\ell_x^f \in C_f^*(x)$. With the same arguments one gets $\ell_x^g \in C_g^*(x)$.

If $\ell_x^f \neq 0_{Y^*}$, another part of the assertion is shown. Otherwise, if $\ell_x^f = 0_{Y^*}$, we conclude from the inequality (3.5)

$$\ell_x^g\big(g(x)\big) > 0.$$

Since $C_f(x) \neq Y$, by a strict separation theorem there exists some continuous linear functional $\bar{\ell}_x^f \in C_f^*(x)\backslash\{0_{Y^*}\}$ (compare the proof of [8, Lemma 3.21,(a), p. 78]). For a sufficiently small $\lambda_x > 0$ we then have

$$\lambda_x \bar{\ell}_x^f\big(f(x) - f(\bar{x})\big) > \underbrace{-\ell_x^g\big(g(x)\big)}_{<0},$$

or

$$\lambda_x \bar{\ell}_x^f\big(f(x) - f(\bar{x})\big) + \ell_x^g\big(g(x)\big) > 0.$$

Consequently, the inequality (3.4) is shown with the continuous linear functional $\lambda_x \bar{\ell}_x^f \in C_f^*(x)\backslash\{0_{Y^*}\}$ instead of ℓ_x^f. □

A strict separation theorem is the fundamental tool for the proof of Theorem 3.2. In duality theory such an approach was already used in [10, Section 5].

Remark 3.3. There are significant differences between the necessary optimality condition of vector optimization problems with variable orderings given in Theorem 3.2 and those in standard nonlinear optimization:

(a) Theorem 3.2 gives necessary optimality conditions without convexity assumptions on f and g.

(b) These new necessary optimality conditions are shown without any constraint qualification (CQ) because the inequality (3.4) is strict, which opens up topological possibilities.

(c) The “Lagrange multipliers” depend on $x \in \hat{S}$, $f(x) \neq f(\bar{x})$, and these are functions of x. Since the union of translated convex cones in (3.1) is generally nonconvex, a separation of $(0_Y, 0_Z)$ and this union of sets can be achieved by using a nonlinear separating functional.

(d) The nonlinear “Lagrange multipliers” make the theory much more complicated in practice.

In general, the number of Lagrange multipliers may be infinite. But for numerical methods one is actually interested in a finite number of multipliers. So, one could start an iteration process with a small number of multipliers and then one could increase this number from step to step. In this case the inequality (3.4) could be helpful from a numerical point of view.

Next, we specialize Theorem 3.2 to the case that the objective function is real-valued with the standard ordering in $\mathbb{R}$.

Corollary 3.4. *Let $\hat{S}$ be a nonempty subset of a real linear space X, and let Z be a locally convex space. Let $f : \hat{S} \to \mathbb{R}$ be a real-valued function and let $g : \hat{S} \to Z$ be a vector function. Let $C_g : \hat{S} \rightrightarrows Z$ be a set-valued map so that for every $x \in \hat{S}$ the image set $C_g(x)$ is a closed convex cone in Z. If $\bar{x}$ is a minimal solution of the optimization problem with variable ordering*

$$\begin{gathered} \min\ f(x) \\ \textit{subject to} \\ g(x) \in -C_g(x) \\ x \in \hat{S}, \end{gathered} \tag{3.6}$$

then for every $x \in \hat{S}$ *with* $f(x) \neq f(\bar{x})$ *and a closed set* $C_g(x)$ *there is a continuous linear functional* $\ell_x^g \in C_g^*(x)$ *with*

$$f(\bar{x}) < f(x) + \ell_x^g\big(g(x)\big). \tag{3.7}$$

Proof. For the application of Theorem 3.2 we set $Y := \mathbb{R}$ and $C_f(x) := \mathbb{R}_+$ for all $x \in \hat{S}$. If $\bar{x}$ is a minimal solution of the optimization problem with variable ordering (3.6), then for every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ and a closed set $C_g(x)$ there are a positive real number μ_x and a continuous linear functional $\ell_x^g \in C_g^*(x)$ with

$$\mu_x\big(f(x) - f(\bar{x})\big) + \ell_x^g\big(g(x)\big) > 0. \tag{3.8}$$

If we set $\bar{\ell}_x^g := \frac{1}{\mu_x}\ell_x^g \in C_g^*(x)$, the inequality (3.8) implies

$$f(\bar{x}) < f(x) + \bar{\ell}_x^g\big(g(x)\big),$$

which has to be shown. □

Corollary 3.4 can also be proved by considering the cases that $x \in \hat{S}$ is feasible, i.e. $g(x) \in -C_g(x)$, or not.

If the optimization problem with variable orderings (2.1) is a discrete problem with a finite number of elements of the set $\hat{S}$, the necessary optimality condition can be simplified.

Corollary 3.5. *Let Assumption 2.1 be satisfied and, in addition, let* Y *and* Z *be locally convex spaces, let* $\hat{S} := \{x^1, \ldots, x^n\}$ *with* $n \in \mathbb{N}$ *and* $x^1, \ldots, x^n \in X$ *be given, and for every* $j \in \{1, \ldots, n\}$ *let* $C_f(x^j) \neq Y$ *and let* $C_f(x^j)$ *and* $C_g(x^j)$ *be closed sets. If* $\bar{x}$ *is a minimal solution of the optimization problem with variable orderings (2.1), then for every* x^j, $j \in \{1, \ldots, n\}$, *with* $f(x^j) \neq f(\bar{x})$ *there are continuous linear functionals* $\ell_{x^j}^f \in C_f^*(x^j)\backslash\{0_{Y^*}\}$ *and* $\ell_{x^j}^g \in C_g^*(x^j)$ *so that the inequality*

$$\min_{\substack{j\in\{1,\ldots,n\}\\ f(x^j)\neq f(\bar{x})}} \Big\{\ell_{x^j}^f\big(f(x^j) - f(\bar{x})\big) + \ell_{x^j}^g\big(g(x^j)\big)\Big\} > 0 \tag{3.9}$$

is fulfilled.

Proof. This result immediately follows from Theorem 3.2 because in this special case the inequality (3.9) is equivalent to the inequalities (3.4) for every x^j, $j \in \{1, \ldots, n\}$, with $f(x^j) \neq f(\bar{x})$. □

This corollary shows that we get a finite number of Lagrange multipliers in this special discrete case. Hence, this necessary optimality condition is much simpler than in the general continuous case. The following theorem shows under special assumptions that even in the continuous case Lagrange multipliers remain constant within a small neigborhood.

Theorem 3.6. *Let Assumption 2.1 be satisfied and, in addition, let* X *be a real normed space, let* Y *and* Z *be real reflexive Banach spaces, and for every* $x \in \hat{S}$ *let* $C_f(x) \neq Y$ *and let the sets* $C_f(x)$ *and* $C_g(x)$ *be closed. Let* $\bar{x}$ *be a minimal solution of the optimization problem with variable orderings (2.1). Let* $x \in \hat{S}$ *with*

$f(x) \neq f(\bar{x})$ be arbitrarily chosen and let there exist a closed ball $B(x,\delta)$ around x with radius $\delta > 0$ so that

$$C_f(\tilde{x}) \times C_g(\tilde{x}) = C_f(x) \times C_g(x) \text{ for all } \tilde{x} \in B(x,\delta). \tag{3.10}$$

Let (f,g) be locally Lipschitz continuous at x. Then there are continuous linear functionals $\ell^f \in C_f^*(x)\backslash\{0_{Y^*}\}$, $\ell^g \in C_g^*(x)$ and a closed ball $B(x,\varepsilon)$ around x with radius $\varepsilon > 0$ so that the inequality (3.4) holds with the continuous linear functionals

$$\ell_{\tilde{x}}^f = \ell^f \text{ and } \ell_{\tilde{x}}^g = \ell^g \text{ for all } \tilde{x} \in B(x,\varepsilon). \tag{3.11}$$

Proof. Let $\bar{x}$ be a minimal solution of the optimization problem with variable orderings (2.1), and choose an arbitrary $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ so that the equality (3.10) holds. By Lemma 3.1 we obtain

$$(0_Y, 0_Z) \notin \left\{ \begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix} \right\} + C_f(x) \times C_g(x).$$

Because of the closedness of the product cone $C_f(x) \times C_g(x)$ there is some $\mu > 0$ with

$$(0_Y, 0_Z) \notin B\left(\begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix}, \mu \right) + C_f(x) \times C_g(x).$$

The ball in this condition is weakly compact and the product cone $C_f(x) \times C_g(x)$ is weakly closed. Consequently, the algebraic sum of these two sets is weakly closed and closed as well because this set is convex. Hence, the origin $(0_Y, 0_Z)$ and the set $B\left(\begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix}, \mu \right) + C_f(x) \times C_g(x)$ can be strictly separated by a nonzero continuous linear functional $(\ell^f, \ell^g) \in C_f^*(x) \times C_g^*(x)$. With an argument used in the proof of Theorem 3.2 one can assume that $\ell^f \neq 0_{Y^*}$. Since (f,g) is locally Lipschitz continuous at x, there are some $\alpha, L > 0$ with

$$\left\| \begin{pmatrix} f(\tilde{x}) - f(x) \\ g(\tilde{x}) - g(x) \end{pmatrix} \right\|_{Y \times Z} \leq L \|\tilde{x} - x\|_X \text{ for all } \tilde{x} \in B(x,\alpha).$$

And this implies

$$\left\| \begin{pmatrix} f(\tilde{x}) - f(\bar{x}) \\ g(\tilde{x}) \end{pmatrix} - \begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix} \right\|_{Y \times Z} \leq L \|\tilde{x} - x\|_X \text{ for all } \tilde{x} \in B(x,\alpha).$$

With the equation (3.10) there is some $\varepsilon \in (0, \delta]$ so that

$$\begin{pmatrix} f(\tilde{x}) - f(\bar{x}) \\ g(\tilde{x}) \end{pmatrix} \in B\left(\begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix}, \mu \right) \text{ for all } \tilde{x} \in B(x,\varepsilon).$$

Hence we obtain

$$\begin{aligned} &\left\{ \begin{pmatrix} f(\tilde{x}) - f(\bar{x}) \\ g(\tilde{x}) \end{pmatrix} \right\} + \underbrace{C_f(\tilde{x})}_{=C_f(x)} \times \underbrace{C_g(\tilde{x})}_{=C_g(x)} \\ \subset\ & B\left(\begin{pmatrix} f(x) - f(\bar{x}) \\ g(x) \end{pmatrix}, \mu \right) + C_f(x) \times C_g(x) \text{ for all } \tilde{x} \in B(x,\varepsilon). \end{aligned}$$

Since the continuous linear functional (ℓ^f, ℓ^g) strictly separates the origin and the set on the right hand side, it also strictly separates the origin and the set on the left hand side. This proves the equation (3.11). □

Finally we show that the necessary optimality condition of Theorem 3.2 is also sufficient under weaker assumptions. This sufficient optimality condition does not need any topological assumptions.

Theorem 3.7. *Let Assumption 2.1 be satisfied. Let $\bar{x}$ be a feasible point of the optimization problem with variable orderings (2.1). For every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ let there exist linear functionals $\ell_x^f \in C_f'(x)$ and $\ell_x^g \in C_g'(x)$ so that the inequality (3.4) is satisfied ($C_f'(x)$ and $C_g'(x)$ denote the dual cone in the algebraic dual space Y' and Z', respectively). Then $\bar{x}$ is a minimal solution of the optimization problem with variable orderings (2.1).*

Proof. Let some $x \in S$ with $f(x) \neq f(\bar{x})$ be arbitrarily chosen. Then the inequality (3.4) implies

$$\ell_x^f\big(f(\bar{x}) - f(x)\big) < \ell_x^g\big(\underbrace{g(x)}_{\in -C_g(x)}\big) \leq 0. \tag{3.12}$$

Assume that

$$f(\bar{x}) \in \{f(x)\} + \big(C_f(x)\backslash\{0_Y\}\big),$$

i.e.

$$f(\bar{x}) - f(x) \in C_f(x)\backslash\{0_Y\}.$$

Because of $\ell_x^f \in C_f'(x)$ we then obtain

$$\ell_x^f\big(f(\bar{x}) - f(x)\big) \geq 0,$$

which contradicts the inequality (3.12). Therefore, we conclude

$$f(\bar{x}) \notin \{f(x)\} + \big(C_f(x)\backslash\{0_Y\}\big).$$

Since $x \in S$ with $f(x) \neq f(\bar{x})$ is arbitrarily chosen, the element $\bar{x} \in S$ is a minimal solution of the optimization problem with variable orderings (2.1). □

In analogy to Corollary 3.4 we specialize Theorem 3.7 to the simpler case of a real-valued objective function with the standard ordering in $\mathbb{R}$.

Corollary 3.8. *Let Assumption 2.1 be satisfied, and let $f : \hat{S} \to \mathbb{R}$ be a real-valued function (i.e. $Y := \mathbb{R}$). Let $\bar{x} \in S$ be a feasible point of the optimization problem with variable orderings (2.1). For every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ let there exist a linear functional $\ell_x^g \in C_g'(x)$ so that the inequality (3.7) is satisfied. Then $\bar{x}$ is a minimal solution of the optimization problem with variable orderings (2.1).*

Proof. It is evident that the condition (3.7) can also be written as

$$f(x) - f(\bar{x}) + \ell_x^g\big(g(x)\big) > 0$$

and then Theorem 3.7 is applicable. □

4. Application to conditional vector optimization problems

The optimality conditions presented in the previous section are now applied to vector optimization problems with objectives and constraints, which are only valid under a condition. If this condition is not fulfilled, then the corresponding constraint or objective vanishes. We call this type of problems *conditional vector optimization problems.*

For the problem investigated in this section we have the following standard assumption.

Assumption 4.1. Let X be a real linear space, and for $k, m \in \mathbb{N}$, let $F_1, \ldots, F_k$, $G_1, \ldots, G_m$ and $\hat{S}$ be nonempty subsets of X. Let $Y_1, \ldots, Y_k$, $Z_1, \ldots, Z_m$ be real linear spaces, and let $f_1 : \hat{S} \to Y_1, \ldots,\ f_k : \hat{S} \to Y_k,\ g_1 : \hat{S} \to Z_1, \ldots,\ g_m : \hat{S} \to Z_m$ be given vector functions. Let $K_{Y_1}, \ldots, K_{Y_k}, K_{Z_1}, \ldots, K_{Z_m}$ be convex cones in $Y_1, \ldots, Y_k$ and $Z_1, \ldots, Z_m$, respectively. Let the constraint set

$$S \;:=\; \big\{x \in \hat{S} \mid g_1(x) \in -K_{Z_1},\ \text{iff } x \in G_1 \\ \vdots \\ g_m(x) \in -K_{Z_m},\ \text{iff } x \in G_m\big\}$$

be nonempty.

Under this assumption we formulate the conditional vector optimization problem in a first and more conceptual form

$$\begin{array}{c} \min \begin{pmatrix} f_1(x),\ \text{iff } x \in F_1 \\ \vdots \\ f_k(x),\ \text{iff } x \in F_k \end{pmatrix} \\ \text{subject to} \\ g_1(x) \in -K_{Z_1},\ \text{iff } x \in G_1 \\ \vdots \\ g_m(x) \in -K_{Z_m},\ \text{iff } x \in G_m \\ x \in \hat{S}. \end{array} \tag{4.1}$$

This is not a standard vector optimization problem because one or more objectives/constraints may vanish at some $x \in \hat{S}$. Since there is no unique image space of the objective map, the standard optimality notions in vector optimization cannot be used.

Notice that for $\hat{S} \subset G_i$ (for any $i \in \{1, \ldots, m\}$) the condition $x \in G_i$ can be dropped and then the conditional inequality constraint in problem (4.1) reduces to a standard inequality constraint, i.e.

$$g_i(x) \in -K_{Z_i}.$$

Hence, problem (4.1) may contain a mixture of standard inequality constraints and conditional inequality constraints. In the same way an objective f_i (for an arbitrary $i \in \{1, \ldots, k\}$) may be fixed without considering a condition.

Problems with vanishing constraints can be found in structural optimization [1, Example 1]. Achtziger and Kanzow have been the first who investigated optimality conditions and constraint qualifications for this problem class. Mathematical programs with switching constraints are also related to problems with vanishing constraints, e.g. see Mehlitz [11] and the references therein. Conditional objectives appear in image registration of medical data [4, Subsection 1.3.1]. Wacker [12] (see also [14]) was the first who investigated such problems with a weighting approach where the nonnegative weights may change during the iteration process. Since the weight of an objective may also be zero, this means that the corresponding objective vanishes. Based on these concrete applications in medical technology Eichfelder [3,4] has developed a theory of variable ordering structures in vector optimization. Variable ordering structures are the right tool for the investigation of problem (4.1).

If the sets $G_1, \ldots, G_m$ are described by inequalities, then the conditional constraints in (4.1) can be transformed to constraints of a mathematical program with vanishing constraints (MPVC), which is closely related to a mathematical program with equilibrium constraints (MPEC) (for details see [1]). In this paper we do not assume that the sets $G_1, \ldots, G_m$ have a special structure. This is possible because we work with variable ordering structures.

Using the sets $G_1, \ldots, G_m$ and the convex cones $K_{Z_1}, \ldots, K_{Z_m}$ for every $i \in \{1, \ldots, m\}$ we define the set-valued map $C_{g_i} : \hat{S} \rightrightarrows Z_i$ with

$$C_{g_i}(x) := \left\{ \begin{array}{ll} K_{Z_i}, & \text{if } x \in G_i \\ Z_i, & \text{if } x \notin G_i \end{array} \right\} \text{ for all } x \in \hat{S}. \tag{4.2}$$

For every $i \in \{1, \ldots, m\}$ and every $x \in \hat{S}$ the set $C_{g_i}(x)$ is a convex cone. The map C_{g_i} defines a variable ordering structure in Z_i. For arbitrary $i \in \{1, \ldots, m\}$ and $x \in \hat{S}$ we then have the equivalence

$$g_i(x) \in -K_{Z_i}, \text{ iff } x \in G_i \iff g_i(x) \in -C_{g_i}(x)$$

So, the constraints in problem (4.1) can be written as normal inequality constraints with respect to a variable ordering. In other words, the complexity of the constraints has been transformed to the ordering structure.

If K'_{Z_i} for any $i \in \{1, \ldots, m\}$ denotes the dual cone of K_{Z_i} in the algebraic dual space Z'_i, then for any $x \in \hat{S}$ the dual cone $C'_{g_i}(x)$ can be written as

$$C'_{g_i}(x) := \left\{ \begin{array}{ll} K'_{Z_i}, & \text{if } x \in G_i \\ \{0_{Z'_i}\}, & \text{if } x \notin G_i. \end{array} \right.$$

The objectives in problem (4.1) can be treated in a similar way. For every $i \in \{1, \ldots, k\}$ we define the set-valued map $C_{f_i} : \hat{S} \rightrightarrows Y_i$ by

$$C_{f_i}(x) := \left\{ \begin{array}{ll} K_{Y_i}, & \text{if } x \in F_i \\ Y_i, & \text{if } x \notin F_i \end{array} \right\} \text{ for all } x \in \hat{S}. \tag{4.3}$$

Finally we extend Assumption 4.1.

Assumption 4.2. Let Assumption 4.1 be satisfied, and let the set-valued maps $C_{f_1}, \ldots, C_{f_k}, C_{g_1}, \ldots, C_{g_m}$ be defined by (4.3) and (4.2), respectively. For every feasible $x \in S$ let at least one objective function remain in the list of objectives (i.e., objectives do not vanish all together). Let $K_{Y_i} \neq Y_i$ for all $i \in \{1, \ldots, k\}$.

Under this extended assumption we now ask for minimal solutions of the *conditional vector optimization problem*

$$\begin{aligned} &\min_{C_{f_1}(x)\times\ldots\times C_{f_k}(x)} \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix} \\ &\text{subject to} \\ &g_1(x) \in -C_{g_1}(x) \\ &\vdots \\ &g_m(x) \in -C_{g_m}(x) \\ &x \in \hat{S}. \end{aligned} \tag{4.4}$$

Although constraints and objectives do not vanish in problem (4.4), the complexity of problem (4.1) is now hidden in the variable ordering structures.

The conditional vector optimization problem (4.4) is a special vector optimization problem with variable orderings as given in (2.1). Here we set the product spaces $Y := Y_1 \times \ldots \times Y_k$ and $Z := Z_1 \times \ldots \times Z_m$, we define the objective vector function $f : \hat{S} \to Y$ by $f := (f_1, \ldots, f_k)$ and the constraint vector function $g : \hat{S} \to Z$ by $g := (g_1, \ldots, g_m)$, and the set-valued maps $C_f : \hat{S} \rightrightarrows Y$ and $C_g : \hat{S} \rightrightarrows Z$ are given by

$$C_f(x) := C_{f_1}(x) \times \ldots \times C_{f_k}(x) \text{ for all } x \in \hat{S}$$

and

$$C_g(x) := C_{g_1}(x) \times \ldots \times C_{g_m}(x) \text{ for all } x \in \hat{S}.$$

The necessary optimality condition given in Theorem 3.2 can be easily applied to the conditional vector optimization problem (4.4).

Corollary 4.3. *Let Assumption 4.2 be satisfied, and in addition, let $Y_1, \ldots, Y_k$, $Z_1, \ldots, Z_m$ be locally convex spaces and let $K_{Y_1}, \ldots, K_{Y_k}, K_{Z_1}, \ldots, K_{Z_m}$ be closed. If $\bar{x}$ is a minimal solution of the conditional vector optimization problem (4.4), then for every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ there are continuous linear functionals $\ell_x^{f_1} \in C_{f_1}^*(x), \ldots, \ell_x^{f_k} \in C_{f_k}^*(x), \ell_x^{g_1} \in C_{g_1}^*(x), \ldots, \ell_x^{g_m} \in C_{g_m}^*(x)$ with $\left(\ell_x^{f_1}, \ldots, \ell_x^{f_k}\right) \neq \left(0_{Y_1^*}, \ldots, 0_{Y_k^*}\right)$ and*

$$\sum_{i=1}^{k} \ell_x^{f_i}\big(f_i(x) - f_i(\bar{x})\big) + \sum_{i=1}^{m} \ell_x^{g_i}\big(g_i(x)\big) > 0. \tag{4.5}$$

Moreover, for $x \in \hat{S} \backslash F_i$ (with $i \in \{1, \ldots, k\}$) we have $\ell_x^{f_i} = 0_{Y_i^*}$ and for $x \in \hat{S} \cap F_i$ it holds $\ell_x^{f_i} \in K_{Y_i}^*$; and for $x \in \hat{S} \backslash G_i$ (with $i \in \{1, \ldots, m\}$) we have $\ell_x^{g_i} = 0_{Z_i^*}$ and for $x \in \hat{S} \cap G_i$ it holds $\ell_x^{g_i} \in K_{Z_i}^*$.

Proof. The first part of this corollary follows from Theorem 3.2, if we notice that the continuous linear functional $\ell_x^f \in C_f^*(x)\backslash\{0_{Y^*}\}$ can be written as

$$\ell_x^f(y_1,\dots,y_k) = \sum_{i=1}^{k} \ell_x^{f_i}(y_i) \text{ for all } (y_1,\dots,y_k) \in Y_1 \times \ldots \times Y_k$$

with appropriate $\ell_x^{f_1} \in C_{f_1}^*(x),\dots,\ell_x^{f_k} \in C_{f_k}^*(x)$ with $\big(\ell_x^{f_1},\dots,\ell_x^{f_k}\big) \neq \big(0_{Y_1^*},\dots,0_{Y_k^*}\big)$. In analogy, the Lagrange multipliers of the constraints are treated.
It remains to prove the additional part of the assertion. If $x \in \hat{S}\backslash F_i$ for $i \in \{1,\dots,k\}$, we have $C_{f_i}(x) = Y_i$ implying $C_{f_i}^*(x) = \{0_{Y_i^*}\}$. Hence, we have $\ell_x^{f_i} = 0_{Y_i^*}$. For the case $x \in \hat{S} \cap F_i$ we conclude $C_{f_i}(x) = K_{Y_i}$ and $\ell_x^{f_i} \in K_{Y_i}^*$. In analogy, one gets the corresponding result for the constraints. □

Corollary 3.4 already presents an optimality condition in the case of only one real-valued objective function. This result can be easily specialized to a conditional optimization problem with only one real-valued objective function with the standard ordering in $\mathbb{R}$.

Corollary 4.4. *Let X be a real linear space, and for $m \in \mathbb{N}$ let $G_1,\dots,G_m$ and $\hat{S}$ be nonempty subsets of X. Let $Z_1,\dots,Z_m$ be locally convex spaces, and let $f : \hat{S} \to \mathbb{R}$, $g_1 : \hat{S} \to Z_1,\dots,$ $g_m : \hat{S} \to Z_m$ be given functions. Let $K_{Z_1},\dots,K_{Z_m}$ be closed convex cones in $Z_1,\dots,Z_m$, and let $C_{g_1},\dots,C_{g_m}$ be defined by (4.2). If $\bar{x}$ is a minimal solution of the conditional optimization problem*

$$\begin{gathered} \min\ f(x) \\ \textit{subject to} \\ g_1(x) \in -C_{g_1}(x) \\ \vdots \\ g_m(x) \in -C_{g_m}(x) \\ x \in \hat{S}, \end{gathered}$$

then for every $x \in \hat{S}$ with $f(x) \neq f(\bar{x})$ there are continuous linear functionals $\ell_x^{g_1} \in C_{g_1}^*(x),\dots,\ell_x^{g_m} \in C_{g_m}^*(x)$ with

$$f(\bar{x}) < f(x) + \sum_{i=1}^{m} \ell_x^{g_i}\big(g_i(x)\big).$$

Moreover, for $x \in \hat{S}\backslash G_i$ (with $i \in \{1,\dots,m\}$) we have $\ell_x^{g_i} = 0_{Z_i^*}$ and for $x \in \hat{S} \cap G_i$ it holds $\ell_x^{g_i} \in K_{Z_i}^*$.

Now we apply Corollary 4.4 to a very simple example.

Example 4.5. We investigate the conditional integer optimization problem

$$\begin{gathered} \min\ \sin x \\ \text{subject to} \\ 1 \le x \le 2\pi \\ \cos x \ge 0.4,\ \text{iff } x \in (\pi, 2\pi) \\ x \in \mathbb{Z}. \end{gathered}$$

Here we set $\hat{S} := \mathbb{Z}$, $G_3 := (\pi, 2\pi)$,

$$f(x) := \sin x \text{ for all } x \in \mathbb{Z},$$

$$g_1(x) := 1 - x \text{ and } C_{g_1}(x) := \mathbb{R}_+ \text{ for all } x \in \mathbb{Z},$$

$$g_2(x) := x - 2\pi \text{ and } C_{g_2}(x) := \mathbb{R}_+ \text{ for all } x \in \mathbb{Z},$$

and

$$g_3(x) := 0.4 - \cos x \text{ and } C_{g_3}(x) := \left\{ \begin{array}{ll} \mathbb{R}_+, & \text{if } x \in (\pi, 2\pi) \\ \mathbb{R}, & \text{if } x \notin (\pi, 2\pi) \end{array} \right\} \text{ for all } x \in \mathbb{Z}.$$

It is obvious that the set S of feasible points is given as $S := \{1, 2, 3, 6\}$, and $\bar{x} := 6$ is the only minimal solution of this problem (compare Figure 1). By Corollary 4.4,

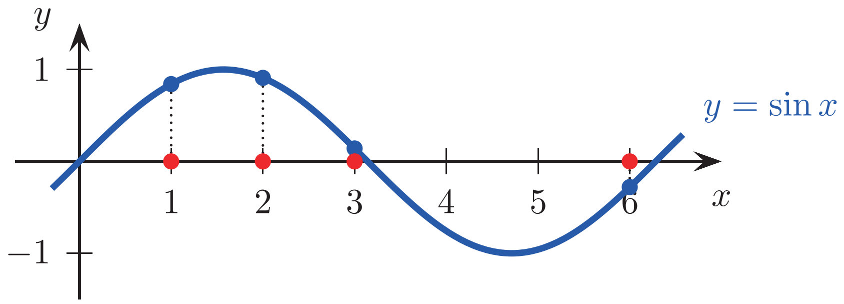

FIGURE 1. Illustration of the objective function values at the four feasible points in Example 4.5.

for every $x \in \mathbb{Z}$ with $\sin x \neq \sin 6$ there exist nonnegative real numbers $\ell_x^{g_1}$, $\ell_x^{g_2}$ and $\ell_x^{g_3}$ with $\ell_x^{g_3} = 0$, if $x \notin (\pi, 2\pi)$, and

$$\begin{aligned} \sin 6 &< \sin x + \ell_x^{g_1} \cdot g_1(x) + \ell_x^{g_2} \cdot g_2(x) + \ell_x^{g_3} \cdot g_3(x) \\ &< \sin x + \ell_x^{g_1} \cdot (1 - x) + \ell_x^{g_2} \cdot (x - 2\pi) + \ell_x^{g_3} \cdot (0.4 - \cos x). \end{aligned} \tag{4.6}$$

This inequality is fulfilled, if for $x \in \mathbb{Z}$ with $x < 1$ one chooses the multipliers $\ell_x^{g_1} := 1$ and $\ell_x^{g_2} := \ell_x^{g_3} := 0$. However, if $x \in \mathbb{Z}$ with $x > 2\pi$, then we set $\ell_x^{g_2} := 1$ and $\ell_x^{g_1} := \ell_x^{g_3} := 0$ and the inequality (4.6) is fulfilled. If one checks the sign of the expression on the right side in (4.6) for the points $x = 1, 2, 3$ with the multipliers $\ell_x^{g_1} := \ell_x^{g_2} := \ell_x^{g_3} := 0$, we see that the necessary optimality condition given in Corollary 4.4 is satisfied. For the remaining points $x = 4$ and $x = 5$ the inequality (4.6) is satisfied for $\ell_x^{g_1} := \ell_x^{g_2} := 0$, $\ell_x^{g_3} := 1$ and $\ell_x^{g_1} := \ell_x^{g_2} := 0$, $\ell_x^{g_3} := 6$, respectively. Hence, the necessary optimality condition in Corollary 4.4 is fulfilled.

In Example 4.5 it is shown that the inequality (4.6) is fulfilled for all $x \in \mathbb{Z}$ with $\sin x \neq \sin 6$. With Corollary 3.8 we then get that $\bar{x} := 6$ is a minimal solution of the conditional integer optimization problem in Example 4.5.

Conclusion

Vector optimization problems with variable orderings constitute an interesting class of vector optimization problems, which are hard to handle. In the case of discrete vector optimization problems with variable orderings and finitely many feasible points, one has only finitely many Lagrange multipliers and it would be interesting to see how standard Karush-Kuhn-Tucker (KKT) conditions look like. For problems with continuous variables Lagrange multipliers are not constants but functions. Therefore, KKT conditions are much more complicated to formulate. Such a KKT theory is still an open problem.

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