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INVERSE PROBLEM OF ESTIMATING THE STOCHASTIC FLEXURAL RIGIDITY IN FOURTH-ORDER MODELS

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Abstract. This work focuses on the inverse problem of estimating a stochastic parameter in fourth-order partial differential equations with random data. In the setting of stochastic Sobolev spaces, we establish the Lipschitz continuity of the solution map and give a new derivative characterization. We investigate the inverse problem in a stochastic optimization framework using the output least-squares (OLS) functional and a new energy least-squares (ELS) functional. We develop a regularization framework and give existence results for the regularized stochastic optimization problems. We also prove the smoothness and the convexity of the ELS objective functional. For the OLS-based stochastic optimization problem, we develop an adjoint approach to compute the derivative of the OLS-functional. In the finite-dimensional noise setting, we give a parameterization of the inverse problem. We develop a computational framework using the stochastic-Galerkin discretization scheme and derive explicit discrete formulas for the considered objective functionals and their gradient. We provide computational results to illustrate the feasibility and efficacy of the developed inversion framework.

1. Introduction

Our focus is on the inverse problem of estimating a random coefficient in the stochastic fourth-order boundary value problem (BVP). Assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space, and $D \subset \mathbb{R}^\ell$ is a bounded domain and ∂D is its sufficiently smooth boundary. Given random fields $q : \Omega \times D \to \mathbb{R}$ and $f : \Omega \times D \to \mathbb{R}$, the direct problem in this work consists of finding a random field $u : \Omega \times D \to \mathbb{R}$ that almost surely satisfies the following BVP:

$$\Delta(q(\omega, x)\Delta u(\omega, x)) = f(\omega, x), \quad \text{in } D, \tag{1.1a}$$

$$u(\omega, x) = 0, \quad \text{on } \partial D, \tag{1.1b}$$

$$\partial_n u(\omega, x) = 0, \quad \text{on } \partial D, \tag{1.1c}$$

where Δ is the Laplace operator and $\partial_n u$ is the normal derivative.

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Fourth-order BVPs, such as (1.1), model the pure bending of a Kirchhoff plate occupying the region D and have been extensively explored in the deterministic setting, see [5, 17]. The parameter q is the flexural rigidity and the solution u is the lateral deflection under the force f per unit area. The parameter q is associated with the thickness of the plate, Young's modulus of elasticity, and the Poisson ratio of the material. Boundary conditions (1.1b) and (1.1c) are the so-called clamped boundary conditions. However, the identification process that we develop in this work can easily be extended to the following pinned boundary conditions:

$$u(\omega, x) = \Delta u(\omega, x) = 0, \text{ on } \partial D.$$

In this work, we propose two stochastic optimization formulations for the inverse problem. The first one is the output least-squares formulation, which is the most commonly used optimization strategy. The second one is an energy norm-based optimization formulation. However, before presenting an outline of our main contributions, we briefly summarize some related research. We begin by noting that in one of the earlier works, Narayanan and Zabaras [2] investigate the inverse problem in the presence of uncertainties in the material data and devise an adjoint-based identification framework. They compute the gradient of the OLS-type objective functional and use a conjugate gradient method to give promising numerical results. In [39], the authors develop a scalable methodology for the stochastic inverse problem using a sparse grid collocation approach. In [35], the authors devise a robust approach by employing generalized polynomial chaos expansion for identifying uncertain elastic parameters from experimental modal data. Morzfeld, Tu, Wilkening, and Chorin [27] study an implicit sampling approach for parameter identification. Warner, Aquino, and Grigoriucite [38] propose an abstract framework for solving inverse problems under uncertainty using the stochastic reduced-order models, and as an application, identify random material parameters in elasto-dynamical systems. Rosic and Matthies [33] study identification problems in a Bayesian setting for the elastoplastic problem. In [21], the authors focus on determining the optimal thickness subjected to stochastic force. In [1], the authors investigate the impact of errors and uncertainties of the conductivity on the electrocardiography imaging (ECGI) solution. Some of the related developments are available in [3,4,7,10–12,20,22,23,26,28,29,34,36,37]. For an overview of common techniques for inverse problems, see [6,8,9,13–15,18,19]. Interesting results for more general stochastic variational inequalities can be found in the recent papers Rockafellar and Sun [30, 31] and Rockafellar and Wets [32].

The main contributions of this work are as follows: We study the topological properties of the parameter-to-solution map. In particular, we establish its Lipschitz continuity and give a new derivative characterization. We propose two stochastic optimization formulations for the inverse problem. The first one is an analog of the classical OLS objective, which is typically nonconvex, and the second one is a new convex objective functional. We provide a regularization framework and give existence results for stochastic optimization problems. We develop a stochastic adjoint approach for the efficient derivative computation of the OLS objective. Assuming the finite-dimensional noise, we give a parameterization of the stochastic variational problem and the optimization problems. We provide a Stochastic-Galerkin-based discretization scheme for the direct and the inverse problem. We provide explicit discrete formulas for the OLS and the ELS functionals and their gradients. We provide detailed computational results.

We divide the contents of this paper into eight sections. Section 2 describes the variational formulation of the PDE with random data. In Section 3, we discuss the properties of the solution map. Section 4 is devoted to the stochastic optimization formulations of the inverse problem. We parameterize the stochastic inverse problem and present the adjoint approach in Section 5. We develop the computational framework in Section 6 and give numerical examples in Section 7. The paper concludes with some general remarks and future research goals.

2. Inverse problem formulation

A convenient analytical setting to study variational problems emerging from BVPs with random data is provided by Bochner spaces, for details see [16, 25]. Given a real Banach space X, a measure space $(\Omega, \mathcal{F}, \mu)$, and an integer $p \in [1, +\infty)$, the Bochner space $L^p(\Omega, X)$ consists of Bochner integrable functions $u : \Omega \to X$ with finite p-th moment, that is,

$$\|u\|_{L^p(\Omega,X)} := \left(\int_\Omega \|u(\omega)\|_X^p d\mu(\omega) \right)^{1/p} = \int_\Omega \|u(\omega)\|_X^p \, d\mu(\omega)^{1/p} < +\infty.$$

If $p = +\infty$, then $L^\infty(\Omega, X)$ is the space of Bochner integrable functions $u : \Omega \to X$ such that

$$\operatorname{ess\,sup}_{\omega \in \Omega} \|u(\omega)\|_X < +\infty.$$

Useful properties of $L^p(D)$ spaces of Lebesgue integrable functions extend naturally to Bochner spaces $L^p(\Omega, X)$. It is known that $L^\infty(\Omega, L^\infty(D)) \subset L^\infty(\Omega \times D)$, but $L^\infty(\Omega, L^\infty(D) \neq L^\infty(\Omega \times D))$, in general. Furthermore, the space $L^p(\Omega, L^q(D))$, for $p, q \in [1, +\infty)$, is isomorphic to

$$\left\{ v : \Omega \times D \to \mathbb{R}^\ell | \int_\Omega \left(\int_D |v(\omega, x)|^q \, dx \right)^{p/q} d\mu(\omega) < +\infty \right\}.$$

For the variational form of BVP (1.1), we will use $\widehat{V} := L^2(\Omega, H^2(D))$ which is a Hilbert space with the inner product defined by

$$\langle u, v \rangle = \int_\Omega \langle u(\omega, x), v(\omega, x) \rangle_{H^2(\Omega)} \, d\mu(\omega).$$

To incorporate the boundary conditions, we will use $V = L^2(\Omega, H_0^2(\Omega)) \subset \widehat{V}$.

A derivation of the variational formulation is similar to the deterministic case. If we take $u \in L^2(\Omega, H^4(D))$ and multiply (1.1) by a test function $v \in V$ and by integrating the product on both sides, invoking the Green's identity twice, and using the boundary conditions, we obtain

$$\begin{aligned} &\int_\Omega \int_D q(\omega, x) \Delta u(\omega, x) \Delta v(\omega, x) dx d\mu(\omega) \\ &\qquad = \int_\Omega \int_D f(\omega, x) v(\omega, x) \, dx d\mu(\omega), \text{ for every } v \in V. \end{aligned} \tag{2.1}$$

Therefore, we are looking for elements $u \in V$ such that (2.1) holds for all $v \in V$.

Given constants κ_0 and κ_1, we introduce the set of feasible parameters:

$$K := \{q(\omega, x) \in L^\infty(\Omega \times D) \mid \ 0 < \kappa_0 \leq q(\omega, x) \leq \kappa_1 < +\infty\}. \tag{2.2}$$

Remark 2.1. In (2.2), the feasible random parameters are bounded above and below by constants. Consequently, many results from the deterministic framework extend naturally to the stochastic framework. The drawback, however, is that andom Log-normal fields are not bounded on $\Omega \times D$, and hence it would be of interest to relax this condition to cover more general random variables.

We have the following existence result:

Theorem 2.2. *There is a unique solution of* (2.1) *in* V.

Proof. We introduce the following notation

$$s(u, v) = \int_\Omega \int_D q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)\, dx d\mu(\omega), \tag{2.3}$$

$$m(v) = \int_\Omega \int_D f(\omega, x)v(\omega, x)\, dx d\mu(\omega), \tag{2.4}$$

and, for a fixed $q(\omega, x)$, write (2.1) as the problem of finding $u \in V$ with

$$s(u, v) = m(v), \quad \text{for every } v \in V. \tag{2.5}$$

Since $q(\omega, x) \in L^\infty(\Omega \times D)$ and $V \subset L^2(\Omega, L^2(D)) \cong L^2(\Omega \times D)$, by the Cauchy-Schwartz inequality, for every $u, v \in V$, we have

$$\begin{aligned}\int_{\Omega\times D} &|q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)|\, dx d\mu(\omega)\\ &\leq \|q(\omega, x)\|_{L^\infty(\Omega\times D)}\|\Delta u(\omega, x)\|_{L^2(\Omega,L^2(\Omega))}\|\Delta v(\omega, x)\|_{L^2(\Omega,L^2(D))},\end{aligned}$$

implying $q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x) \in L^1(\Omega \times D)$. Then, Fubini's theorem for iterated integrals yields

$$\begin{aligned}|s(u, v)| &= \left|\int_\Omega \int_D q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)\, dx d\mu(\omega)\right|\\ &= \left|\int_{\Omega\times D} q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)\, dx d\mu(\omega)\right|.\end{aligned}$$

Using repeatedly Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|s(u, v)| &\leq \int_{\Omega\times D} |q(\omega, x)\Delta u(\omega, x)\Delta v(\omega, x)|\, dx d\mu(\omega)\\ &\leq \|q(\omega, x)\|_{L^\infty(\Omega\times D)} \int_{\Omega\times D} |\Delta u(\omega, x)\Delta v(\omega, x)|\, dx d\mu(\omega)\\ &\leq \|q(\omega, x)\|_{L^\infty(\Omega\times D)}\|u(\omega, x)\|_V\|v(\omega, x)\|_V,\end{aligned}$$

which proves the continuity of the bilinear form s.

By using Fubini's theorem once again, we obtain

$$\begin{aligned}s(v, v) &= \int_\Omega \int_D q(\omega, x)\Delta v(\omega, x)\Delta v(\omega, x) dx d\mu(\omega)\\ &= \int_{\Omega\times D} q(\omega, x)\Delta v(\omega, x)\Delta v(\omega, x)\, dx d\mu(\omega).\end{aligned}$$

Since $q(\omega, x)$ is bounded from below by κ_0, almost surely, we have

$$s(v, v) \geq \kappa_0 \int_{\Omega \times D} \Delta v(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) = \kappa_0 \int_\Omega \|\Delta v(\omega, \cdot)\|^2_{L^2(D)} d\mu(\omega),$$

which, due to the fact that the map $v(\omega, \cdot) \mapsto |\Delta v(\omega, \cdot)|$ defines a norm which is equivalent to the original norm of $H^2(D)$, further implies that for some constant $c > 0$, we have

$$s(v, v) \geq c\kappa_0 \int_\Omega \|v(\omega, x)\|^2_{H_2(D)} d\mu(\omega) = c\kappa_0 \|v(\omega, x)\|^2_V,$$

which proves the coercivity of s.

Analogously, for the given $f(\omega, x) \in L^2(\Omega; H^2(D)^*)$ and for any $v(\omega, x) \in V$, for the functional

$$m(v) = \int_\Omega \langle f(\omega, \cdot), v(\omega, \cdot)\rangle d\mu(\omega),$$

we have

$$\begin{aligned} |m(v)| &= \left| \int_\Omega \langle f(\omega . \cdot), v(\omega, \cdot)\rangle d\mu(\omega) \right| \\ &\leq \|f(\omega, x)\|_{H^2(\Omega, H^2(D)^*)} \|v(\omega, x)\|_{L^2(\Omega; H^2(D)^*)}, \end{aligned}$$

which proves the continuity of m.

Consequently, the unique solvability of (2.1) ensues from the Lax-Milgram lemma (see [24, Remark 2.2]). □

3. Lipschitz continuity and smoothness of the solution map

We begin with the following assertion, where, for $q(\omega, x) \in K$, $u_q(\omega, x) \in V$ denotes the solution of (2.1).

Theorem 3.1. *The map $K \ni q(\omega, x) \mapsto u_q(\omega, x)$ is Lipschitz continuous.*

Proof. For $q_1(\omega, x), q_2(\omega, x) \in K$, let $u_{q_1}(\omega, x), u_{q_2}(\omega, x) \in V$ be the corresponding solutions of (2.1). Then, for every $v \in V$, we have

$$\int_\Omega \int_D q_1(\omega, x) \Delta u_{q_1}(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) = \int_\Omega \int_D f(\omega, x) v(\omega, x)\, dx d\mu(\omega),$$
$$\int_\Omega \int_D q_2(\omega, x) \Delta u_{q_2}(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) = \int_\Omega \int_D f(\omega, x) v(\omega, x)\, dx d\mu(\omega).$$

We rearrange the above equations to get that for every $v \in V$, we have

$$\begin{aligned} &\int_\Omega \int_D q_1(\omega, x) \Delta u_{q_1}(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) \\ &\qquad - \int_\Omega \int_D q_2(\omega, x) \Delta u_{q_2}(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) = 0. \end{aligned}$$

We rearrange the above equation as follows

$$\begin{aligned} &\int_\Omega \int_D q_1(\omega, x) \Delta (u_{q_1}(\omega, x) - u_{q_2}(\omega, x)) \Delta v(\omega, x)\, dx d\mu(\omega) \\ &\qquad + \int_\Omega \int_D (q_1(\omega, x) - q_2(\omega, x)) \Delta u_{q_2}(\omega, x) \Delta v(\omega, x)\, dx d\mu(\omega) = 0, \end{aligned}$$

which holds for all $v \in V$, and by taking $v = u_{q_1}(\omega, x) - u_{q_2}(\omega, x)$, we obtain

$$\begin{aligned}
&\kappa_0 \int_\Omega \|\Delta(u_{q_1}(\omega, \cdot) - u_{q_2}(\omega, \cdot))\|^2_{L^2(D)} d\mu(\omega) \\
&\leq \int_\Omega \int_D q_1(\omega, x)\Delta(u_{q_1}(\omega, x) - u_{q_2}(\omega, x))\Delta(u_{q_1}(\omega, x) - u_{q_2}(\omega, x)) dx d\mu(\omega) \\
&= -\int_\Omega \int_D (q_1(\omega, x) - q_2(\omega, x))\Delta u_{q_2}(\omega, x)\Delta(u_{q_1}(\omega, x) - u_{q_2}(\omega, x)) dx d\mu(\omega) \\
&\leq \int_\Omega \int_D |(q_1(\omega, x) - q_2(\omega, x))\Delta u_{q_2}(\omega, x)\Delta(u_{q_1}(\omega, x) - u_{q_2}(\omega, x)) dx|\, d\mu(\omega) \\
&\leq \|q_1(\omega, x) - q_2(\omega, x)\|_{L^\infty(\Omega\times D)} \\
&\times \int_\Omega \int_D |\Delta u_{q_2}(\omega, x)\Delta(u_{q_1}(\omega, x) - u_{q_2}(\omega, x)) dx|\, d\mu(\omega) \\
&\leq \|q_1(\omega, x) - q_2(\omega, x)\|_{L^\infty(\Omega\times D)} \|u_{q_2}(\omega, x)\|_V \|u_{q_1}(\omega, x) - u_{q_2}(\omega, x)\|_V,
\end{aligned}$$

and by evident inequality $\|u_{q_2}(\omega, x)\|_V \leq \|f(\omega, x)\|_{L^2(\Omega, H^2(D)^*)}$, we obtain

$$\|u_{q_1}(\omega, x) - u_{q_2}(\omega, x)\|_V \leq c\|q_1(\omega, x) - q_2(\omega, x)\|_{L^\infty(\Omega\times D)},$$

for a constant $c > 0$. The proof is complete. □

We now provide a derivative characterization of the solution map. In some of our assertions, we are dealing with the interior of K in $L^\infty(\Omega \times D)$ with respect to the norm topology.

Theorem 3.2. *Let $q(\omega, x)$ be in the interior of K. Then, the derivative $\delta u_q(\omega, x) := Du_q(\delta q(\omega, x))$ of $u_q(\omega, x)$ in the direction $\delta q(\omega, x)$ is the unique solution of the stochastic variational problem: Find $\delta u_q(\omega, x) \in V$ such that*

$$\begin{aligned}
(3.1) \qquad &\int_\Omega \int_D q(\omega, x)\Delta\delta u_q(\omega, x)\Delta v(\omega, x) dx\, d\mu(\omega) \\
&\quad = -\int_\Omega \int_D \delta q \Delta u_q(\omega, x)\Delta v(\omega, x)\, dx d\mu(\omega), \quad \textit{for every } v \in V.
\end{aligned}$$

Proof. The existence of $\delta u_q(\omega, x) \in V$ satisfying (3.1) is a direct consequence of Theorem 2.2. For $q(\omega, x)$ belonging to the interior of K, let $\delta q(\omega, x)$ be sufficiently small so that $q(\omega, x) + \delta q(\omega, x) \in K$. Consequently, the quantity

$$\delta w(\omega, x) = u_{q+\delta q}(\omega, x) - u_q(\omega, x),$$

is well-defined.

By the definition of $u_q(\omega, x)$ and $u_{q+\delta q}(\omega, x)$, for every $v \in V$, we get

$$\begin{aligned}
&\int_\Omega \int_D q(\omega, x)\Delta u_q(\omega, x)\Delta v(\omega, x) dx\, d\mu(\omega) \\
&\quad = \int_\Omega \int_D f(\omega, x)v(\omega, x)\, dx\, d\mu(\omega), \quad \text{for every } v \in V,
\end{aligned}$$

and

$$\int_\Omega \int_D (q(\omega, x) + \delta q(\omega, x))\Delta u_{q+\delta q}(\omega, x)\Delta v(\omega, x) dx\, d\mu(\omega)$$

$$= \int_\Omega \int_D f(\omega,x)v(\omega,x)\,dx\,d\mu(\omega), \quad \text{for every } v \in V.$$

We combine the above identities to obtain

$$\int_\Omega \int_D (q(\omega,x)+\delta q(\omega,x))\Delta\delta w(\omega,x)\Delta v(\omega,x)dx\,d\mu(\omega) = -\int_\Omega \int_D \delta q(\omega,x)\Delta u_q(\omega,x)\Delta v(\omega,x)dx\,d\mu(\omega), \tag{3.2}$$

and by subtracting (3.1) from (3.2), we get

$$\int_\Omega \int_D q(\omega,x)\Delta\left(\delta w(\omega,x)-\delta u_q(\omega,x)\right)\Delta v(\omega,x)dx\,d\mu(\omega) = -\int_\Omega \int_D \delta q(\omega,x)\Delta\delta w(\omega,x)\Delta v(\omega,x)dx\,d\mu(\omega).$$

We now choose $v = \delta w(\omega,x) - \delta u_q(\omega,x)$, in the above equation, and obtain

$$\int_\Omega \int_D q(\omega,x)\Delta\left(\delta w(\omega,x)-\delta u_q(\omega,x)\right)\Delta\left(\delta w(\omega,x)-\delta u_q(\omega,x)\right)dx\,d\mu(\omega) = -\int_\Omega \int_D \delta q(\omega,x)\Delta\delta w(\omega,x)\Delta\left(\delta w(\omega,x)-\delta u_q(\omega,x)\right)dx\,d\mu(\omega),$$

and the above identity confirms that there is a constant $c > 0$, such that

$$\begin{aligned}\|\delta w(\omega,x)-\delta u_q(\omega,x)\|_V &\le c\|\delta w(\omega,x)\|\|\delta q(\omega,x)\|_{L^\infty(\Omega\times D)}\\ &\le c\|\delta q(\omega,x)\|^2_{L^\infty(\Omega\times D)},\end{aligned}$$

where we employed the Lipschitz continuity of the solution map. Therefore,

$$\frac{\|u_{q+\delta q}(\omega,x)-u_q(\omega,x)-\delta u(\omega,x)\|_V}{\|\delta q(\omega,x)\|_{L^\infty(\Omega\times D)}} = o\left(\|\delta q(\omega,x)\|_{L^\infty(\Omega\times D)}\right),$$

and by taking $\|\delta q(\omega,x)\|_{L^\infty(\Omega\times D)} \to 0$, we confirm that $\delta u(\omega,x)$ is the sought derivative and that (3.2) holds. The proof is complete. □

4. Optimization framework for the inverse problem

We first define the output least-squares (OLS) functional:

$$\widehat{J}(q) := \frac{1}{2}\int_\Omega \int_D |u_q(\omega,x)-z(\omega,x))|^2 dx\,d\mu(\omega),$$

where $u_q(\omega,x)$ solves (2.1) and $z(\omega,x) \in L^2(\Omega, L^2(D))$ is the data. In the numerical experiments, besides the L^2-norm, we will also use the H^2-norm.

We next define the energy least-squares (ELS) functional

$$J(q) = \frac{1}{2}\int_\Omega \int_D q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

where $u_q(\omega,x)$ solves (2.1) and $z(\omega,x) \in L^2(\Omega, H_0^2(D))$ is the data.

Remark 4.1. Note that the ELS functional requires higher regularity on the data as it is defined by using the energy associated to the variational problem. For a noisy data set, this might cause a problem, and some data smoothing might be necessary to alleviate it.

One of the major deficiencies of the OLS formulation is its nonconvexity, which results in theoretical and computational challenges and poses the risk of locating only local solutions of the OLS-based stochastic optimization problem. The ELS-functional, on the other hand, is convex:

Theorem 4.2. *Let $q(\omega, x)$ be an element in the interior of K. Then:*
(1) *The first derivative of J at q is given by*

$$DJ(q)(\delta q)$$
$$= -\frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u_q(\omega,x)+z(\omega,x))\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega).$$

(2) *The second derivative of J at q is given by*

$$D^2J(q)(\delta q,\delta q) = \int_\Omega\int_D q(\omega,x)\Delta\delta u_q(\omega,x)\Delta\delta u_q(\omega,x)dx\,d\mu(\omega).$$

The ELS functional is convex in the interior of the set K.

Proof. We proceed by to compute the first derivative by the chain rule

$$DJ(q)(\delta q)$$
$$= \frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u(\omega,x)-z(\omega,x))\Delta(u(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$+ \frac{1}{2}\int_\Omega q(\omega,x)\Delta\delta u_q(\omega,x)\Delta(u(q,\omega)-z(\omega,x))dx\,d\mu(\omega)$$
$$+ \frac{1}{2}\int_\Omega q(\omega,x)\Delta\delta u_q(\omega,x)\Delta(u(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= \frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u(\omega,x)-z(\omega,x))\Delta(u(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$+ \int_\Omega\int_D q(\omega,x)\Delta\delta u_q(\omega,x)\Delta(u(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

where $\delta u_q(\omega, x)$ is the derivative of $u_q(\omega, x)$ in the direction δq.

Since,

$$\int_\Omega\int_D q(\omega,x)\Delta\delta u_q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= -\int_\Omega\int_D \delta q(\omega,x)\Delta u_q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

we obtain

$$DJ(q)(\delta q)$$
$$= \frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$- \int_\Omega\int_D \delta q(\omega,x)\Delta u_q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= -\frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u_q(\omega,x)+z(\omega,x))\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega).$$

For the second-order derivative, we continue as follows:

$$
\begin{aligned}
D^2J(q)(\delta q,\delta q) &= -\frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta\delta u_q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega)\\
&\quad -\frac{1}{2}\int_\Omega\int_D \delta q(\omega,x)\Delta(u_q(\omega,x)+z(\omega,x))\Delta\delta u_q(\omega,x)dx\,d\mu(\omega)\\
&= -\int_\Omega\int_D \delta q(\omega,x)\Delta u(q)(\omega,x)\Delta\delta u_q(\omega,x)dx\,d\mu(\omega)\\
&= \int_\Omega\int_D q(\omega,x)\Delta\delta u_q(\omega,x)\Delta\delta u_q(\omega,x)dx\,d\mu(\omega),
\end{aligned}
$$

where we used Theorem 3.2.

For convexity of the ELS, we note that there is a constant $\alpha > 0$ such that the following inequality holds for all $q(\omega,x)$ in the interior of K

$$
D^2J(q)(\delta q,\delta q) \geq \alpha\|\delta u_q(\omega,x)\|_V^2, \tag{4.1}
$$

and consequently J is a convex functional. □

Since the inverse problems are severely ill-posed, some type of regularization is essential. For this, we introduce the following admissible set:

$$
K := \left\{q\in H = L^2(\Omega, H(D)) :\ 0<\kappa_0\leq q(\omega,x)\leq\kappa_1 \text{ a.s. } \Omega\times D\right\},
$$

where H is a Hilbert space compactly embedded into $B := L^\infty(\Omega, L^\infty(\Omega))$, and $H(D)$ is continuously embedded in $L^\infty(\Omega)$.

Remark 4.3. Note that the random parameters above satisfy the same bounds as in (2.2), but have higher regularity dictated by the space H. As a slight abuse of notation, we denote both sets by K.

We introduce the following optimization problems:

1. Solve the following regularized OLS-based optimization problem:

$$
\min_{q\in K}\widehat{J}_\kappa(q) := \int_\Omega\int_D |u_q(\omega,x)-z(\omega,x)|^2dx\,d\mu(\omega)+\kappa\|q(\omega,x)\|_H^2, \tag{4.2}
$$

where $u_q(\omega,x)$ solves (2.1) for $q(\omega,x)$, $z(\omega,x)\in L^2(\Omega, L^2(D))$ is the data, $\kappa>0$ is a fixed regularization parameter, and $\|\cdot\|_H^2$ is the regularizer.

2. Solve the following regularized ELS-based optimization problem:

$$
\begin{aligned}
\min_{q\in K} J_\kappa(q) := &\int_\Omega\int_D q(\omega,x)\Delta(u_q(\omega,x)-z(\omega,x))\Delta(u_q(\omega,x)-z(\omega,x))dx\,d\mu(\omega)\\
&+\kappa\|q(\omega,x)\|_H^2,
\end{aligned}\tag{4.3}
$$

where $u_q(\omega,x)$ solves (2.1) for $q(\omega,x)$, $z(\omega,x)\in L^2(\Omega, H^2(D))$ is the data, $\kappa>0$ is a fixed regularization parameter, and $\|\cdot\|_H^2$ is the regularizer.

We have the following existence result:

Theorem 4.4. *For $\kappa>0$, optimization problem (4.3) has a unique solution.*

Proof. Since $J_\kappa(q) \geq 0$, for every $q \in K$, there is a minimizing sequence $\{q_n(\omega, x)\}$ in K such that

$$\lim_{n\to\infty} J_\kappa(q_n) = \inf\{J_\kappa(q) |\, q(\omega, x) \in K\}.$$

Therefore, $\{J_\kappa(q_n)\}$ is bounded, and hence $\{q_n\}$ is bounded in $\|\cdot\|_H$. Since H is reflexive, $\{q_n(\omega, x)\}$ has a subsequence, which weakly converges to some $\bar{q}(\omega, x) \in K$. Retaining the same notation for subsequences, let $u_n(\omega, x)$ be the solution of the variational problem for to $q_n(\omega, x)$. That is,

$$\int_\Omega \int_D q_n(\omega, x)\Delta u_n(\omega, x)\Delta v(\omega, x) dx\, d\mu(\omega)$$
$$= \int_\Omega \int_D f(\omega, x)v(\omega, x)\, dx\, d\mu(\omega), \text{ for all } v \in V.$$

We take $v = u_n(\omega, x)$ and obtain

$$\int_\Omega \int_D q_n(\omega, x)\Delta u_n(\omega, x)\Delta u_n(\omega, x) dx\, d\mu(\omega) = \int_\Omega \int_D f(\omega, x)u_n(\omega, x)\, dx\, d\mu(\omega),$$

which leads to the boundedness of $u_n(\omega, x) = u_{q_n}(\omega, x)$. Therefore, $\{u_n(\omega, x)\}$ has a subsequence that converges weakly to some $\bar{u}(\omega, x) \in V$. We claim that $\bar{u}(\omega, x) = u_{\bar{q}}(\omega, x)$. The equation

$$\int_\Omega \int_D q_n(\omega, x)\Delta u_n(\omega, x)\Delta v(\omega, x) dx\, d\mu(\omega)$$
$$= \int_\Omega \int_D f(\omega, x)v(\omega, x)\, dx\, d\mu(\omega), \quad \text{for every } v \in K,$$

after a simple rearrangement of terms, implies that

$$\int_\Omega \int_D [\bar{q}(\omega, x)\Delta\bar{u}(\omega, x)\Delta v(\omega, x) - f(\omega, x)v(\omega, x)\, d\mu(\omega)\, dx]$$
$$= -\int_\Omega \int_D (q_n(\omega, x) - \bar{q}(\omega, x))\Delta u_n(\omega, x)\Delta v(\omega, x))\, dx\, d\mu(\omega)$$
$$- \int_\Omega \int_D \bar{q}(\omega, x)\Delta(u_n(\omega, x) - \bar{u}(\omega, x))\Delta v(\omega, x)\, dx\, d\mu(\omega). \tag{4.4}$$

Notice that

$$\left|\int_\Omega \int_D (q_n(\omega, x) - \bar{q}(\omega, x))\Delta u_n(\omega, x)\Delta v(\omega, x))\, dx\, d\mu(\omega)\right|$$
$$\leq \left(\int_\Omega \int_D |q_n(\omega, x) - \bar{q}(\omega, x))|\, |\Delta u_n(\omega, x)|^2\, dx\, d\mu(\omega)\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \int_D |q_n(\omega, x) - \bar{q}(\omega, x)|\ |\Delta v(\omega, x)|^2\ dx\, d\mu(\omega)\right)^{\frac{1}{2}}$$
$$\to 0$$

by the Dominated Convergence Theorem. Since the second term on the right-hand side of (4.4) also converges to zero, we have

$$\int_\Omega \int_D [\bar{q}(\omega, x)\Delta\bar{u}(\omega, x)\Delta v(\omega, x) - f(\omega, x)v(\omega, x)\, d\mu(\omega)] = 0.$$

Since $v \in V$ is arbitrary, and the variational problem is uniquely solvable, we get $\bar{u}(\omega,x) = u_{\bar{q}}(\omega,x)$.

We claim that $J(q_n) \to J(\bar{q})$. The identities

$$\int_\Omega \int_D q_n(\omega,x)\Delta(u_n(\omega,x)-z(\omega,x))\Delta(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= \int_\Omega \int_D f(\omega,x)(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$- \int_\Omega \int_D q_n(\omega,x)\Delta z(\omega,x)\Delta(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

and

$$\int_\Omega \int_D \bar{q}(\omega,x)\Delta(\bar{u}(\omega,x)-z(\omega,x))\Delta(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= \int_\Omega \int_D f(\omega,x)(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$- \int_\Omega \int_D \bar{q}(\omega,x)\Delta z(\omega,x)\Delta(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

in view of the rearrangement

$$\int_\Omega \int_D q_n(\omega,x)\Delta z(\omega,x)\Delta(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$- \int_\Omega \int_D \bar{q}(\omega,x)\Delta z(\omega,x)\Delta(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$= \int_\Omega \int_D (q_n(\omega,x)-\bar{q}(\omega,x))\Delta z(\omega,x)\Delta(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$- \int_\Omega \int_D \bar{q}(\omega,x)\Delta z(\omega,x)\Delta(u_n(\omega,x)-\bar{u}(\omega,x))dx\,d\mu(\omega),$$

imply that

$$\int_\Omega \int_D q_n(\omega,x)\Delta(u_n(\omega,x)-z(\omega,x))(u_n(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$\to \int_\Omega \int_D \bar{q}(\omega,x)\Delta(\bar{u}(\omega,x)-z(\omega,x))\Delta(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega),$$

and consequently,

$$J_\kappa(\bar{q}) = \frac{1}{2}\int_\Omega \int_D \bar{q}(\omega,x)\Delta(\bar{u}(\omega,x)-z(\omega,x))\Delta(\bar{u}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$+ \kappa\|\bar{q}(\omega,x)\|_H^2$$
$$\leq \lim_{n\to\infty} \frac{1}{2}\int_\Omega \int_D q_n(\omega,x)\Delta(u_{q_n}(\omega,x)-z(\omega,x))\Delta(u_{q_n}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)$$
$$+ \liminf_{n\to\infty} \kappa\|q_n(\omega,x)\|_H^2$$
$$\leq \liminf_{n\to\infty} \left\{\frac{1}{2}\int_\Omega \int_D q_n(\omega,x)\Delta(u_{q_n}(\omega,x)-z(\omega,x))\Delta(u_{q_n}(\omega,x)-z(\omega,x))dx\,d\mu(\omega)\right.$$
$$\left.+\kappa\|q_n(\omega,x)\|_H^2\right\}$$

$$= \inf \{J_\kappa(q) \mid q(\omega, x) \in K\},$$

confirming that $\bar{q}$ is a solution of (4.3). Taking into account (4.1), the solution of (4.3) is unique such that the proof is complete. □

5. Finite-dimensional noise

An essential aspect of the study of stochastic PDEs and stochastic optimization problems is the representation of the random fields by a finite number of mutually independent random variables. For this, we recall [25]:

Definition 5.1. *Let $\xi_k : \Omega \mapsto \Gamma_k \subset \mathbb{R}$, for $k = 1, \ldots, M$, be real-valued random variables with $M < \infty$. A function $v \in L^2(\Omega, L^2(D))$ of the form $v(\xi(\omega), x)$ for $x \in D$ and $\omega \in \Omega$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma \subset \mathbb{R}^M$ and $\Gamma := \Gamma_1 \times \Gamma_2 \cdots \times \Gamma_M$ is called a* ***finite-dimensional noise.***

If a random field $v(\xi, x)$ is finite-dimensional noise, a change of variable can be made for computing expectations. For instance, denoting by σ, the joint density of ξ, we have

$$\|v\|^2_{L^2(\Omega, L^2(D))} = \int_\Omega \|v\|^2_{L^2(D)}\, d\mu(\omega) = \int_\Gamma \sigma(y) \|v(y, \cdot)\|^2_{L^2(D)} dy.$$

Consequently, by defining $y_k := \xi_k(\Omega)$ and setting $y = (y_1, y_2, \ldots, y_M)$, we associate a random field $v(x, \xi)$ with a finite-dimensional noise by a function $v(x, y)$ in the weighted L^2 space

$$L^2_\rho(\Gamma, L^2(\Omega)) := \left\{ v : \Omega \times \Gamma \to \mathbb{R} : \int_\Gamma \rho(y) \|v(y, \cdot)\|^2_{L^2(D)} dy < +\infty \right\},$$

where ρ is a nonnegative and bounded function.

In what follows, we assume that q and f are finite dimensional noise and given through the expansions:

$$q(\omega, x) := E[q](x) + \sum_{k=1}^{P} q_k(x) \xi_k(\omega),$$
$$f(\omega, x) := E[f](x) + \sum_{k=1}^{N} f_k(x) \xi_k(\omega),$$

where the real-valued functions q_k and f_k are uniformly bounded.

It follows from the Doob-Dynkin lemma that a solution of (2.1) is finite-dimensional noise and u is a function of ξ where $\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \Omega \mapsto \Gamma$ and $M := \max\{P, N\}$.

Then, variational problem (2.1) reduces to the parametric deterministic variational problem: Find $u \in V_\sigma := L^2_\sigma(\Gamma, H^2_0(D))$ such that

$$\int_\Gamma \sigma(y) \int_D q(y, x) \Delta u(y, x) \Delta v(y, x) dx\, dy = \int_\Gamma \sigma(y) \int_D f(y, x) v(y, x)\, dx\, dy, \quad \text{for every } v \in V_\sigma. \tag{5.1}$$

For the inverse problem, we will assume that the data z depends, via ξ, on the finite-dimensional noise variables $\{\xi_i\}_{i=1}^M$. Therefore, we will assume that the parameter q is also the function of the variables $\{\xi_i\}_{i=1}^M$. That is,

$$q(x, \xi) = q(x, \xi_1(\omega), \xi_2(\omega), \dots, \xi_M(\omega)) \in \widetilde{H}(\Omega) := L^2_\sigma(\Gamma, H(D)).$$

The finite-dimensional noise variants of the OLS/ELS objectives read:

$$\widehat{J}(q) := \int_\Gamma \sigma(y) \int_D (u_q(y,x) - z(y,x)) \cdot (u_q(y,x) - z_n(y,x))\, dx\, dy,$$
$$J(q) := \int_\Gamma \sigma(y) \int_D q(y,x) \Delta (u_q(y,x) - z(y,x))\, \Delta (u_q(y,x) - z(y,x))\, dx\, dy,$$

where $u_q(y,x)$ solves (5.1) and $z(y,x)$ is the finite-dimensional noise data.

Following Theorem 3.2, we get a derivative characterization of the finite-dimensional noise solution map and the derivative of the ELS functional:

Theorem 5.2. *Let δq be in the interior of K. Then, the derivative $\delta u_q := Du_q(\delta q)$ of u_q in the direction δq is the unique solution of the following parameterized variational problem:*

$$\int_\Gamma \sigma(y) \int_D q(y,x) \Delta \delta u_q(y,x) \Delta v(y,x) dx\, dy$$
$$= - \int_\Gamma \sigma(y) \int_D \delta q \Delta u_a(y,x) \Delta v(y,x)\, dx\, dy, \quad \textit{for all } v \in V_\sigma.$$

Furthermore, the derivative of the finite-dimensional noise ELS reads:

$$DJ(q)(\delta q) = \frac{1}{2} \int_\Gamma \sigma(y) \int_D \delta q \Delta (u_q + z) \Delta (u_q - z) dx\, dy.$$

For a derivative characterization for the OLS objective, it follows from

$$\widehat{J}(q) = \frac{1}{2} \int_\Gamma \sigma(y) \int_D (u(y,x) - z(y,x))^2 dx dy,$$

by a direct computation that

$$D\widehat{J}(q)(\delta q) = \int_\Gamma \sigma(y) \int_D \delta u (u(y,x) - z(y,x)) dx\, dy,$$

where the derivative $\delta u = Du(q)(\delta q)$ can be computed by Theorem 5.2.

By using the adjoint approach, it can be shown

$$DJ(a)(\delta a) = \int_\Gamma \sigma(y) \int_D \delta q(y,x) \Delta u(y,x) \Delta w(y,x) dx\, dy, \tag{5.2}$$

where $w = w(y,x) \in V_\sigma$ is such that for every $v \in V_\sigma$:

$$\int_\Gamma \sigma(y) \int_D q(y,x) \Delta w(y,x) \Delta v(y,x) dx dy$$
$$= \int_\Gamma \sigma(y) \int_D (z(y,x) - u(y,x)) v(y,x) dx\, dy. \tag{5.3}$$

Remark 5.3. In the following section, we discretize the finite-dimensional noise problems by using the Stochastic Galerkin approach. However, in this work, we confine to giving discrete formulae only and don't study the convergence of the finite-dimensional problems. The critical ideas of such convergence theory are similar to those used in [3, 25, 28] and would be studied in a future work via developing errors estimates.

6. Computational framework

In the following, we will use Stochastic Galerkin approach for providing discrete formulae for the direct problem, the OLS and the ELS functionals, and their gradients. The variational problem that needs to be discretized reads: Find $u \in V := L^2_\sigma(\Gamma, H^2_0(D))$ such that

$$\int_\Gamma \sigma(y) \int_D q(y,x)\Delta u(y,x)\Delta v(y,x)dxdy = \int_\Gamma \sigma(y) \int_D f(y,x)v(y,x)dxdy, \quad \text{for all } v \in V.$$

Let V_{hk} be a finite-dimensional subspace of V. An element $u_{hk} \in V_{hk}$ is the stochastic Galerkin solution, if

$$\int_\Gamma \sigma(y) \int_D q(y,x)\Delta u_{hk}(y,x)\Delta v(y,x)dxdy = \int_\Gamma \sigma(y) \int_D f(y,x)v(y,x)dxdy, \quad \text{for every } v \in V_{hk}. \tag{6.1}$$

Let V_h be a N-dimensional subspace of $H^2_0(\Omega)$ and S_k be a G-dimensional subspace of $L^2_\sigma(\Gamma)$ with

$$V_h = \text{span}\{\Phi_1, \Phi_2, \dots, \Phi_N\},$$
$$S_k = \text{span}\{\Psi_1, \Psi_2, \dots, \Psi_G\}.$$

We assume that $\{\Psi_1, \Psi_2, \dots, \Psi_G\}$ is orthogonal with respect to σ, that is,

$$\int_\Gamma \sigma(y)\Psi_n(y)\Psi_m(y)dy = \delta_{nm},$$

where δ_{nm} stands for the Kronecker delta, $\delta_{nm} = 1$ for $n = m$, $\delta_{nm} = 0$ for $n \neq m$. We construct a finite-dimensional subspace of V by tensorising the basic functions Φ_i and Ψ_j. That is, the following NG-dimensional space will be the finite-dimensional test space for the discrete variational problem:

$$V_{hk} := V_h \otimes S_k := \text{span}\{\Phi_i\Psi_j \mid i = 1, \dots, N,\ j = 1, \dots, G\}.$$

Therefore, any $v \in V_h \otimes S_k$ has the representation

$$v(y,x) = \sum_{i=1}^N \sum_{j=1}^G V_{ij}\Phi_i(x)\Psi_j(y) = \sum_{j=1}^G \left[\sum_{i=1}^N V_{ij}\Phi_i(x)\right] \Psi_j(y) = \sum_{j=1}^G V_j \Psi_j(y),$$

with $V_j(x) \equiv \sum_{i=1}^N V_{ij}\Phi_i(x)$.

It is convenient to introduce the following notation

$$V = \mathrm{vec}(V_{ij}) = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_G \end{bmatrix} \in \mathbb{R}^{NG}, \quad \text{where} \quad V_j(x) \equiv \begin{bmatrix} V_{1j} \\ \vdots \\ V_{Nj} \end{bmatrix}. \tag{6.2}$$

We will assume that the random field has a finite linear expansion:

$$q(y,x) = q_0(x) + \sum_{s=1}^{M} y_s q_s(x) = \sum_{s=0}^{M} y_s q_s(x), \tag{6.3}$$

where, by convention, $y_0 = 1$. The spatial components q_s are discretized by using another P-dimensional space $Q_h = \mathrm{span}\{\varphi_1, ..., \varphi_P\}$. Therefore,

$$q(y,x) = \sum_{i=1}^{P} Q_{0i}\varphi_i(x) + \sum_{s=1}^{M}\left(\sum_{i=1}^{P} Q_{si}\varphi_i(x)\right) y_s = \sum_{s=0}^{M} Q_s y_s,$$

where the vectors $Q_s(x) \equiv (Q_{si}) \in \mathbb{R}^P$ for $s = 0, ..., M$,

$$Q = \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_M \end{bmatrix} \in \mathbb{R}^{P(M+1)\times 1},$$

will be the unknowns for the optimization problems.

Using (6.1), for every $i = 1, \dots, N$, $n = 1, \dots, G$, we have

$$\int_\Gamma \sigma(y)\Psi_n(y)\left(\int_D q(y,x)\Delta\left(\sum_{k=1}^{N}\sum_{m=1}^{G} U_{km}\Phi_k(x)\Psi_m(y)\right)\Delta\Phi_i(x)dx\right)dy$$
$$= \int_\Gamma \sigma(y)\Psi_n(y)\left(\int_D f(y,x)\Phi_i(x)dx\right)dy,$$

or equivalently, and hence for every $i = 1, \dots, N,\ n = 1, \dots, G$, we have

$$\sum_{k=1}^{N}\sum_{m=1}^{G} U_{km}\int_\Gamma \sigma(y)\Psi_n(y)\Psi_m(y)\left(\int_D q(y,x)\Delta\Phi_k(x)\Phi_i(x)dx\right)dy$$
$$= \int_\Gamma \sigma(y)\Psi_n(y)\left(\int_D f(y,x)\Phi_i(x)dx\right)dy.$$

We define $K(Q_s) \in \mathbb{R}^{N\times N}$, and $g^s_{nm} \in \mathbb{R}$, for each $s \in \{0, ..., M\}$, by

$$K(Q_s)_{i,k} = \int_D Q_s(x)\Delta\Phi_k(x)\Delta\Phi_i(x)dx,$$
$$g^s_{nm} = \int_\Gamma \sigma(y)\Psi_n(y)\Psi_m(y)y_s dy.$$

Therefore, for each $s \in \{0, ..., M\}$, we have

$$G^s = (g^s_{nm}) \in \mathbb{R}^{G\times G},$$

where, for $s = 0$, in view of the orthogonality, we get

$$G^0 = \left(\int_\Gamma \sigma(y)\Psi_n(y)\Psi_m(y)dy \right) = I.$$

We also define

$$(F_n)_i = \int_\Gamma \sigma(y)\Psi_n(y) \int_D f(y,x)\Phi_i(x)dxdy, \quad \text{for every } n = 1, \dots, G.$$

Summarizing, we obtain the following form of the discrete system:

$$\left(K(Q_0) + \sum_{s=1}^{M} g_{nn}^s K(Q_s) \right) U_n + \sum_{m \neq n} \sum_{s=1}^{M} g_{nm}^s K(Q_s) U_m = F_n,$$

for every $n = 1, \dots, G$, which can also be written as

$$\mathbb{K}(Q)U = F,$$

where

$$\mathbb{K}(Q) :=$$

$$\begin{bmatrix} K(Q_0) + \sum\limits_{s=1}^{M} g_{11}^s K(Q_s) & \sum\limits_{s=1}^{M} g_{12}^s K(Q_s) & \cdots & \sum\limits_{s=1}^{M} g_{1G}^s K(Q_s) \\ \sum\limits_{s=1}^{M} g_{21}^s K(Q_s) & K(Q_0) + \sum\limits_{s=1}^{M} g_{22}^s K(Q_s) & & \sum\limits_{s=1}^{M} g_{2G}^s K(Q_s) \\ \vdots & & \ddots & \vdots \\ \sum\limits_{s=1}^{M} g_{G1}^s K(Q_t) & \cdots & \cdots & K(Q_0) + \sum\limits_{s=1}^{M} g_{GG}^s K(Q_s), \end{bmatrix}$$

and

$$U := \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_G \end{bmatrix}, \quad \text{and} \quad F := \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_G \end{bmatrix}.$$

By using Kronecker product $\otimes$, $\mathbb{K}(Q) = \sum\limits_{s=0}^{M} G^s \otimes K(Q_s)$ and we can express this system in the following form

$$\left[\sum_{s=0}^{M} G^s \otimes K(Q_s) \right] U = F. \tag{6.4}$$

6.1. **Discrete ELS.** The ELS-based optimization problem reads

$$\begin{aligned} \min_{q \in K} J_\kappa(q) = & \frac{1}{2} \int_\Gamma \sigma(y) \int_D q(y,x)\Delta(u_q - z)\Delta(u_q - z)dxdy \\ & + \frac{\kappa}{2} \int_\Gamma \sigma(y) \int_D \|q(y,x)\|_{H^1(\Omega)}^2 \, dxdy, \end{aligned}$$

where $u_q(y,x)$ is the solution to (5.1).

For simplicity, we set $v(y,x) := u(y,x) - z(y,x)$, and by using (6.3), we obtain

$$J(q) = \frac{1}{2}\sum_{s=0}^{M}\int_\Gamma \sigma(y) y_s \int_\Omega q_s(x)\Delta v(y,x)\Delta v(y,x)\,dxdy.$$

Taking $v(y,x) = \sum_{k=1}^{N}\sum_{m=1}^{G} V_{km}\Phi_k(x)\Psi_m(y)$, and using the notation (6.2), it can be shown that

$$J(Q) = \frac{1}{2}(U-Z)^\top \left(\sum_{s=0}^{M} G^s \otimes K(Q^s)\right)(U-Z).$$

Moreover,

$$R(Q) = \frac{\kappa}{2} Q^\top \left(\Psi \otimes (H_Q + K_Q)\right) Q,$$

where by $\Psi \in \mathbb{R}^{(M+1)\times(M+1)}$, $K_Q,\ H_Q \in \mathbb{R}^{P\times P}$ are given by

$$\begin{aligned}
\Psi_{s,t} &= \int_\Gamma \sigma(y) y_s y_t d\mathbf{y} \quad \text{for every } s,t = 0\dots,M,\\
(H_Q)_{i,j} &= \int_D \varphi_j(x)\varphi_i(x)dx \text{ for every } i,j = 1,\dots,P,\\
(K_Q)_{i,j} &= \int_D \nabla\varphi_j(x)\nabla\varphi_i(x)dx \text{ for every } i,j = 1,\dots,P.
\end{aligned}$$

Summarizing, we have

$$J_\kappa(Q) = \frac{1}{2}(U-Z)^\top \left[\sum_{s=0}^{M} G^s \otimes K(Q^s)\right](U-Z) + \frac{\kappa}{2}\, Q^\top \left(\Psi \otimes (H_Q + K_Q)\right) Q.$$

We recall that the continuous derivative formula is given by

$$DJ(q)(\delta q) = -\frac{1}{2}\int_\Gamma \sigma(y)\int_D \delta q(y,x)\Delta(u_q + z)\Delta(u_q - z)dxdy.$$

By employing the same reasoning for discretization, we have

$$DJ(Q)(\delta Q) = -\frac{1}{2}(U+Z)^\top \left[\sum_{s=0}^{M} G^s \otimes K(\delta Q_s)\right](U-Z)\,. \tag{6.5}$$

For an explicit gradient formula, we will use the notion of the adjoint matrix. We recall that a matrix $L(V) \in \mathbb{R}^{N\times P}$ is called adjoint stiffness matrix if

$$L(V)B = K(B)V, \quad \text{for every } B \in \mathbb{R}^P, V \in \mathbb{R}^N.$$

Then, it can be shown that

$$\begin{aligned}
\nabla J(Q) = -\frac{1}{2}\big[\,&(U+Z)^\top \left(G^0 \otimes I_N\right) L(U-Z)\\
&\cdots (U+Z)^\top \left(G^M \otimes I_N\right) L\,(U-Z)\,\big],
\end{aligned}$$

where we used the following vectorized notation:

$$L(U-Z) = \begin{bmatrix} L(U_1 - Z_1) \\ L(U_2 - Z_2) \\ \vdots \\ L(U_G - Z_G) \end{bmatrix}.$$

Summarizing,

$$\nabla J_k(Q) = -\frac{1}{2}[(U+Z)^\top \left(G^0 \otimes I_N\right) L(U-Z) \\ \cdots (U+Z)^\top \left(G^M \otimes I_N\right)(U-Z)] + \kappa Q^\top \left(\Psi \otimes (H_Q + K_Q)\right).$$

6.2. **Discrete OLS.** For the numerical experiments, we will consider the following variant of the OLS:

$$\min_{q\in K} \widehat{J}_\kappa(q) = \frac{1}{2}\int_\Gamma \sigma(y)\int_D (u(y,x) - z(y,x))^2 dxdy + \\ \frac{1}{2}\int_\Gamma \sigma(y)\int_D |\Delta\left(u(y,x) - z(y,x)\right)|^2 \, dxdy + \frac{\kappa}{2}\int_\Gamma \sigma(y)\int_D \|q(y,x)\|^2_{H^1(\Omega)} \, dxdy.$$

Setting $v = u(y,x) - z(y,x)$, the discrete OLS (without the regularizer) is:

$$\widehat{J}(Q) = V^\top \left(I_Q \otimes (K_U + K_U)\right) V,$$

where,

$$(H_U)_{i,j} = \int_D \Phi_j(x)\Phi_i(x)dx,$$
$$(K_U)_{i,j} = \int_D \Delta\Phi_i(x)\Delta\Phi_j(x)dx.$$

Consequently, adding the discrete regularizing term, we obtain

$$\widehat{J}_k(Q) = \frac{1}{2}(U-Z)^\top \left(I_Q \otimes (K_U + H_U)\right)(U-Z) + \frac{\kappa}{2}Q^\top(\Psi \otimes (K_Q + H_Q))Q.$$

By using the adjoint approach, the continuous derivative is given by

$$DJ(q)(\delta q) = \int_\Gamma \sigma(y)\int_D \delta q(y,x)\Delta u(y,x)\Delta w(y,x)dxdy, \tag{6.6}$$

where $w \in V$ is the adjoint solution so that for every $v \in V$, we have

$$\int_\Gamma \sigma(y)\int_D q(y,x)\Delta w(y,x)\Delta v(y,x)dxdy \\ = \int_\Gamma \sigma(y)\int_D \left(z(y,x) - u(y,x) + \Delta\left(z(y,x) - u(y,x)\right)\right) v(y,x)dxdy.$$

As before, the discrete adjoint equation can be shown to be

$$\left[\sum_{s=0}^{M} G^s \otimes K(Q_s)\right] W = P,$$

where W is the discrete adjoint solution and

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_G \end{bmatrix} \in \mathbb{R}^{NG}$$

for every $n \in \{1, ..., G\}$, is given by

$$(P_n)_i = \int_\Gamma \sigma(y)\Psi_n(y) \int_\Omega \left(z(y,x) - u(y,x) + \Delta\left(z(y,x) - u(y,x)\right)\right)\Phi_i(x)dxdy \ .$$

By a standard discretization,

$$(P_n)_i = \sum_{k=1}^{N} \left(\int_D \Phi_k(x)\Phi_i(x)dx + \int_D \Delta\Phi_k(x)\Delta\Phi_i(x)dx \right)(Z_{kn} - U_{kn}),$$

implying that

$$P_n = (Q_U + K_U)(Z_n - U_n).$$

Denoting this by $P = (I_N \otimes (Q_U + K_U))(Z - U)$, we obtain

$$\left[\sum_{s=0}^{M} G^s \otimes K(Q_s)\right] W = \left[I_N \otimes (Q_U + K_U)\right](Z - U).$$

Analogously as in (6.5), the discrete version of (6.6) is given by

$$DJ(Q)(\delta Q) = \sum_{s=0}^{M} \left(\sum_{i,j=1}^{Q} g_{ij}^s (U_i + Z_i)^\top L(U_j - Z_j) \right) \delta Q_s,$$

while, the corresponding gradient, by

$$\nabla J(Q) = \left[\ U^\top \left(G^0 \otimes I_N\right) L(W) \quad \cdots \quad U^\top \left(G^M \otimes I_N\right) L\left(W\right)\ \right].$$

Summarizing, we have

$$\begin{aligned} \nabla J_k(Q) = &\left[\ U^\top \left(G^0 \otimes I_N\right) L(W) \quad \cdots \quad U^\top \left(G^M \otimes I_N\right) L\left(W\right)\ \right] \\ &+ \kappa Q^\top \left(\Psi \otimes (H_Q + K_Q)\right). \end{aligned}$$

7. Numerical examples

Following (6.3), we assume that q admits a finite linear combination:

$$q(\omega, x) = q_0(x) + \sum_{i=1}^{M} q_i(x) Y_i(\omega). \tag{7.1}$$

In the examples below, the distribution of random variables $\{Y_i(\omega)\}_{i=1}^{Q}$ is assumed to be known a priori. We consider piecewise linear elements for V_h and Q_h over the same nodes, where V_h is defined on interior nodes and Q_h also incorporates the boundary nodes.

We test two different optimization approaches, the ELS and the OLS objectives. In the examples, we consider exact data $z = \bar{u}$. All the experiments were carried out on a computer with Intel(R) Core(TM) i5-8250U CPU at 1.60GHz and 8 GB of memory by using Matlab (2019). In particular, each optimization problem is solved by using interior point algorithm implementation provided by Matlab `fmincon` routine (Matlab 2019).

We measure the expectation and variance of the identification (relative) error functional. For example, for the particular case of the ELS objective, we measure the identification error by the quantities

$$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(q) = \frac{\mathbb{E}\left(\sqrt{\int_D (\bar{q}_h(\omega,z) - q^{\mathrm{M}}_h(\omega,x))dx}\right)}{\mathbb{E}\left(\sqrt{\int_D \bar{q}_h(\omega,x)^2 dx}\right)},$$

$$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(q) = \frac{\mathrm{Var}\left(\sqrt{\int_D (\bar{q}_h(\omega,x) - q^{\mathrm{M}}_h(\omega,x)dx)}\right)}{\mathrm{Var}\left(\sqrt{\int_D \bar{q}_h(\omega,x)^2 dx}\right)}$$

where q^{M}_h corresponds with the estimated parameter using the ELS-approach. Similarly, we measure the simulated data error by the quantities

$$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(u) = \frac{\mathbb{E}\left(\sqrt{\int_D (\bar{u}_h(\omega,x) - u_h(q^{\mathrm{M}}_h)(\omega,x))^2 dx}\right)}{\mathbb{E}\left(\sqrt{\int_D \bar{u}_h(\omega,x)^2 dx}\right)},$$

$$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(u) = \frac{\mathrm{Var}\left(\sqrt{\int_D (\bar{q}_h(\omega,x) - u_h(q^{\mathrm{M}}_h)(\omega,x))^2 dx)}\right)}{\mathrm{Var}\left(\sqrt{\int_D \bar{u}_h(\omega,x)^2 dx}\right)},$$

where $\bar{q}_h$ and $\bar{u}_h$ corresponds with the interpolants of parameter $\bar{q}$ and $\bar{u}$.

Example 1. For $D = (0,1)$, we take

$$\bar{q}(\omega,x) = 1 + Y_1(\omega),$$
$$\bar{u}(\omega,x) = x^2(1-x)^2 + Y_1(\omega)(1-\cos(2\pi x)),$$

which correspond to

$$\begin{aligned} f(\omega,x) &= \frac{d^2}{dx^2}((1+Y_1(\omega))\frac{d^2}{dx^2}(x^2(1-x)^2 + Y_1(\omega)(1-\cos(2\pi x))) \\ &= -\left(16\pi^4 Y_1(\omega)\cos 2\pi x - 24\right)\left(Y_1(\omega)+1\right), \end{aligned}$$

where $Y_1(\omega) \sim U[0,1]$ is uniformly distributed over interval $[0,1]$.

Since we only have one dimension of stochasticity, we consider $\sigma(y) = 1$ and take orthonormal Legendre polynomials on $[0,1]$. As mentioned above, we use the Stochastic-Galerkin method for discretization. In Table 1, we check its accuracy on (6.4) for exact interpolated $\bar{q}$ and f.

We fix κ =1e-06, which gives a stable reconstruction for the considered discretization levels. In general, reconstruction is excellent for both approaches, ELS and H^2-OLS, and both are comparable, as we can see in Tables 2 and 3. We can visually check the quality of reconstruction in Figures 1 and Figures 2. The samples of the estimated parameter were randomly generated by taking into account representation (7.1). Reconstruction in simulated data is visually nearly flawless, see Figures 3 and 4.

Remark 7.1. We chose the value of the regularization parameter κ based on extensive but heuristic computational experimentation. However, it is of interest to conduct a systematic regularization parameter search by employing specially designed methods such as the L-curve or the Morozov principle. However, it should be noted that a precise theory for choosing a regularization parameter for nonlinear inverse problems, such as ours, is somewhat limited (as opposed to the linear inverse problems), and most selection criteria remain largely heuristic.

TABLE 1. Stochastic-Galerkin discretization Error for Example 1.

$\dim V_h$	$\frac{\mathbb{E}\left(\sqrt{\int_D(\bar{u}(\omega,x)-\bar{u}_h(y,x))^2dx}\right)}{\mathbb{E}\left(\sqrt{\int_D(\bar{u}(\omega,x)^2dx}\right)}$	$\frac{Var\left(\sqrt{\int_D(\bar{u}(\omega,x)-\bar{u}_h(\omega,x)^2dx)}\right)}{Var\left(\sqrt{\int_D\bar{u}(\omega,x))^2dx}\right)}$
50	3.8294e-04	7.6832e-04
100	5.1911e-05	1.0391e-04
150	1.5808e-05	3.1619e-05
200	6.7612e-06	1.3519e-05

TABLE 2. Example 1. Numerical errors for $\kappa = 1e - 06$ (ELS).

$\dim V_h$	$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(a)$	$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(a)$	$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(u)$	$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(u)$	CPU time
50	6.7654e-04	2.0241e-03	4.2899e-07	9.9110e-07	0.95 s.
100	1.2484e-04	3.7481e-04	2.2944e-08	2.9078e-08	1.97 s.
150	4.5998e-05	1.3841e-04	8.8905e-09	1.3484e-08	4.28 s.
200	2.2681e-05	7.2722e-05	8.1806e-09	1.8736e-08	7.31 s.

TABLE 3. Example 1. Numerical errors for $\kappa = 1e - 06$ (H^2-OLS).

$\dim V_h$	$\varepsilon^{H^2-OLS}_{\mathrm{mean}}(a)$	$\varepsilon^{H^2-OLS}_{\mathrm{var}}(a)$	$\varepsilon^{H^2-OLS}_{\mathrm{mean}}(u)$	$\varepsilon^{H^2-OLS}_{\mathrm{var}}(u)$	CPU time
50	6.7650e-04	2.0240e-03	4.2884e-07	9.9307e-07	1.04 s.
100	1.2483e-04	3.7481e-04	2.3690e-08	2.9018e-08	2.56 s.
150	4.6009e-05	1.3837e-04	1.1723e-08	1.0608e-08	5.24 s.
200	2.2660e-05	6.8360e-05	1.1375e-08	9.4175e-09	10.4 s.

Example 2. We take $D = (0, 1)$ and consider the parameter two degrees of stochasticity

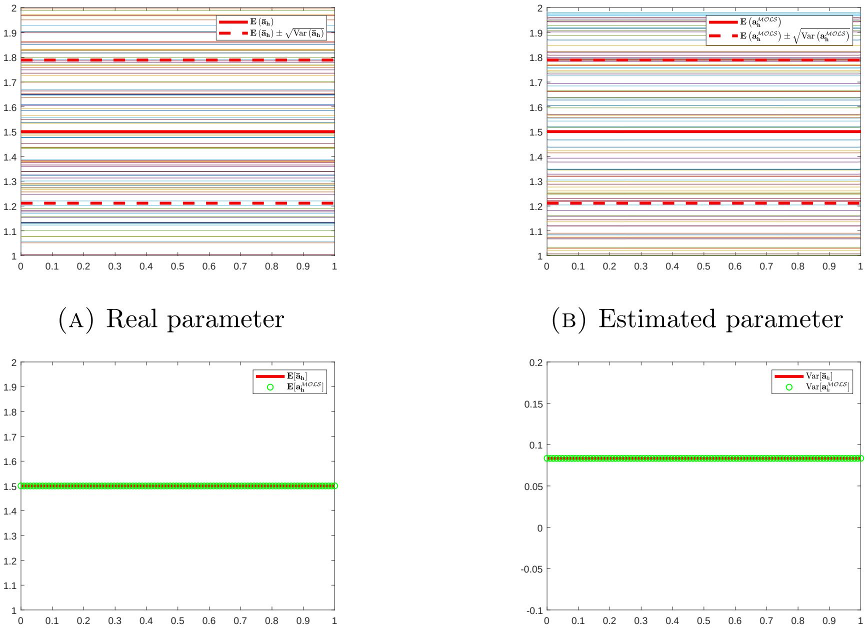
$$q(w, x) = 3 + x^2 + Y_1(\omega)\cos(\pi x) + Y_2(\omega)\sin(2\pi x),$$

and the solution

$$u(\omega, x) = x^2(1 - x)^2 Y_1(\omega)$$

where $Y_1(\omega)$, $Y_2(\omega) \sim U[0, 1]$ are uniformly distributed over $[0, 1]$, and

$$f(y_1, y_2, x) = p_1(x)Y_1(\omega) + p_2(x)Y_1(\omega)Y_2(\omega) + p_3(x)Y_2^2(\omega)$$



(A) Real parameter (B) Estimated parameter

(C) Expected value comparison (D) Variance comparison

FIGURE 1. Example 1. Comparison between the exact and the estimated parameter (ELS). M=100 samples.

where

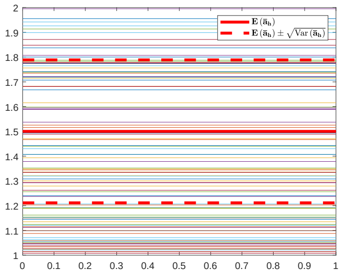
$$
\begin{aligned}
p_1(x) &= 76 - 72x + 144x^2, \\
p_2(x) &= 24\sin(2\pi x) - 8\pi^2\sin(2\pi x) - 48\pi\cos(2\pi x) + 48\pi^2 x\sin(2\pi x) \\
&\quad - 48\pi^2x^2\sin(2\pi x) + 96\pi x\cos(2\pi x), \\
p_3(x) &= 24\cos\pi x + 24\pi\sin\pi x - 2\pi^2\cos\pi x - 12\pi^2x^2\cos\pi x \\
&\quad - 48\pi x\sin\pi x + 12\pi^2 x\cos(\pi x).
\end{aligned}
$$

Here the stochastic domain is given by $\Gamma = [0,1]\times[0,1]$. We have $\sigma(y_1, y_2) = 1$, and orthonormal Legendre polynomials on $[0,1]\times[0,1]$, which are defined as tensorial product of the one dimensional ones. In Table 4 we show the accuracy of the stochastic Galerkin approach. .

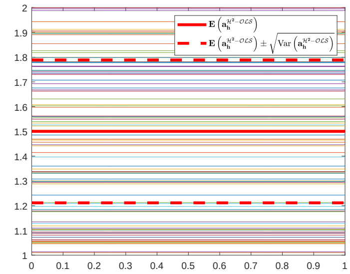
We again choose the regularization parameter κ =1e-06. Numerical results are given in Tables 3 and 5. In this case, for the parameters and optimization solver considered, both approaches give very good reconstruction for the parameter, check Figures 5 and 6, and nearly perfect for the simulated data, see Figures 7 and 8.

Summarizing, for these two examples, both the ELS and the H^2-OLS are quite comparable, but the ELS approach gives a slightly better reconstruction and significantly faster.

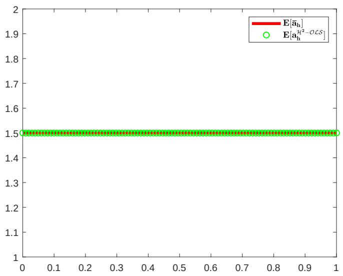
Remark 7.2. However, we should emphasize that the experiments are synthetic, and the data vector are computed and measured. In other words, there is some



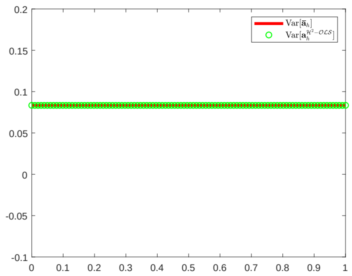
(A) Real parameter



(B) Estimated parameter



(C) Expected value comparison



(D) Variance comparison

FIGURE 2. Example 1. Comparison between the exact and the estimated parameter (H^2-OLS). M=100 samples.

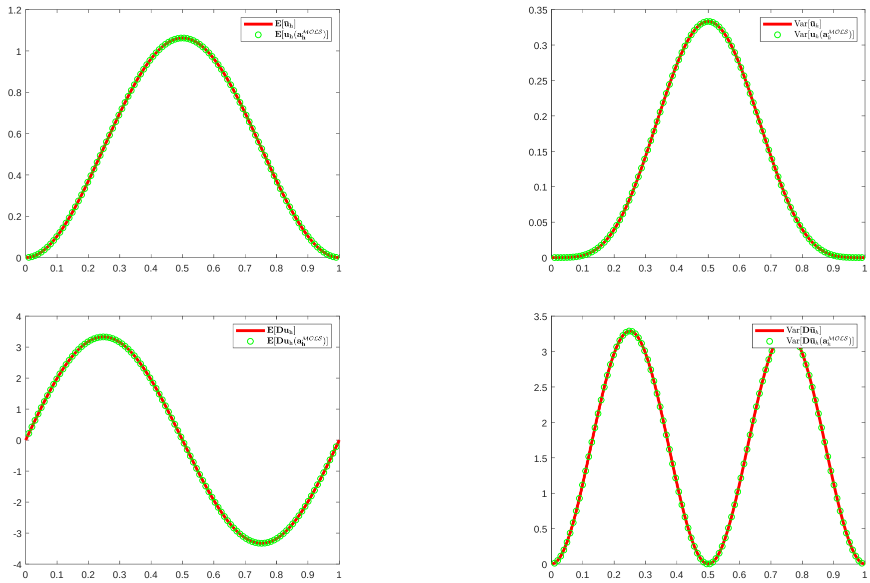


FIGURE 3. Example 1. Comparison between the exact and the simulated data (ELS).

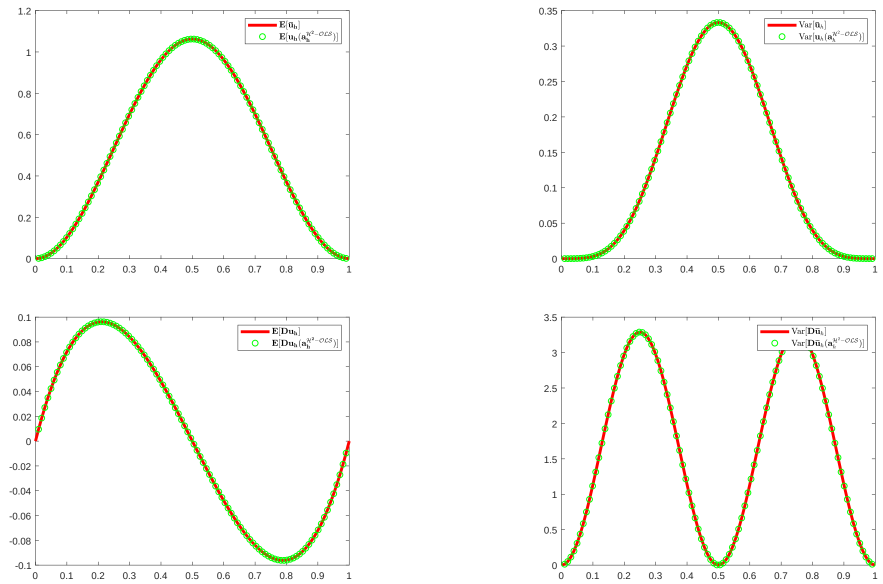


FIGURE 4. Example 1. Comparison between the exact and the simulated data (H^2-OLS).

modeling error in the data but no measurement error. It would be of interest to conduct detailed numerical experimentation using noisy data.

TABLE 4. Stochastic-Galerkin discretization Error for Example 2.

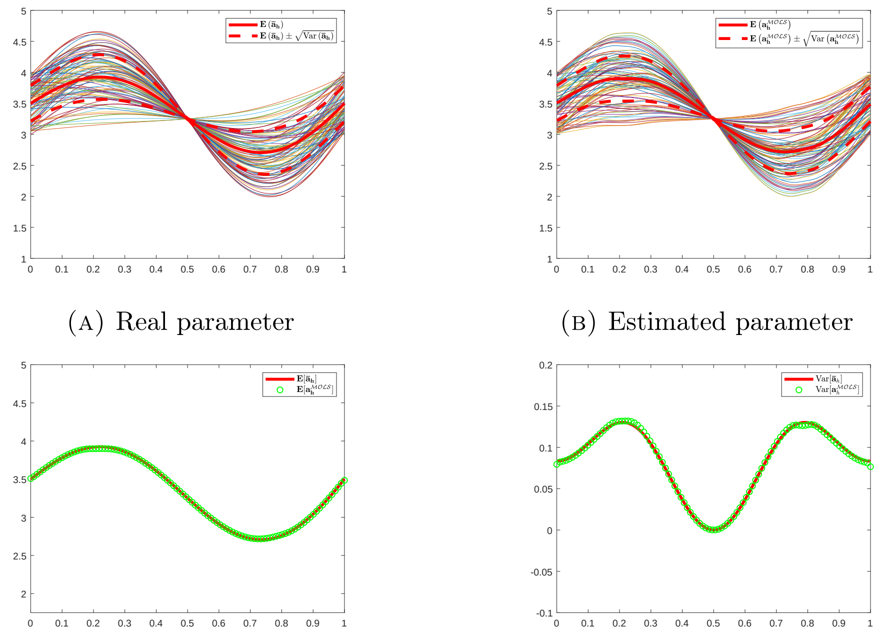
$\dim V_h$	$\frac{\mathbb{E}\left(\sqrt{\int_D (\bar{u}(\omega,x)-\bar{u}_h(y,x))^2 dx}\right)}{\mathbb{E}\left(\sqrt{\int_D (\bar{u}(\omega,x)^2 dx}\right)}$	$\frac{Var\left(\sqrt{\int_D (\bar{u}(\omega,x)-\bar{u}_h(\omega,x)^2 dx)}\right)}{Var\left(\sqrt{\int_D (\bar{u}(\omega,x))^2 dx)}\right)}$
50	1.3079e-03	3.4486e-03
100	3.0580e-04	8.3439e-04
150	1.3220e-04	3.6552e-04
200	7.3253e-05	2.0397e-04

TABLE 5. Example 2. Numerical errors for κ =1-e06 (ELS).

$\dim V_h$	$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(a)$	$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(a)$	$\varepsilon^{\mathrm{M}}_{\mathrm{mean}}(u)$	$\varepsilon^{\mathrm{M}}_{\mathrm{var}}(u)$	CPU time
50	2.2279e-03	2.1234e-02	4.4820e-05	1.1115e-04	12.8 s.
100	2.3245e-03	1.5148e-02	4.6350e-05	1.2426e-04	8.4 s.
150	2.3953e-03	1.5834e-02	4.9231e-05	1.4682e-04	19.2 s.
200	2.4360e-03	1.7765e-02	5.2233e-05	1.6908e-04	32.4 s.

TABLE 6. Example 2. Numerical errors for κ =1e-06 (H^2-OLS).

$\dim V_h$	$\varepsilon_{\text{mean}}^{OLS}(a)$	$\varepsilon_{\text{var}}^{\mathcal{OLS}}(a)$	$\varepsilon_{\text{mean}}^{OLS}(u)$	$\varepsilon_{\text{var}}^{OLS}(u)$	CPU time
50	4.6957e-03	3.3054e-02	1.8273e-04	3.3926e-04	11.3 s.
100	4.9910e-03	2.8096e-02	1.9164e-04	3.9573e-04	10.5 s.
150	5.0516e-03	2.9961e-02	2.0315e-04	4.6662e-04	18.7 s.
200	5.1228e-03	3.4130e-02	2.1376e-04	5.2858e-04	35.2 s.



(A) Real parameter

(B) Estimated parameter

(C) Expected value comparison

(D) Variance comparison

FIGURE 5. Example 2. Comparison between real and estimated parameter (ELS). M=100 samples.

8. CONCLUDING REMARKS

We investigated the inverse problem of estimating the stochastic flexural rigidity in the fourth-order boundary-value problems. It is of evident significance to derive error estimates for the inverse problem and, as a result, prove the convergence of the discrete problems to the continuous ones.

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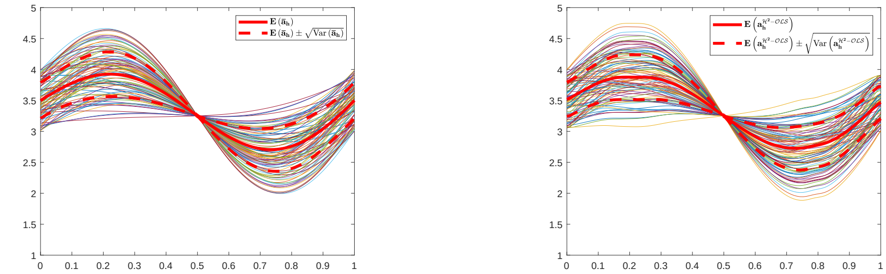
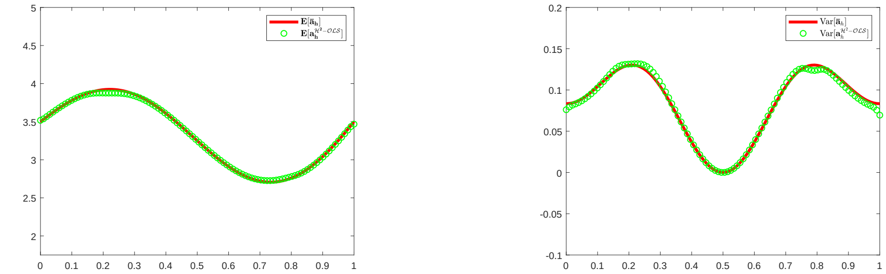


(A) Real parameter (B) Estimated parameter



(C) Expected value comparison (D) Variance comparison

FIGURE 6. Example 2. Comparison between the exact and the estimated parameter (H^2-ELS). M=100 samples.

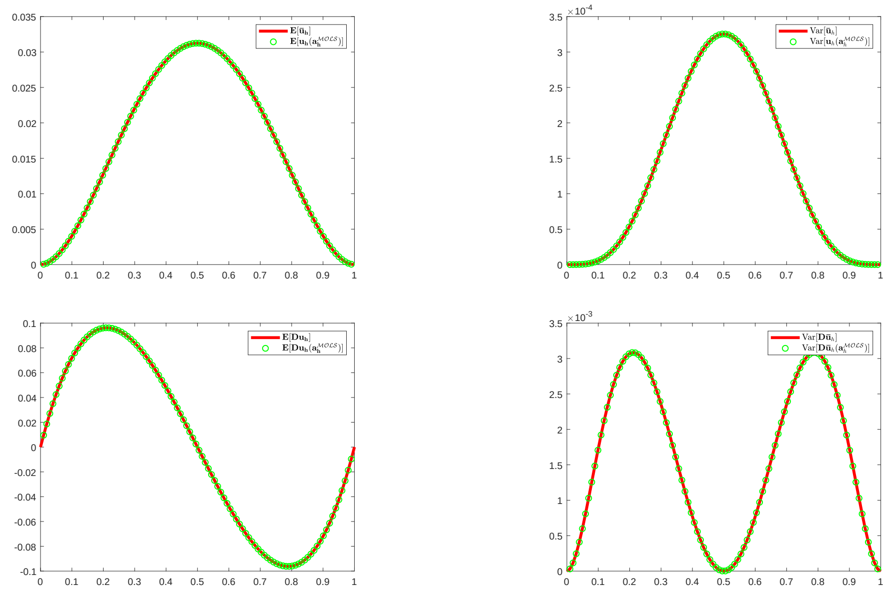


FIGURE 7. Example 2. Comparison between real and simulated data (ELS).

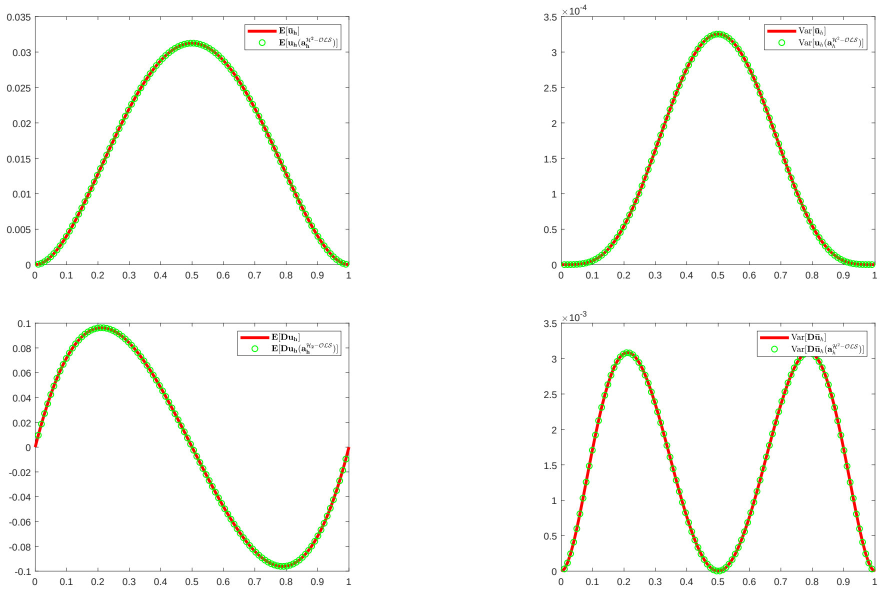


FIGURE 8. Example 2. Comparison between the exact and the simulated data (H^2-OLS).

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