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SADDLE-POINT EQUILIBRIUM SEQUENCE IN A SINGULAR FINITE HORIZON ZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAME WITH DELAYED DYNAMICS

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Abstract. A finite horizon zero-sum linear-quadratic game with point-wise and distributed state delays in the dynamics is considered. The cost functional of this game does not contain a control cost of the minimizing player, which means that the game under the consideration is singular. For this game, we propose definitions of the saddle point equilibrium and the game value. The game is solved by a regularization method. Namely, we replace the original game with a new game for the same delayed dynamics. The functional in this new game is the sum of the functional of the original game and a finite horizon integral of the quadratic form of the minimizing player's control with a small positive coefficient. The new game is regular, and it is a cheap control game. An asymptotic analysis of this cheap control game is carried out in the frames of the singular perturbation theory. This analysis is applied to derive the saddle-point equilibrium and the value of the original singular game.

1. Introduction

A game, which is unsolvable in the frames of first-order solvability conditions, is called singular. A singular zero-sum dynamic game can be solved neither by application of the Isaacs MinMax principle [20], nor by application of the Bellman–Isaacs equation approach [2,20]. Singular zero-sum dynamic games appear in various applications (see e.g. [17,19,24–26,29,30]). In solution of singular zero-sum dynamic games, higher order necessary/sufficient solvability conditions can be helpful (see e.g. [6, 22, 26] and references therein). In some cases, to derive a solution of a singular game, one can use a geometric analysis of the set of all its candidate optimal trajectories (see e.g. [18, 24]). A singular game also can be solved numerically [5]. However, all these methods fail to yield a solution of a singular zero-sum game, having no players' saddle-point controls in the class of non-generalized (regular) functions. Games with such a feature and with *undelayed dynamics* have been studied extensively in the literature (see e.g. [1, 13–15, 25, 28]). To the best of our knowledge, a singular zero-sum game *with delayed dynamics* was studied only in the single work [12] in the literature. In this work, a singular *infinite horizon* zero-sum linear-quadratic differential game with state delays in the dynamics was solved. In

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the present paper, a *finite horizon* zero-sum linear-quadratic game with point-wise and distributed state delays in the equation of dynamics is considered. It should be noted that this equation is more general than the one considered in [12]. The cost functional of the game, considered in the present paper, does not contain a control cost of the minimizing player (the minimizer). This means that the game cannot be solved by application of the Isaacs MinMax principle and by the Bellman-Isaacs equation approach, i.e., this game is singular. Moreover, this game does not have, in general, a saddle-point equilibrium in the class of regular functions. For this game, we propose novel definitions of the saddle-point equilibrium and game value. We solve this game by a regularization method, consisting in an approximate replacing the original singular game with a new regular game. Namely, the dynamic equation of the new game is the same as in the original game, while the cost functional in the new game is the sum of the original cost functional and a finite horizon integral of the square of the minimizer's control with a small weight coefficient (a small positive parameter). Due to the latter, the new game is a cheap control game. For this new game, we derive solvability conditions, as well as a saddle-point equilibrium and a game value. The derived solvability conditions reduce the solution of the regular cheap control game to solution of a hybrid system of Riccati-type matrix ordinary and partial differential equations subject to boundary conditions. The matrix of the coefficients in the quadratic terms of these equations is symmetric but indefinite. The system of the Riccati-type equations and the boundary conditions (the boundary-value problem) are perturbed by the small parameter appearing in the cost functional of the cheap control game. Subject to a proper assumption, an asymptotic solution to this boundary-value problem is constructed and justified. Using this asymptotic solution, the existence of the saddle-point equilibrium and the value of the original singular game is established, and their expressions are derived.

The paper is organized as follows. In the next section, the game is rigorously formulated, including main notions and definitions. Objectives of the paper are outlined. In Section 3, some auxiliary results are obtained. The regularization of the original singular game is carried out in Section 4, yielding a cheap control zero-sum differential game with state delays in the dynamics. The solvability conditions of this game in a class of state-feedback players' controls are established. The saddle-point equilibrium and the game value are derived. In Section 5, the zero-order asymptotic solution of the hybrid system of the Riccati-type equations, arising in the solvability conditions for the cheap control game, is constructed and justified in the frames of the singular perturbations theory. In Section 6, the cheap control game is transformed equivalently to a game with a singularly perturbed dynamics. Based on the latter game, a reduced zero-sum differential game is derived. A saddle-point state-feedback solution of the reduced game is obtained. In Section 7, using the results of the previous sections, the existence of the saddle-point equilibrium sequence and the game value in the original singular game is established. The expressions of these sequence and value are derived. Conclusions are placed in Section 8.

2. Game Formulation: Main Notions and Definitions

We consider the following differential time delay system describing the dynamics of the game:

$$\frac{dx(t)}{dt} = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\eta)x(t+\eta)d\eta + C_1 v(t), \tag{2.1}$$

$$\frac{dy(t)}{dt} = A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h) + H_{22}y(t-h) + \int_{-h}^{0} \big[G_{21}(\eta)x(t+\eta) + G_{22}(\eta)y(t+\eta)\big]d\eta + u(t) + C_2 v(t), \tag{2.2}$$

where $t \in [0, t_f]$, ($t_f > 0$ is a prescribed time instant); $h > 0$ is a given constant time delay; $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$; $\big(x(t), x(t+\eta)\big)$, $\big(y(t), y(t+\eta)\big)$, $\eta \in [-h, 0)$, are state variables; $u(t) \in \mathbb{R}^m$ and $v(t) \in \mathbb{R}^s$ are players' controls; A_{ij}, H_{i1}, C_i, $(i = 1, 2;\ j = 1, 2)$, and H_{22} are given constant matrices of corresponding dimensions; $G_{i1}(\eta)$, $(i = 1, 2)$, and $G_{22}(\eta)$ are given matrix-valued functions of corresponding dimensions continuous in the interval $[-h, 0]$.

The system (2.1)-(2.2) is considered subject to the initial conditions

$$x(\eta) = \varphi_x(\eta), \quad y(\eta) = \varphi_y(\eta), \quad \eta \in [-h, 0); \qquad x(0) = x_0, \quad y(0) = y_0, \tag{2.3}$$

where $\varphi_x(\eta) \in L^2[-h, 0; \mathbb{R}^n]$ and $\varphi_y(\eta) \in L^2[-h, 0; \mathbb{R}^m]$ are given vector-valued functions; $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$ are given vectors.

The cost functional of the game, to be minimized by the control u (the minimizer's control) and maximized by the control v (the maximizer's control), is

$$J(u, v) = \int_0^{t_f} \Big[x^T(t)D_1x(t) + y^T(t)D_2y(t) - v^T(t)Mv(t)\Big]dt, \tag{2.4}$$

where D_1, D_2 and M are symmetric matrices of corresponding dimensions; D_1 is positive semi-definite, while D_2 and M are positive definite.

Since a quadratic control cost of the minimizer does not appear in the functional (2.4), the game (2.1)-(2.4) cannot be solved by the Isaacs's MinMax principle and by the Bellman-Isaacs equation method, i.e., it is singular. Moreover, this game does not have, in general, a minimizer's saddle-point control among regular (non-generalized) functions. In what follows, the game (2.1)-(2.4) is called the Singular Differential Game (SDG).

Denote:

$$z \stackrel{\triangle}{=} \mathrm{col}(x, y), \quad z_0 \stackrel{\triangle}{=} \mathrm{col}(x_0, y_0), \quad \varphi(\eta) \stackrel{\triangle}{=} \mathrm{col}\big(\varphi_x(\eta), \varphi_y(\eta)\big), \tag{2.5}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, while x_0, y_0, $\varphi_x(\eta)$, $\varphi_y(\eta)$ are given in (2.3).

Let $\psi(\eta)$ be any vector-valued function from the space $L^2[-h, 0; \mathbb{R}^{n+m}]$. For all $t \in [0, t_f]$, let us consider the t-parametric set $\mathcal{U}_t$ of all vector-valued continuous functionals $u\big[z, \psi(\eta), t\big] : \mathbb{R}^{n+m} \times L^2[-h, 0; \mathbb{R}^{n+m}] \to \mathbb{R}^m$, and the t-parametric set $\mathcal{V}_t$ of all vector-valued continuous functionals $v\big[z,\psi(\eta),t\big] : \mathbb{R}^{n+m} \times L^2[-h,0;\mathbb{R}^{n+m}] \to \mathbb{R}^s$.

Definition 2.1. Let us denote by UV the set of all pairs $\big(u[z_t,t],v[z_t,t]\big)$, $z_t \overset{\triangle}{=} \big\{\big(z(t),z(\theta)\big),\theta \in [t-h,t)\big\}$, satisfying the following conditions: (i) for any fixed $t \in [0,t_f]$, $\big(z(t),z(t+\eta)\big) \in \mathbb{R}^{n+m}\times L^2[-h,0;\mathbb{R}^{n+m}]$ and $u[z_t,t] \in \mathcal{U}_t$, $v[z_t,t] \in \mathcal{V}_t$; (ii) the initial-value problem (2.1)–(2.3) for $u(t) = u[z_t,t]$, $v(t) = v[z_t,t]$, $t \in [0,t_f]$, and any $\varphi_x(\eta) \in L^2[-h,0;\mathbb{R}^n]$, $\varphi_y(\eta) \in L^2[-h,0;\mathbb{R}^m]$, $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$ has the unique absolutely continuous solution $z_{uv}\big(t;z_0,\varphi(\eta)\big) = \mathrm{col}\big(x_{uv}\big(t;z_0,\varphi(\eta)\big),y_{uv}\big(t;z_0,\varphi(\eta)\big)\big)$ in the interval $[0,t_f]$, where for any $t \in [0,t_f]$, $x_{uv}\big(t;z_0,\varphi(\eta)\big) \in \mathbb{R}^n$ and $y_{uv}\big(t;z_0,\varphi(\eta)\big) \in \mathbb{R}^m$; (iii) the time realization $u[z_{uv,t},t]$ of the control $u[z_t,t]$ along the solution $z_{uv}\big(t;z_0,\varphi(\eta)\big)$ belongs to $L^2[0,t_f;\mathbb{R}^m]$; (iv) the time realization $v[z_{uv,t},t]$ of the control $v[z_t,t]$ along the solution $z_{uv}\big(t;z_0,\varphi(\eta)\big)$ belongs to $L^2[0,t_f;\mathbb{R}^s]$.
In what follows, UV is called the set of all admissible pairs of the players' state-feedback controls (strategies) in the SDG.

For any given $u_0[z_t,t] \in \mathcal{U}_t$ and $v_0[z_t,t] \in \mathcal{V}_t$, $t \in [0,t_f]$, we consider the sets

$$\begin{aligned} \mathcal{F}_v\big(u_0[z_t,t]\big) &\overset{\triangle}{=} \{v[z_t,t] \in \mathcal{V}_t, t \in [0,t_f] : \big(u_0[z_t,t],v[z_t,t]\big) \in UV\},\\ \mathcal{F}_u\big(v_0[z_t,t]\big) &\overset{\triangle}{=} \{u[z_t,t] \in \mathcal{U}_t, t \in [0,t_f] : \big(u[z_t,t],v_0[z_t,t]\big) \in UV\}. \end{aligned} \tag{2.6}$$

Let

$$\begin{aligned} \mathcal{H}_u &\overset{\triangle}{=} \{u[z_t,t] \in \mathcal{U}_t, t \in [0,t_f] : \mathcal{F}_v\big(u[z_t,t]\big) \neq \emptyset\},\\ \mathcal{H}_v &\overset{\triangle}{=} \{v[z_t,t] \in \mathcal{V}_t, t \in [0,t_f] : \mathcal{F}_u\big(v[z_t,t]\big) \neq \emptyset\}. \end{aligned} \tag{2.7}$$

Definition 2.2. For any prechosen $u[z_t,t] \in \mathcal{H}_u$, the value

$$J_u\big(u[z_t,t];z_0,\varphi(\eta)\big) = \sup_{v[z_t,t]\in\mathcal{F}_v\left(u[z_t,t]\right)} J\big(u[z_t,t],v[z_t,t]\big) \tag{2.8}$$

is called the guaranteed result of $u[z_t,t]$ in the SDG.

Definition 2.3. For any prechosen $v[z_t,t] \in \mathcal{H}_v$, the value

$$J_v\big(v[z_t,t];z_0,\varphi(\eta)\big) = \inf_{u[z_t,t]\in\mathcal{F}_u\left(v[z_t,t]\right)} J\big(u[z_t,t],v[z_t,t]\big) \tag{2.9}$$

is called the guaranteed result of $v[z_t,t]$ in the SDG.

Consider a sequence of the pairs $\big\{\big(u_q^*[z_t,t],v_q^*[z_t,t]\big)\big\} \in UV$, $(q = 1,2,...)$.

Definition 2.4. We call the sequence $\big\{\big(u_q^*[z_t,t],v_q^*[z_t,t]\big)\big\}_{q=1}^{+\infty}$ a saddle-point equilibrium sequence of the SDG if: (i) the limit value $J^*\big(z_0,\varphi(\eta)\big) \overset{\triangle}{=} \lim_{q\to+\infty} J\big(u_q^*[z_t,t],v_q^*[z_t,t]\big)$ exists and is finite; (ii) for any $u_q[z_t,t] \in \mathcal{F}_u\big(v_q^*[z_t,t]\big)$ and $v_q[z_t,t] \in \mathcal{F}_v\big(u_q^*[z_t,t]\big)$, $(q = 1,2,...)$, the following inequality is satisfied:

$$\limsup_{q\to+\infty} J\big(u_q^*[z_t,t],v_q[z_t,t]\big) \le J^*\big(z_0,\varphi(\eta)\big)$$

$$\leq \liminf_{q\to+\infty} J\big(u_q[z_t,t], v_q^*[z_t,t]\big).$$

(2.10)

The value $J^*\big(z_0, \varphi(\eta)\big)$ is called a value of the SDG.

In what follows, we are going:
(a) to establish conditions, subject to which the saddle-point equilibrium sequence and the value of the SDG exist;
(b) to obtain expressions for these sequence and value.

3. Auxiliary Results

Let us introduce into the consideration the following block matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad H = \begin{pmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{pmatrix}, \quad G(\eta) = \begin{pmatrix} G_{11}(\eta) & 0 \\ G_{21}(\eta) & G_{22}(\eta) \end{pmatrix}, \tag{3.1}$$

$$B = \begin{pmatrix} 0 \\ I_m \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}. \tag{3.2}$$

The matrices A, H, $G(\eta)$ and D are of the dimension $(n+m)\times(n+m)$, while the matrices B and C are of the dimensions $(n+m)\times m$ and $(n+m)\times s$, respectively.

Due to the notations (2.5) and (3.1),(3.2), the system (2.1)-(2.2), the initial conditions (2.3) and the cost functional (2.4) can be rewritten in the equivalent form as:

$$\frac{dz(t)}{dt} = Az(t) + Hz(t-h) + \int_{-h}^{0} G(\eta)z(t+\eta)d\eta + Bu(t) + Cv(t), \quad t \geq 0, \tag{3.3}$$

$$z(\eta) = \varphi(\eta), \qquad \eta \in [-h, 0); \qquad z(0) = z_0, \tag{3.4}$$

$$J(u,v) = \int_0^{t_f} \Big[z^T(t)Dz(t) - v^T Mv(t)\Big]dt. \tag{3.5}$$

Thus, the SDG can be represented in the form (3.3)-(3.5).

In the SDG (3.3)-(3.5), consider the following state-feedback minimizer's control

$$u = u_K[z_t,t] \overset{\triangle}{=} K_1(t)z(t) + \int_{-h}^{0} K_2(t,\eta)z(t+\eta)d\eta, \qquad t \geq 0, \tag{3.6}$$

where $K_1(t)$ is a given $m\times(n+m)$-matrix-values function, continuous in the interval $[0,t_f]$; $K_2(t,\eta)$ is a given $m\times(n+m)$-matrix-valued function, continuous for $(t,\eta) \in [0,t_f]\times[-h,0]$.

Also, consider the hybrid system of one ordinary and two partial differential equations of Riccati type with respect to unknown matrices $\mathcal{P}(t)$, $\mathcal{Q}(t,\eta)$ and $\mathcal{R}(t,\eta,\rho)$ in the domain $\Omega \overset{\triangle}{=} \{(t,\eta,\rho) : t\in[0,t_f],\ \eta\in[-h,0],\ \rho\in[-h,0]\}$:

$$\begin{aligned}\frac{d\mathcal{P}(t)}{dt} &= -\mathcal{P}(t)\big(A + BK_1(t)\big) - \big(A+BK_1(t)\big)^T\mathcal{P}(t) \\ &\quad -\mathcal{P}(t)CM^{-1}C^T\mathcal{P}(t) - \mathcal{Q}(t,0) - \mathcal{Q}^T(t,0) - D,\end{aligned} \tag{3.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}\right)\mathcal{Q}(t,\eta)=-\big(A+BK_1(t)\big)^T\mathcal{Q}(t,\eta)$$
$$-\mathcal{P}(t)CM^{-1}C^T\mathcal{Q}(t,\eta)-\mathcal{P}(t)\big(G(\eta)+BK_2(t,\eta)\big)$$
$$-\mathcal{R}(t,0,\eta), \tag{3.8}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\rho}\right)\mathcal{R}(t,\eta,\rho)=-\big(G(\eta)+BK_2(t,\eta)\big)^T\mathcal{Q}(t,\rho)$$
$$-\mathcal{Q}^T(t,\eta)\big(G(\rho)+BK_2(t,\rho)\big)-\mathcal{Q}^T(t,\eta)CM^{-1}C^T\mathcal{Q}(t,\rho). \tag{3.9}$$

The system (3.7)-(3.9) is subject to the boundary conditions

$$\mathcal{P}(t_f)=0,\quad \mathcal{Q}(t_f,\eta)=0,\quad \mathcal{R}(t_f,\eta,\rho)=0,\quad \eta\in[-h,0],\quad \rho\in[-h,0], \tag{3.10}$$

$$\mathcal{Q}(t,-h)=\mathcal{P}(t)H,$$
$$\mathcal{R}(t,-h,\eta)=H^T\mathcal{Q}(t,\eta),\qquad \mathcal{R}(t,\eta,-h)=\mathcal{Q}^T(t,\eta)H,$$
$$t\in[0,t_f],\quad \eta\in[-h,0]. \tag{3.11}$$

Lemma 3.1. *Let the problem (3.7)-(3.9),(3.10)-(3.11) have a continuous solution* $\{\mathcal{P}_K(t),\mathcal{Q}_K(t,\eta),\mathcal{R}_K(t,\eta,\rho)\}$, $(t,\eta,\rho)\in\Omega$, *such that* $\mathcal{P}_K^T(t)=\mathcal{P}_K(t)$, $\mathcal{R}_K^T(t,\eta,\rho)=\mathcal{R}_K(t,\rho,\eta)$. *Then,*
(i) $u_K[z_t,t]\in H_u$;
(ii) *the guaranteed result of* $u_K[z_t,t]$ *in the SDG is*

$$J_u\big(u_K[z_t,t];z_0,\varphi(\eta)\big)=z_0^T\mathcal{P}_K(0)z_0+2z_0^T\int_{-h}^0\mathcal{Q}_K(0,\eta)\varphi(\eta)d\eta$$
$$+\int_{-h}^0\int_{-h}^0\varphi^T(\eta)\mathcal{R}_K(0,\eta,\rho)\varphi(\rho)d\eta d\rho; \tag{3.12}$$

(iii) *this result is attained for the maximizer's control*

$$v[z_t,t]=v_K[z_t,t]\stackrel{\triangle}{=}M^{-1}C^T\left[\mathcal{P}_K(t)z(t)+\int_{-h}^0\mathcal{Q}_K(t,\eta)z(t+\eta)d\eta\right]; \tag{3.13}$$

(iv) *for any* $z_0\in\mathbb{R}^{n+m}$ *and any* $\varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}]$, *the guaranteed result of* $u_K[z_t,t]$ *in the SDG is nonnegative.*

Proof. We start with the first item. To prove the item (i), it is sufficiently to show the existence of $v[z_t,t]\in\mathcal{V}_t$ for all $t\in[0,t_f]$, such that $\big(u_K[z_t,t],v[z_t,t]\big)\in UV$. We choose $v[z_t,t]=v_K[z_t,t]$. For any $t\in[0,t_f]$, the vector-valued functional $v_K[z_t,t]$ is linear with respect to the pair $\big(z(t),z(t+\eta)\big)\in\mathbb{R}^{n+m}\times L^2[-h,0;\mathbb{R}^{n+m}]$. Moreover, since $\mathcal{Q}_K(t,\eta)$ is continuous with respect to $\eta\in[-h,0]$ for any $t\in[0,t_f]$, this vector-valued functional is continuous with respect to $\big(z(t),z(t+\eta)\big)\in\mathbb{R}^{n+m}\times L^2[-h,0;\mathbb{R}^{n+m}]$ for any $t\in[0,t_f]$.

Substitution of the control (3.6) into the original system in the form (3.3) instead of $u(t)$ yields

$$\frac{dz(t)}{dt}=(A+BK_1(t))z(t)+Hz(t-h)$$

$$+\int_{-h}^{0}\big(G(\eta)+BK_2(t,\eta)\big)z(t+\eta)d\eta+Cv(t),\quad t\geq 0. \tag{3.14}$$

By virtue of the results of [4], for any given vector $z_0\in\mathbb{R}^{n+m}$ and function $\varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}]$, the equation (3.14) for $v(t)=v_K[z_t,t]$ and subject to the initial conditions $z(\eta)=\varphi(\eta)$, $\eta\in[-h,0)$, $z(0)=z_0$ has the unique absolutely continuous solution $z_K\big(t;z_0,\varphi(\eta)\big)$, $t\in[0,t_f]$. Hence, $z_K\big(t;z_0,\varphi(\eta)\big)\in L^2[0,t_f;\mathbb{R}^{n+m}]$ and $u_K[z_{K,t},t]\in L^2[0,t_f;E^m]$, $v_K[z_{K,t},t]\in L^2[0,t_f;E^s]$. Due to Definition 2.1, these inclusions mean that
$\big(u_K[z_t,t],v_K[z_t,t]\big)\in UV$. Thus, the item (i) is proven.

Proceed to the items (ii) and (iii). For any $t\in[0,t_f]$, consider the Lyapunov-Krasovskii-like functional

$$V[z_t,t]=z^T(t)\mathcal{P}_K(t)z(t)+2z^T(t)\int_{t-h}^{t}\mathcal{Q}_K(t,\tau-t)z(\tau)d\tau$$
$$+\int_{t-h}^{t}\int_{t-h}^{t}z^T(\tau)\mathcal{R}_K(t,\tau-t,\theta-t)z(\theta)d\tau d\theta. \tag{3.15}$$

For any given $v[z_t,t]\in\mathcal{F}_v\big(u_K[z_t,t]\big)$, and any given $z_0\in\mathbb{R}^{n+m}$ and $\varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}]$, we consider the solution $z_{Kv}\big(t;z_0,\varphi(\eta)\big)$, $t\in[0,t_f]$, of the problem (3.3)-(3.4) with $u(t)=u_K[z_t,t]$, $v(t)=v[z_t,t]$. By $V_{Kv}(t)$, let us denote the time realization of the functional (3.15) along this solution, i.e., $V_{Kv}(t)\stackrel{\triangle}{=}V[z_{Kv,t},t]$. Also, by $v[z_{Kv,t},t]$ and $v_K[z_{Kv,t},t]$, we denote the time realizations of the controls $v[z_t,t]$ and $v_K[z_t,t]$, respectively, along the solution $z_{Kv}\big(t;z_0,\varphi(\eta)\big)$.

Now, differentiating the function $V_{Kv}(t)$, we obtain the following expression for its derivative for all $t\in[0,t_f]$ (in this expression for the sake of simplicity we omit the designation of the dependence of $z_{Kv}\big(t;z_0,\varphi(\eta)\big)$ on z_0 and $\varphi(\eta)$):

$$\begin{aligned}\frac{dV_{Kv}(t)}{dt}&=2\left(\frac{dz_{Kv}(t)}{dt}\right)^T\left(\mathcal{P}_K(t)z_{Kv}(t)+\int_{t-h}^{t}\mathcal{Q}_K(t,\tau-t)z_{Kv}(\tau)d\tau\right)\\&+z_{Kv}^T(t)\frac{d\mathcal{P}_K(t)}{dt}z_{Kv}(t)\\&+2z_{Kv}^T(t)\bigg[\mathcal{Q}_K(t,0)z_{Kv}(t)-\mathcal{Q}_K(t,-h)z_{Kv}(t-h)\\&+\int_{t-h}^{t}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}\right)\mathcal{Q}_K(t,\eta)\bigg|_{\eta=\tau-t}z_{Kv}(\tau)d\tau\bigg]\\&+2z_{Kv}^T(t)\int_{t-h}^{t}\mathcal{R}_K(t,0,\theta-t)z_{Kv}(\theta)d\theta\\&-2z_{Kv}^T(t-h)\int_{t-h}^{t}\mathcal{R}_K(t,-h,\theta-t)z_{Kv}(\theta)d\theta\\&+\int_{t-h}^{t}\int_{t-h}^{t}z_{Kv}^T(\tau)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\rho}\right)\mathcal{R}_K(t,\eta,\rho)\bigg|_{\eta=\tau-t,\rho=\theta-t}z_{Kv}(\theta)d\tau d\theta.\end{aligned}\tag{3.16}$$

Remember that $z_{Kv}(t)$ satisfies the equation (3.14) for $v(t) = v[z_t, t] \in \mathcal{F}_v\big(u_K[z_t, t]\big)$. Taking into account this fact, changing the variables $\tau - t = \eta$, $\theta - t = \rho$ in the integrals of the equation (3.16), and using the boundary-value problem (3.7)-(3.9),(3.10)-(3.11), we can rewrite (3.16) as:

$$\begin{aligned}\frac{dV_{Kv}(t)}{dt} &= -z_{Kv}^T(t)\big(D + \mathcal{P}_K(t)CM^{-1}C^T\mathcal{P}_K(t)\big)z_{Kv}(t)\\ &+ 2\big(v[z_{Kv,t}, t]\big)^T C^T\left(\mathcal{P}_K(t)z_{Kv}(t) + \int_{-h}^{0}\mathcal{Q}_K(t,\eta)z_{Kv}(t+\eta)d\eta\right)\\ &- 2z_{Kv}^T(t)\mathcal{P}_K(t)CM^{-1}C^T\int_{-h}^{0}\mathcal{Q}_K(t,\eta)z_{Kv}(t+\eta)d\eta\\ &- \int_{-h}^{0} z_{Kv}^T(t+\eta)\mathcal{Q}_K^T(t,\eta)d\eta CM^{-1}C^T\int_{-h}^{0}\mathcal{Q}_K(t,\rho)z_{Kv}(t+\rho)d\rho,\end{aligned}$$

$$t \in [0, t_f]. \tag{3.17}$$

Finally, using (3.13), the equation (3.17) can be rewritten in the form

$$\begin{aligned}\frac{dV_{Kv}(t)}{dt} &= -z_{Kv}^T(t)Dz_{Kv}(t)\\ &- \big(v[z_{Kv,t}.t] - v_K[z_{Kv,t}, t]\big)^T M\big(v[z_{Kv,t}, t] - v_K[z_{Kv,t}, t]\big)\\ &+ \big(v[z_{Kv,t}, t]\big)^T Mv[z_{Kv,t}, t], \quad t \in [0, t_f].\end{aligned} \tag{3.18}$$

Since the matrix M is positive definite, the equation (3.18) directly yields the inequality

$$\frac{dV_{Kv}(t)}{dt} + z_{Kv}^T(t)Dz_{Kv}(t) - \big(v[z_{Kv,t}, t]\big)^T Mv[z_{Kv,t}, t] \le 0, \quad t \in [0, t_f],$$

or

$$z_{Kv}^T(t)Dz_{Kv}(t) - \big(v[z_{Kv,t}, t]\big)^T Mv[z_{Kv,t}, t] \le -\frac{dV_{Kv}(t)}{dt}, \quad t \in [0, t_f].$$

Let us integrate the latter with respect to t in the interval $[0, t_f]$. Then, using the equations (3.5),(3.10),(3.15) and the absolute continuity of $z_{Kv}(t)$ yield the inequality

$$\begin{aligned}J\big(u_K[z_t, t], v[z_t, t]\big) &\le z_0^T\mathcal{P}_K(0)z_0 + 2z_0^T\int_{-h}^{0}\mathcal{Q}_K(0,\eta)\varphi(\eta)d\eta\\ &+ \int_{-h}^{0}\int_{-h}^{0}\varphi^T(\eta)\mathcal{R}_K(0,\eta,\rho)\varphi(\rho)d\eta d\rho\end{aligned}$$

$$\forall\ z_0 \in \mathbb{R}^{n+m}, \quad \varphi(\eta) \in L^2[-h, 0; \mathbb{R}^{n+m}], \quad v[z_t, t] \in \mathcal{F}_v\big(u_K[z_t, t]\big). \tag{3.19}$$

Now, the substitution of $v[z_t, t] = v_K[z_t, t]$ into the equation (3.18) and the integration of the resulting equation in the interval $[0, t_f]$ yield

$$J\big(u_K[z_t,t],v_K[z_t,t]\big)=z_0^T\mathcal{P}_K(0)z_0+2z_0^T\int_{-h}^{0}\mathcal{Q}_K(0,\eta)\varphi(\eta)d\eta$$
$$+\int_{-h}^{0}\int_{-h}^{0}\varphi^T(\eta)\mathcal{R}_K(0,\eta,\rho)\varphi(\rho)d\eta d\rho$$
$$\forall\ z_0\in\mathbb{R}^{n+m},\quad \varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}].$$

This equality, along with the inequality (3.19), directly yields the validity of the items (ii) and (iii).

Let us prove the item (iv). Consider the control $v_0[z_t,t]\equiv 0$. It is clear that $v_0[z_t,t]\in\mathcal{V}_t$ for all $t\in[0,t_f]$. By virtue of the results of [4], for any given $z_0\in\mathbb{R}^{n+m}$ and $\varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}]$, the equation (3.14) for $v(t)=v_0[z_t,t]$ and subject to the initial conditions $z(\eta)=\varphi(\eta)$, $\eta\in[-h,0)$, $z(0)=z_0$ has the unique absolutely continuous solution $z_{K0}\big(t;z_0,\varphi(\eta)\big)$, $t\in[0,t_f]$. Therefore, $v_0[z_t,t]\in\mathcal{F}_v\big(u_K[z_t,t]\big)$.

By $V_{K0}(t)$, we denote the time realization of the functional (3.15) along $z_{K0}\big(t;z_0,\varphi(\eta)\big)$, i.e., $V_{K0}(t)\overset{\triangle}{=}V[z_{K0,t},t]$. Replacing in (3.18) the control $v[z_t,t]$ with the control $v_0[z_t,t]$, we obtain

$$\frac{dV_{K0}(t)}{dt}=-\Big(z_{K0}\big(t;z_0,\varphi(\eta)\big)\Big)^TDz_{K0}\big(t;z_0,\varphi(\eta)\big)$$
$$-\big(v_K[z_{K0,t},t]\big)^TMv_K[z_{K0,t},t],\quad t\in[0,t_f],$$

which yields the inequality

$$\frac{dV_{K0}(t)}{dt}\le 0,\quad t\in[0,t_f].$$

Integration of this inequality in the interval $[0,t_f]$, and use of the equations (3.5), (3.10), (3.15) and the absolute continuity of $z_{K0}\big(t;z_0,\varphi(\eta)\big)$ yield the inequality

$$J\big(u_K[z_t,t],v_0[z_t,t]\big)\ge 0.$$

The latter, along with Definition 2.2, directly implies the validity of the item (iv). Thus, the lemma is proven. □

4. Regularization of the SDG

4.1. **Cheap Control Differential Game.** We analyze the SDG by regularization method. This method consists in replacing the original singular game with a regular differential game, which depends on a small positive parameter. When this parameter is replaced with zero, the new game becomes the SDG. Based on this observation, we construct the regular differential game, associated with the SDG, in the following way. We keep for the new (regular) game the dynamics (3.3) and the initial conditions (3.4) of the SDG, while the cost functional of the new game has the regular form

$$J_\varepsilon(u,v)=\int_0^{t_f}\big[z^T(t)Dz(t)+\varepsilon^2u^T(t)u(t)-v^T(t)Mv(t)\big]dt,\tag{4.1}$$

where $\varepsilon>0$ is a small parameter.

Remark 4.1. The regularization method was used in many works in the literature for solution of singular optimal control problems with un-delayed and delayed dynamics (see e.g. [3, 9, 10, 16, 21] and references therein), and for solution of singular differential games with un-delayed dynamics (see e.g. [11, 13–15, 25]). However, to the best of our knowledge, a singular differential game with time delay dynamics was studied only in a single work (see [12]) where an infinite horizon game was considered.

Remark 4.2. Since the parameter $\varepsilon > 0$ is small, the game (3.3)–(3.4),(4.1) is a differential game with a cheap control of the minimizer. In what follows, we call the game (3.3)–(3.4),(4.1) the Cheap Control Differential Game (CCDG). Differential games with a cheap control of at least one of the players and with un-delayed dynamics were studied in the literature in a number of works (see e.g. [8, 11, 13–15, 23, 25, 27, 30, 31]). However, to the best of our knowledge, a differential game with a cheap control and with delays in the dynamics was studied only in the work [12]. Since for any $\varepsilon > 0$ the weight matrix for the minimizer's control cost in the cost functional (4.1) is positive definite, the CCDG is a regular differential game. The set of all admissible pairs of players' state-feedback controls (strategies) in this game is the same as in the SDG, namely, it is UV.

4.2. **State-Feedback Saddle-Point Equilibrium in the CCDG.** Consider the following hybrid system of Riccati-type ordinary and partial differential equations with respect to unknown matrices $P(t)$, $Q(t,\eta)$ and $R(t,\eta,\rho)$ in the domain Ω:

$$\begin{aligned}\frac{dP(t)}{dt} = -P(t)A - A^T P(t) + P(t)\big(S_u(\varepsilon) - S_v\big)P(t)\\ -Q(t,0) - Q^T(t,0) - D,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right) Q(t,\eta) = -A^T Q(t,\eta) + P(t)\big(S_u(\varepsilon) - S_v\big)Q(t,\eta)\\ -P(t)G(\eta) - R(t,0,\eta),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \rho}\right) R(t,\eta,\rho) = -G^T(\eta)Q(t,\rho) - Q^T(t,\eta)G(\rho)\\ +Q^T(t,\eta)\big(S_u(\varepsilon) - S_v\big)Q(t,\rho),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}S_u(\varepsilon) = \frac{1}{\varepsilon^2}BB^T = \begin{pmatrix} 0 & 0 \\ 0 & (1/\varepsilon^2)I_m \end{pmatrix}, \quad S_v = CM^{-1}C^T = \begin{pmatrix} S_{v1} & S_{v2} \\ S_{v2}^T & S_{v3} \end{pmatrix},\\ S_{v1} = C_1 M^{-1} C_1^T, \quad S_{v2} = C_1 M^{-1} C_2^T, \quad S_{v3} = C_2 M^{-1} C_2^T.\end{aligned} \tag{4.5}$$

We consider the following boundary conditions for the system (4.2)-(4.4):

$$P(t_f) = 0, \quad Q(t_f,\eta) = 0, \quad R(t_f,\eta,\rho) = 0, \quad (\eta,\rho) \in [-h,0] \times [-h,0], \tag{4.6}$$

$$\begin{aligned}Q(t,-h) = P(t)H, \quad R(t,-h,\eta) = H^T Q(t,\eta), \quad R(t,\eta,-h) = Q^T(t,\eta)H,\\ \eta \in [-h,0].\end{aligned}$$

(4.7)

In what follows of this subsection, we assume:
A1. For a given $\varepsilon > 0$, the system (4.2)-(4.4) subject to the boundary conditions (4.6)-(4.7) has a continuous solution $\{P^*(t,\varepsilon), Q^*(t,\eta,\varepsilon), R^*(t,\eta,\rho,\varepsilon)\}$, $(t,\eta,\rho) \in \Omega$, such that $\big(P^*(t,\varepsilon)\big)^T = P^*(t,\varepsilon)$, $\big(R^*(t,\eta,\rho,\varepsilon)\big)^T = R^*(t,\rho,\eta,\varepsilon)$.

For all $t \in [0,t_f]$, consider the following t-dependent vector-valued functionals:

$$u_\varepsilon^*[z_t,t] \triangleq -\frac{1}{\varepsilon^2}B^T\left(P^*(t,\varepsilon)z(t) + \int_{-h}^{0} Q^*(t,\eta,\varepsilon)z(t+\eta)d\eta\right) \in \mathcal{U}_t, \tag{4.8}$$

$$v_\varepsilon^*[z_t,t] \triangleq M^{-1}C^T\left(P^*(t,\varepsilon)z(t) + \int_{-h}^{0} Q^*(t,\eta,\varepsilon)z(t+\eta)d\eta\right) \in \mathcal{V}_t. \tag{4.9}$$

Theorem 4.3. *Let the assumption A1 be valid. Then:*
(a) the pair $\big(u_\varepsilon^*[z_t,t], v_\varepsilon^*[z_t,t]\big) \in UV$, *i.e., it is admissible in the CCDG;*
(b) for any $u[z_t,t] \in \mathcal{F}_u\big(v_\varepsilon^*[z_t,t]\big)$ *and any* $v[z_t,t] \in \mathcal{F}_v\big(u_\varepsilon^*[z_t,t]\big)$*, the admissible pair* $\big(u_\varepsilon^*[z_t,t], v_\varepsilon^*[z_t,t]\big)$ *satisfies the following inequality:*

$$J_\varepsilon\big(u_\varepsilon^*[z_t,t], v[z_t,t]\big) \le J_\varepsilon\big(u_\varepsilon^*[z_t,t], v_\varepsilon^*[z_t,t]\big) \le J_\varepsilon\big(u[z_t,t], v_\varepsilon^*[z_t,t]\big),$$

i.e., this pair is a saddle-point equilibrium in the regular CCDG;
(c) the value of the CCDG is

$$J_\varepsilon^*\big(z_0,\varphi(\eta)\big) \triangleq J_\varepsilon\big(u_\varepsilon^*[z_t,t], v_\varepsilon^*[z_t,t]\big) = z_0^T P^*(0,\varepsilon)z_0$$
$$+2z_0^T\int_{-h}^{0} Q^*(0,\eta,\varepsilon)\varphi(\eta)d\eta + \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\eta)R^*(0,\eta,\rho,\varepsilon)\varphi(\rho)d\eta d\rho; \tag{4.10}$$

(d) for any $z_0 \in \mathbb{R}^{n+m}$ *and any* $\varphi(\eta) \in L^2[-h,0;\mathbb{R}^{n+m}]$*, the value of the CCDG is nonnegative.*

Proof. We start with the item (a). Let us substitute the controls (4.8) and (4.9) into the equation (3.3) instead of $u(t)$ and $v(t)$, respectively. Due to this substitution, the equation (3.3) becomes as:

$$\frac{dz(t)}{dt} = \Big(A - \big(S_u(\varepsilon) - S_v\big)P^*(t,\varepsilon)\Big)z(t) + Hz(t-h)$$
$$+\int_{-h}^{0}\Big(G(\tau) - \big(S_u(\varepsilon) - S_v\big)Q^*(t,\tau,\varepsilon)\Big)z(t+\tau)d\tau, \quad t \in [0,t_f]. \tag{4.11}$$

Since $P^*(t,\varepsilon)$ is a continuous function of $t \in [0,t_f]$ and $Q^*(t,\tau,\varepsilon)$ is a continuous function of $(t,\tau) \in [0,t_f] \times [-h,0]$, then for any $\varphi(\eta) \in L^2[-h,0;\mathbb{R}^{n+m}]$ and $z_0 \in \mathbb{R}^{n+m}$ the linear time delay equation (4.11) subject to the initial conditions (3.4) has the unique absolutely continuous solution in the interval $[0,t_f]$. This fact, along with Definition 2.1, directly yields the validity of the item (a).

Proceed to the items (b) and (c). For any $t \in [0,t_f]$, consider the Lyapunov-Krasovskii-like functional

$$V_\varepsilon[z_t,t] = z^T(t)P^*(t,\varepsilon)z(t) + 2z^T(t)\int_{t-h}^{t} Q^*(t,\tau-t,\varepsilon)z(\tau)d\tau$$
$$+\int_{t-h}^{t}\int_{t-h}^{t} z^T(\eta)R^*(t,\tau-t,\theta-t,\varepsilon)z(\theta)d\tau d\theta. \qquad (4.12)$$

For any given pair $\big(u[z_t,t],v[z_t,t]\big) \in UV$, and any given function $\varphi(\eta) \in L^2[-h,0;\mathbb{R}^{n+m}]$ and vector $z_0 \in \mathbb{R}^{n+m}$, we consider the unique absolutely continuous solution $z_{uv}\big(t;z_0,\varphi(\eta)\big)$, $t \in [0,t_f]$ of the problem (3.3)-(3.4) with $u(t) = u[z_t,t]$, $v(t) = v[z_t,t]$. By $V_{\varepsilon,uv}(t)$, we denote the time realization of the functional (4.12) along this solution, i.e., $V_{\varepsilon,uv}(t) \triangleq V_\varepsilon[z_{uv,t},t]$. Also, by $u[z_{uv,t},t]$ and $v[z_{uv,t},t]$, let us denote the time realizations of the controls $u[z_t,t]$ and $v[z_t,t]$, respectively, along the solution $z_{uv}\big(t;z_0,\varphi(\eta)\big)$. Now, the calculation of the derivative $dV_{\varepsilon,uv}(t)/dt$, $t \in [0,t_f]$ yields the following expression (in this expression for the sake of simplicity we omit the designation of the dependence of $z_{uv}\big(t;z_0,\varphi(\eta)\big)$ on z_0 and $\varphi(\eta)$):

$$\begin{aligned}\frac{dV_{\varepsilon,uv}(t)}{dt} &= 2\left(\frac{dz_{uv}(t)}{dt}\right)^T\left(P^*(t,\varepsilon)z_{uv}(t) + \int_{t-h}^{t} Q^*(t,\tau-t,\varepsilon)z_{uv}(\tau)d\tau\right)\\
&+ z_{uv}^T(t)\frac{dP^*(t,\varepsilon)(t)}{dt}z_{uv}(t) + 2z_{uv}^T(t)\Big[Q^*(t,0,\varepsilon)z_{uv}(t)\\
&- Q^*(t,-h,\varepsilon)z_{uv}(t-h) + \int_{t-h}^{t}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}\right)Q^*(t,\eta,\varepsilon)\Big|_{\eta=\tau-t} z_{uv}(\tau)d\tau\Big]\\
&+ 2z_{uv}^T(t)\int_{t-h}^{t} R^*(t,0,\theta-t,\varepsilon)z_{uv}(\theta)d\theta\\
&- 2z_{uv}^T(t-h)\int_{t-h}^{t} R^*(t,-h,\theta-t,\varepsilon)z_{uv}(\theta)d\theta\\
&+ \int_{t-h}^{t}\int_{t-h}^{t} z_{uv}^T(\tau)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\rho}\right)R^*(t,\eta,\rho)\Big|_{\eta=\tau-t,\rho=\theta-t} z_{uv}(\theta)d\tau d\theta.\end{aligned} \qquad (4.13)$$

Changing the variables $\tau - t = \eta$, $\theta - t = \rho$ in the integrals in the equation (4.13), and using the problem (4.2)-(4.4),(4.6)-(4.7), we can rewrite (4.13) as:

$$\begin{aligned}\frac{dV_{\varepsilon,uv}(t)}{dt} &= -z_{uv}^T(t)Dz_{uv}(t) + z_{uv}^T(t)P^*(t,\varepsilon)\big(S_u(\varepsilon)-S_v\big)P^*(t,\varepsilon)z_{uv}(t)\\
&+ 2\Big(Bu[z_{uv,t},t] + Cv[z_{uv,t},t]\Big)^T\Big(P^*(t,\varepsilon)z_{uv}(t)\\
&+ \int_{-h}^{0} Q^*(t,\eta,\varepsilon)z_{uv}(t+\eta)d\eta\Big)\\
&+ 2z_{uv}^T(t)P^*(t,\varepsilon)\big(S_u(\varepsilon)-S_v\big)\int_{-h}^{0} Q^*(t,\eta,\varepsilon)z_{uv}(t+\eta)d\eta\\
&+ \int_{-h}^{0} z_{uv}^T(t+\eta)\big(Q^*(t,\eta,\varepsilon)\big)^T d\eta\big(S_u(\varepsilon)-S_v\big)\int_{-h}^{0} Q^*(t,\rho,\varepsilon)z_{uv}(t+\rho)d\rho.\end{aligned} \qquad (4.14)$$

Using (4.8) and (4.9), we can rewrite the equation (4.14) it the form:

$$
\begin{aligned}
\frac{dV_{\varepsilon,uv}(t)}{dt} &= -z_{uv}^T(t)Dz_{uv}(t) - \varepsilon^2\big(u[z_{uv,t},t]\big)^T u[z_{uv,t},t] \\
&+ \big(v[z_{uv,t},t]\big)^T M v[z_{uv,t},t] \\
&+ \varepsilon^2\big(u[z_{uv,t},t] - u_\varepsilon^*[z_{uv,t},t]\big)^T\big(u[z_{uv,t},t] - u_\varepsilon^*[z_{uv,t},t]\big) \\
&- \big(v[z_{uv,t},t] - v_\varepsilon^*[z_{uv,t},t]\big)^T M\big(v[z_{uv,t},t] - v_\varepsilon^*[z_{uv,t},t]\big), \quad t \in [0,t_f],
\end{aligned}
\tag{4.15}
$$

where $u_\varepsilon^*[z_{uv,t},t]$ and $v_\varepsilon^*[z_{uv,t},t]$ are the time realizations of the controls $u_\varepsilon^*[z_t,t]$ and $v_\varepsilon^*[z_t,t]$, respectively, along $z_{uv}\big(t; z_0, \varphi(\eta)\big)$.

Integration of (4.15) in the interval $[0,t_f]$, and use of the equations (3.5), (4.6), (4.10) and (4.12) yield after a routine algebra the equation

$$
\begin{aligned}
J_\varepsilon\big(u[z_t,t], v[z_t,t]\big) &= J_\varepsilon^*\big(z_0,\varphi(\eta)\big) \\
&+ \int_0^{t_f}\Big(\varepsilon^2\big(u[z_{uv,t},t] - u_\varepsilon^*[z_{uv,t},t]\big)^T\big(u[z_{uv,t},t] - u_\varepsilon^*[z_{uv,t},t]\big) \\
&- \big(v[z_{uv,t},t] - v_\varepsilon^*[z_{uv,t},t]\big)^T M\big(v[z_{uv,t},t] - v_\varepsilon^*[z_{uv,t},t]\big)\Big)dt.
\end{aligned}
\tag{4.16}
$$

Taking into account the positive definiteness of the matrix M, the equation (4.16) immediately yields the validity of the items (b) and (c).

Finally, proceed to the item (d). Consider the pair of the players' controls, consisting of $u = u_\varepsilon^*[z_t,t]$ and $v = v_0[z_t,t] \equiv 0$. Similarly to the proof of the item (a), it is proven the inclusion $\big(u_\varepsilon^*[z_t], v_0[z_t]\big) \subset UV$. Let, for any $\varphi(\eta) \in L^2[-h,0;\mathbb{R}^{n+m}]$ and $z_0 \in \mathbb{R}^{n+m}$, the function $z_{\varepsilon 0}^*\big(t; z_0, \varphi(\eta)\big)$, $t \in [0,t_f]$ be the solution of the problem (3.3)-(3.4) with $u(t) = u_\varepsilon^*[z_t,t]$, $v(t) = v_0[z_t,t]$. By $V_{\varepsilon,u0}(t)$, we denote the time realization of the functional (4.12) along this solution, i.e., $V_{\varepsilon,u0}(t) \overset{\triangle}{=} V_\varepsilon[z_{\varepsilon 0,t}^*,t]$. Using these notations and the equation (4.15), we obtain the equation

$$
\begin{aligned}
\frac{dV_{\varepsilon,u0}(t)}{dt} &= -\Big(z_{\varepsilon 0}^*\big(t; z_0,\varphi(\tau)\big)\Big)^T D z_{\varepsilon 0}^*\big(t; z_0,\varphi(\tau)\big) \\
&- \varepsilon^2\big(u_\varepsilon^*[z_{\varepsilon 0,t}^*,t]\big)^T u_\varepsilon^*[z_{\varepsilon 0,t}^*,t] \\
&- \big(v_\varepsilon^*[z_{\varepsilon 0,t}^*,t]\big)^T M v_\varepsilon^*[z_{\varepsilon 0,t}^*,t], \quad t \in [0,t_f],
\end{aligned}
\tag{4.17}
$$

where $u_\varepsilon^*[z_{\varepsilon 0,t}^*,t]$ and $v_\varepsilon^*[z_{\varepsilon 0,t}^*,t]$ are the time realizations of the controls $u_\varepsilon^*[z_t,t]$ and $v_\varepsilon^*[z_t,t]$, respectively, along the solution $z_{\varepsilon 0}^*\big(t; z_0,\varphi(\eta)\big)$.

The equation (4.17) yields the inequality $dV_{\varepsilon,u0}(t)/dt \le 0$, $t \in [0,t_f]$. Integration of this inequality in the interval $[0,t_f]$, and use of the equations (4.6), (4.10) and (4.12) directly yield the validity of the item (d). Thus, the theorem is proven. □

5. Asymptotic Analysis of the Problem (4.2)-(4.4),(4.6)-(4.7)

5.1. **Transformation of (4.2)-(4.4),(4.6)-(4.7).** Due to the structure of the matrix $S_u(\varepsilon)$ (see the equation (4.5)), the equations (4.2), (4.3) and (4.4) have singularities at $\varepsilon = 0$ in their right-hand sides. To remove these singularities, we look for the solution $\{P(t,\varepsilon), Q(t,\eta,\varepsilon), R(t,\eta,\rho,\varepsilon)\}$ of the problem (4.2)-(4.4),(4.6)-(4.7) in the block form

$$P(t,\varepsilon) = \begin{pmatrix} P_1(t,\varepsilon) & \varepsilon P_2(t,\varepsilon) \\ \varepsilon P_2^T(t,\varepsilon) & \varepsilon P_3(t,\varepsilon) \end{pmatrix}, \tag{5.1}$$

$$Q(t,\eta,\varepsilon) = \begin{pmatrix} Q_1(t,\eta,\varepsilon) & Q_2(t,\eta,\varepsilon) \\ \varepsilon Q_3(t,\eta,\varepsilon) & \varepsilon Q_4(t,\eta,\varepsilon) \end{pmatrix}, \tag{5.2}$$

$$R(t,\eta,\rho,\varepsilon) = \begin{pmatrix} R_1(t,\eta,\rho,\varepsilon) & R_2(t,\eta,\rho,\varepsilon) \\ R_2^T(t,\rho,\eta,\varepsilon) & R_3(t,\eta,\rho,\varepsilon) \end{pmatrix}, \tag{5.3}$$

where $P_j(t,\varepsilon)$, $R_j(t,\eta,\rho,\varepsilon)$, $(j = 1,2,3)$ are matrices of the dimensions $n\times n$, $n\times m$, $m \times m$, respectively; $Q_i(t,\eta,\varepsilon)$, $(i = 1,...,4)$ are matrices of the dimensions $n \times n$, $n \times m$, $m \times n$, $m \times m$, respectively.

Substitution of (5.1)-(5.3) and the block representations for the matrices A, H, $G(\tau)$, D, $S_u(\varepsilon)$, S_v (see the equations (3.1), (3.2), (4.5)) into (4.2)-(4.4),(4.6)-(4.7) transforms this boundary-value problem to the following equivalent problem with respect to $P_j(t,\varepsilon)$, $Q_i(t,\eta,\varepsilon)$, $R_j(t,\eta,\rho,\varepsilon)$, $(j = 1,2,3;\ i = 1,...,4)$ (in this new problem, for simplicity, we omit the designation of the dependence of the unknown matrices on ε):

$$\begin{aligned}\frac{dP_1(t)}{dt} = {} & -P_1(t)A_{11} - A_{11}^T P_1(t) - \varepsilon P_2(t)A_{21} - \varepsilon A_{21}^T P_2^T(t) + P_2(t)P_2^T(t) \\ & -P_1(t)S_{v1}P_1(t) - \varepsilon P_2(t)S_{v2}^T P_1(t) - \varepsilon P_1(t)S_{v2}P_2^T(t) \\ & -\varepsilon^2 P_2(t)S_{v3}P_2^T(t) - Q_1(t,0) - Q_1^T(t,0) - D_1,\end{aligned} \tag{5.4}$$

$$\begin{aligned}\varepsilon\frac{dP_2(t)}{dt} = {} & -P_1(t)A_{12} - \varepsilon P_2(t)A_{22} - \varepsilon A_{11}^T P_2(t) - \varepsilon A_{21}^T P_3(t) + P_2(t)P_3(t) \\ & -\varepsilon P_1(t)S_{v1}P_2(t) - \varepsilon^2 P_2(t)S_{v2}^T P_2(t) - \varepsilon P_1(t)S_{v2}P_3(t) \\ & -\varepsilon^2 P_2(t)S_{v3}P_3(t) - Q_2(t,0) - \varepsilon Q_3^T(t,0),\end{aligned} \tag{5.5}$$

$$\begin{aligned}\varepsilon\frac{dP_3(t)}{dt} = {} & -\varepsilon P_2^T(t)A_{12} - \varepsilon A_{12}^T P_2(t) - \varepsilon P_3(t)A_{22} - \varepsilon A_{22}^T P_3(t) + \big(P_3(t)\big)^2 \\ & -\varepsilon^2 P_2^T(t)S_{v1}P_2(t) - \varepsilon^2 P_3(t)S_{v2}^T P_2(t) - \varepsilon^2 P_2^T(t)S_{v2}P_3(t) \\ & -\varepsilon^2 P_3(t)S_{v3}P_3(t) - \varepsilon Q_4(t,0) - \varepsilon Q_4^T(t,0) - D_2,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right)Q_1(t,\eta) = {} & -A_{11}^T Q_1(t,\eta) - \varepsilon A_{21}^T Q_3(t,\eta) + P_2(t)Q_3(t,\eta) \\ & -P_1(t)S_{v1}Q_1(t,\eta) - \varepsilon P_2(t)S_{v2}^T Q_1(t,\eta) - \varepsilon P_1(t)S_{v2}Q_3(t,\eta) \\ & -\varepsilon^2 P_2(t)S_{v3}Q_3(t,\eta) - P_1(t)G_{11}(\eta) - \varepsilon P_2(t)G_{21}(\eta) - R_1(t,0,\eta),\end{aligned} \tag{5.7}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}\right)Q_2(t,\eta)=-A_{11}^TQ_2(t,\eta)-\varepsilon A_{21}^TQ_4(t,\eta)+P_2(t)Q_4(t,\eta)$$
$$-P_1(t)S_{v1}Q_2(t,\eta)-\varepsilon P_2(t)S_{v2}^TQ_2(t,\eta)-\varepsilon P_1(t)S_{v2}Q_4(t,\eta)$$
$$-\varepsilon^2P_2(t)S_{v3}Q_4(t,\eta)-\varepsilon P_2(t)G_{22}(\eta)-R_2(t,0,\eta), \tag{5.8}$$

$$\varepsilon\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}\right)Q_3(t,\eta)=-A_{12}^TQ_1(t,\eta)-\varepsilon A_{22}^TQ_3(t,\eta)+P_3(t)Q_3(t,\eta)$$
$$-\varepsilon P_2^T(t)S_{v1}Q_1(t,\eta)-\varepsilon P_3(t)S_{v2}^TQ_1(t,\eta)-\varepsilon^2P_2^T(t)S_{v2}Q_3(t,\eta)$$
$$-\varepsilon^2P_3(t)S_{v3}Q_3(t,\eta)-\varepsilon P_2^T(t)G_{11}(\eta)$$
$$-\varepsilon P_3(t)G_{21}(\eta)-R_2^T(t,\eta,0), \tag{5.9}$$

$$\varepsilon\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}\right)Q_4(t,\eta)=-A_{12}^TQ_2(t,\eta)-\varepsilon A_{22}^TQ_4(t,\eta)+P_3(t)Q_4(t,\eta)$$
$$-\varepsilon P_2^T(t)S_{v1}Q_2(t,\eta)-\varepsilon P_3(t)S_{v2}^TQ_2(t,\eta)-\varepsilon^2P_2^T(t)S_{v2}Q_4(t,\eta)$$
$$-\varepsilon^2P_3(t)S_{v3}Q_4(t,\eta)-\varepsilon P_3(t)G_{22}(\eta)-R_3(t,0,\eta), \tag{5.10}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_1(t,\eta,\rho)=-G_{11}^T(\eta)Q_1(t,\rho)-Q_1^T(t,\eta)G_{11}(\rho)$$
$$-\varepsilon G_{21}^T(\eta)Q_3(t,\rho)-\varepsilon Q_3^T(t,\eta)G_{21}(\rho)+Q_3^T(t,\eta)Q_3(t,\rho)$$
$$-Q_1^T(t,\eta)S_{v1}Q_1(t,\rho)-\varepsilon Q_3^T(t,\eta)S_{v2}^TQ_1(t,\rho)$$
$$-\varepsilon Q_1^T(t,\eta)S_{v2}Q_3(t,\rho)-\varepsilon^2Q_3^T(t,\eta)S_{v3}Q_3(t,\rho), \tag{5.11}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_2(t,\eta,\rho)=-G_{11}^T(\eta)Q_2(t,\rho)-\varepsilon G_{21}^T(\eta)Q_4(t,\rho)$$
$$-\varepsilon Q_3^T(t,\eta)G_{22}(\rho)+Q_3^T(t,\eta)Q_4(t,\rho)-Q_1^T(t,\eta)S_{v1}Q_2(t,\rho)$$
$$-\varepsilon Q_3^T(t,\eta)S_{v2}^TQ_2(t,\rho)-\varepsilon Q_1^T(t,\eta)S_{v2}Q_4(t,\rho)$$
$$-\varepsilon^2Q_3^T(t,\eta)S_{v3}Q_4(t,\rho), \tag{5.12}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_3(t,\eta,\rho)=-\varepsilon G_{22}^T(\eta)Q_4(t,\rho)-\varepsilon Q_4^T(t,\eta)G_{22}(\rho)$$
$$+Q_4^T(t,\eta)Q_4(t,\rho)-Q_2^T(t,\eta)S_{v1}Q_2(t,\rho)-\varepsilon Q_4^T(t,\eta)S_{v2}^TQ_2(t,\rho)$$
$$-\varepsilon Q_2^T(t,\eta)S_{v2}Q_4(t,\rho)-\varepsilon^2Q_4^T(t,\eta)S_{v3}Q_4(t,\rho), \tag{5.13}$$

$$P_j(t_f)=0,\quad Q_i(t_f,\eta)=0,\quad R_j(t_f,\eta,\rho)=0,$$
$$j=1,2,3,\quad i=1,...,4, \tag{5.14}$$

$$Q_1(t,-h)=P_1(t)H_{11}+\varepsilon P_2(t)H_{21},\quad Q_2(t,-h)=\varepsilon P_2(t)H_{22}, \tag{5.15}$$

$$Q_3(t,-h)=P_2^T(t)H_{11}+P_3(t)H_{21},\quad Q_4(t,-h)=P_3(t)H_{22}, \tag{5.16}$$

$$R_1(t,-h,\eta)=H_{11}^TQ_1(t,\eta)+\varepsilon H_{21}^TQ_3(t,\eta),$$
$$R_1(t,\eta,-h)=Q_1^T(t,\eta)H_{11}+\varepsilon Q_3^T(t,\eta)H_{21},$$

(5.17)
$$R_2(t,-h,\eta) = H_{11}^T Q_2(t,\eta) + \varepsilon H_{21}^T Q_4(t,\eta),$$
$$R_2(t,\eta,-h) = \varepsilon Q_3^T(t,\eta) H_{22},$$

(5.18)

(5.19) $$R_3(t,-h,\eta) = \varepsilon H_{22}^T Q_4(t,\eta), \quad R_3(t,\eta,-h) = \varepsilon Q_4^T(t,\eta) H_{22}.$$

The problem (5.4)-(5.19) is a singularly perturbed boundary-value problem. In order to construct its asymptotic solution, we apply in the next section the boundary function method [32].

5.2. **Asymptotic Solution of the System (5.4)-(5.19): Formal Construction.** We look for the zero-order asymptotic solution of (5.4)-(5.19) in the form

(5.20) $$\{P_{j0}(t,\varepsilon), Q_{i0}(t,\eta,\varepsilon), R_{j0}(t,\eta,\rho,\varepsilon)\}, \quad j=1,2,3, \quad i=1,...,4$$

where

(5.21) $$P_{10}(t,\varepsilon) = P_{10}^o(t), \quad P_{l0}(t,\varepsilon) = P_{l0}^o(t) + P_{l0}^t(\xi), \quad l=2,3, \quad \xi = (t-t_f)/\varepsilon,$$

$$Q_{k0}(t,\eta,\varepsilon) = Q_{k0}^o(t,\eta),$$
$$Q_{p0}(t,\eta,\varepsilon) = Q_{p0}^o(t,\eta) + Q_{p0}^t(\xi,\eta) + Q_{p0}^\eta(t,\zeta) + Q_{p0}^{t,\eta}(\xi,\zeta),$$
$$k=1,2, \quad p=3,4, \quad \zeta = (\eta+h)/\varepsilon,$$

(5.22)

(5.23) $$R_{j0}(t,\eta,\rho,\varepsilon) = R_{j0}^o(t,\eta,\rho).$$

Here the terms with the superscript o are the so called outer solution terms, the terms with the superscript t are the boundary layer correction terms in a neighborhood of the boundary $t=t_f$, the terms with the superscript η are the boundary layer correction terms in a neighborhood of the boundary $\eta=-h$, and the terms with the superscript t,η are the boundary layer correction terms in a neighborhood of the boundary $(t=t_f, \eta=-h)$ of the domain Ω.

5.2.1. *Outer Solution Terms.* Equations and conditions for these terms are obtained in the following way. First, we set formally $\varepsilon=0$ in the problem (5.4)-(5.19) and redenote $P_j(t)$, $Q_i(t,\eta)$, $R_j(t,\eta,\rho)$ with $P_{j0}^o(t)$, $Q_{i0}^o(t,\eta)$, $R_{j0}^o(t,\eta,\rho)$, $(j=1,2,3;\ i=1,...,4)$. Then, we remove from the resulting problem the boundary conditions for each term, which does not satisfy a differential equation. Thus we have the following problem in the domain Ω:

$$\frac{dP_{10}^o(t)}{dt} = -P_{10}^o(t)A_{11} - A_{11}^T P_{10}^o(t) + P_{20}^o(t)\big(P_{20}^o(t)\big)^T$$
(5.24) $$-P_{10}^o(t)S_{v1}P_{10}^o(t) - Q_{10}^o(t,0) - \big(Q_{10}^o(t,0)\big)^T - D_1,$$

(5.25) $$0 = -P_{10}^o(t)A_{12} + P_{20}^o(t)P_{30}^o(t) - Q_{20}^o(t,0),$$

(5.26) $$0 = \big(P_{30}^o(t)\big)^2 - D_2,$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}\right)Q_{10}^o(t,\eta) = -A_{11}^TQ_{10}^o(t,\eta) + P_{20}^o(t)Q_{30}^o(t,\eta)$$
$$-P_{10}^o(t)S_{v1}Q_{10}^o(t,\eta) - P_{10}^o(t)G_{11}(\eta) - R_{10}^o(t,0,\eta), \tag{5.27}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}\right)Q_{20}^o(t,\eta) = -A_{11}^TQ_{20}^o(t,\eta) + P_{20}^o(t)Q_{40}^o(t,\eta)$$
$$-P_{10}^o(t)S_{v1}Q_{20}^o(t,\eta) - R_{20}^o(t,0,\eta), \tag{5.28}$$

$$0 = -A_{12}^TQ_{10}^o(t,\eta) + P_{30}^o(t)Q_{30}^o(t,\eta) - \big(R_{20}^o(t,\eta,0)\big)^T, \tag{5.29}$$

$$0 = -A_{12}^TQ_{20}^o(t,\eta) + P_{30}^o(t)Q_{40}^o(t,\eta) - R_{30}^o(t,0,\eta), \tag{5.30}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_{10}^o(t,\eta,\rho) = -G_{11}^T(\eta)Q_{10}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^TG_{11}(\rho)$$
$$+\big(Q_{30}^o(t,\eta)\big)^TQ_{30}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^TS_{v1}Q_{10}^o(t,\rho), \tag{5.31}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_{20}^o(t,\eta,\rho) = -G_{11}^T(\eta)Q_{20}^o(t,\rho)$$
$$+\big(Q_{30}^o(t,\eta)\big)^TQ_{40}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^TS_{v1}Q_{20}^o(t,\rho), \tag{5.32}$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \rho}\right)R_{30}^o(t,\eta,\rho) = \big(Q_{40}^o(t,\eta)\big)^TQ_{40}^o(t,\rho)$$
$$-\big(Q_{20}^o(t,\eta)\big)^TS_{v1}Q_{20}^o(t,\rho), \tag{5.33}$$

$$P_{10}^o(t_f) = 0, \quad Q_{k0}^o(t_f,\eta) = 0, \quad R_{j0}^o(t_f,\eta,\rho) = 0,$$
$$j = 1,2,3, \quad k = 1,2, \tag{5.34}$$

$$Q_{10}^o(t,-h) = P_{10}^o(t)H_{11}, \quad Q_{20}^o(t,-h) = 0, \tag{5.35}$$

$$R_{10}^o(t,-h,\eta) = H_{11}^TQ_{10}^o(t,\eta), \quad R_{10}^o(t,\eta,-h) = \big(Q_{10}^o(t,\eta)\big)^TH_{11}, \tag{5.36}$$

$$R_{20}^o(t,-h,\eta) = H_{11}^TQ_{20}^o(t,\eta), \quad R_{20}^o(t,\eta,-h) = 0, \tag{5.37}$$

$$R_{30}^o(t,-h,\eta) = 0, \quad R_{30}^o(t,\eta,-h) = 0. \tag{5.38}$$

It is verified directly that we can set

$$Q_{20}^o(t,\eta) \equiv 0,\ Q_{40}^o(t,\eta) \equiv 0,\ R_{20}^o(t,\eta,\rho) \equiv 0,\ R_{30}^o(t,\eta,\rho) \equiv 0,\ (t,\eta,\rho) \in \Omega \tag{5.39}$$

without a formal contradiction with the problem (5.24)-(5.38). In what follows, we look for the solution of this problem satisfying the condition (5.39). Substitution of (5.39) into (5.24)-(5.38) yields a new system. In this system, the equations (5.24), (5.26),(5.27), (5.31) and (5.36) remain unchanged. However, for the sake of the integrity of the new system, we write these equations in the new system. Thus, we have the following problem in the domain Ω:

$$\frac{dP_{10}^o(t)}{dt} = -P_{10}^o(t)A_{11} - A_{11}^TP_{10}^o(t) + P_{20}^o(t)\big(P_{20}^o(t)\big)^T$$

$$-P_{10}^o(t)S_{v1}P_{10}^o(t) - Q_{10}^o(t,0) - \big(Q_{10}^o(t,0)\big)^T - D_1, \tag{5.40}$$

$$0 = -P_{10}^o(t)A_{12} + P_{20}^o(t)P_{30}^o(t), \tag{5.41}$$

$$0 = \big(P_{30}^o(t)\big)^2 - D_2, \tag{5.42}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right)Q_{10}^o(t,\eta) = -A_{11}^T Q_{10}^o(t,\eta) + P_{20}^o(t)Q_{30}^o(t,\eta)$$
$$-P_{10}^o(t)S_{v1}Q_{10}^o(t,\eta) - P_{10}^o(t)G_{11}(\eta) - R_{10}^o(t,0,\eta), \tag{5.43}$$

$$0 = -A_{12}^T Q_{10}^o(t,\eta) + P_{30}^o(t)Q_{30}^o(t,\eta), \tag{5.44}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \rho}\right) R_{10}^o(t,\eta,\rho) = -G_{11}^T(\eta)Q_{10}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^T G_{11}(\rho)$$
$$+\big(Q_{30}^o(t,\eta)\big)^T Q_{30}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^T S_{v1}Q_{10}^o(t,\rho), \tag{5.45}$$

$$P_{10}^o(t_f) = 0, \quad Q_{10}^o(t_f,\eta) = 0, \quad R_{10}^o(t_f,\eta,\rho) = 0, \tag{5.46}$$

$$Q_{10}^o(t,-h) = P_{10}^o(t)H_{11}, \tag{5.47}$$

$$R_{10}^o(t,-h,\eta) = H_{11}^T Q_{10}^o(t,\eta), \quad R_{10}^o(t,\eta,-h) = \big(Q_{10}^o(t,\eta)\big)^T H_{11}, \tag{5.48}$$

The equation (5.42) yields the solution

$$P_{30}^o(t) = P_{30}^{o*}(t) \stackrel{\triangle}{=} D_2^{1/2}, \quad t \in [0,t_f], \tag{5.49}$$

where the superscript "1/2" denotes the unique positive definite square root of the corresponding positive definite matrix.

By virtue of (5.49), the equations (5.41) and (5.44) yield

$$P_{20}^o(t) = P_{10}^o(t)A_{12}D_2^{-1/2}, \quad Q_{30}^o(t,\eta) = D_2^{-1/2}A_{12}^T Q_{10}^o(t,\eta),$$
$$(t,\eta) \in [0,t_f] \times [-h,0], \tag{5.50}$$

where $D_2^{-1/2}$ is the inverse matrix for $D_2^{1/2}$.

Substitution of (5.49) and (5.50) into the equations (5.40), (5.43), (5.45) yields after a routine algebra the following system:

$$\frac{dP_{10}^o(t)}{dt} = -P_{10}^o(t)A_{11} - A_{11}^T P_{10}^o(t) + P_{10}^o(t)S_0P_{10}^o(t)$$
$$-Q_{10}^o(t,0) - \big(Q_{10}^o(t,0)\big)^T - D_1, \tag{5.51}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right) Q_{10}^o(t,\eta) = -A_{11}^T Q_{10}^o(t,\eta) + P_{10}^o(t)S_0Q_{10}^o(t,\eta)$$
$$-P_{10}^o(t)G_{11}(\eta) - R_{10}^o(t,0,\eta), \tag{5.52}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \rho}\right) R_{10}^o(t,\eta,\rho) = -G_{11}^T(\eta)Q_{10}^o(t,\rho) - \big(Q_{10}^o(t,\eta)\big)^T G_{11}(\rho)$$
$$+\big(Q_{10}^o(t,\eta)\big)^T S_0Q_{10}^o(t,\rho), \tag{5.53}$$

where $S_0 = A_2 D_2^{-1} A_2^T - S_{v1}$.

Thus, to obtain the solution of the problem (5.40)-(5.48), one has to solve the system (5.51)-(5.53) subject to the boundary conditions (5.46)-(5.48). If the latter has a solution $P_{10}^o(t)$, $Q_{10}^o t, \eta)$, $R_{10}^o(t,\eta,\rho)$ in the domain Ω, then the other terms of the solution to (5.40)-(5.48) are immediately obtained by (5.49) and (5.50).

In what follows, we assume:

A2. The system (5.51)-(5.53) subject to the boundary conditions (5.46)-(5.48) has a continuous solution $\{P_{10}^{o*}(t), Q_{10}^{o*}(t,\eta), R_{10}^{o*}(t,\eta,\rho)\}$, $(t,\eta,\rho) \in \Omega$, such that $\big(P_{10}^{o*}(t)\big)^T = P_{10}^{o*}(t)$, $\big(R_{10}^{o*}(t,\eta,\rho)\big)^T = R_{10}^{o*}(t,\rho,\eta)$.

Using the equation (5.50) and the solution of the problem (5.51)-(5.53), (5.46)-(5.48), mentioned in the assumption A2, we obtain the corresponding components of the solution to the problem (5.40)-(5.48) as:

$$P_{20}^o(t) = P_{20}^{o*}(t) \stackrel{\triangle}{=} P_{10}^{o*}(t) A_{12} D_2^{-1/2}, \quad t \in [0, t_f],$$
$$Q_{30}^o(t,\eta) = Q_{30}^{o*}(t,\eta) \stackrel{\triangle}{=} D_2^{-1/2} A_{12}^T Q_{10}^{o*}(t,\eta), \quad (t,\eta) \in [0,t_f] \times [-h, 0]. \tag{5.54}$$

This equation, along with the equation (5.34), yields

$$P_{20}^{o*}(t_f) = 0, \quad Q_{30}^o(t_f,\eta) = 0, \quad \eta \in [-h,0]. \tag{5.55}$$

5.2.2. *Boundary Layer Correction Terms in a Neighborhood of $t = t_f$.* To obtain equations for these terms, we substitute the expressions for $P_{j0}(t,\varepsilon)$, $(j = 1,2,3)$ (see the equation (5.21)), the expressions for $Q_{i0}(t,\eta,\varepsilon)$, $(i = 1,...,4)$ (see the equation (5.22)) and the expressions for $R_l(t,\eta,\rho,\varepsilon)$, $(l = 2,3)$ (see the equation (5.23)) into the equations (5.5)-(5.6) and (5.9)-(5.10) instead of $P_j(t)$, $Q_i(t,\eta)$ and $R_l(t,\eta,\rho)$, $(j = 1,2,3;\ l = 2,3;\ i = 1,...,4)$. Then, we equate the coefficients for ε^0, depending on ξ and (ξ,η), on both sides of the resulting equations. Thus, taking into account (5.39), (5.49) and (5.55), we have

$$\frac{dP_{20}^t(\xi)}{d\xi} = P_{20}^t(\xi) D_2^{1/2} + P_{20}^t(\xi) P_{30}^t(\xi), \quad \xi \le 0, \tag{5.56}$$

$$\frac{dP_{30}^t(\xi)}{d\xi} = P_{30}^t(\xi) D_2^{1/2} + D_2^{1/2} P_{30}^t(\xi) + \big(P_{30}^t(\xi)\big)^2, \quad \xi \le 0, \tag{5.57}$$

$$\frac{\partial Q_{p0}^t(\xi,\eta)}{\partial \xi} = D_2^{1/2} Q_{p0}^t(\xi,\eta) + P_{30}^t(\xi) Q_{p0}^t(\xi,\eta), \quad \xi \le 0, \quad \eta \in [-h,0], \quad p = 3,4, \tag{5.58}$$

Conditions for these differential equations are obtained by substitution of the expressions for $P_{l0}(t,\varepsilon)$, $(l = 2,3)$ and $Q_{p0}(t,\eta,\varepsilon)$, $(p = 3,4)$ into the corresponding terminal conditions in (5.14) and equating the coefficients for ε^0 on both sides of the resulting equations. This procedure immediately yields

$$P_{20}^t(0) = 0, \quad P_{30}^t(0) = -D_2^{-1/2},$$
$$Q_{p0}^t(0,\eta) = 0, \quad \eta \in [-h,0], \quad p = 3,4. \tag{5.59}$$

Solving the problem (5.56)-(5.59), we obtain

$$P_{20}^t(\xi) \equiv 0, \quad Q_{p0}^t(\xi,\eta) \equiv 0, \quad p = 3,4,$$
$$P_{30}^t(\xi) = P_{30}^{t*}(\xi) \stackrel{\triangle}{=} -2D_2^{1/2}\exp(2D_2^{1/2}\xi)\big[I_m + \exp(2D_2^{1/2}\xi)\big]^{-1},$$
$$\xi \le 0, \quad \eta \in [-h,0]. \tag{5.60}$$

Since $D_2^{1/2}$ is a positive definite matrix, then

$$\big\|P_{30}^{t*}(\xi)\big\| \le a\exp(\beta\xi), \quad \xi \le 0, \tag{5.61}$$

where $\|\cdot\|$ denotes the Euclidean norm of a matrix; $a > 0$ and $\beta > 0$ are some constants.

5.2.3. *Boundary Layer Correction Terms in a Neighborhood of* $\eta = -h$. To obtain equations for these terms, we substitute the expressions for $P_{l0}(t,\varepsilon)$, $(l = 2,3)$ (see the equation (5.21)), the expressions for $Q_{i0}(t,\eta,\varepsilon)$, $(i = 1,...,4)$ (see the equation (5.22)) and the expressions for $R_l(t,\eta,\rho,\varepsilon)$, $(l = 2,3)$ (see the equation (5.23)) into the equations (5.9)-(5.10) instead of $P_l(t)$, $Q_i(t,\eta)$ and $R_l(t,\eta,\rho)$, $(l = 2,3;\ i = 1,...,4)$. Then, we equate the coefficients for ε^0, depending on (t,ζ), on both sides of the resulting equations. Thus, taking into account (5.49), we have

$$\frac{\partial Q_{p0}^{\eta}(t,\zeta)}{\partial\zeta} = -D_2^{1/2}Q_{p0}^{\eta}(t,\zeta), \quad t \in [0,t_f], \quad \zeta \ge 0, \quad p = 3,4. \tag{5.62}$$

Conditions for these differential equations are obtained by substitution of the expressions for $Q_{p0}(t,\eta,\varepsilon)$, $(p = 3,4)$ into the boundary conditions (5.16) and equating the coefficients for ε^0, depending on t, on both sides of the resulting equations, yielding

$$Q_{30}^{\eta}(t,0) = \big(P_{20}^{o}(t)\big)^T H_{11} + P_{30}^{o}(t)H_{21} - Q_{30}^{o}(t,-h), \quad t \in [0,t_f],$$
$$Q_{40}^{\eta}(t,0) = P_{30}^{o}(t)H_{22} - Q_{40}^{o}(t,-h), \quad t \in [0,t_f].$$

Using the equations (5.35),(5.39),(5.49),(5.54), we can transform these conditions as:

$$Q_{30}^{\eta}(t,0) = D_2^{1/2}H_{21}, \quad Q_{40}^{\eta}(t,0) = D_2^{1/2}H_{22}, \quad t \in [0,t_f]. \tag{5.63}$$

The initial-value problem (5.62),(5.63) has the unique solution

$$Q_{30}^{\eta}(t,\zeta) = Q_{30}^{\eta*}(t,\zeta) \stackrel{\triangle}{=} \exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{21}, \quad t \in [0,t_f], \ \ \zeta \ge 0,$$
$$Q_{40}^{\eta}(t,\zeta) = Q_{40}^{\eta*}(t,\zeta) \stackrel{\triangle}{=} \exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{22}, \quad t \in [0,t_f], \ \ \zeta \ge 0, \tag{5.64}$$

satisfying the inequality

$$\Big\|Q_{p0}^{\eta*}(t,\zeta)\Big\| \le a\exp(-\beta\zeta), \quad t \in [0,t_f], \ \ \zeta \ge 0, \ \ p = 3,4, \tag{5.65}$$

where $a > 0$ and $\beta > 0$ are some constants.

5.2.4. *Boundary Layer Correction Terms in a Neighborhood of* $(t = t_f, \eta = -h)$. Similarly to the equations and conditions for the above considered boundary layer correction terms, using (5.39),(5.49),(5.55),(5.60) and (5.64), we obtain the following equations and conditions for $Q_{30}^{t,\eta}(\xi,\zeta)$ and $Q_{40}^{t,\eta}(\xi,\zeta)$:

$$\Big(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\Big)Q_{30}^{t,\eta}(\xi,\zeta) = \big[D_2^{1/2} + P_{30}^{t*}(\xi)\big]Q_{30}^{t,\eta}(\xi,\zeta)$$

$$+P_{30}^{t*}(\xi)\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{21}, \quad \xi \le 0, \quad \zeta \ge 0, \tag{5.66}$$

$$\Big(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\zeta}\Big)Q_{40}^{t,\eta}(\xi,\zeta) = \big[D_2^{1/2} + P_{30}^{t*}(\xi)\big]Q_{40}^{t,\eta}(\xi,\zeta)$$

$$+P_{30}^{t*}(\xi)\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{22}, \quad \xi \le 0, \quad \zeta \ge 0, \tag{5.67}$$

$$Q_{30}^{t,\eta}(0,\zeta) = -\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{21}, \quad \zeta \ge 0,$$
$$Q_{40}^{t,\eta}(0,\zeta) = -\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{22}, \quad \zeta \ge 0, \tag{5.68}$$

$$Q_{30}^{t,\eta}(\xi,0) = P_{30}^{t*}(\xi)H_{21}, \quad Q_{40}^{t,\eta}(\xi,0) = P_{30}^{t*}(\xi)H_{22}, \quad \xi \le 0, \tag{5.69}$$

where $P_{30}^{t*}(\xi)$ is given in (5.60).

Solving the boundary-value problem (5.66)-(5.69) and using the results of [7], we obtain

$$Q_{30}^{t,\eta*}(\xi,\zeta) = \Phi(\xi)\Psi_3(\xi+\zeta)$$

$$+(1/2)P_{30}^{t}(\xi)D_2^{-1/2}\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{21}, \quad \xi \le 0, \quad \zeta \ge 0,$$

$$Q_{40}^{t,\eta*}(\xi,\zeta) = \Phi(\xi)\Psi_4(\xi+\zeta)$$

$$+(1/2)P_{30}^{t}(\xi)D_2^{-1/2}\exp\big(-D_2^{1/2}\zeta\big)D_2^{1/2}H_{22}, \quad \xi \le 0, \quad \zeta \ge 0,$$

where $\Phi(\xi)$ is a unique solution of the problem

$$\frac{d\Phi(\xi)}{d\xi} = \big[D_2^{1/2} + P_{30}^{t*}(\xi)\big]\Phi(\xi), \qquad \Phi(0) = I_m,$$

and $\Psi_p(\chi)$, $(p = 3,4)$ have the form

$$\Psi_3(\chi) = \begin{cases} -(1/2)D_2^{1/2}\exp\big(D_2^{1/2}\chi\big)H_{21}, & \chi \le 0, \\ -(1/2)D_2^{1/2}\exp\big(-D_2^{1/2}\chi\big)H_{21}, & \chi > 0, \end{cases}$$

$$\Psi_4(\chi) = \begin{cases} -(1/2)D_2^{1/2}\exp\big(D_2^{1/2}\chi\big)H_{22}, & \chi \le 0, \\ -(1/2)D_2^{1/2}\exp\big(-D_2^{1/2}\chi\big)H_{22}, & \chi > 0. \end{cases}$$

It is clear that $\Psi_p(\chi), (p = 3,4)$ are continuous at $\chi = 0$, and $Q_{p0}^{t,\eta}(\xi,\zeta), (p = 3,4)$ are exponentially decaying as $|\xi| + \zeta \to +\infty$.

5.3. **Justification of the asymptotic solution.** Denote

$$\begin{aligned} P_{30}^*(t,\varepsilon) &\stackrel{\triangle}{=} P_{30}^{o*}(t) + P_{30}^{t*}\big(t-t_f)/\varepsilon\big), \\ Q_{30}^*(t,\eta,\varepsilon) &\stackrel{\triangle}{=} Q_{30}^{o*}(t,\eta) + Q_{30}^{\eta*}\big(t,(\eta+h)/\varepsilon\big) \\ &\quad + Q_{30}^{t,\eta*}\big((t-t_f)/\varepsilon,(\eta+h)/\varepsilon\big), \\ Q_{40}^*(t,\eta,\varepsilon) &\stackrel{\triangle}{=} Q_{40}^{\eta*}\big(t,(\eta+h)/\varepsilon\big) + Q_{40}^{t,\eta*}\big((t-t_f)/\varepsilon,(\eta+h)/\varepsilon\big), \\ &\quad (t,\eta)\in[0,t_f]\times[-h,0]. \end{aligned} \tag{5.70}$$

Lemma 5.1. *Let the assumption A2 be valid. Then, there exists a positive number* ε_0 *such that for all* $\varepsilon\in(0,\varepsilon_0]$*, the problem (5.4)-(5.19) has a continuous solution* $\{P_j^*(t,\varepsilon),Q_i^*(t,\eta,\varepsilon),R_j^*(t,\eta,\rho,\varepsilon),\ j=1,2,3,\ i=1,2,3\}$ *in the domain* Ω*. For all* $(t,\eta,\rho,\varepsilon)\in\Omega\times(0,\varepsilon_0]$*, this solution satisfies the symmetry properties* $\big(P_1^*(t,\varepsilon)\big)^T = P_1^*(t,\varepsilon)$*,* $\big(P_3^*(t,\varepsilon)\big)^T = P_3^*(t,\varepsilon)$*,* $\big(R_1^*(t,\eta,\rho,\varepsilon)\big)^T = R_1^*(t,\rho,\eta,\varepsilon)$*,* $\big(R_3^*(t,\eta,\rho,\varepsilon)\big)^T = R_3^*(t,\rho,\eta,\varepsilon)$*, and the inequalities*

$$\begin{aligned} &\Big\|P_k^*(t,\varepsilon)-P_{k0}^{o*}(t)\Big\|\le a\varepsilon, \qquad \Big\|P_3^*(t,\varepsilon)-P_{30}^*(t,\varepsilon)\Big\|\le a\varepsilon, \\ &\Big\|Q_1^*(t,\eta,\varepsilon)-Q_{10}^{o*}(t,\eta)\Big\|\le a\varepsilon, \qquad \Big\|Q_2^*(t,\eta,\varepsilon)\Big\|\le a\varepsilon, \\ &\Big\|Q_p^*(t,\eta,\varepsilon)-Q_{p0}^*(t,\eta,\varepsilon)\Big\|\le a\varepsilon, \\ &\Big\|R_1^*(t,\eta,\rho,\varepsilon)-R_{10}^{o*}(t,\eta,\rho)\Big\|\le a\varepsilon, \qquad \Big\|R_l^*(t,\eta,\rho,\varepsilon)\Big\|\le a\varepsilon, \\ &k=1,2, \qquad p=3,4, \qquad l=2,3, \end{aligned} \tag{5.71}$$

where $a>0$ *is some constant independent of* ε*.*

Proof. The proof is carried out quite similarly to the work [7], where a problem similar to the problem (5.4)-(5.19) is analyzed. The only essential difference between the problems of [7] and (5.4)-(5.19) is that the former is associated with a linear-quadratic cheap control problem with state delays, while the latter is associated with a zero-sum linear-quadratic cheap control game with state delays. Therefore, in the problem of [7] the matrix of coefficients for the quadratic terms is symmetric positive semi-definite, while in (5.4)-(5.19) such a matrix is symmetric but indefinite. Due to this indefiniteness, we introduce an additional assumption (the assumption A2), while the other assumptions in the present paper are similar to those in [7] yielding the similar results. Namely, the existence of the solution to the problem (5.4)-(5.19) for all sufficiently small $\varepsilon>0$, which satisfies the symmetry properties and the inequalities (5.71). □

Corollary 5.2. *Let the assumption A2 be valid. Then, for any* $\varepsilon\in(0,\varepsilon_0]$*, all the statements of Theorem 4.3 are valid.*

Proof. Due to Lemma 5.1, for any $\varepsilon\in(0,\varepsilon_0]$, the problem (4.2)-(4.4),(4.6)-(4.7) has the solution $\big\{P^*(t,\varepsilon),Q^*(t,\eta,\varepsilon),R^*(t,\eta,\rho,\varepsilon)\big\}$, $(t,\eta,\rho)\in\Omega$. The components $P^*(\varepsilon)$, $Q^*(\tau,\varepsilon)$ and $R^*(\tau,\rho,\varepsilon)$ of this solution have the block form (5.1),(5.2) and

(5.3), respectively, where $P_j(t,\varepsilon) = P_j^*(t,\varepsilon)$, $Q_i(t,\eta,\varepsilon) = Q_i^*(t,\eta,\varepsilon)$, $R_j(t,\eta,\rho,\varepsilon) = R_j^*(t,\eta,\rho,\varepsilon)$, $(j = 1,2,3;\ i = 1,...,4)$. For any $\varepsilon \in (0,\varepsilon_0]$, this solution satisfies all the conditions of the assumption A1, which means the validity of Theorem 4.3 for any such ε. □

Consider the value

$$J_0^*\big(x_0,\varphi_x(\eta)\big) \overset{\triangle}{=} x_0^T P_{10}^*(0)x_0 + 2x_0^T \int_{-h}^{0} Q_{10}^{o*}(0,\eta)\varphi_x(\eta)d\eta + \int_{-h}^{0}\int_{-h}^{0} \varphi_x^T(\eta)R_{10}^{o*}(0,\eta,\rho)\varphi_x(\rho)d\eta d\rho, \tag{5.72}$$

where x_0 and $\varphi_x(\eta)$ are the upper blocks of the vectors z_0 and $\varphi(\eta)$, respectively, (see the equation (2.5)).

Corollary 5.3. *Let the assumption A2 be valid. Then, for all $\varepsilon \in (0,\varepsilon_0]$, the value of the CCDG $J_\varepsilon^*\big(z_0,\varphi(\eta)\big)$ satisfies the inequality*

$$\big\|J_\varepsilon^*\big(z_0,\varphi(\eta)\big) - J_0^*\big(x_0,\varphi_x(\eta)\big)\big\| \le c\big(z_0,\varphi(\eta)\big)\varepsilon, \tag{5.73}$$

where $c\big(z_0,\varphi(\eta)\big) > 0$ is some constant independent of ε but depending on z_0 and $\varphi(\eta)$.

Proof. The corollary is a direct consequence of the item (c) of Theorem 4.3, Lemma 5.1, Corollary 5.2 and its proof, and the equation (5.72). □

6. Reduced Differential Game

6.1. **Transformation of the CCDG.** Let us transform the minimizer's control of this game as:

$$u(t) = (1/\varepsilon)w(t), \qquad t \in [0,t_f], \tag{6.1}$$

where $w(t)$ is a new control of the minimizer.

Due to this transformation and the equations (2.5),(3.2), the dynamic system and the cost functional of the game (2.1)-(2.3),(4.1) become as:

$$\frac{dx(t)}{dt} = A_{11}x(t) + A_{12}y(t) + H_{11}x(t-h) + \int_{-h}^{0} G_{11}(\eta)x(t+\eta)d\eta + C_1v(t), \tag{6.2}$$

$$\varepsilon\frac{dy(t)}{dt} = \varepsilon\Big[A_{21}x(t) + A_{22}y(t) + H_{21}x(t-h) + H_{22}y(t-h) + \int_{-h}^{0}\big(G_{21}(\eta)x(t+\eta) + G_{22}(\eta)y(t+\eta)\big)d\eta + C_2v(t)\Big] + w(t), \tag{6.3}$$

$$\mathcal{J}(w,v) = \int_0^{t_f} \big[x^T(t)D_1x(t) + y^T(t)D_2y(t)$$

$$+w^T(t)w(t) - v^T(t)Mv(t)\big]dt. \tag{6.4}$$

Note, that the dynamics of the transformed game (6.2)-(6.4),(2.3) is singularly perturbed, while the controls' costs are not small, i.e., are not cheap. We call this game the Singularly Perturbed Game (SPG).

6.2. **Derivation of the Reduced Differential Game.** The dynamic system and the cost functional of this game are obtained from (6.2)-(6.4) by setting there formally $\varepsilon = 0$ and redenoting x, y, v, w and $\mathcal{J}$ with $x_{\rm r}$, $y_{\rm r}$, $v_{\rm r}$, $w_{\rm r}$ and $\mathcal{J}_{\rm r}$, respectively. Thus, we have

$$\begin{aligned}\frac{dx_{\rm r}(t)}{dt} &= A_{11}x_{\rm r}(t) + A_{12}y_{\rm r}(t) + H_{11}x_{\rm r}(t-h)\\ &+ \int_{-h}^{0} G_{11}(\eta)x_{\rm r}(t+\eta)d\eta + C_1 v_{\rm r}(t), \qquad t \in [0, t_f],\end{aligned} \tag{6.5}$$

$$0 = w_{\rm r}(t), \qquad t \in [0, t_f], \tag{6.6}$$

$$\begin{aligned}\mathcal{J}_{\rm r} = \int_0^{t_f} &\big[x_{\rm r}^T(t)D_1x_{\rm r}(t) + y_{\rm r}^T(t)D_2y_{\rm r}(t)\\ &+w_{\rm r}^T(t)w_{\rm r}(t) - v_{\rm r}^T(t)Mv_{\rm r}(t)\big]dt.\end{aligned} \tag{6.7}$$

Due to (6.6), the functional (6.7) becomes:

$$\mathcal{J}_{\rm r} = \int_0^{t_f} \big[x_{\rm r}^T(t)D_1x_{\rm r}(t) + y_{\rm r}^T(t)D_2y_{\rm r}(t) - v_{\rm r}^T(t)Mv_{\rm r}(t)\big]dt. \tag{6.8}$$

Since the variable $y_{\rm r}(t)$ does not satisfy any equation for $t \in [0, t_f]$, it can be chosen to satisfy a desirable property of the system (6.5) and the functional (6.8), i.e., $y_{\rm r}(t)$ can be chosen as a control in these system and functional. Moreover, since the control of the maximizer $v_{\rm r}$ is present in (6.5),(6.8), while a minimizer's control does not appear in these system and functional, then it is reasonable to choose $y_{\rm r}(t)$ as a minimizer's control. This observation means that the functional (6.8) depends on $y_{\rm r}$ and $v_{\rm r}$, i.e., $\mathcal{J}_{\rm r} = \mathcal{J}_{\rm r}(y_{\rm r}, v_{\rm r})$. Thus, the functional (6.8) is minimized by $y_{\rm r}$ and maximized by $v_{\rm r}$. The initial conditions for the system (6.5) are obtained from (2.3) by removing the initial conditions for $y(\cdot)$, which yields

$$x_{\rm r}(\eta) = \varphi_x(\eta), \quad \eta \in [-h, 0); \quad x_{\rm r}(0) = x_0. \tag{6.9}$$

Thus, the Reduced Differential Game (RDG) consists of the dynamic system (6.5), the initial conditions (6.9) and the functional (6.8). Since D_2 and M are positive definite matrices, the RDG is regular.

6.3. **State-Feedback Saddle-Point Equilibrium in the RDG.** Let $\phi(\eta)$ be any function belonging to $L^2[-h, 0; \mathbb{R}^n]$. For all $t \in [0, t_f]$, let us consider the t-parametric set $\mathcal{Y}_{{\rm r},t}$ of all vector-valued continuous functionals $y_{\rm r}\big[x_{\rm r}, \phi(\eta), t\big] : \mathbb{R}^n \times L^2[-h, 0; \mathbb{R}^n] \to \mathbb{R}^m$, and the t-parametric set $\mathcal{V}_{{\rm r},t}$ of all vector-valued continuous functionals $v_{\rm r}\big[x_{\rm r}, \phi(\eta), t\big] : \mathbb{R}^n \times L^2[-h, 0; \mathbb{R}^n] \to \mathbb{R}^s$.

Definition 6.1. By $\big(YV\big)_{\rm r}$, we denote the set of all pairs $\big(y_{\rm r}[x_{{\rm r},t},t],v_{\rm r}[x_{{\rm r},t},t]\big)$, $x_{{\rm r},t} \stackrel{\triangle}{=} \big\{\big(x_{\rm r}(t),x_{\rm r}(\theta)\big),\theta\in[t-h,t)\big\}$, satisfying the following conditions: (i) for any fixed $t\in[0,t_f]$, $\big(x_{\rm r}(t),x_{\rm r}(t+\eta)\big)\in\mathbb{R}^n\times L^2[-h,0;\mathbb{R}^n]$ and $y_{\rm r}[x_{{\rm r},t},t]\in\mathcal{Y}_{{\rm r},t}$, $v_{\rm r}[x_{{\rm r},t},t]\in\mathcal{V}_{{\rm r},t}$; (ii) the initial-value problem (6.5),(6.9) for $y_{\rm r}(t)=y_{\rm r}[x_{{\rm r},t},t]$, $v_{\rm r}(t)=v_{\rm r}[x_{{\rm r},t},t]$, $t\in[0,t_f]$, and any $\varphi_x(\eta)\in L^2[-h,0;\mathbb{R}^n]$, $x_0\in\mathbb{R}^n$ has the unique absolutely continuous solution $x_{{\rm r},yv}\big(t;x_0,\varphi_x(\eta)\big)$ in the interval $[0,t_f]$; (iii) the time realization of the minimizer's state-feedback control $y_{\rm r}[x_{{\rm r},t},t]$ along the solution $x_{{\rm r},yv}\big(t;x_0,\varphi_x(\eta)\big)$ belongs to $L^2[0,t_f;\mathbb{R}^m]$; (iv) the time realization of the maximizer's state-feedback control $v_{\rm r}[x_{{\rm r},t},t]$ along $x_{{\rm r},yv}\big(t;x_0,\varphi_x(\eta)\big)$ belongs to $L^2[0,t_f;\mathbb{R}^s]$.

In what follows, $\big(YV\big)_{\rm r}$ is called the set of all admissible pairs of the players' state-feedback controls (strategies) in the RDG.

For any given $y_{{\rm r},0}[x_{{\rm r},t},t]\in\mathcal{Y}_{{\rm r},t}$ and $v_{{\rm r},0}[x_{{\rm r},t},t]\in\mathcal{V}_{{\rm r},t}$, consider the sets

$$\begin{aligned}\mathcal{F}_{{\rm r},v}\big(y_{{\rm r},0}[x_{{\rm r},t},t]\big) &\stackrel{\triangle}{=} \{v_{\rm r}[x_{{\rm r},t},t]\in\mathcal{V}_{{\rm r},t}:\big(y_{{\rm r},0}[x_{{\rm r},t},t],v_{\rm r}[x_{{\rm r},t},t]\big)\in(YV)_{\rm r}\},\\ \mathcal{F}_{{\rm r},y}\big(v_{{\rm r},0}[x_{{\rm r},t},t]\big) &\stackrel{\triangle}{=} \{y_{\rm r}[x_{{\rm r},t},t]\in\mathcal{Y}_{{\rm r},t}:\big(y_{\rm r}[x_{{\rm r},t},t],v_{{\rm r},0}[x_{{\rm r},t},t]\big)\in(YV)_{\rm r}\}.\end{aligned} \tag{6.10}$$

Consider the following t-dependent vector-valued functionals:

$$\begin{aligned}y_{\rm r}^*[x_{{\rm r},t},t] &\stackrel{\triangle}{=} -D_2^{-1}A_{12}^T\left(P_{10}^{o*}(t)x_{\rm r}(t)+\int_{-h}^{0}Q_{10}^{o*}(t,\eta)x_{\rm r}(t+\eta)d\eta\right)\in\mathcal{Y}_{\rm r,t},\\ v_{\rm r}^*[x_{{\rm r},t},t] &\stackrel{\triangle}{=} M^{-1}C_1^T\left(P_{10}^{o*}(t)x_{\rm r}(t)+\int_{-h}^{0}Q_{10}^{o*}(t,\eta)x_{\rm r}(t+\eta)d\eta\right)\in\mathcal{V}_{\rm r,t}.\end{aligned} \tag{6.11}$$

Note, that both inclusions in (6.11) are valid for all $t\in[0,t_f]$.

Similarly to Theorem 4.3, we have the following assertion.

Theorem 6.2. *Let the assumption A2 be valid. Then:*
(a) the pair $\big(y_{\rm r}^*[x_{{\rm r},t},t],v_{\rm r}^*[x_{{\rm r},t},t]\big)\in(YV)_{\rm r}$*, i.e., it is admissible in the RDG;*
(b) for any $y_{\rm r}[x_{{\rm r},t},t]\in\mathcal{F}_{{\rm r},y}\big(v_{\rm r}^*[x_{{\rm r},t},t]\big)$ *and any* $v_{\rm r}[x_{{\rm r},t},t]\in\mathcal{F}_{{\rm r},v}\big(y_{\rm r}^*[x_{{\rm r},t},t]\big)$*, the admissible pair* $\big(y_{\rm r}^*[x_{{\rm r},t},t],v_{\rm r}^*[x_{{\rm r},t},t]\big)$ *satisfies the following inequality:*

$$\mathcal{J}_{\rm r}\big(y_{\rm r}^*[x_{{\rm r},t},t],v_{\rm r}[x_{{\rm r},t},t]\big)\le\mathcal{J}_{\rm r}\big(y_{\rm r}^*[x_{{\rm r},t},t],v_{\rm r}^*[x_{{\rm r},t},t]\big)\le\mathcal{J}_{\rm r}\big(y_{\rm r}[x_{{\rm r},t},t],v_{\rm r}^*[x_{{\rm r},t},t]\big),$$

i.e., this pair is a saddle-point equilibrium in the regular RDG;
(c) the value of the RDG is $\mathcal{J}_{\rm r}^*\big(x_0,\varphi_x(\eta)\big)\stackrel{\triangle}{=}\mathcal{J}_{\rm r}\big(y_{\rm r}^*[x_{{\rm r},t},t],v_{\rm r}^*[x_{{\rm r},t},t]\big)=J_0^*\big(x_0,\varphi_x(\eta)\big)$*, given by (5.72);*
(d) for any $x_0\in\mathbb{R}^n$ *and any* $\varphi_x(\eta)\in L^2[-h,0;\mathbb{R}^n]$*, the value of the RDG is non-negative.*

Remark 6.3. Theorem 6.2 presents a game-theoretic interpretation of the problem (5.51)-(5.53),(5.46)-(5.48), arising in the asymptotic solution of the problem (4.2)-(4.4),(4.6)-(4.7). Namely, the property of the problem (5.51)-(5.53),(5.46)-(5.48), required in the assumption A2 (the existence of the continuous and symmetric solution), is sufficient for the existence of the saddle-point equilibrium in the RDG.

7. Main Results

For any given $\varepsilon \in (0, \varepsilon_0]$, consider the vector-valued functional

$$u^*_{\varepsilon,0}[z_t, t] \stackrel{\triangle}{=} -\frac{1}{\varepsilon}\left[\big(P^{o*}_{20}(t)\big)^T x(t) + P^{o*}_{30}(t)y(t) + \int_{-h}^{0} Q^{o*}_{30}(t,\eta)x(t+\eta)d\eta\right], \tag{7.1}$$

where $t \in [0, t_f]$; $z_t = \mathrm{col}\big(x_t, y_t\big)$.

Note that the vector-valued functional (7.1) is linear with respect to $\big(x(t), x(t+\eta), y(t)\big)$. Moreover, the matrix-valued coefficients in this functional are continuous functions for $(t, \eta) \in [0, t_f] \times [-h, 0]$. Therefore,

$$u^*_{\varepsilon,0}[z_t, t] \in \mathcal{U}_t \quad \forall t \in [0, t_f]. \tag{7.2}$$

Lemma 7.1. *Let the assumption A2 be valid. Then, for all $\varepsilon \in (0, \varepsilon_0]$, the following inclusion is fulfilled: $\big(u^*_{\varepsilon,0}[z_t, t], v^*_{\mathrm{r}}[x_t, t]\big) \in UV$, where $v^*_{\mathrm{r}}[\cdot, \cdot]$ is given in (6.11).*

Proof. First of all, let us note the following. Since $v^*_{\mathrm{r}}[x_{\mathrm{r},t}, t] \in \mathcal{V}_{\mathrm{r},t}$ for all $t \in [0, t_f]$, then $v^*_{\mathrm{r}}[x_t, t] \in \mathcal{V}_t$ for all $t \in [0, t_f]$. Now, the substitution of $u^*_{\varepsilon,0}[z_t, t]$ and $v^*_{\mathrm{r}}[x_t, t]$ into the system (2.1)-(2.2) instead of $u(t)$ and $v(t)$, respectively, yields a linear functional-differential system with continuous coefficients. By virtue of the results of [4], this linear system subject to the initial conditions (2.3) has the unique absolutely continuous solution. Therefore, due to Definition 2.1, the inclusion stated in the lemma is fulfilled. □

Lemma 7.2. *Let the assumption A2 be valid. Then, there exists a positive number $\varepsilon_1 \le \varepsilon_0$ such that, for all $\varepsilon \in (0, \varepsilon_1]$, the guaranteed result $J_u\big(u^*_{\varepsilon,0}[z_t, t]; z_0, \varphi(\eta)\big)$ of the minimizer's state-feedback control $u^*_{\varepsilon,0}[z_t, t]$ in the SDG satisfies the inequality*

$$\Big|J_u\big(u^*_{\varepsilon,0}[z_t, t]; z_0, \varphi(\eta)\big) - J^*_0\big(x_0, \varphi_x(\eta)\big)\Big| \le c\big(z_0, \varphi(\eta)\big)\varepsilon, \tag{7.3}$$

where $c\big(z_0, \varphi(\eta)\big)$ is some positive constant independent of ε, while depending on z_0 and $\varphi(\eta)$; $J^*_0\big(x_0, \varphi_x(\eta)\big)$ is the RDG value given by (5.72).

Proof. For a given $\varepsilon \in (0, \varepsilon_0]$, consider the following two $m \times (n+m)$-matrices:

$$K_1(t) = -\frac{1}{\varepsilon}\Big(\big(P^{o*}_{20}(t)\big)^T, P^{o*}_{30}(t)\Big), \qquad K_2(t,\eta) = -\frac{1}{\varepsilon}\Big(Q^{o*}_{30}(t,\eta), 0\Big),$$
$$(t,\eta) \in [0, t_f] \times [-h, 0]. \tag{7.4}$$

Using these matrices, we can represent the control $u^*_{\varepsilon,0}[z_t, t]$ in the form (3.6).

Due to Lemma 3.1, we can conclude the following. If, for some $\varepsilon \in (0, \varepsilon_0]$, the problem (3.7)-(3.9),(3.10)-(3.11),(7.4) has a continuous solution $\mathcal{P}_K(t) = \mathcal{P}^*_K(t, \varepsilon)$, $\mathcal{Q}_K(t, \eta) = \mathcal{Q}^*_K(t, \eta, \varepsilon)$, $\mathcal{R}_K(t, \eta, \rho) = \mathcal{R}^*_K(t, \eta, \rho, \varepsilon)$, $(t, \eta, \rho) \in \Omega$, such that $\big(\mathcal{P}^*_K(t, \varepsilon)\big)^T = \mathcal{P}^*_K(t, \varepsilon)$, $\big(\mathcal{R}^*_K(t, \eta, \rho, \varepsilon)\big)^T = \mathcal{R}^*_K(t, \rho, \eta, \varepsilon)$, then $u^*_{\varepsilon,0}[z_t, t] \in \mathcal{H}_u$ and its guaranteed result in the SDG has the form (3.12).

Similarly to Section 5, by constructing and justifying an asymptotic solution of the problem (3.7)-(3.9),(3.10)-(3.11),(7.4), we obtain the existence of a positive number $\varepsilon_1 \le \varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_1]$ this problem has the solution with the above mentioned properties. Moreover, this solution has the block form similar to (5.1)-(5.3):

$$\mathcal{P}_K^*(t,\varepsilon) = \begin{pmatrix} \mathcal{P}_{K,1}^*(t,\varepsilon) & \varepsilon\mathcal{P}_{K,2}^*(t,\varepsilon) \\ \varepsilon\big(\mathcal{P}_{K,2}^*(t,\varepsilon)\big)^T & \varepsilon\mathcal{P}_{K,3}^*(t,\varepsilon) \end{pmatrix},$$

$$\mathcal{Q}_K^*(t,\eta,\varepsilon) = \begin{pmatrix} \mathcal{Q}_{K,1}^*(t,\eta,\varepsilon) & \mathcal{Q}_{K,2}^*(t,\eta,\varepsilon) \\ \varepsilon\mathcal{Q}_{K,3}^*(t,\eta,\varepsilon) & \varepsilon\mathcal{Q}_{K,4}^*(t,\eta,\varepsilon) \end{pmatrix},$$

$$\mathcal{R}_K^*(t,\eta,\rho,\varepsilon) = \begin{pmatrix} \mathcal{R}_{K,1}^*(t,\eta,\rho,\varepsilon) & \mathcal{R}_{K,2}^*(t,\eta,\rho,\varepsilon) \\ \big(\mathcal{R}_{K,2}^*(t,\rho,\eta,\varepsilon)\big)^T & \mathcal{R}_{K,3}^*(t,\eta,\rho,\varepsilon) \end{pmatrix},$$

(7.5)

the matrices $\mathcal{P}_{K,j}^*(t,\varepsilon)$, $\mathcal{Q}_{K,i}^*(t,\eta,\varepsilon)$, $\mathcal{R}_{K,j}^*(t,\tau,\rho,\varepsilon)$, $(j = 1,2,3;\ i = 1,...,4)$ are bounded for all $(t,\eta,\rho,\varepsilon) \in \Omega \times (0,\varepsilon_1]$, and the following inequalities are satisfied: $\big\|\mathcal{P}_{K,1}^*(t,\varepsilon) - P_{10}^{o*}(t)\big\| \le a\varepsilon$, $\big\|\mathcal{Q}_{K,1}^*(t,\eta,\varepsilon) - Q_{10}^{o*}(t,\eta)\big\| \le a\varepsilon$, $\big\|\mathcal{Q}_{K,2}^*(t,\eta,\varepsilon)\big\| \le a\varepsilon$, $\big\|\mathcal{R}_{K,1}^*(t,\eta,\rho,\varepsilon) - R_{10}^{o*}(t,\eta,\rho)\big\| \le a\varepsilon$, $\big\|\mathcal{R}_{K,l}^*(t,\eta,\rho,\varepsilon)\big\| \le a\varepsilon$, $(l = 2,3)$, where $P_{10}^{o*}(t)$, $Q_{10}^{o*}(t,\eta)$, $R_{10}^{o*}(t,\eta,\rho)$ are the components of the solution to the problem (5.51)-(5.53),(5.46)-(5.48) mentioned in the assumption A2; $a > 0$ is some constant independent of ε. These inequalities, along with the representation (3.12) for the guaranteed result of $u_{\varepsilon,0}^*[z_t,t]$ and the equations (2.5),(5.72),(7.5), directly yield the statement of the lemma. □

Lemma 7.3. *Let the assumption A2 be valid. Then, the guaranteed result* $J_v\big(v_{\mathrm{r}}^*[x_t,t]; z_0,\varphi(\eta)\big)$ *of the maximizer's state-feedback control* $v_{\mathrm{r}}^*[x_t,t]$ *in the SDG is*

$$J_v\big(v_{\mathrm{r}}^*[x_t,t]; z_0,\varphi(\eta)\big) = J_0^*(x_0,\varphi_x(\tau)). \tag{7.6}$$

Proof. First of all let us note that, by virtue of Lemma 7.1, the control $v_{\mathrm{r}}^*[x_t,t] \in \mathcal{H}_v$. Moreover, due to Definition 2.3, the value $J_v\big(v_{\mathrm{r}}^*[x_t,t]; z_0,\varphi(\eta)\big)$ represents an optimal value of the cost functional in the optimal control problem, obtained from the SDG by substitution of $v(t) = v_{\mathrm{r}}^*[x_t,t]$ into the equation of dynamics (3.3) and the cost functional (3.5). The equation of dynamics in this optimal control problem has the form

$$\frac{dz(t)}{dt} = \widetilde{A}(t)z(t) + Hz(t-h) + \int_{-h}^{0} \widetilde{G}(t,\eta)z(t+\eta)d\eta + Bu(t), \quad t \in [0,t_f], \tag{7.7}$$

and the cost functional is

$$\widetilde{J}(u) \triangleq \int_0^{t_f} \bigg(z^T(t)\widetilde{D}(t)z(t) - 2z^T(t)\int_{-h}^{0}\widetilde{F}(t,\eta)z(t+\eta)d\eta - \int_{-h}^{0}\int_{-h}^{0} z^T(t+\eta)\widetilde{L}(t,\eta,\rho)z(t+\rho)d\eta d\rho \bigg) dt \to \inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)},$$

(7.8)

where

$$\widetilde{A}(t) = \begin{pmatrix} A_{11} + S_{v1}P_{10}^{o*}(t) & A_{12} \\ A_{21} + S_{v2}^T P_{10}^{o*}(t) & A_{22} \end{pmatrix},$$

$$\widetilde{G}(t,\eta) = \begin{pmatrix} G_{11}(\eta) + S_{v1}Q_{10}^{o*}(t,\eta) & 0 \\ G_{21}(\eta) + S_{v2}^{T}Q_{10}^{o*}(t,\eta) & G_{22}(\eta) \end{pmatrix},$$

$$\widetilde{D}(t) = \begin{pmatrix} D_1 - P_{10}^{o*}(t)S_{v1}P_{10}^{o*}(t) & 0 \\ 0 & D_2 \end{pmatrix},$$

$$\widetilde{F}(t,\eta) = \begin{pmatrix} P_{10}^{o*}(t)S_{v1}Q_{10}^{o*}(t,\eta) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{L}(t,\eta,\rho) = \begin{pmatrix} \big(Q_{10}^{o*}(t,\eta)\big)^{T}S_{v1}Q_{10}^{o*}(t,\rho) & 0 \\ 0 & 0 \end{pmatrix}.$$

The system (7.7) is subject to the initial conditions (3.4).

The problem (7.7)-(7.8),(3.4) is a singular linear-quadratic optimal control problem with state delays in the dynamics and in the cost functional. Moreover,

$$J_v\big(v_{\mathrm{r}}^*[x_t,t]; z_0,\varphi(\eta)\big) = \inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)} \widetilde{J}(u). \tag{7.9}$$

The value in the right-hand side of (7.9) can be calculated in the way, similar to that of [9] for solution of a singular stochastic linear-quadratic optimal control problem with state delays in the dynamics. To keep the paper to be self-contained as much as possible, we present here this calculation adapted to the problem (7.7)-(7.8),(3.4). However, not to overload the paper and thus to keep its readability, we present this calculation in a brief form.

The further proof consists of three stages.

Stage 1: Regularization of (7.7)-(7.8),(3.4).

We solve this singular optimal control problem by the regularization method, i.e., we replace it by a regular optimal control problem. The latter consists of the same equation of dynamics (7.7) and initial conditions (3.4). However, the cost functional of the new problem has the regular form

$$\widetilde{J}_{\varepsilon}(u) \triangleq \int_0^{t_f}\bigg(z^T(t)\widetilde{D}(t)z(t) - 2z^T(t)\int_{-h}^{0}\widetilde{F}(t,\eta)z(t+\eta)d\eta - \int_{-h}^{0}\int_{-h}^{0} z^T(t+\eta)\widetilde{L}(t,\eta,\rho)z(t+\rho)d\eta d\rho + \varepsilon^2 u^T(t)u(t)\bigg)dt \to \min_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}, \tag{7.10}$$

where $\varepsilon > 0$ is a small parameter.

Consider the following hybrid system of Riccati-type ordinary and partial differential equations with respect to unknown matrices $\widetilde{P}(t)$, $\widetilde{Q}(t,\eta)$ and $\widetilde{R}(t,\eta,\rho)$ in the domain Ω:

$$\frac{d\widetilde{P}(t)}{dt} = -\widetilde{P}(t)\widetilde{A}(t) - \widetilde{A}^T(t)\widetilde{P}(t) + \widetilde{P}(t)S_u(\varepsilon)\widetilde{P}(t) - \widetilde{Q}(t,0) - \widetilde{Q}^T(t,0) - \widetilde{D}(t), \tag{7.11}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta}\right)\widetilde{Q}(t,\eta) = -\widetilde{A}^T(t)\widetilde{Q}(t,\eta) + \widetilde{P}(t)S_u(\varepsilon)\widetilde{Q}(t,\eta)$$

$$-\widetilde{P}(t)\widetilde{G}(t,\eta) - \widetilde{R}(t,0,\eta) + \widetilde{F}(t,\eta), \tag{7.12}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \rho}\right)\widetilde{R}(t,\eta,\rho) = -\widetilde{G}^T(t,\eta)\widetilde{Q}(t,\rho) - \widetilde{Q}^T(t,\eta)\widetilde{G}(t,\rho)$$
$$+\widetilde{Q}^T(t,\eta)S_u(\varepsilon)\widetilde{Q}(t,\rho) + \widetilde{L}(t,\eta,\rho), \tag{7.13}$$

where $S_u(\varepsilon)$ is given in (4.5).

The system (7.11)-(7.13) is subject to the boundary conditions

$$\widetilde{P}(t_f) = 0, \quad \widetilde{Q}(t_f,\eta) = 0, \quad \widetilde{R}(t_f,\eta,\rho) = 0, \tag{7.14}$$

$$\widetilde{Q}(t,-h) = \widetilde{P}(t)H, \quad \widetilde{R}(t,-h,\eta) = H^T\widetilde{Q}(t,\eta), \quad \widetilde{R}(t,\eta,-h) = \widetilde{Q}^T(t,\eta)H, \tag{7.15}$$

where $(t,\eta,\rho) \in [0,t_f]\times[-h,0]\times[-h,0]$.

The following assertion is proven quite similarly to the items (i)-(iii) of Lemma 3.1.

Assertion. *Let for a given $\varepsilon > 0$, the problem (7.11)-(7.15) have a continuous solution $\{\widetilde{P}^*(t,\varepsilon), \widetilde{Q}^*(t,\eta,\varepsilon), \widetilde{R}^*(t,\eta,\rho,\varepsilon)\}$, $(t,\eta,\rho) \in \Omega$, such that $\big(\widetilde{P}^*(t,\varepsilon)\big)^T = \widetilde{P}^*(t,\varepsilon)$, $\big(\widetilde{R}^*(t,\eta,\rho,\varepsilon)\big)^T = \widetilde{R}^*(t,\rho,\eta,\varepsilon)$. Based on this solution, let us construct the following control for the system (7.7):*

$$u(t) = \tilde{u}^*_\varepsilon[z_t,t] \stackrel{\triangle}{=} -\frac{1}{\varepsilon^2}B^T\left[\widetilde{P}^*(t,\varepsilon)z(t) + \int_{-h}^{0}\widetilde{Q}^*(t,\eta,\varepsilon)z(t+\eta)d\eta\right].$$

Then: (a) $\tilde{u}^*_\varepsilon[z_t,t] \in \mathcal{F}_u\big(v^*_{\mathrm{r}}[x_t,t]\big)$; (b) this control solves the optimal control problem (7.7),(7.10),(3.4); (c) the optimal value $\widetilde{J}^*_\varepsilon\big(z_0,\varphi(\eta)\big)$ of the cost functional in this problem has the form

$$\widetilde{J}^*_\varepsilon\big(z_0,\varphi(\eta)\big) \stackrel{\triangle}{=} \widetilde{J}_\varepsilon\big(\tilde{u}^*_\varepsilon[z_t,t]\big) = z_0^T\widetilde{P}^*(0,\varepsilon)z_0 + 2z_0^T\int_{-h}^{0}\widetilde{Q}^*(0,\eta,\varepsilon)\varphi(\eta)d\eta$$
$$+\int_{-h}^{0}\int_{-h}^{0}\varphi^T(\eta)\widetilde{R}^*(0,\eta,\rho,\varepsilon)\varphi(\rho)d\eta d\rho.$$

Stage 2: Asymptotic solution of the problem (7.11)-(7.15).

The asymptotic solution of this problem is constructed and justified similarly to the constructing and justifying the asymptotic solution to the problem (4.2)-(4.4),(4.6)-(4.7) in Section 5. Moreover, constructing and justifying the asymptotic solution to (7.11)-(7.15), we obtain the existence of a positive number $\tilde{\ }\hat{\varepsilon}_0$ such that for all $\varepsilon \in (0,\hat{\varepsilon}_0]$ the solution of this problem, mentioned in Assertion, exists and has the block form similar to (5.1)-(5.3):

$$\widetilde{P}^*(t,\varepsilon) = \begin{pmatrix} \widetilde{P}^*_1(t,\varepsilon) & \varepsilon\widetilde{P}^*_2(t,\varepsilon) \\ \varepsilon\big(\widetilde{P}^*_2(t,\varepsilon)\big)^T & \varepsilon\widetilde{P}^*_3(t,\varepsilon) \end{pmatrix},$$

$$\widetilde{Q}^*(t,\eta,\varepsilon) = \begin{pmatrix} \widetilde{Q}^*_1(t,\eta,\varepsilon) & \widetilde{Q}^*_2(t,\eta,\varepsilon) \\ \varepsilon\widetilde{Q}^*_3(t,\eta,\varepsilon) & \varepsilon\widetilde{Q}^*_4(t,\eta,\varepsilon) \end{pmatrix},$$

$$\widetilde{R}^*(t,\eta,\rho,\varepsilon) = \begin{pmatrix} \widetilde{R}^*_1(t,\eta,\rho,\varepsilon) & \widetilde{R}^*_2(t,\eta,\rho,\varepsilon) \\ \big(\widetilde{R}^*_2(t,\rho,\eta,\varepsilon)\big)^T & \widetilde{R}^*_3(t,\eta,\rho,\varepsilon) \end{pmatrix},$$

(7.16)

the matrices $\widetilde{P}_j^*(t,\varepsilon)$, $\widetilde{Q}_i^*(t,\eta,\varepsilon)$, $\widetilde{R}_j^*(t,\eta,\rho,\varepsilon)$, $(j=1,2,3;\ i=1,...,4)$ are bounded for all $(t,\eta,\rho,\varepsilon)\in\Omega\times(0,\hat{\varepsilon}_0]$, and the following inequalities are satisfied: $\big\|\widetilde{P}_1^*(t,\varepsilon)-P_{10}^{o*}(t)\big\|\le a\varepsilon$, $\big\|\widetilde{Q}_1^*(t,\eta,\varepsilon)-Q_{10}^{o*}(t,\eta)\big\|\le a\varepsilon$, $\big\|\widetilde{Q}_2^*(t,\eta,\varepsilon)\big\|\le a\varepsilon$, $\big\|\widetilde{R}_1^*(t,\eta,\rho,\varepsilon)-R_{10}^{o*}(t,\eta,\rho)\big\|\le a\varepsilon$, $\big\|\widetilde{R}_l^*(t,\eta,\rho,\varepsilon)\big\|\le a\varepsilon$, $(l=2,3)$, where $P_{10}^{o*}(t)$, $Q_{10}^{o*}(t,\eta)$, $R_{10}^{o*}(t,\eta,\rho)$ are the components of the solution to the problem (5.51)-(5.53),(5.46)-(5.48) mentioned in the assumption A2; $a>0$ is some constant independent of ε. Using these inequalities, as well as the statement (c) of Assertion and the equations (5.72),(7.16), we immediatelly obtain the inequality

$$\big|\widetilde{J}_\varepsilon^*\big(z_0,\varphi(\eta)\big)-J_0^*\big(x_0,\varphi_x(\eta)\big)\big|\le c\big(z_0,\varphi(\eta)\big)\varepsilon,\quad \varepsilon\in(0,\hat{\varepsilon}_0] \tag{7.17}$$

where $c\big(z_0,\varphi(\eta)\big)$ is some positive constant independent of ε, while depending on z_0 and $\varphi(\eta)$.

Let us remember that $J_0^*\big(x_0,\varphi_x(\eta)\big)$ is the value of the RDG given by (5.72).

Stage 3: Obtaining an expression for $\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)$**.**

First, let us rewrite the inequality (7.17) in the equivalent form

$$\begin{aligned}&J_0^*\big(x_0,\varphi_x(\eta)\big)-c\big(z_0,\varphi(\eta)\big)\varepsilon\le\widetilde{J}_\varepsilon^*\big(z_0,\varphi(\eta)\big)\\&\le J_0^*\big(x_0,\varphi_x(\eta)\big)+c\big(z_0,\varphi(\eta)\big)\varepsilon,\quad \varepsilon\in(0,\hat{\varepsilon}_0].\end{aligned}$$

(7.18)

Using this inequality and the statement (a) of Assertion, we obtain for any $\varepsilon\in(0,\hat{\varepsilon}_0]$:

$$\begin{aligned}\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)&\le\widetilde{J}\big(\tilde{u}_\varepsilon^*[z_t,t]\big)\le\widetilde{J}_\varepsilon\big(\tilde{u}_\varepsilon^*[z_t,t]\big)=\widetilde{J}_\varepsilon^*\big(z_0,\varphi(\eta)\big)\\&\le J_0^*\big(x_0,\varphi_x(\eta)\big)+c\big(z_0,\varphi(\eta)\big)\varepsilon,\end{aligned}$$

yielding $\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)\le J_0^*\big(x_0,\varphi_x(\eta)\big)$.

Now, we are going to show that

$$\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)=J_0^*\big(x_0,\varphi_x(\tau)\big). \tag{7.19}$$

To prove the equality (7.19), we assume the opposite, i.e.,

$$\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)<J_0^*\big(x_0,\varphi_x(\tau)\big). \tag{7.20}$$

This inequality means the existence of $\hat{u}[z_t,t]\in\mathcal{F}_u\big(v_{\mathrm{r}}^*[x_t,t]\big)$, such that

$$\inf_{u[z_t,t]\in\mathcal{F}_u\left(v_{\mathrm{r}}^*[x_t,t]\right)}\widetilde{J}(u)<\widetilde{J}\big(\hat{u}[z_t,t]\big)<J_0^*\big(x_0,\varphi_x(\eta)\big). \tag{7.21}$$

Since $\tilde{u}_\varepsilon^*[z_t,t]$ is the optimal control in the problem (7.7),(7.10),(3.4), and (7.18) holds, we obtain for any $\varepsilon\in(0,\hat{\varepsilon}_0]$, any $z_0\in\mathbb{R}^{n+m}$ and any $\varphi(\eta)\in L^2[-h,0;\mathbb{R}^{n+m}]$:

$$\begin{aligned}&J_0^*\big(x_0,\varphi_x(\eta)\big)-c\big(z_0,\varphi_x(\eta)\big)\varepsilon\le\widetilde{J}_\varepsilon^*\big(z_0,\varphi(\eta)\big)\\&=\widetilde{J}_\varepsilon\big(\tilde{u}_\varepsilon^*[z_t,t]\big)\le\widetilde{J}_\varepsilon\big(\hat{u}[z_t,t]\big)=\widetilde{J}\big(\hat{u}[z_t,t]\big)+b\varepsilon^2,\end{aligned}$$

(7.22)

where $b = \int_0^{t_f} \big(\hat{u}[\hat{z}_t, t]\big)^T \hat{u}[\hat{z}_t, t]dt$; $\hat{z}(t)$, $t \in [0, t_f]$, is the solution of the initial-value problem (7.7),(3.4) generated by the control $u(t) = \hat{u}[z_t, t]$, and $\hat{z}_t = \big(\hat{z}(t), \hat{z}(t+\eta)\big)$, $t \in [0, t_f]$, $\eta \in [-h, 0)$.

The chain of the inequalities and the equalities (7.22) implies the inequality $J_0^*\big(x_0, \varphi_x(\eta)\big) \le \widetilde{J}\big(\hat{u}[z_t, t]\big)$, which contradicts the right-hand inequality in (7.21). Thus, the inequality (7.20) is wrong, meaning the validity of the equality (7.19). The latter, along with (7.9), directly yields the statement of the lemma. □

Corollary 7.4. *Let the assumption A2 be valid. Then, the following inequality is satisfied in the SDG:*

$$\big|J\big(u_{\varepsilon,0}^*[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big) - J_0^*\big(x_0, \varphi_x(\eta)\big)\big| \le c\big(z_0, \varphi(\eta)\big)\varepsilon, \qquad \varepsilon \in (0, \varepsilon_1], \tag{7.23}$$

where the positive constants ε_1 and $c\big(z_0, \varphi(\eta)\big)$ have been introduced in Lemma 7.2.

Proof. First, let us rewrite the inequality (7.3) in the equivalent form

$$\begin{aligned} J_0^*\big(x_0, \varphi_x(\eta)\big) - c\big(z_0, \varphi(\eta)\big)\varepsilon &\le J_u\big(u_{\varepsilon,0}^*[z_t, t]; z_0, \varphi(\eta)\big) \\ &\le J_0^*\big(x_0, \varphi_x(\eta)\big) + c\big(z_0, \varphi(\eta)\big)\varepsilon, \qquad \varepsilon \in (0, \varepsilon_1]. \end{aligned}$$

This inequality, along with Definition 2.2 and Lemma 7.1, yields for all $\varepsilon \in (0, \varepsilon_1]$
(7.24)

$$J\big(u_{\varepsilon,0}^*[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big) \le J_u\big(u_{\varepsilon,0}^*[z_t, t]; z_0, \varphi(\eta)\big) \le J_0^*\big(x_0, \varphi_x(\eta)\big) + c\big(z_0, \varphi(\eta)\big)\varepsilon.$$

Furthermore, by virtue of Definition 2.3, Lemma 7.1 and Lemma 7.3, we have for all $\varepsilon \in (0, \varepsilon_1]$

$$\begin{aligned} J\big(u_{\varepsilon,0}^*[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big) \ge J_v\big(v_{\mathrm{r}}^*[x_t, t]; z_0, \varphi(\eta)\big) &= J_0^*\big(x_0, \varphi_x(\eta)\big) \\ &\ge J_0^*\big(x_0, \varphi_x(\eta)\big) - c\big(z_0, \varphi(\eta)\big)\varepsilon. \end{aligned}$$

The latter inequality and the inequality (7.24) directly yield the inequality (7.23), which completes the proof of the corollary. □

Let $\{\varepsilon_q\}$, $(q = 1, 2, ...)$, be a sequence of numbers, satisfying the following conditions: (1_ε) $\varepsilon_q \in (0, \varepsilon_1]$, $(q = 1, 2, ...)$; (2_ε) $\varepsilon_q \to +0$ for $q \to +\infty$.

Theorem 7.5. *Let the assumption A2 be valid. Then, for any given $z_0 \in \mathbb{R}^{n+m}$ and $\varphi(\tau) \in L^2[-h, 0; \mathbb{R}^{n+m}]$, the sequence $\big\{\big(u_{\varepsilon_q,0}^*[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big)\big\}_{q=1}^{+\infty}$ is the saddle-point equilibrium sequence of the SDG, i.e., for any sequences $u_q[z_t, t] \in \mathcal{F}_u\big(v_{\mathrm{r}}^*[x_t, t]\big)$ and $v_q[z_t, t] \in \mathcal{F}_v\big(u_{\varepsilon_q,0}^*[z_t, t]\big)$, $(q = 1, 2, ...)$, the following inequality is satisfied in the SDG:*

$$\begin{aligned} \limsup_{q \to +\infty} J\big(u_{\varepsilon_q,0}^*[z_t, t], v_q[z_t, t]\big) &\le \lim_{q \to +\infty} J\big(u_{\varepsilon_q,0}^*[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big) \\ &\le \liminf_{q \to +\infty} J\big(u_q[z_t, t], v_{\mathrm{r}}^*[x_t, t]\big). \end{aligned}$$

Moreover, the value of the SDG $J^*\big(z_0, \varphi(\eta)\big)$ equals to the value of the RDG $J_0^*\big(x_0, \varphi_x(\eta)\big)$.

Proof. Using Definition 2.2 and Definition 2.3, we obtain for all $q = 1, 2, ...$

$$J_u\big(u^*_{\varepsilon_q,0}[z_t,t]; z_0, \varphi(\eta)\big) \geq J\big(u^*_{\varepsilon_q,0}[z_t,t], v_q[z_t,t]\big),$$
$$J_v\big(v^*_{\mathrm{r}}[x_t,t]; z_0, \varphi(\eta)\big) \leq J\big(u_q[z_t,t], v^*_{\mathrm{r}}[x_t,t]\big). \tag{7.25}$$

The inequalities (7.3),(7.23) and the first inequality in (7.25) yield

$$\begin{aligned}\lim_{q\to+\infty} J\big(u^*_{\varepsilon_q,0}[z_t,t], v^*_{\mathrm{r}}[x_t,t]\big) &= J^*_0\big(x_0, \varphi_x(\eta)\big)\\ &= \lim_{q\to+\infty} J_u\big(u^*_{\varepsilon_q,0}[z_t,t]; z_0, \varphi(\eta)\big)\\ &\geq \limsup_{q\to+\infty} J\big(u^*_{\varepsilon_q,0}[z_t,t], v_q[z_t,t]\big).\end{aligned} \tag{7.26}$$

Using the equality (7.6), the inequality (7.23) and the second inequality in (7.25), we obtain

$$\begin{aligned}\lim_{q\to+\infty} J\big(u^*_{\varepsilon_q,0}[z_t,t], v^*_{\mathrm{r}}[x_t,t]\big) = J^*_0\big(x_0, \varphi_x(\eta)\big) &= J_v\big(v^*_{\mathrm{r}}[x_t,t]; z_0, \varphi(\eta)\big)\\ &\leq \liminf_{q\to+\infty} J\big(u_q[z_t,t], v^*_{\mathrm{r}}[x_t,t]\big).\end{aligned} \tag{7.27}$$

Now, the statements of the theorem immediately follow from (7.26) and (7.27). Thus, the theorem is proven. □

Remark 7.6. Theorems 7.5 and 6.2 yield the following conclusions. To construct the saddle-point equilibrium sequence of the SDG and obtain the value of this game, we have to solve the lower Euclidean dimension regular RDG, and calculate the gain matrices $P^{o*}_{20}(t)$, $P^{o*}_{30}(t)$ and $Q^{o*}_{30}(t,\eta)$, $(t,\eta) \in [0, t_f] \times [-h, 0]$, using the equations (5.49),(5.54).

8. Conclusions

In this paper, the finite horizon zero-sum linear-quadratic differential game with state delays in the equation of dynamics was considered. The delays are of both, point-wise and distributed, types. The case where the cost functional of the game does not contain a control cost of the minimizing player (the minimizer) was treated. The absence of the minimizer's control cost in the cost functional means that the game is singular. For this game, the novel definitions of the saddle-point equilibrium (the saddle-point equilibrium sequence) and the game value were proposed. The original singular game was solved by its regularisation. The latter means that this game was approximated by auxiliary regular game with the same equation of dynamics. However, in contrast with the original game, the cost functional of the approximating game has an additional addend. Namely, this addend is a finite horizon integral of the square of the minimizer's control with a small positive weight (small positive parameter). Thus, the approximating game is a finite horizon zero-sum linear-quadratic time delay differential game with cheap control of the minimizer. Solvability conditions of the approximating game were derived. The asymptotic analysis of the boundary-value problem for the hybrid system of Riccati-type matrix equations, arising in these solvability conditions, was carried out in the frames of the singular perturbations theory. Using this analysis, the saddle-point equilibrium sequence in the original singular game was designed. The expression for the value of this game was derived. It was shown that the obtaining the saddle-point sequence and the value of the singular game is based on the solution of a lower Euclidean dimension regular zero-sum differential game – the reduced differential game.

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