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# TOWARDS COMPETITIVE EQUILIBRIUM BY DOUBLE AUCTIONS

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Abstract. This paper inquires whether iterated *double auctions* could lead economic agents towards *competitive equilibrium*. Each participant contemplates various exchanges of commodities for money - itself a commodity here. Thereby everybody forms his own indifference criterion, silently submitted to the auctioneer. It's an extended real-valued function which reflects threshold payments for bundles of commodities. The auctioneer convolutes the received functions so as to find clearing prices and quantities.

What quantifiable value could a double auction add? How might that value, if any, be shared? Will it dwindle when the auction is iterated? Ultimately, if the added value is nil, does competitive equilibrium then prevail?

Each of these questions is constructively addressed, using convex analysis as tool-kit - and the core and competitive equilibrium as solution concepts.

## 1. Introduction

Spurred by advances of internet and computerized platforms, auctions steadily gain economic importance. Essentially, each instance amounts to a market institution where assets, contracts or goods are traded for money. Auction designs and procedures vary greatly in rules for legitimate allocations, messages and payments [15].

This paper considers prevalent institutions called *double auctions* [12].[1] Abstracting from detail, these are modelled here - in conceptual and idealized form - as follows: Each participant already holds some bundle of commodities. After contemplating changes in his holding, and after valuing each potential change in money, he silently submits to the auctioneer an *indifference criterion* - an extended real-valued function which reflects threshold payments for commodities. The auctioneer convolutes the received functions and clears the exchange by linear pricing.

Considering this institution and its *raison d'être*, three questions come up:
First, *what added, quantifiable value could a double auction generate, and how might that value, if any, be shared*?
Second, *will the added value dwindle by way of repeated auctions*?
Third, and ultimately, *if the added amount is nil, then, does competitive equilibrium already prevail*?

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*Key words and phrases.* Double auction, money, convolution, core, competitive equilibrium.

[1]Day-ahead electricity pools [20] illustrate the mechanism. So does also foreign exchange swaps in central banks - and deposits or loans there.

These three *theoretical* questions motivate the paper. It seeks to answer them all; see corresponding Theorems 4.3, 6.1 and 6.2, each offering novelties. Additional but *empirical* motivation comes from the findings of Vernon Smith [27]. Long experimenting with double auctions and similar market mechanisms, he saw a "scientific mystery." Even when participants were few and imperfectly informed, results were remarkably stable and insensitive to experimental details.

This paper returns to the said mystery, apparently still unsolved. It shows, by way of analysis, that a double auction, executed just *once*, may already bring out a *price-supported core solution*. This result foreshadows the first *fundamental welfare theorem* on competitive equilibrium [17], [19]. A main novelty is that *iterated* auctions can lead participants towards such equilibrium.[2] Thus, double auctions may eventually focalize *price-taking behavior* as long-run limit. In the interim, such behavior is neither realistic nor needed as presumption [26].[3]

As in Vernon Smith's experiments, the participants can be fairly few. This fact contrasts the common view that "perfect competition" in markets requires many and minor parties. Here, nobody comes duplicated, triplicated.... And certainly, the agent set isn't an atomless continuum [3]. Nonetheless, as argued below, repeated double auctions "shrinks" the core towards competitive equilibrium [19].

Comparison with Walrasian *tâtonnement* also brings out stark differences. That process presumes that *all* agents invariably be price-taking optimizers, and it permits *no* trade out of equilibrium. By contrast, double auctions need *no* agents of such sort, and *all* trade happens out of equilibrium.

Along the same line, what fits best as solution concept - and as steady state here - is Debreu's *valuation equilbrium* [4]. It incorporates *perfect competition* but differs, except by chance, from the Walrasian version in that each agent's wealth equals the value of his *final* holding - as opposed to that of his *initial* counterpart.

As upshot, double auctions - besides operating as important, practical and welfare-improving procedures - also speak for themselves as *algorithms*. In fact, they may approximate or eventually compute competitive equilibrium. Yet they can hardly be construed as aimed at optimizing some system criterion. Rather, what eventually emerges is a solution to a *fixed point problem* - one which no party addressed or stated. Somewhat surprisingly, the process has a *potential* - or *Lyapunov function* - which invites use of Zangwill's convergence theorem [30].

The subsequent analysis rests on two primitives. First, agents' preferences are closed convex. Second, a money commodity denominates all marginal rates of exchange or substitution [14], [17]. Conditioned by his holding, each agent comes up with an *indifference criterion* - an extended real-valued *function* along which he willingly trades goods for no less than threshold payments.

[2]Feldman [7] let repeated, bilateral deals bring agents, via *core solutions* for *two agents at a time*, to competitive equlibrium - whence indirectly to Pareto optimality [9]. By contrast, here repeated, double auctions lead participants, via *core solutions* for *all agents*, directly towards competitive equilibrium. Either approach emphasizes cooperative apects; nothing is said on evolutionary or strategic underpinnings of competitive equilibrium [13], [18], [26].

[3]Double auctions facilitate price discovery. They also unbundle valuation of different commodities, mitigate or curb strategic behavior, and comply with incentives. Most important: they encourage revelation of true values.

On a mathematical note, for construction of an indifference criterion, utility needs neither be represented nor smooth. On an economic note, utility isn't necessarily transferable or quasi-linear. Anyway, the auctioneer convolutes the received critera. Invoking duality, he cares for efficient clearing, executed by linear pricing.

The paper addresses mathematicians motivated by economics as well as mathematically inclined economists. It presumes negligible knowledge of economic theory and brings little of mathematical or technical novelties. Apart from Theorems 4.3, 6.1 and 6.2, most contributions come by way of analysis and modelling.

The paper is planned as follows. Section 2 fixes notations and recalls some preliminaries. Section 3 clarifies the construction and nature of any individual agent's indifference criterion. Section 4 studies how the auctioneer could handle the criteria he receives. Section 5 identifies each steady state, in which the auction has no effect, with competitive equilibrium of valuation sort. Section 6 inquires whether repeated auctions could bring the agents thereto, and Section 7 concludes.

## 2. Notations and preliminaries

By assumption, privately held commodities - all of homogenous, known qualities - are perfectly divisible, marketable and transferable, without any friction or transaction cost. No consumption, production or transfer has external impacts.

Any bundle of commodities is represented by a vector in a locally convex, real and separated vector space $\mathbb{X}$.[4] The shorthand expression $x^* \in \mathbb{X}^*$ means that $x^*$ is a continuous linear functional, mapping $\mathbb{X}$ into the set $\mathbb{R}$ of real numbers. It's convenient then to write simply $x^*x$ instead of $x^*(x)$. Any such functional may serve as a price regime.

A (cost) criterion $c : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ is *proper* iff it has non-empty *effective domain* $dom c := c^{-1}(\mathbb{R})$. The same $c$ is declared *closed* (resp. *convex*) iff its *epigraph* $\{(x,r) \in \mathbb{X} \times \mathbb{R} : c(x) \le r\}$ is likewise. For any proper $c$, its *conjugate function*

$$x^* \in \mathbb{X}^* \mapsto c^*(x^*) := \sup\{x^*x - c(x) : x \in \mathbb{X}\} \in \mathbb{R} \cup \{+\infty\} \tag{2.1}$$

is closed convex. So defined, $c^*(x^*)$ equals the **added value** or profit, potentially obtained, under an exogenous price regime $x^*$. Call $x^* \in \mathbb{X}^*$ a *subgradient* of $c$ at $x \in dom c$, as signalled by writing

$$x^* \in \partial c(x) \quad \text{iff} \quad x \in \arg\max\{x^* - c\}. \tag{2.2}$$

(2.2) holds precisely when *Fenchel's equality*

$$x^*x = c^*(x^*) + c(x) \tag{2.3}$$

confirms that total revenue $x^*x$ be split between profit $c^*(x^*)$ (2.1) and cost $c(x)$. If $c$ is closed convex and bounded above near $x$, the *subdifferential* $\partial c(x)$ is non-empty.

[4]Euclidean spaces suit computational or practical purposes, $\dim \mathbb{X}$ then being the number of marketable commodities. However, finance and insurance often require infinite dimensions.

## 3. Preferences, money, and indifference

This section considers just *one* generic agent. His *preferences* are represented by a binary relation $\succsim$ in $\mathbb{X} \times \mathbb{X}$. A choice $x \in \mathbb{X}$ belongs to the *effective domain* of $\succsim$, denoted *dom* $\succsim$ and supposed non-empty, iff the *preferred set*

$$\{\succsim x\} := \{\hat{x} \in \mathbb{X} : \hat{x} \succsim x\} \tag{3.1}$$

contains $x$. Consequently, $\succsim$ is *reflexive* on its domain but not necessarily *complete. Transitivity* is invoked later when required. *Strict preference,* $\hat{x} \succ x$ means $\hat{x} \succsim x$ while $x$ *not* $\succsim \hat{x}$. A *utility function* $u : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ would *represent* $\succsim$ when

$$u(\hat{x}) \geq u(x) > -\infty \iff \hat{x} \succsim x \in dom \succsim . \tag{3.2}$$

No such representation (3.2) is needed though. Provided all preferred sets (3.1) be closed convex, any representing function $u$ is upper semicontinuous and quasi-concave with effective domain $domu := u^{-1}(\mathbb{R}) = dom \succsim$.[5]

**Money.** *One* special good (say, gold) $g \in \mathbb{X} \diagdown 0$, referred to as *money*, serves as commonly accepted unit of account, means of payment (numeraire) and intermediary medium of exchange [1], [7], [26].[6]

$\mathbb{X}$ is the *direct sum* $\mathbb{R}g \oplus \mathcal{X}$ of the subspace $\mathbb{R}g$, spanned by money, besides a complementary space $\mathcal{X}$, composed of *money-free,* "*real*" bundles. Instead of $x = rg + \chi$ one may write $x = (r, \chi)$. Any $x^* \in \mathbb{X}^* := \mathbb{R}^* g \oplus \mathcal{X}^*$ takes the corresponding form $x^* = (r^*, \chi^*) \in \mathbb{R}^* \times \mathcal{X}^*$, and it operates by $x^* x = r^* r + \chi^* \chi$.

Trade goes in real goods for money or *quid pro quo.* Money is never exchanged for itself. Being numeraire, a unit of (presently available) money always commands price 1. So, from here on,

$$\text{any } x^* = (r^*, \chi^*) \in \mathbb{X}^*, \text{ intended or used for valuation, has } r^* = 1. \tag{3.3}$$

The component $\chi^* \in \mathcal{X}^*$ just prices "real goods". Conversely, any "price vector" $\chi^* \in \mathcal{X}^*$ extends to a unique valuation regime $x^* = (1, \chi^*) \in \mathbb{X}^*$. Convention (3.3) was motivated by economic arguments. It's also underpinned by Proposition 3.3 below.

$\succsim$ is declared *quasi-linear* when $\hat{x} \succsim x$ iff $\hat{x} + rg \succsim x + rg$ for all $r \in \mathbb{R}_+$. Then, typically, a representation (3.2) of $\succsim$ takes the form $rg + \chi \mapsto r + u(\chi)$; see [25]. Such partial separability implies *transferable utility* - a useful but rather restrictive property, dispensed with here.

**Assumption** (*on money*). More money is *strictly preferable*,[7] meaning that for any $x \in dom \succsim$,

$$r > 0 \implies rg + x \succ x, \tag{3.4}$$

[5] It's convenient that such a function also be concave - as in [14] and [26].

[6] $g$ could be gold, fiat bills or some risk-free security. Arguments using endogenous money date back to Dupuit (1844). Issues revolve around interpersonal comparisons, quantifiable compensations, and aggregate welfare. For extensions, see Luenberger [17]; for valuation in security markets, see [11], [22], [23].

[7] For no bundle or real good $\neq g$ do I assume that more of it is better.

and

$$\hat{x} \succ x \implies \hat{x} - rg \succ x \ \textit{ for sufficiently small } \ r > 0. \tag{3.5}$$

Together, (3.4) and (3.5) open doors to *scalarized* preferences:

**Indifference criterion.** Suppose the agent holds *endowment* $x \in dom \succsim$. If contemplating to "supply" a real-good bundle $\chi \in \mathcal{X}$, he *asks* for no less money compensation than

$$c(\chi \,|x\,) := \inf \{r \in \mathbb{R} : rg - \chi + x \succsim x\}. \tag{3.6}$$

Thus, making money a means of all transactions, the agent's *indifference criterion* (3.6) reflects his idiosyncratic valuations or threshold compensations - all in money and depending on $x$.[8]

By assumption, $c(\cdot\,|x\,) > -\infty$ and $c(0\,|x\,) \le 0$. So, $c(\cdot\,|x\,)$ is proper. Definition (3.6) applies verbatim with $\hat{x} \in \mathbb{X}$ instead of $\chi \in \mathcal{X}$. I emphasize though, that $\chi$ should best be construed as *change* in the real-good component of endowment $x$. Convention (3.3) is supported by the fact that

$$c(rg + \chi \,|x\,) \in \mathbb{R} \implies c(rg + \chi \,|x\,) = r + c(\chi \,|x\,), \tag{3.7}$$

a property shared with many financial measures [11], [24]. The following result derives straightforwardly:

**Proposition 3.1** (on closed convex indifference criteria)**.** *If the preferred set* $\{\succsim x\}$ *(3.1) is closed (resp. convex), then so is also the function* $\chi \mapsto c(\chi\,|x\,)$.[9]

Proposition 3.1 motivates a standing

**Assumption** (*on preferences*). Each preferred set (3.1) is henceforth taken as closed convex.

**Remark 3.2** (on ask versus bid)**.** (3.6) was motivated as "minimal" amount of money *asked* for *supply.* Regarding instead the receiving side, Luenberger [17] considered "maximal", monetary *bid* for *demand*:

$$\chi \mapsto b(\chi \,|x\,) := \sup \{r \in \mathbb{R} : -rg + \chi + x \succsim x\} \in \mathbb{R} \cup \{-\infty\}. \tag{3.8}$$

Under the hypotheses of Prop. 3.1, because $b(\chi\,|x\,) = -c(-\chi\,|x\,)$, Luenberger's *benefit function* $b(\cdot\,|x\,)$ is closed concave. It facilitates interpretation and supplements the narrative. Yet, for formal analysis, it's redundant.

Given $x \in dom \succsim$, a price $x^* \in \mathbb{X}^*$ offers **added value** (2.1):

$$c^*(x^*\,|x\,) := \sup \{x^*\hat{x} - c(\hat{x}\,|x\,) : \hat{x} \in \mathbb{X}\}. \tag{3.9}$$

Notice that the assumption $c(0\,|x\,) \le 0$ implies $c^*(\cdot\,|x\,) \ge 0$. Moreover, then

$$c^*(x^*\,|x\,) = 0 \iff \hat{x} = 0 \text{ solves (3.9) with } c(0\,|x\,) = 0. \tag{3.10}$$

As expected, and in compliance with (3.3) and (3.7), money must always be priced at unit level [1]:

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[8] $g$ measures desire, but satisfaction is never mentioned. It may happen that no real $r$ satisfies $rg - \chi + x \succsim x$. Then, the convention $\inf \emptyset = +\infty$ comes into play.

[9] For related material, see [28], [29].

**Proposition 3.3** (on linear pricing and minimal expenditure)**.** *Added value (3.9) cannot be finite unless $x^*g = 1$. Then, it equals*

$$c^*(x^* \,|x\,) = \sup \{x^*(x - \hat{x}) : \hat{x} \succsim x\} \,. \tag{3.11}$$

*$\hat{x}$ is a best choice in (3.11) iff it solves the problem of* ***minimal expenditure****:*[10]

$$\mathcal{E}(x^* \,|x\,) := \inf \{x^*\hat{x} \ \mid \hat{x} \succsim x\,\} = x^*x - c^*(x^* \,|x\,). \tag{3.12}$$

*Proof.* Simply observe that

$$\begin{aligned} c^*(x^* \,|x\,) &= \sup \{x^*\chi - r \ \mid \hat{x} := rg - \chi + x \succsim x,\, \chi \in \mathbb{X},\ r \in \mathbb{R}\} \\ &= \sup \{x^*(x - \hat{x}) + r(x^*g - 1) : \hat{x} \succsim x,\, r \in \mathbb{R}\} \\ &= \begin{cases} \sup \{x^*(x - \hat{x}) : \hat{x} \succsim x\} & \text{and (3.11) holds if } x^*g = 1, \\ +\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} x^*x - \mathcal{E}(x^* \,|x\,) & \text{if } x^*g = 1, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

□

Producers and consumers - being prime figures in microeconomic theory [17], [19] - will enter Section 6 on competitive equilibrium. Definition (3.6) already described how a producer might ponder his supply. Concluding this section is brief consideration of a consumer, constrained by his budget:

**Proposition 3.4** (on budget-constrained choice of consumption)**.** *If $x \in dom \succsim$ and $c^*(x^* \,|x\,) > 0$, the affordable, strictly preferred set $\{\hat{x} \succ x : x^*\hat{x} \leq x^*x\}$ can not be empty. Conversely, if the said set is indeed non-empty, then, $c^*(x^* \,|x\,) > 0$. In short, given budget $\beta := x^*x$,*

$$\{\hat{x} \succ x : x^*\hat{x} \leq \beta\} \quad \textit{is empty iff} \quad c^*(x^* \,|x\,) = 0.$$

*Proof.* From (3.8) follows follows that

$$\sup \{b(\check{x} \,|x\,) - x^*\check{x} : \check{x} \in \mathbb{X}\} = c^*(x^* \,|x\,). \tag{3.13}$$

So, if $c^*(x^* \,|x\,) > 0$, some $\check{x} \in \mathbb{X}$ satisfies $b(\check{x} \,|x\,) - x^*\check{x} > 0$, and thereby,

$$x^*[\check{x} - b(\check{x} \,|x\,)g + x] < \beta = x^*x.$$

Hence $\hat{x} := \check{x} - [b(\check{x} \,|x\,) + r]g + x$ costs $x^*\hat{x} \leq \beta$ for sufficiently small $r > 0$. At the same time, from (3.4),

$$\hat{x} \succ \check{x} - b(\check{x} \,|x\,)g + x \succsim x.$$

Consequently, $\hat{x}$ is affordable (within budget $x^*x$) and strictly preferred to $x$.

For the converse, suppose some $\hat{x} \in \mathbb{X}$ is such a bundle. Then, by (3.5), for sufficiently small $r > 0$, it holds $-rg + (\hat{x} - x) + x \succ x$ and thereby $b(\hat{x} - x \,|x\,) > 0$. Then,

$$b(\hat{x} - x \,|x\,) - x^*(\hat{x} - x) > 0.$$

In turn, by (3.13), this implies $c^*(x^* \,|x\,) > 0$. □

[10] See [17] or [19].

## 4. The double auction

Accommodated henceforth is fixed finite ensemble $I$ of economic agents; $\#I \geq 2$. The members exchange or trade bundles of assets, claims or commodities. Transactions happen on a common platform, possibly internet-based, and managed by a system operator or auctioneer.

Agent $i \in I$ already holds some "endowment" $x_i \in \mathbb{X}$. Conditioned by that holding, he submits his indifference criterion (3.6):

$$\chi_i \in \mathcal{X} \mapsto c_i(\chi_i \,|x_i\,) \in \mathbb{R} \cup \{+\infty\} \tag{4.1}$$

to the auctioneer. Thereby he commits to "supply" whatever real bundle $\chi_i \in \mathcal{X}$ for no less pecuniary payment than $c_i(\chi_i \,|x_i\,)$.[11] If $c_i(\chi_i \,|x_i\,) < +\infty$, then, by tacit assumption, agent $i$ can indeed honour his commitments. (In particular, he has no concerns with liquidity.) Each endowment $x_i$ remains fixed in this section. So, until other notice, it's convenient to write just $c_i(\cdot)$ instead of $c_i(\cdot\,|x_i\,)$.

Any double auction is a *two-sided market.* While (4.1) was motivated as minimal compensation for *supply*, a twin version reads as maximal expense for *demand*; see Remark 3.2. So, to put diverse items - or parties - on equal footing, henceforth, by convention: *compensation is negative expense*, and s*upply is negative demand.*

**The auctioneer** pursues no self-interest, collects no commissions, and discriminates nobody. Further, to enforce *the law of one price*, and preclude *arbitrage* [16] or second-hand trade, he valuates real bundles by some endogenous functional $\chi^* \in \mathcal{X}^*$, communicated at closure time of the auction. Also, to curb or mitigate strategic behavior, he requires that agents' "bids" be silent, "simultaneous", single-shot, and anonymous - put into "closed envelopes".

Balanced exchange means a *redistribution* $(\chi_i) \in \mathcal{X}^I$, satisfying $\sum_{i\in I} \chi_i = 0$. That is, the auction should serve as *clearing house.* At the same time, for *efficiency*, the mechanism should minimize $\sum_{i\in I} c_i(\chi_i)$.

Besides efficiency, to incite voluntary participation, each agent $i \in I$ ought see some surplus atop his cost - alongside a best choice for him. In short, if finally asked to "supply" real bundle $\chi_i \in \mathcal{X}$, he should receive

$$\text{payment } \ \chi^*\chi_i \geq c_i(\chi_i) \ \text{ determined by a common price } \ \chi^* \in \partial c_i(\chi_i). \tag{4.2}$$

Inequality (4.2) reflects individual *rationality* - or equivalently, that no single agent would reasonably *block* or veto the outcome. Absence of blocking should also apply for coalitions of agents. The outcome better be *stable*; it ought withstand coordinated defections or reneging on commitments.

Can the auctioneer meet all these requirements - seemingly too many? Instead of imposing them as constraints, could they rather come out as consequences? Theorem 4.3 provides affirmative and constructive answers. Propositions 4.1&2 prepare the ground, and - upon doing so - they play down the role of convexity.

What comes next might be called "welfare analysis" of *one* single round of a double auction. It's coached within the frames of *cooperative game theory*, focused

[11] By convention, $c_i(\chi_i \,|x_i\,) = +\infty$ when $i$ cannot "supply" $\chi_i$. In particular, this observation applies to each agent $i$ who deals with fairly few commodities. For him, $dom c_i(\cdot\,|x_i\,)$ has empty interior.

here on *transferable-value* and on the *core* as solution concept [21]. The aim is to identify the auction result as a core outcome, supported by some price $\chi^* \in \mathcal{X}^*$. Thus - for narrative and simplicity - the auctioneer's task will be cast *as though* he solves a particular cooperative game. Arguments revolve around *inf-convolutions*

$$c_{\mathcal{I}}(\chi_{\mathcal{I}}) := \inf \left\{ \sum_{i \in \mathcal{I}} c_i(\chi_i) \;\middle|\; \sum_{i \in \mathcal{I}} \chi_i = \chi_{\mathcal{I}} \right\}, \tag{4.3}$$

taken across various agent ensembles $\mathcal{I} \subseteq I$. For now, fix any $\mathcal{I} \subseteq I$ with $\#\mathcal{I} \geq 2$. If available, any subgradient $\chi^* \in \partial c_{\mathcal{I}}(\chi_{\mathcal{I}})$ (2.2) facilitates decomposition and solution of problem (4.3). On this account, it holds in any linear space:

**Proposition 4.1** (on optimal allocations, coinciding subgradients, and equal "margins" [8])**.** *Along each optimal allocation* $(\chi_i)$ *in (4.3) there is a subdifferential coincidence:*

$$\partial c_{\mathcal{I}}(\chi_{\mathcal{I}}) \subseteq \cap_{i \in \mathcal{I}} \partial c_i(\chi_i).$$

*Conversely, provided* $\sum_{i \in \mathcal{I}} \chi_i = \chi_{\mathcal{I}}$*, it holds*

$$\partial c_{\mathcal{I}}(\chi_{\mathcal{I}}) \supseteq \cap_{i \in \mathcal{I}} \partial c_i(\chi_i).$$

*If moreover,* $\cap_{i \in \mathcal{I}} \partial c_i(\chi_i) \neq \varnothing$*, then* $(\chi_i)$ *is optimal in (4.3).*

*Proof* from [8] is included for completeness. If $\chi^* \in \partial c_{\mathcal{I}}(\chi_{\mathcal{I}})$, and $(\chi_i)$ solves (4.3), then $\sum_{i \in \mathcal{I}} \hat{\chi}_i = \hat{\chi}$ implies

$$\sum_{i \in \mathcal{I}} c_i(\hat{\chi}_i) \geq c_{\mathcal{I}}(\hat{\chi}) \geq c_{\mathcal{I}}(\chi_{\mathcal{I}}) + \chi^*(\hat{\chi} - \chi_{\mathcal{I}}) = \sum_{i \in \mathcal{I}} \left[ c_i(\chi_i) + \chi^*(\hat{\chi}_i - \chi_i) \right].$$

In this string, posit $\hat{\chi}_j = \chi_j$ for each $j \in \mathcal{I} \diagdown i$ to get $c_i(\hat{\chi}_i) \geq c_i(\chi_i) + \chi^*(\hat{\chi}_i - \chi_i)$. Since $i \in \mathcal{I}$ and $\hat{\chi}_i \in \mathcal{X}$ were arbitrary, $\chi^* \in \partial c_i(\chi_i)$ for all $i \in \mathcal{I}$.

Conversely, suppose $\chi^* \in \cap_{i \in \mathcal{I}} \partial c_i(\chi_i)$ and $\sum_{i \in \mathcal{I}} \chi_i = \chi_{\mathcal{I}}$. Since $c_i(\hat{\chi}_i) \geq c_i(\chi_i) + \chi^*(\hat{\chi}_i - \chi_i)$ for any $\hat{\chi}_i \in \mathcal{X}$ and $i \in \mathcal{I}$, summation across $\mathcal{I}$ yields

$$\sum_{i \in \mathcal{I}} c_i(\hat{\chi}_i) \geq \sum_{i \in \mathcal{I}} c_i(\chi_i) + \chi^* \sum_{i \in \mathcal{I}} (\hat{\chi}_i - \chi_i).$$

In the last inequality, let $\sum_{i \in \mathcal{I}} \hat{\chi}_i = \chi_{\mathcal{I}}$ to see that $(\chi_i)$ solves (4.3). For arbitrary $\hat{\chi} \in \mathcal{X}$, the instance $\sum_{i \in \mathcal{I}} \hat{\chi}_i = \hat{\chi}$ entails $c_{\mathcal{I}}(\hat{\chi}) \geq c_{\mathcal{I}}(\chi_{\mathcal{I}}) + \chi^*(\hat{\chi} - \chi_{\mathcal{I}})$, whence $\chi^* \in \partial c_{\mathcal{I}}(\chi_{\mathcal{I}})$. $\square$

By Prop. 4.1, if some $c_i$ is differentiable at $\chi_i$, and $\cap_{i \in \mathcal{I}} \partial c_i(\chi_i) \neq \varnothing$, then agents' "margins" are equal at Pareto optimum. While subgradients (in the sense of convex analysis) support global optimality, *generalized directional derivatives*, satisfying

$$c'(\chi; d) \geq \limsup_{r \to 0^+} \frac{c(\chi + rd) - c(\chi)}{r},$$

offer local, neoclassical perspectives. One such perspective comes next; it's as a digression which can be skipped:

**Proposition 4.2** (on redistribution and directional margins). *Any best solution* $(\chi_i)$ *to allocation problem (4.3) satisfies the variational inequality*

$$\sum_{i\in\mathcal{I}} d_i = 0 \Longrightarrow \sum_{i\in\mathcal{I}} c_i'(\chi_i; d_i) \geq 0. \tag{4.4}$$

*Conversely, if each* $c_i$ *is convex, locally with respect to* $\chi_i$, *and (4.4) holds with* $\sum_{i\in\mathcal{I}} \chi_i = \chi_{\mathcal{I}}$, *then* $(\chi_i)$ *solves (4.3).*

*Proof.* If $(\chi_i)$ solves (4.3), and $\sum_{i\in\mathcal{I}} d_i = 0$, then $\sum_{i\in\mathcal{I}} c_i'(\chi_i; d_i) \geq$

$$\sum_{i\in\mathcal{I}} \limsup_{r\to 0^+} r^{-1}[c_i(\chi_i + rd_i) - c_i(\chi_i)] \geq \limsup_{r\to 0^+} r^{-1} \sum_{i\in\mathcal{I}} [c_i(\chi_i + rd_i) - c_i(\chi_i)] \geq 0.$$

Conversely, if $c_i$ is convex at $\chi_i$, and $(\hat{\chi}_i)$ allocates $\chi_{\mathcal{I}}$, posit $d_i := \hat{\chi}_i - \chi_i$ to have $\sum_{i\in\mathcal{I}} d_i = 0$. So, (4.4) implies

$$\sum_{i\in\mathcal{I}} c_i(\hat{\chi}_i) - \sum_{i\in\mathcal{I}} c_i(\chi_i) = \sum_{i\in\mathcal{I}} [c_i(\chi_i + d_i) - c_i(\chi_i)] \geq \sum_{i\in\mathcal{I}} c_i'(\chi_i; d_i) \geq 0.$$

□

After these preparations, returning now to the auctioneer, note that he seeks to redistribute goods and money efficiently across the entire agent set. Thus, he faces inf-convolution (4.3) for the "*grand coalition*" $\mathcal{I} = I$ with $\chi_{\mathcal{I}} = 0$.

*Might not the members of some strict subset* $\mathcal{I} \subsetneq I$, $\#\mathcal{I} \geq 2$, *rather want to organize an auction or exchange among themselves*? To come to grips with this question - and to emphasize stability and welfare - it's expedient to frame the setting as a *transferable-value, cooperative game* [21] in which *coalition* $\mathcal{I} \subseteq I$ can, by (4.3), obtain no less *value* than

$$v_{\mathcal{I}} := -c_{\mathcal{I}}(0). \tag{4.5}$$

*Henceforth* ***suppose each*** $c_i(0) \leq 0$. Under that condition, by (4.3), clearly, $v_{\mathcal{I}} \geq 0$ in (4.5). For interpretation, $v_{\mathcal{I}}$ is the value coalition $\mathcal{I}$ could shoot at by organizing a self-sufficient auction in "autarky" - without access to outside markets.

Recall that a *"value profile"* $(V_i) \in \mathbb{R}^I$ *belongs to the* ***core*** *of a transferable-value, cooperative game with player set* $I$ *- in which coalition* $\mathcal{I} \subseteq I$ *can secure itself joint, already specified value* $v_{\mathcal{I}} \in \mathbb{R} \cup \{-\infty\}$ *- iff*

$$\sum_{i\in\mathcal{I}} V_i \geq v_{\mathcal{I}} \text{ for each } \mathcal{I} \subseteq I \text{ with equality for the grand coalition } \mathcal{I} = I. \tag{4.6}$$

In the present setting, a double auction makes a core solution come straight up:

**Theorem 4.3** (on price-supported core solutions). *Consider the cooperative game in which coalition* $\mathcal{I} \subseteq I$ *can get no less value than* $v_{\mathcal{I}} = -c_{\mathcal{I}}(0) \geq 0$ *(4.5). Then, for any* ***shadow price*** $\chi^* \in \partial c_I(0)$ *(2.2), by offering agent* $i \in I$ ***added value*** $V_i := c_i^*(\chi^*)$ *(2.1), the game generates a core solution (4.6). If moreover,* $(\chi_i)$ *solves (4.3) for* $\mathcal{I} = I$ *and* $\chi_I = 0$, *then* $\chi^*\chi_i \geq c_i(\chi_i)$.

*Thus, for whatever shadow price* $\chi^* \in \partial c_I(0)$, *the double auction may, in toto,* ***add value***

$$V_I := -c_I(0) = c_I^*(\chi^*) \geq 0. \tag{4.7}$$

*The latter is constructively and explicitly shared because* $V_I = \sum_{i\in I} V_i$. *It follows that* $V_I = 0$ *iff all* $V_i = 0$, *and then, necessarily, each* $c_i(0) = 0$.

*Proof.* (2.1) implies, for *any* $\chi^* \in \mathcal{X}^*$, that coalition $\mathcal{I} \subseteq I$ would get aggregate value

$$V_{\mathcal{I}} := \sum_{i\in\mathcal{I}} V_i = \sum_{i\in\mathcal{I}} c_i^*(\chi^*) = c_{\mathcal{I}}^*(\chi^*) \geq -c_{\mathcal{I}}(0) = v_{\mathcal{I}}. \tag{4.8}$$

In particular, $\chi^* \in \partial c_I(0)$ iff $V_I = c_I^*(\chi^*) = -c_I(0) = v_I$.[12] Then, by Proposition 4.1, $\chi^* \in \partial c_i(\chi_i)$ so that $\chi^*\chi_i = c_i^*(\chi^*) + c_i(\chi_i)$ for each $i$ (2.3). Because $c_i(0) \leq 0$, it holds $c_i^*(\chi^*) \geq 0$, hence $\chi^*\chi_i \geq c_i(\chi_i)$. The assertion on shared value, derives from (4.8) when $\mathcal{I} = I$. $\square$

Apart from the possibility that $|c_I(0)| > 0$ - and apart then, from positive profits or added values - the outcome resembles competitive equilibrium. This is exemplified next:

**Example 4.4** (production economies)**.** Suppose "producer" $i \in I$, has already "supplied" $x_i \in \mathbb{X}$. He incurs cost $c_i(\chi_i)$ for putting out additional "supply" $\chi_i \in \mathcal{X}$, $c_i(0) = 0$. Then, each shadow price $\chi^* \in \partial c_I(0)$ on products, alongside any optimal solution $(\chi_i)$ to problem $c_I(0)$ (4.3) "comes close" to *competitive equilibrium.* Indeed, *markets clear*: $\sum_{i\in I} \chi_i = 0$. Further, agents' *profits are acceptable and maximal*: $c_i^*(\chi^*) = \chi^*\chi_i - c_i(\chi_i) \geq 0$ with $\chi^* \in \partial c_i(\chi_i)$ $\forall i$.

**Some remarks** conclude this section. None are essential; all can be skipped.
(*On the auction*). Neither Theorem 4.3. nor Example 4.4 should lure one into believing that double auctions provide a direct, two-way link to competitive equilibrium. Certainly, as modelled, the auction promotes efficiency and welfare. Its result depends though, on the (ex ante) holdings; there are *endowment effects.* The (ex post) updated holdings may well invite another round of auction - as will be explored later.

It deserves emphasis that exchange is anonymous - and indirect when $\#I \geq 3$. No bargaining, matching, price prediction or search is needed. By connecting agents via an automated hub, the auction dispenses with brokers, networks and topological complexity. Moreover, money mitigates agents' incentives to act strategically.
(*On coordination*). Note that every agent participates voluntarily. And no strict subset $\mathcal{I} \subset I$ stands to gain by organizing a double auction among its members. Also, the more numerous the participants, the merrier they are. Indeed, if $\mathcal{I}, \overline{\mathcal{I}} \subset I$ are disjoint and non-empty, then $c_{\mathcal{I}\cup\overline{\mathcal{I}}}(0) \leq c_{\mathcal{I}}(0) + c_{\overline{\mathcal{I}}}(0)$, each term being $\leq 0$.[13] In short, the double auction coordinates payments and quantities; it's an allocative and integrative mechanism. It channels goods from parties who have relative high abundance or low appreciation of some goods, to other parties who lack or like those goods.
(*On convexity*). So far, no assumptions were made as to properties of the criteria $c_i$

[12] Inequalities $\sum_{i\in\mathcal{I}} V_i \geq v_{\mathcal{I}}$ $\forall \mathcal{I} \subseteq I$, reflect *weak duality.* By contrast, the equality for $\mathcal{I} = I$ captures *strong duality.* The latter points to convexity, possibly in weakened form; see Prop. 4.1 and the final remark of this section.

[13] It benefits a coalition to admit extra members for which $c_i(0) < 0$.

in (4.3) - apart from Theorem 4.3 naturally requiring that each $c_i(0) \leq 0$. Remarkably, no convexity conditions came up. The reason is simple: Although subgradients (2.2) entered as chief objects, their existence were never considered an issue.

On the last account, returning to Prop. 4.1, suppose inf-convolution $c_{\mathcal{I}}(\cdot)$ (4.3) be convex and bounded above near $\chi_{\mathcal{I}}$. Then, the subdifferential $\partial c_{\mathcal{I}}(\chi_{\mathcal{I}})$ is indeed non-empty. More generally, by Prop. 3.3, instead of a convex $c_{\mathcal{I}}(\cdot)$, it suffices that this function coincides with its closed convex envelope at $\chi_{\mathcal{I}}$. On a qualitative note, admittance of many, minor participants contributes towards convexity of $c_{\mathcal{I}}(\cdot)$; see [2], [3] or [6].

## 5. Competitive equilibrium

The double auction provides *Pareto improvement* iff the resulting added value $V_I$ (4.7) is strictly positive. By Theorem 4.3, the particular outcome $V_I = 0$ merits special attention. *Henceforth suppose* $c_i(0\,|x_i) = 0$ *for each* $i \in I$ *and* $x_i \in dom \succsim_i$. Let $x_i^0 \in dom \succsim_i$ be the initial holding of agent $i$. Write

$$\mathbf{X} := \left\{ \mathbf{x} = (x_i) \in \mathbb{X}^I : x_i \succsim_i x_i^0 \ \forall i \text{ and } \sum_{i\in I} x_i = \sum_{i \in I} x_i^0 \right\} \tag{5.1}$$

for the set of feasible allocations.

**Definition 5.1** (competitive equilibrium). A price cum allocation pair $[\chi^*, \mathbf{x} = (x_i)] \in \mathcal{X}^* \times \mathbf{X}$ (5.1) constitutes a **competitive equilibrium** iff

$$\chi^* \in \partial c_I(0\,|\mathbf{x}) \quad \textit{with} \quad V_I = c_I^*(\chi^*\,|\mathbf{x}) = 0. \tag{5.2}$$

In such equilibrium, there is shadow pricing, market clearing, annulment of added values - and no further trade. From Prop. 4.1 and Theorem 4.3 follows forthwith:

**Proposition 5.2** (on competitive equilibrium). $[\chi^*, (x_i)] \in \mathcal{X}^* \times \mathbf{X}$ *is a competitive market equilibrium iff* $\chi^* \in \cap_{i\in I} \partial c_i(0\,|x_i)$ *and each* $V_i = c_i^*(\chi^*\,|x) = 0$. *There can be no competitive equilibrium unless* $c_I(0\,|\mathbf{x}) = 0$. *Then, necessarily each* $c_i(0\,|x_i) = 0$.

As explained in Section 3, any price $\chi^* \in \mathcal{X}^*$ on real bundles extends to a price $x^* = (1, \chi^*)$ on $\mathbb{X}$. Modulo such extension, to view competitive equilibrium in more standard manner, invoke customary consumers and producers. From Propositions 3.1-3.4 follows:

**Proposition 5.3** (market clearing and agents' choice). *Competitive equilibrium* $[x^*, \mathbf{x}]$ *prevails iff markets clear and*

$$\textit{for consumer } i\textit{:} \quad \hat{x}_i \succ_i x_i \quad \Longrightarrow \quad x^*\hat{x}_i > x^* x_i,$$

$$\textit{for producer } j\textit{:} \quad x^*\hat{x}_j > x^* x_j \quad \Longrightarrow \quad x_j \textit{ is infeasible.}$$

A special yet most studied, classical instance of competitive equilibrium, $[x^*, (x_i)] \in \mathbb{X}^* \times \mathbf{X}$ is declared *Walrasian* if, besides all other properties, it also holds that $x^* x_i = x^* x_i^0 \ \forall i$. These budget conditions pinpoint problems with the Walrasian concept; see footnote 1 in [26]. As said above, the more appropriate solution concept is here rather that of *valuation equilibrium* [4]. It differs from the Walrasian version if $x^* x_i \neq x^* x_i^0$ for at least one $i \in I$.

## 6. Convergence to competitive equilibrium

Suppose agent ensemble $I$ organizes a double auction among its members. Let these actually hold a feasible profile $\mathbf{x} = (x_i) \in \mathbf{X}$ (5.1). *Henceforth suppose the inf-convolution $c_I(0\,|\mathbf{x})$ be attained with non-empty subdifferential $\partial c_I(0\,|\mathbf{x})$.* Then, member $i \in I$ may exit from the auction with *Pareto-improved, updated holding*

$$x_i^{+1} := \pi_i g - \chi_i + x_i \succsim_i x_i, \tag{6.1}$$

featuring his "supply" $\chi_i \in \mathcal{X}$ of "real" goods, for which he received *payment*

$$\pi_i := \chi^* \chi_i \geq c_i(\chi_i\,|x_i\,), \tag{6.2}$$

thereby getting *added value* $c_i^*(\chi^*\,|x_i\,) \geq 0$, based on a common *shadow price*

$$\chi^* \in \partial c_I(0\,|\mathbf{x}\,) = \cap_{i \in I} \partial c_i(\chi_i\,|x_i\,) \ \text{ with } \sum_{i \in I} \chi_i = 0. \tag{6.3}$$

In short, qua market mechanism, the double auction generates an *algorithm:*

$$\mathbf{x} \in \mathbf{X} \rightrightarrows \mathbf{A}(\mathbf{x}) := \left\{ \mathbf{x}^{+1} \in \mathbf{X} \,\middle|\, \mathbf{x}^{+1} = (x_i^{+1}) \text{ satisfies (6.1)-(6.3)} \right\}. \tag{6.4}$$

Since the preferences $\succsim_i$ are invariant, any effective double auction offers Pareto improvement. Hence, voluntary participation complies with individual rationality and incentives. Moreover, the attending improvements, defined by (6.1)-(6.3), are concrete, specific and quantifiable.[14]

If each $\succsim_i$ is represented by some utility function $u_i(\cdot)$ (3.2), then $(x_i) \mapsto \sum_{i \in I} u_i(x_i)$ becomes a *potential* or *Lyapunov function* for iterated double auctions - that is, algorithm (6.4) makes $\sum_{i \in I} u_i(x_i)$ increase. Representation (3.2) implies, however, that $\succsim_i$ be complete and transitive on its domain.

For greater appeal and generality, without complete preferences, granted only transitivity, there is a more informative and easily observable potential. Indeed, upon iterating the double auction, *added value $V_I$ steadily decreases*:

**Theorem 6.1** (on reduction of value added)**.** *Suppose each preference $\succsim_i$ be transitive. Then, an iteration of the double auction reduces the addition of overall value (4.7). That is, for consecutive clearing prices $\chi^* \in \partial c_I(0\,|\mathbf{x})$, $\chi^{*+1} \in \partial c_I(0\,|\mathbf{x}^{+1}\,)$ it holds:*

$$V_I^{+1} := c_I^*(\chi^{*+1}\,|\mathbf{x}^{+1}\,) \leq c_I^*(\chi^*\,|\mathbf{x}\,) =: V_I. \tag{6.5}$$

*Proof.* A notational issue must first be clarified. Conjugation (2.1) of a function $c : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is denoted $c^*(\chi^*)$. If $c : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, write $c^*(x^*)$. For any indifference criterion (3.6), by (3.3), (3.7) and Proposition 3.3, $c_I^*(x^*\,|\mathbf{x}\,) < +\infty \Longrightarrow x^* = (1, \chi^*)$ & $c_I^*(x^*\,|\mathbf{x}) = c_I^*(\chi^*\,|\mathbf{x})$.

Note that $\chi^* \in \partial c_I(0\,|\mathbf{x}\,)$ implies $0 \in \partial c_I^*(\chi^*\,|\mathbf{x}\,)$ (2.3). Thus, for any clearing price $\chi^*$ of the double auction, $\chi^* \in \arg\min c_I^*(\cdot\,|\mathbf{x})$, and $x^* = (1, \chi^*) \in \arg\min c_I^*(\cdot\,|\mathbf{x})$. Now, from (3.12) follows that

$$c_I^*(\chi^*\,|\mathbf{x}\,) = c_I^*(x^*\,|\mathbf{x}\,) = x^* \sum_{i \in I} x_i^0 - \sum_{i \in I} \mathcal{E}_i(x^*\,|x_i\,).$$

[14]When $c_I^*(\chi^*\,|\mathbf{x}\,) > 0$, any payment profile $i \in I \mapsto \hat{\pi}_i > 0$ atop costs $c_i(\chi_i\,|x_i\,)$, and satisfying $\sum_{i \in I} \hat{\pi}_i \leq c_I^*(\chi^*\,|\mathbf{x}\,)$, would entail strict Pareto improvement. It does, however, not necessarily comply with (4.2). If moreover, $\sum_{i \in I} \hat{\pi}_i < c_I^*(\chi^*\,|\mathbf{x}\,)$, the auction wouldn't offer a core solution.

By (6.1) $x_i^{+1} \succsim_i x_i$. Hence, granted transitive preferences $\succsim_i$, expenditures increase: $\mathcal{E}_i(x^* \left| x_i^{+1} \right.) \geq \mathcal{E}_i(x^* \left| x_i \right.)$ for all $x^* \in \mathbb{X}^*$ and each $i \in I$; see Prop. 3.3. Consequently, writing $x_I^0 := \sum_{i \in I} x_i^0$, it follows that $c_I^*(\chi^{*+1} \left| \mathbf{x}^{+1} \right.) =$

$$c_I^*(x^{*+1} \left| \mathbf{x}^{+1} \right.) = \inf_{\hat{x}^*} \left\{ \hat{x}^* x_I^0 - \sum_{i \in I} \mathcal{E}_i(\hat{x}^* \left| x_i^{+1} \right.) \right\}$$

$$\leq \inf_{\hat{x}^*} \left\{ \hat{x}^* x_I^0 - \sum_{i \in I} \mathcal{E}_i(\hat{x}^* \left| x_i \right.) \right\} = c_I^*(x^* \left| \mathbf{x} \right.) = c_I^*(\chi^* \left| \mathbf{x} \right.).$$

□

Theorem 6.1 links to received, qualitative results on the "shrinking of the core" which obtains when agents be duplicated, triplicated ... - in short, "replicated." It adds to such results - albeit constructively, explicitly and quantitatively - by stressing that iterated double auctions may substitute for replication of agents [19], [21].

**Theorem 6.2** (Convergence to competitive equilibrium)**.** *Let each preference relation $\succsim_i$ be transitive and form a closed subset of $\mathbb{X} \times \mathbb{X}$. Suppose the correspondence $\mathbf{x} \in \mathbf{X} \rightrightarrows \partial c_I(0 \left| \mathbf{x} \right.)$ has non-empty values and sequentially compact graph. Then, for each cluster point $\mathbf{x}$ of a sequence $\mathbf{x}^{k+1} \in \mathbf{A}(\mathbf{x}^k)$, $k = 0, 1, ...$, emanating from $\mathbf{x}^0 \in \mathbf{X}$, there is a price $\chi^* \in \mathcal{X}^*$ such that $(\chi^*, \mathbf{x})$ is a competitive equilibrium.*

*Proof.* Consider a sequence $(\chi^{*k}, \mathbf{x}^k)$ with $\chi^{*k} \in \partial c_I(0 \left| \mathbf{x}^k \right.)$. Posit $V^k := c_I^*(\chi^{*k} \left| \mathbf{x}^k \right.)$. From (6.5) follows that $V^k \searrow V$ for some limit $V \geq 0$. By sequential compactness, some subsequence $(\chi^{*k}, \mathbf{x}^k)$, $k \in K$, converges to $(\chi^*, \mathbf{x})$ with $\chi^* \in \partial c_I(0 \left| \mathbf{x} \right.)$. Suppose $V > 0$ and consider a subsequence $\mathcal{K} \subseteq K$ such that $(\chi^{*k+1}, \mathbf{x}^{k+1}) \to_{\mathcal{K}} (\hat{\chi}, \hat{\mathbf{x}})$ with $\hat{\chi}^* \in \partial c_I(0 \left| \hat{\mathbf{x}} \right.)$. Then $V(\chi^* \left| \hat{\mathbf{x}} \right.) < V$. This contradicts the fact that $V^{k+1} \to_{\mathcal{K}} V$. □

Theorem 6.2 singles out steady states in which another auction has no effect. Granted differentiability in the aggregate, such states are competitive equilibria:

**Theorem 6.3** (Steady states as competitive equilibria)**.** *Suppose the set $\mathbf{X}$ of feasible allocations is non-empty compact (5.1). Also, for every $\mathbf{x} \in \mathbf{X}$, suppose the infimal value $c_I(0 \left| \mathbf{x} \right.)$ (4.3) be attained, and that the subdifferential $\partial c_I(0 \left| \mathbf{x} \right.)$ reduces to a singleton.*
*Then, the double auction has at least one steady state $\mathbf{x} \in \mathbf{X}$, defined by the fixed point condition $\mathbf{x} \in \mathbf{A}(\mathbf{x})$ (6.4) - or equivalently, by $\chi^* \in \partial c_I(0 \left| \mathbf{x} \right.)$ and $c_I^*(\chi^* \left| \mathbf{x} \right.) = 0$. Any such fixed point yields a competitive equilibrium.*

*Proof.* Correspondence $\mathbf{A}(\cdot)$ (6.4) has non-empty convex values and closed graph. Hence, by Kakutani's theorem, it admits a fixed point $\mathbf{x} \in \mathbf{A}(\mathbf{x})$. If $c_I^*(\chi^* \left| \mathbf{x} \right.) > 0$ for some $\chi^* \in \partial c_I(0 \left| \mathbf{x} \right.)$, then $\mathbf{x}^{+1} \in \mathbf{A}(\mathbf{x})$ implies $\mathbf{x}^{+1} \neq \mathbf{x}$. In fact, $c_i^*(\chi^* \left| \mathbf{x} \right.) > 0$ implies $x_i^{+1} \succ_i x_i$. Consequently, for any $\chi^* \in \partial c_I(0 \left| \mathbf{x} \right.)$ it holds $c_I^*(x^* \left| \mathbf{x} \right.) = 0 \Longleftrightarrow \mathbf{x} \in \mathbf{A}(\mathbf{x})$. □

Ignoring entry or exit of agents - as well as possible lack of strict convexity/ differentiability of the indifference criteria - equilibrium is definite. Then, there is no *path dependence*; the process turns on initial holdings, and a specific equilibrium is *selected*.

## 7. Concluding remarks

It's fitting to compare and contrast iterated double auctions with classical *Walrasian tâtonnement* [19]. The latter lets each price change have the same sign as the corresponding aggregate excess demand. Both processes feature a central auctioneer or system operator. Otherwise they differ greatly in *decisions*, *messages*, *practicality, stability* and *trade*:
As to *decisions*, the operator of a double auction sets prices *and* he redistributes commodities as well as money; the Walrasian auctioneer just changes prices.
Regarding *messages*, participants in double auctions communicate criteria that relate prices to quantities; in Walrasian tâtonnement they report no such relations, just intended quantities.
While double auctions are *practical* and much used, Walrasian tâtonnement is neither.
Most important: double auctions display remarkable *stability*; Walrasian tâtonnement can generate instability, even chaos [18], [19].[15]
Double auctions permit *trade* out of equilibrium; classical tâtonnement does not.

Finally, arguing for themselves by way of experiments [27], double auctions support theoretical foundations for competitive equilibrium. Thereby they furnish a basis for applied studies.

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