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A LOCAL ANALYSIS FOR EIGENVALUE COMPLEMENTARITY PROBLEMS

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Abstract. Eigenvalue complementarity problems still have interesting theoretical properties that are not fully understood so far. Here, we are interested in local Lipschitzian error bounds. We first introduce the new class of ostensibly affine complementarity problems and derive conditions that characterize the existence of such an error bound for this class. These results are applied to eigenvalue complementarity problems. In particular, this allows us to prove the existence of a local Lipschitzian error bound for the symmetric linear eigenvalue complementarity problem.

1. Introduction

This paper is devoted to a local analysis of the *Eigenvalue Complementarity Problem* (EiCP)

$$\text{Find } (x,\lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R} \text{ such that} \quad x \geq 0, \quad M(\lambda)x \geq 0, \quad x^\top M(\lambda)x = 0,$$

where $M : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuously differentiable (matrix-valued) mapping. The notation $y \geq 0$ indicates that each component of y is nonnegative, i.e., y belongs to the nonnegative orthant $\mathbb{R}^n_+$. Throughout, we assume that the problem is solvable.

For given matrices $A, B \in \mathbb{R}^{n\times n}$, the EiCP with

$$M(\lambda) := \lambda B - A, \tag{1.1}$$

is frequently studied. We refer the reader to [7, 19, 31, 36, 43] for basic properties, to [1, 3, 4, 6, 9, 21, 30, 35] for algorithms, and to [13, 14] and the monographs [2, 11] for applications. In [4, 15, 22, 42] and [19] extensions to more general cones and tensor eigenvalue complementarity problems can be found.

In addition to EiCPs with (1.1), the case

$$M(\lambda) := \lambda^2 C + \lambda B - A, \tag{1.2}$$

gained interest (see [8, 20, 41] for instance), where the matrices $A, B, C \in \mathbb{R}^{n\times n}$ are given. An EiCP with M given by (1.2) is known as *quadratic* EiCP. In contrast to this, we will speak here of a *linear* EiCP if M is given by (1.1). Notice that in the literature the linear EiCP is rather known as EiCP.

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As first contribution of our paper (Section 2), we define and investigate the class of *ostensibly affine* complementarity problems. Then, as second contribution, we will show under mild assumptions, that a *complementary eigenvalue* λ can be substituted such that the EiCP is locally equivalent to an ordinary complementarity problem possessing some valuable properties (see Section 3). Based on these developments, we will finally prove in Section 3.4 that, for an arbitrary but fixed solution of this complementarity problem, there exists a local Lipschitzian error bound if the EiCP is linear with symmetric matrices A, B, whereas B is additionally positive definite in (1.1).

To explain and emphasize the meaning of the latter result, let us consider an ordinary complementarity problem

$$\text{Find } x \in \mathbb{R}^n \text{ such that} \quad x \geq 0, \quad H(x) \geq 0, \quad x^\top H(x) = 0 \tag{1.3}$$

with some given continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$. (For the local reduction of an EiCP to such a complementarity problem, the properties of H will be specified later on.) It is well known, that the complementarity problem (1.3) can be equivalently written as system of equations

$$F(x) = 0, \tag{1.4}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is given by the nonsmooth mapping

$$F(x) := \min\{x, H(x)\} \tag{1.5}$$

with $\min\{\cdot,\cdot\}$ taken componentwise. For more details and other possible reformulations, see [18]. Let x^* be any fixed element of the solution set $F^{-1}(0)$ of (1.4). Then, it is said that F provides a *local Lipschitzian error bound* at x^*, if there are constants $c, \varepsilon > 0$ such that

$$c\,\text{dist}[x, F^{-1}(0)] \leq \|F(x)\| \quad \text{for all } x \in x^* + \varepsilon\mathbb{B}, \tag{1.6}$$

where $\|\cdot\|$ stands for the Euclidean norm, $\mathbb{B} := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is the unit ball, and $\text{dist}[x, Z] := \inf\{\|x - y\| \mid y \in Z\}$ denotes the point-to-set distance of x to a nonempty set $Z \subset \mathbb{R}^n$. For brevity, we will omit the term "Lipschitzian" throughout.

Recently, necessary and sufficient conditions for the existence of a local error bound for certain nonsmooth systems of equations were studied in [24]. An application to complementarity systems can be found in [25]. However, the sufficient conditions therein turn out to be too restrictive for the purpose of analyzing EiCPs. Within the present article, we will derive new sufficient conditions for the existence of a local error bound for the solution set of (1.3). The new conditions shall relax those in [24, 25], whereas the mapping H is assumed to be of a special kind, termed *ostensibly affine* (see Section 2 below).

The knowledge of error bounds is important in several areas of mathematical programming. We just mention the analysis of the convergence speed of algorithms, e.g., see [32, 33]. Another important benefit of error bounds is their use in the design of algorithms for systems of (possibly nonsmooth) equations whose solutions are nonisolated. In such cases, (generalized) Jacobians at a nonisolated solution are singular. For classical Newton methods, this may destroy superlinear convergence. Advanced Newton-type methods that employ local error bounds can avoid these difficulties, see [17, 23, 44, 45], for instance. By now, superlinear convergence of existing Newton-type methods for linear EiCPs [3, 4, 34] relies on assumptions that guarantee the nonsingularity of certain (generalized) Jacobians.

2. Ostensibly affine complementarity problems

In a first part of this section, we will provide some preliminaries. In particular, we give some selected results recently obtained in [24, 25]. The second part is devoted to a new class of ordinary complementarity problems and its study regarding local error bounds.

2.1. **Preliminaries.** As before, we use the reformulation (1.4) of the complementarity problem (1.3) with the nonsmooth mapping F given by (1.5). Since the mapping H defining the complementarity problem (1.3) is assumed to be continuous at least, it follows that F is continuous and $F^{-1}(0)$ is closed. With reference to [24, 29], the mapping H is said to be *strictly differentiable at* $x \in F^{-1}(0)$ *with respect to* $F^{-1}(0)$, if H is differentiable at x with Jacobian $H'(x) \in \mathbb{R}^{n\times n}$ and

$$\|H(y+d) - H(y) - H'(x)d\| = o(\|d\|) \quad \text{as } y \overset{F^{-1}(0)}{\to} x \text{ and } d \to 0 \tag{2.1}$$

holds. Clearly, the latter is particularly fulfilled provided x is isolated in $F^{-1}(0)$ and H is differentiable at x, or if H is differentiable near x with the derivative being continuous at x, see [39, Proposition 3.4.2].

The following notion is inspired by [27, Definition 8.22].

Definition 2.1. A set $C \subset \mathbb{R}^n$ is called *closed semilinear near* $x \in C$, if there exist closed (convex) polyhedra $P_1, ..., P_N \subset \mathbb{R}^n$ along with a constant $\varepsilon > 0$ such that

$$C \cap (x + \varepsilon\mathbb{B}) = \bigcup_{l=1}^{N} P_l \cap (x + \varepsilon\mathbb{B}) \quad \text{and} \quad x \in \bigcap_{l=1}^{N} P_l.$$

Remark 2.2. Let $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ be given. Then, similarly to the proof of [32, Lemma 3.1], it can be shown that the set $\{x \in \mathbb{R}^n \mid \min\{x, Mx+q\} = 0\}$ is closed semilinear near any of its elements. □

Definition 2.3. A set $C \subset \mathbb{R}^n$ is called *T-conical near* $x \in C$, if there exists $\delta > 0$ such that

$$C \cap (x + \delta\mathbb{B}) = x + (T_C(x) \cap \delta\mathbb{B}),$$

where

$$T_C(x) := \{d \in \mathbb{R}^n \mid \exists (t_k) \downarrow 0 \exists (d^k) \to d : \ (x + t_k d^k) \subset C\}$$

is the *tangent cone to C at x*.

The name for the latter conicity concept was suggested recently in [24] to distinguish it from the conicity concept in [5], where the radial cone is used instead of the tangent cone. A related concept [26] relies on an arbitrary closed convex cone.

Lemma 2.4 (Proposition 8.24, [27])**.** *If C is closed semilinear near $x \in C$, then C is T-conical near x.*

Evidently, the converse implication of the previous lemma is not true in general. In the remainder of this section, let $x^* \in F^{-1}(0)$ be fixed. In order to approximate $T_{F^{-1}(0)}(x^*)$ later on, we will make use of the cone

$$\mathfrak{D}F(x^*) := \{d \in \mathbb{R}^n \mid \exists (t_k) \downarrow 0 \exists (d^k) \to d : \|F(x^* + t_k d^k)\| = o(t_k)\}. \tag{2.2}$$

It can be easily seen [24, Lemma 8] that

$$T_{F^{-1}(0)}(x^*) \subset \mathfrak{D}F(x^*). \tag{2.3}$$

Lemma 2.5. *Suppose that H is differentiable at x^*. If F provides a local error bound at x^*, then*

$$T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*)$$

is satisfied.

Proof. To show that the differentiability of H at x^* implies semidifferentiability of F at x^*, arguments as in the proof of [25, Lemma 1] can be used. Now, according to [24, Lemma 2 a)], we further have that $\|F(\cdot)\|$ is semidifferentiable at x^*. Therefore, [24, Lemma 5 a)] and [24, Theorem 1 a)] yield

$$\mathfrak{D}F(x^*) = \{d \in \mathbb{R}^n \mid \exists (t_k) \downarrow 0 : \|F(x^* + t_k d)\| = o(t_k)\} \subset T_{F^{-1}(0)}(x^*)(x^*).$$

In combination with (2.3), the proof is complete. □

Lemma 2.6. *Suppose that H is differentiable at x^*. Then, for any $d \in \mathbb{R}^n$, we have $d \in \mathfrak{D}F(x^*)$ if and only if there exists some constant $\tau > 0$ such that*

$$\min\left\{x^* + td, H(x^*) + tH'(x^*)d\right\} = 0$$

is valid for all $t \in [0, \tau]$.

Proof. The lemma can be shown by arguments in the proof of [25, Lemma 1]. □

Lemma 2.7. *Suppose that H is differentiable at x^*, $F^{-1}(0)$ is T-conical near x^*, and $d \in \mathfrak{D}F(x^*)$. If there exists some $\tau > 0$ (possibly smaller than the one obtained in Lemma 2.6) so that the implication*

$$H_i(x^*) = H_i'(x^*)d = 0 \quad \Longrightarrow \quad H_i(x^* + td) = 0 \tag{2.4}$$

holds for all $(i, t) \in \{1, \ldots, n\} \times [0, \tau]$, then $d \in T_{F^{-1}(0)}(x^*)$ is valid.

Proof. According to Lemma 2.6, we have

$$\min\left\{x^* + td, H(x^*) + tH'(x^*)d\right\} = 0 \quad \text{for all } t \in [0, \tau]. \tag{2.5}$$

Now, take any $i \in \{1, \ldots, n\}$. If $H_i(x^*) = H_i'(x^*)d = 0$, then (2.4)–(2.5) provide

$$\min\{(x^* + td)_i, H_i(x^* + td)\} = 0 \quad \text{for all } t \in [0, \tau]. \tag{2.6}$$

If $H_i(x^*) > 0$, the continuity of H_i and (2.5) imply (2.6), where we assume that $\tau > 0$ is reduced when needed. Finally, if $H_i'(x^*)d \neq 0 = H_i(x^*)$, then (2.5) yields

$$(x^* + td)_i = 0 < H_i(x^*) + tH_i'(x^*)d \quad \text{for all } t \in (0, \tau].$$

This and the differentiability of H_i at x^* show that (2.6) is satisfied, where again $\tau > 0$ has to be reduced if necessary. Altogether, we have that (2.6) holds for all $i \in \{1, \ldots, n\}$. Hence, $x^* + td \in F^{-1}(0)$ for all $t > 0$ small enough. Due to the T-conicity of $F^{-1}(0)$ near x^*, it follows that $d \in T_{F^{-1}(0)}(x^*)$. □

2.2. **Local error bounds.** This subsection is devoted to a special class of complementarity problems and the question when F provides a local error bound for complementarity problems of this class.

Definition 2.8. The mapping H is called *ostensibly affine near* x^*, if H is strictly differentiable at x^* with respect to $F^{-1}(0)$ and if there exists $\varepsilon > 0$ such that

$$H(x) = H(x^*) + H'(x^*)(x - x^*) \quad \text{for all } x \in F^{-1}(0) \cap (x^* + \varepsilon\mathbb{B}).$$

If H is ostensibly affine near x^*, the complementarity problem (1.3) is called *ostensibly affine near* x^*.

Remark 2.9. Suppose that H is differentiable at x^*. If x^* is an isolated solution, i.e., there is some $\varepsilon > 0$, such that $F^{-1}(0) \cap (x^* + \varepsilon\mathbb{B}) = \{x^*\}$, then it is easily seen that H is ostensibly affine near x^*. □

Remark 2.10. Because $F^{-1}(0)$ is not necessarily a linear subspace, ostensibly affine mappings are not necessarily affine. Hence, ostensibly affine complementarity problems are not necessarily linear complementarity problems (which are well studied with respect to our current interests [32, 37]): Indeed, let us consider the nonlinear complementarity problem with $H(x) := \min\{x, 0\}^2$ for $x \in \mathbb{R}$. Then, $F^{-1}(0) = \mathbb{R}_+$ is a halfspace, H is strictly differentiable at $x^* := 0$ with $H'(x^*) = 0$, and $0 = H(x) = H(x^*) + H'(x^*)(x - x^*)$ holds only for $x \in F^{-1}(0)$. Thus, H is ostensibly affine near x^*, but it is not an affine mapping. □

In what follows, if H is differentiable at x^*, we will use the set

$$L_H(x^*) := \left\{x \in \mathbb{R}^n \mid \min\left\{x, H(x^*) + H'(x^*)(x - x^*)\right\} = 0\right\}. \tag{2.7}$$

Lemma 2.11. *Suppose that H is ostensibly affine near x^*. Then, the following statements are valid:*

a) *There exists some $\varepsilon > 0$ such that, for all $x \in x^* + \varepsilon\mathbb{B}$,*

$$x \in F^{-1}(0) \quad \Longrightarrow \quad x \in L_H(x^*).$$

b) *If F provides a local error bound at x^*, then the converse of the implication in statement a) is also fulfilled for some (possibly different) $\varepsilon > 0$ and thus, $F^{-1}(0)$ is closed semilinear near x^*.*

Proof. To see that statement a) is valid, let $\varepsilon > 0$ be the constant given by Definition 2.8. Then, for any $x \in F^{-1}(0) \cap (x^* + \varepsilon\mathbb{B})$ we notice that

$$0 = \min\{x, H(x)\} = \min\{x, H(x^*) + H'(x^*)(x - x^*)\} \tag{2.8}$$

because H is ostensibly affine near x^*.

Statement b) is shown by contradiction. Hence, there exists a sequence $(x^k) \subset \mathbb{R}^n \setminus F^{-1}(0)$ with $(x^k) \to x^*$ and

$$\min\{x^k, H(x^*) + H'(x^*)(x^k - x^*)\} = 0 \quad \text{for all } k \in \mathbb{N}. \tag{2.9}$$

Since $F^{-1}(0)$ is closed (see Section 2.1), there is a sequence $(\bar{x}^k) \subset F^{-1}(0)$ with $(\bar{x}^k) \to x^*$ such that

$$\operatorname{dist}[x^k, F^{-1}(0)] = \|x^k - \bar{x}^k\| \quad \text{for all } k \in \mathbb{N}. \tag{2.10}$$

Because F provides a local error bound at x^*, there are $c > 0$ and $K \in \mathbb{N}$, so that

$$\begin{array}{rcl} c\mathrm{dist}[x^k, F^{-1}(0)] & \leq & \|\min\{x^k, H(x^k)\}\| \\ & = & \|\min\{x^k, H(\bar{x}^k) + H'(x^*)(x^k - \bar{x}^k)\}\| + o(\|x^k - \bar{x}^k\|) \end{array}$$

is valid for all $k \geq K$, where the equality is due to the strict differentiability of H at x^*, see Definition 2.8. Hence, this definition for $x = \bar{x}^k$, (2.9), and (2.10) yield

$$\begin{array}{rcl} c\mathrm{dist}[x^k, F^{-1}(0)] & \leq & \|\min\{x^k, H(x^*) + H'(x^*)(x^k - x^*)\}\| + o(\mathrm{dist}[x^k, F^{-1}(0)]) \\ & = & o(\mathrm{dist}[x^k, F^{-1}(0)]) \end{array}$$

for all $k \geq K$, perhaps with some different $K \in \mathbb{N}$. This contradicts $(x^k) \subset \mathbb{R}^n \setminus F^{-1}(0)$. Finally, Remark 2.2 ensures that $F^{-1}(0)$ is closed semilinear near x^*. ☐

The following example demonstrates that $F^{-1}(0)$ is not necessarily closed semilinear near some x^* even if H is ostensibly affine near x^*.

Example 2.12. Consider $H(x) := (x_1(x_2 - x_1^2), 0)^\top$ for $x = (x_1, x_2) \in \mathbb{R}^2$. Then,

$$F^{-1}(0) = (\{0\} \times \mathbb{R}_+) \cup \left\{(x_1, x_2) \mid x_2 = x_1^2,\, x_1 \geq 0\right\}.$$

Whereas $F^{-1}(0)$ is not closed semilinear near $x^* := 0$, it is easy to verify that $H'(x^*)x = 0 = H(x)$ for all $x \in F^{-1}(0)$. Thus, H is ostensibly affine near x^*. ☐

Theorem 2.13. *Suppose that H is ostensibly affine near x^*. Then, the following statements are equivalent:*

a) *F provides a local error bound at x^*.*
b) *$F^{-1}(0)$ is T-conical near x^* and $T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*)$.*
c) *There exists some $\varepsilon > 0$ such that, for all $x \in x^* + \varepsilon\mathbb{B}$,*

$$x \in L_H(x^*) \quad \Longrightarrow \quad x \in F^{-1}(0).$$

Proof. a) implies b): Just combine Lemma 2.11 with Lemma 2.4 and Lemma 2.5.

b) implies c): Since $F^{-1}(0)$ is T-conical near x^*, we obtain by part a) of Lemma 2.11 that there exists $\delta > 0$ with

$$x^* + \left(T_{F^{-1}(0)}(x^*) \cap \delta\mathbb{B}\right) = F^{-1}(0) \cap (x^* + \delta\mathbb{B}) \subset L_H(x^*) \cap (x^* + \delta\mathbb{B}).$$

Therefore, to prove statement c), it suffices to show that there is $\varepsilon \in (0, \delta]$ satisfying

$$L_H(x^*) \cap (x^* + \varepsilon\mathbb{B}) \subset x^* + \left(T_{F^{-1}(0)}(x^*) \cap \varepsilon\mathbb{B}\right). \tag{2.11}$$

According to Remark 2.2, the set $L_H(x^*)$ is closed semilinear near x^*. Hence, there exist (convex) polyhedra $P_1, ..., P_N$ along with some $\varepsilon \in (0, \delta]$ so that

$$L_H(x^*) \cap (x^* + \varepsilon\mathbb{B}) = \bigcup_{l=1}^{N} P_l \cap (x^* + \varepsilon\mathbb{B}) \quad \text{and} \quad x^* \in \bigcap_{l=1}^{N} P_l \tag{2.12}$$

are valid. Now, for any $x \in L_H(x^*) \cap (x^* + \varepsilon\mathbb{B})$, (2.7) and (2.12) give

$$0 = \min\left\{x^* + t(x - x^*), H(x^*) + tH'(x^*)(x - x^*)\right\} \quad \text{for all } t \in [0, 1].$$

Thus, by Lemma 2.6, we have $x - x^* \in \mathfrak{D}F(x^*) \cap \varepsilon\mathbb{B}$ and hence,

$$x \in x^* + \left(T_{F^{-1}(0)}(x^*) \cap \varepsilon\mathbb{B}\right)$$

follows from statement b). In other words, (2.11) is true.

c) implies a): To see that F provides a local error bound at x^*, let us assume the contrary. Therefore, a sequence $(x^k) \subset \mathbb{R}^n \setminus F^{-1}(0)$ exists with $x^k \to x^*$ and $\operatorname{dist}[x^k, F^{-1}(0)]\|F(x^k)\|^{-1} \to \infty$. Since $F^{-1}(0)$ coincides with $L_H(x^*)$ near x^* by statement c), [37, Proposition 1] shows that there is a constant $c > 0$ so that, for all $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} c\operatorname{dist}[x^k, F^{-1}(0)] &= c\operatorname{dist}[x^k, L_H(x^*)] \\ &\le \|\min\{x^k, H(x^*) + H'(x^*)(x^k - x^*)\}\| \end{aligned} \tag{2.13}$$

holds. Because $F^{-1}(0)$ is closed, there exists a sequence $(\bar{x}^k) \subset F^{-1}(0)$ satisfying $\operatorname{dist}[x^k, F^{-1}(0)] = \|x^k - \bar{x}^k\|$ for all $k \in \mathbb{N}$. Thus, the strict differentiability of H at x^*, that H is ostensibly affine near x^*, and (2.13) yield, for all $k \in \mathbb{N}$ large enough,

$$\begin{aligned} \|F(x^k)\| &= \|\min\{x^k, H(x^k)\}\| \\ &= \|\min\{x^k, H(\bar{x}^k) + H'(x^*)(x^k - \bar{x}^k)\}\| + o(\|x^k - \bar{x}^k\|) \\ &= \|\min\{x^k, H(x^*) + H'(x^*)(x^k - x^*)\}\| + o(\|x^k - \bar{x}^k\|) \\ &\ge c\|x^k - \bar{x}^k\| + o(\|x^k - \bar{x}^k\|). \end{aligned}$$

This contradicts the fact that $\operatorname{dist}[x^k, F^{-1}(0)]\|F(x^k)\|^{-1}$ tends to infinity. □

By Lemma 2.11 a), the implication in Theorem 2.13 c) is actually an equivalence. Nonetheless, it suffices to check only the implication in order to confirm (or to refute) the validity of statements a) and b) of the theorem.

Remark 2.14. Statement b) in Theorem 2.13 is equivalent to the existence of $\delta > 0$ such that

$$F^{-1}(0) \cap (x^* + \delta\mathbb{B}) = x^* + \left(T_{F^{-1}(0)}(x^*) \cap \delta\mathbb{B}\right) = x^* + (\mathfrak{D}F(x^*) \cap \delta\mathbb{B})\,. \tag{2.14}$$

In contrast to this, a solution x^* is termed *noncritical* [25, Definition 1] if

$$\widehat{T}_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*), \tag{2.15}$$

where $\widehat{T}_{F^{-1}(0)}(x^*)$ denotes Clarke's (regular) tangent cone to $F^{-1}(0)$ at x^*, cf. [12, p. 11]. Under a certain strict semidifferentiability assumption, condition (2.15) is known to be necessary and sufficient for F to provide a local error bound at x^*, see [24, Corollary 1]. Suppose that F is (at least) semidifferentiable at x^*. Then, according to the discussion in [24, Section 4], equality (2.15) implies

$$\widehat{T}_{F^{-1}(0)}(x^*) = T_{F^{-1}(0)}(x^*) \tag{2.16}$$

and (2.15) becomes equivalent to

$$\widehat{T}_{F^{-1}(0)}(x^*) = T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*). \tag{2.17}$$

Further, $T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*)$ does obviously hold if and only if there exists some $\delta > 0$ satisfying

$$x^* + \left(T_{F^{-1}(0)}(x^*) \cap \delta\mathbb{B}\right) = x^* + (\mathfrak{D}F(x^*) \cap \delta\mathbb{B})\,.$$

Therefore, (2.14) and (2.17) differ in the equations on their most-left-hand side. For complementarity problems, however, equation (2.16) can naturally be violated, see [25, Remark 3]. Hence, (2.17) (and thus (2.15)) can often not be employed to ensure that F provides a local error bound, even if mapping H is affine. □

The next result is an immediate consequence of Remark 2.9 and Theorem 2.13.

Corollary 2.15. *Suppose that H is differentiable at x^*. If x^* is isolated in $F^{-1}(0)$, then the following statements are equivalent:*

a) *F provides a local error bound at x^*.*
b) *$\mathfrak{D}F(x^*) = \{0\}$.*
c) *x^* is isolated in $L_H(x^*)$.*

The corollary shows in particular, that the two conditions in statement b) of Theorem 2.13 reduce to one, if x^* is isolated in $F^{-1}(0)$.

Example 2.16. For $H(x) := x^2$ and $x^* := 0$, we have $F^{-1}(0) = T_{F^{-1}(0)}(x^*) = \{x^*\}$. Thus, $F^{-1}(0)$ is T-conical near x^*. According to Remark 2.9, H is ostensibly affine near x^*. However, an application of Lemma 2.6 yields $\mathfrak{D}F(x^*) = \mathbb{R}_+$. Hence, $T_{F^{-1}(0)}(x^*) \subsetneq \mathfrak{D}F(x^*)$ follows. Due to Corollary 2.15, we conclude that F does not provide a local error bound at x^*. □

The previous example shows that T-conicity does not necessarily imply the equality $T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*)$. The next example demonstrates that the reverse implication need not be true as well. Therefore, in general, the two conditions in statement b) of Theorem 2.13 do not imply each other.

Example 2.17. Consider $H(x) := x^3 \sin(1/x)$ for $x \neq 0$ and $H(x) := 0$, otherwise. Then, H is continuously differentiable with the derivative,

$$H'(x) := \begin{cases} x\,(3x\sin(1/x)) - \cos(1/x))\,, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly, $F^{-1}(0) = \{(k\pi)^{-1} \mid k \in \mathbb{N}\} \cup \{0\}$. For $x^* = 0$, we observe that $H'(x^*) = 0$. Thus, since $H(x) = 0$ for all $x \in F^{-1}(0)$, H is evidently ostensibly affine near x^*. Now, it can be immediately seen that $T_{F^{-1}(0)}(x^*) = \mathbb{R}_+$. Moreover, again taking into account Lemma 2.6 yields $\mathfrak{D}F(x^*) = \mathbb{R}_+$. Therefore, we have $T_{F^{-1}(0)}(x^*) = \mathfrak{D}F(x^*)$. However, $F^{-1}(0)$ is clearly not T-conical near x^*. Finally, with Theorem 2.13 in mind, we conclude that F does not provide a local error bound at x^*. □

3. A local study of Eigenvalue complementarity problems

The first subsection provides conditions that allow to locally remove a complementary eigenvalue and to reformulate EiCP as an ordinary complementarity problem. Based on the results in Section 2.2, we investigate in Section 3.2 conditions under which F yields an error bound for this complementarity problem. The third subsection discusses several examples and in the last subsection we apply our results to the linear EiCP.

3.1. **Local Reformulation of EiCPs.** This subsection shows how an EiCP can be locally reformulated as ordinary complementarity problem (1.3). With the previous section in mind, we first note that the solution set of an EiCP is nothing else than

$$\text{SOL} := \Big\{(x, \lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R} \mid \min\{x, M(\lambda)x\} = 0\Big\}. \tag{3.1}$$

Projecting SOL onto $\mathbb{R}^n$ and $\mathbb{R}$, respectively, we get

$$X^* := \{x \in \mathbb{R}^n \mid \exists \lambda \in \mathbb{R} : \quad (x, \lambda) \in \text{SOL}\},$$
$$\Lambda^* := \{\lambda \in \mathbb{R} \mid \exists x \in \mathbb{R}^n : \quad (x, \lambda) \in \text{SOL}\}.$$

Moreover, we define the set-valued mapping $S : X^* \rightrightarrows \Lambda^*$ by

$$S(x) := \{\lambda \in \Lambda^* \mid (x, \lambda) \in \text{SOL}\}. \tag{3.2}$$

Lemma 3.1. *For any $x \in X^*$, the set $S(x)$ is nonempty and closed.*

Proof. Obviously, $S(x)$ is nonempty by definition for any $x \in X^*$. Since the matrix-valued mapping $\lambda \mapsto M(\lambda)$ introduced in Section 1 is continuous, the set

$$S(x) = \left\{\lambda \in \mathbb{R} \mid \; M(\lambda)x \geq 0, \; x^\top M(\lambda)x = 0\right\}$$

is closed. □

Assumption 1. The pair $(x, \lambda) \in \text{SOL}$ satisfies $\lim\limits_{y \overset{X^*}{\to} x} \operatorname{dist}[\lambda, S(y)] = 0$.

Lemma 3.2. *The pair $(x, \lambda) \in$ SOL fulfills Assumption 1 if and only if there is a function $s : X^* \to \Lambda^*$ with the following properties:*

a) *s is continuous at x,*
b) *$s(x) = \lambda$,*
c) *$s(y) \in S(y)$ for all $y \in X^*$ near x.*

Proof. Suppose first that Assumption 1 is valid for the pair $(x, \lambda) \in$ SOL. Then, due to Lemma 3.1, for any $y \in X^*$ there exists some $s(y) \in S(y) \subset \Lambda^*$ with

$$\operatorname{dist}[\lambda, S(y)] = \min\{|\lambda - \gamma| \mid \gamma \in S(y)\} = |\lambda - s(y)|. \tag{3.3}$$

Since λ belongs to $S(x)$, the latter equations give $s(x) = \lambda$. From (3.3) and Assumption 1, we obtain

$$0 \leq \liminf_{y \overset{X^*}{\to} x} |\lambda - s(y)| \leq \limsup_{y \overset{X^*}{\to} x} |\lambda - s(y)| = \limsup_{y \overset{X^*}{\to} x} \operatorname{dist}[\lambda, S(y)] = 0.$$

Therefore, $s(y)$ tends to $\lambda = s(x)$ as $y \in X^*$ tends to x and hence, the function $s : X^* \to \Lambda^*$ as defined above fulfills properties a)–c).

To prove the converse implication, let $s : X^* \to \Lambda^*$ be a function with properties a)–c). Then, $s(y) \in S(y)$ holds for all $y \in X^*$ near x according to c). Thus, b) gives

$$\operatorname{dist}[\lambda, S(y)] \leq |\lambda - s(y)| = |s(x) - s(y)|$$

for all $y \in X^*$ sufficiently close to x. Therefore, according to a), Assumption 1 is valid for the pair $(x, \lambda) \in$ SOL. □

To reformulate EiCP as ordinary complementarity problem, we will substitute the complementary eigenvalue λ by means of Dini's classical implicit function theorem, see [16] for instance. For this purpose, the next assumption is needed.

Assumption 2. The pair $(x, \lambda) \in$ SOL satisfies $x^\top M'(\lambda)x \neq 0$.

Lemma 3.3. *Suppose that the pair $(x^*, \lambda^*) \in$ SOL fulfills Assumption 2. Then, there are open neighborhoods $U \subset \mathbb{R}^n$ of x^* and $V \subset \mathbb{R}$ of λ^* along with a continuously differentiable function $f : U \to V$ with the following properties:*

a) $f(x^*) = \lambda^*$,

b) *For all* $x \in U$*, it holds that* $x^\top M(f(x))x = 0$.

c) *For all* $x \in U$*, it holds that*

$$\nabla f(x) = -\frac{1}{x^\top M'(f(x))x} \cdot \left(M(f(x)) + M(f(x))^\top \right) x.$$

d) *If* $x^\top M(\lambda)x = 0$ *holds for some* $(x, \lambda) \in U \times V$*, then* $\lambda = f(x)$ *follows.*

e) *For all* $x \in U$ *and all* $\mu > 0$ *with* $\mu x \in U$*, it holds that* $f(\mu x) = f(x)$.

Moreover, U *and* V *can be chosen such that each pair* $(x, \lambda) \in (U \times V) \cap \mathrm{SOL}$ *satisfies Assumption 2.*

Proof. Assertions a)–d) directly follow from [16, Theorem 1B.1], applied to the function $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with $\phi(x, \lambda) := x^\top M(\lambda)x$. To prove e), let us pick $\mu > 0$ and $x \in U$ with $\mu x \in U$. Set $\lambda := f(x)$ and notice that $\lambda \in V$. Then, assertion b) yields $(\mu x)^\top M(\lambda)(\mu x) = 0$. Since d) gives $\lambda = f(\mu x)$, we have $f(\mu x) = f(x)$. □

Theorem 3.4. *In the setting of Lemma 3.3, the following statements are valid:*

a) *The mapping* $\Phi : U \to \mathbb{R}^n$*, defined by*

$$\Phi(x) := M(f(x))x, \tag{3.4}$$

is continuously differentiable.

b) *If the pair* (x^*, λ^*) *fulfills Assumption 1, then there is an open neighborhood* $U_0 \subset U$ *of* x^* *so that, for all* $x \in U_0$*, it holds that* $x \in X^*$ *if and only if*

$$x \geq 0, \quad \Phi(x) \geq 0, \quad \text{and} \quad x^\top \Phi(x) = 0. \tag{3.5}$$

In particular, U_0 *can be chosen such that both Assumption 1 and Assumption 2 are fulfilled for each pair* $(x, f(x))$ *with* $x \in X^* \cap U_0$.

Proof. Statement a) is obvious, so we only prove statement b). We first notice that, by Lemma 3.1,

$$\mathrm{dist}[\lambda^*, S(x)] = \min\{|\lambda^* - \lambda| \mid \lambda \in S(x)\} \quad \text{for } x \in X^*$$

is valid. Since Assumption 1 at the pair (x^*, λ^*) was requested to hold, Lemma 3.2 implies that function $x \in X^* \to d(x) := \mathrm{dist}[\lambda^*, S(x)]$ is continuous at x^* with $d(x^*) = 0$. Hence, there exists $\varepsilon > 0$ such that

$$x^* + \varepsilon\mathbb{B} \subset U \quad \text{and} \quad \lambda^* \pm d(x) \in V \text{ for all } x \in X^* \cap (x^* + \varepsilon\mathbb{B}) \tag{3.6}$$

are satisfied. To show that EiCP is locally equivalent to the ordinary complementarity problem (3.5), first note that Lemma 3.3 b) and (3.4) imply

$$x^\top \Phi(x) = 0 \quad \text{for all } x \in U. \tag{3.7}$$

Let us now pick any $x \in x^* + \varepsilon\mathbb{B}$ that solves the complementarity problem (3.5). Due to (3.6), we have $x \in U$ and, thus, (3.4) gives $\Phi(x) = M(f(x))x$. Hence, from (3.7) and (3.5), we obtain

$$x \geq 0, \quad M(f(x))x \geq 0, \quad x^\top M(f(x))x = 0.$$

Thus, $(x, f(x)) \in \mathrm{SOL}$ and $x \in X^*$ follow.

To see the reverse, namely that $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$ implies (3.5), we first note that Lemma 3.1 yields the existence of some $\lambda(x) \in S(x)$ satisfying

$$d(x) = |\lambda^* - \lambda(x)|.$$

Therefore, (3.6) implies $\lambda(x) \in V$ and, according to Lemma 3.3 d), we observe that $\lambda(x) = f(x)$. Combining the latter with (3.4) yields $\Phi(x) = M(\lambda(x))x$ and, since $(x, \lambda(x)) \in \mathrm{SOL}$, we notice that x solves (3.5).

In particular, we have just proven that

$$f(x) \in S(x) \quad \text{for all } x \in X^* \cap (x^* + \varepsilon\mathbb{B}). \tag{3.8}$$

Hence, to prove the final statement in part b) of the theorem, pick any $\varepsilon' \in (0, \varepsilon)$ and any $\hat{x} \in X^* \cap (x^* + \varepsilon'\mathbb{B})$. Because of (3.8), we have $(\hat{x}, f(\hat{x})) \in \mathrm{SOL}$. Thanks to Lemma 3.3, we can assume ε' to be small enough such that $(\hat{x}, f(\hat{x}))$ fulfills Assumption 2. Moreover, we can assume that there exists some $\hat{\varepsilon} \in (0, \varepsilon')$ such that

$$\hat{x} + \hat{\varepsilon}\mathbb{B} \subset x^* + \varepsilon'\mathbb{B}. \tag{3.9}$$

By Lemma 3.3, f is continuous at $\hat{x}$. In addition, (3.8)–(3.9) yield $f(x) \in S(x)$ for all $x \in X^* \cap (\hat{x} + \hat{\varepsilon}\mathbb{B})$. Thus, applying Lemma 3.2 yields that Assumption 1 is fulfilled for the pair $(\hat{x}, f(\hat{x})) \in \mathrm{SOL}$. □

3.2. **Local error bounds.** In Definition 2.1, we recalled the notion that a set is closed semilinear near one of its elements. Moreover, we have seen in Section 2 that this notion in the case of an ostensibly affine complementarity problem is closely tied to the existence of a local error bound. Therefore, we begin with a sufficient condition for the solution set X^* of an EiCP to be closed semilinear near any of its elements. In the main part of this subsection, we study the local behavior of the mapping Φ introduced in Theorem 3.4. Particularly, we provide conditions under which a continuous extension of Φ is ostensibly affine and show how a local error bound for the complementarity problem (3.5) can be derived.

Consider the set of matrices

$$\mathcal{M} := \Big\{ M(\lambda) \mid \lambda \in \Lambda^* \Big\}, \tag{3.10}$$

and denote its cardinality by $|\mathcal{M}|$.

Lemma 3.5. *Suppose that $|\mathcal{M}| < \infty$. Then, $X^* \cup \{0\}$ is closed semilinear near any of its elements.*

Proof. If $|\mathcal{M}| < \infty$, there are matrices $M_1, ..., M_K \in \mathcal{M}$ such that $\{M_1, ..., M_K\} = \mathcal{M}$. By Remark 2.2, for each $l \in \{1, ..., K\}$, the set $\{x \in \mathbb{R}^n \mid \min\{x, M_l x\} = 0\}$ is closed semilinear near any of its elements. Thus, the lemma is true because

$$\begin{aligned} X^* \cup \{0\} &= \bigcup_{l=1}^{K} \{x \in \mathbb{R}^n \mid \min\{x, M_l x\} = 0\} \\ &= \big\{x \in \mathbb{R}^n \mid \exists \lambda \in \Lambda^* : \min\big\{x, M(\lambda)x\big\} = 0\big\}. \end{aligned}$$

□

The condition $|\mathcal{M}| < \infty$ is known to hold for some common EiCPs. For instance, it can be shown that EiCPs with the mapping M defined by (1.1) or (1.2) have the property that $|\Lambda^*| < \infty$ is valid under appropriate assumptions on the matrices A, B, C, see [28, Section 1] and Section 3.4 below. Then, $|\mathcal{M}| < \infty$ follows immediately. However, notice that $|\mathcal{M}| < \infty$ does not imply $|\Lambda^*| < \infty$ in general. To see this, consider the one-dimensional case with $M(\lambda) = \sin(\lambda)$, which implies $|\Lambda^*| = \infty$, while $\mathcal{M} = \{0\}$.

Lemma 3.6. *Suppose that the pair $(x^*, \lambda^*) \in \mathrm{SOL}$ fulfills Assumptions 1–2 and that $|\mathcal{M}| < \infty$. Let the set U_0 and the mappings f and Φ be as in Theorem 3.4. Then, for some $\varepsilon > 0$ with $x^* + \varepsilon\mathbb{B} \subset U_0$,*

$$\Phi(x) = M(\lambda^*)x \tag{3.11}$$

is satisfied for all $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$.

Proof. Recall that (3.4) in Theorem 3.4 yields

$$\Phi(x) = M(f(x))x \quad \text{for all } x \in U_0. \tag{3.12}$$

The proof is by contradiction. Then, because $X^* \cup \{0\}$ is closed semilinear near x^* (Lemma 3.5) and by (3.12), we can assume without loss of generality, that there is a polyhedron $P \subset X^*$ along with a sequence $(x^k) \subset P$ that converges to x^* so that

$$\Phi(x^k) = M(f(x^k))x^k \neq M(\lambda^*)x^k \quad \text{for all } k \in \mathbb{N}.$$

Thus, it immediately follows that

$$M(f(x^k)) \neq M(\lambda^*) \quad \text{for all } k \in \mathbb{N}. \tag{3.13}$$

Taking into account that $x \mapsto M(f(x))$ is continuous at all $x \in U_0$ (Lemma 3.3), we see that (3.13) implies

$$|\{M(f(x^k)) \mid k \in \mathbb{N}\}| = \infty. \tag{3.14}$$

Furthermore, using part b) in Theorem 3.4, we conclude that $f(x^k) \in \Lambda^*$ for all k sufficiently large. Thus, (3.14) contradicts the assumption $|\mathcal{M}| < \infty$. $\square$

Lemma 3.7. *In the setting of Lemma 3.6, there exists some $\varepsilon > 0$ (possibly smaller than the one obtained in Lemma 3.6) with $x^* + \varepsilon\mathbb{B} \subset U_0$, such that*

$$\Phi(x) = \Phi(x^*) + \Phi'(x^*)(x - x^*)$$

is satisfied for all $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$.

Proof. Let us first define the index sets

$$I_+(x) := \{i \mid \Phi_i(x) + x_i > 0\}, \quad \text{and} \quad I_0 := \{i \mid \Phi_i(x^*) = x_i^* = 0\}.$$

Obviously, we have

$$I_+(x^*) \cup I_0 = \{1, \dots, n\} \quad \text{and} \quad I_+(x^*) \cap I_0 = \varnothing.$$

As Φ is continuous (see Theorem 3.4), there is some $\varepsilon > 0$, smaller or equal to the one in Lemma 3.6, so that

$$x \in U_0 \quad \text{and} \quad I_+(x^*) \subset I_+(x) \qquad \text{for all } x \in X^* \cap (x^* + \varepsilon\mathbb{B}).$$

For any $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$, these observations, with Theorem 3.4 b) in mind, yield

$$\Phi(x^*)^\top x = \sum_{i \in I_+(x^*)} \Phi_i(x^*) x_i + \sum_{i \in I_0} \Phi_i(x^*) x_i = 0. \tag{3.15}$$

In the same way, we obtain,

$$\Phi(x)^\top x^* = 0. \tag{3.16}$$

Now, Theorem 3.4 a) and Lemma 3.3 a) imply

$$\begin{aligned} \Phi'(x^*) &= M(f(x^*)) + M'(f(x^*))\, x^* \, \nabla f(x^*)^\top \\ &= M(\lambda^*) + M'(\lambda^*)\, x^* \, \nabla f(x^*)^\top. \end{aligned} \tag{3.17}$$

Hence, combining the latter with Lemma 3.6, Lemma 3.3 a) and c), Theorem 3.4 a), and (3.15)–(3.16) gives, for any $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$,

$$\begin{aligned} \Phi'(x^*)x &= \Phi(x) + M'(\lambda^*)\, x^* \, \nabla f(x^*)^\top x \\ &= \Phi(x) - \frac{1}{x^* M'(\lambda^*) x^*} M'(\lambda^*) x^* \left(M(\lambda^*) x^* + M(\lambda^*)^\top x^* \right)^\top x \\ &= \Phi(x) - \frac{1}{x^* M'(\lambda^*) x^*} M'(\lambda^*) x^* \left(\Phi(x^*)^\top x + \Phi(x)^\top x^* \right) \\ &= \Phi(x). \end{aligned} \tag{3.18}$$

Since (3.18) also yields $\Phi(x^*) = \Phi'(x^*) x^*$, the statement of the lemma is true. □

Theorem 3.8. *In the setting of Lemma 3.7, there exists a continuous mapping $H : \mathbb{R}^n \to \mathbb{R}^n$ that is ostensibly affine near x^* and coincides with Φ on the nonempty and closed set $x^* + \varepsilon\mathbb{B}$ (with $\varepsilon > 0$ from Lemma 3.7).*

Proof. The existence of a continuous extension $H : \mathbb{R}^n \to \mathbb{R}^n$ of mapping Φ is a consequence of Tietze's extension theorem, see [10, Theorem 9.23] for instance. Due to Theorem 3.4 a), H is therefore continuously differentiable near x^*, hence strictly differentiable at x^*. Moreover, from Lemma 3.7 we now obtain that $H(x) = H(x^*) + H'(x^*)(x - x^*)$ is valid for all $x \in X^* \cap (x^* + \varepsilon\mathbb{B})$. It remains to show that $F^{-1}(0) \cap (x^* + \varepsilon\mathbb{B}) = X^* \cap (x^* + \varepsilon\mathbb{B})$ holds, with F defined according to (1.5), i.e., $F(x) = \min\{x, H(x)\}$ for $x \in \mathbb{R}^n$. This, however, follows immediately from Theorem 3.4 b) since H coincides with Φ on $x^* + \varepsilon\mathbb{B}$. Hence, H is ostensibly affine near x^*. □

Corollary 3.9. *In the setting of Lemma 3.7, let $H : \mathbb{R}^n \to \mathbb{R}^n$ be the continuous mapping determined by Theorem 3.8 and $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined according to* (1.5). *Then, the following statements are equivalent:*

a) *F provides a local error bound at x^*.*
b) *$T_{X^*}(x^*) = \mathfrak{D}F(x^*)$.*
c) *There exists some $\varepsilon > 0$ such that, for each $x \in x^* + \varepsilon\mathbb{B}$,*

$$\min\left\{ x, M(\lambda^*)x - \frac{(x^*)^\top \left(M(\lambda^*) + M(\lambda^*)^\top \right) x}{(x^*)^\top M'(\lambda^*) x^*} \cdot M'(\lambda^*)\, x^* \right\} = 0 \tag{3.19}$$

implies $(x, \lambda^*) \in \mathrm{SOL}$.

Proof. To apply Theorem 2.13 for the particular mapping H from Theorem 3.8, we first note that, by the latter theorem, H is ostensibly affine near x^*. Statement a) of the present corollary is statement a) of Theorem 2.13. By Lemma 3.5 and Lemma 2.4, we see that, for the concrete mapping H, statement b) of Theorem 2.13 reduces to statement b) of the present corollary. To verify that statement c) of Theorem 2.13 for the current mapping H is statement c) of the corollary, let us consider an arbitrarily chosen $x \in x^* + \varepsilon\mathbb{B}$ (with $\varepsilon > 0$ from Theorem 3.8). Then, by Theorem 3.8, $x \in L_H(x^*)$ is equivalent to $\min\{x, \Phi(x^*) + \Phi'(x^*)(x - x^*)\} = 0$. From (3.18), we have $\Phi(x^*) = \Phi'(x^*)x^*$. This, (3.17), and parts a) and c) of Lemma 3.3 yield

$$\Phi(x^*) + \Phi'(x^*)(x - x^*) = M(\lambda^*)x - \frac{(x^*)^\top \left(M(\lambda^*) + M(\lambda^*)^\top\right) x}{(x^*)^\top M'(\lambda^*) x^*} \cdot M'(\lambda^*)\, x^*.$$

Hence, $x \in L_H(x^*)$ in c) of Theorem 2.13 coincides with (3.19). Moreover, $x \in F^{-1}(0)$ in part c) of this theorem is nothing else than $\min\{x, H(x)\} = 0$. Due to $x \in x^* + \varepsilon B$, the latter is equivalent to $\min\{x, \Phi(x)\} = 0$ and, by Theorem 3.4, to $x \in X^*$. Then, on the one hand, $(x, \lambda^*) \in \mathrm{SOL}$ immediately implies $x \in X^*$ and thus, $\min\{x, \Phi(x)\} = 0$. On the other hand, if $\min\{x, \Phi(x)\} = 0$ (or equivalently $x \in X^*$), then (3.11) in Lemma 3.6 yields $0 = \min\{x, M(\lambda^*)x\}$, i.e., $(x, \lambda^*) \in \mathrm{SOL}$. Hence, for the mapping H given by Theorem 3.8, statement c) of Theorem 2.13 is nothing else than statement c) in the corollary. This completes the proof. □

By Theorem 3.4 a) and Theorem 3.8, we observe that $F(x) = \min\{x, M(f(x))x\}$ holds for all x near x^*. However, in general, it can be difficult to compute function values of the implicit function f given by Lemma 3.3. Nonetheless, in the case of a symmetric linear EiCP, $f(x)$ can be computed easily, see Section 3.4 below.

3.3. **Discussion of assumptions.** In this section, we have used Assumptions 1–2 and the assumption that $|\mathcal{M}| < \infty$. We demonstrate by examples that none of these assumptions implies one of the others.

Example 3.10. Let us consider $M(\lambda) := \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$. For $\tilde{x} := (0, 1)$, we find that $(\tilde{x}, \lambda) \in \mathrm{SOL}$ for all $\lambda \in \mathbb{R}$. Thus, $|\mathcal{M}| = \infty$. Moreover, for $\hat{x} := (1, 0)$, we see that the pair $(\hat{x}, 0) \in \mathrm{SOL}$ fulfills Assumption 2. Finally, we notice that $0 \in S(x)$ for all $x \in X^*$. Since the function $x \mapsto s(x) = 0$ is continuous at any $x \in X^*$, Lemma 3.2 implies that Assumption 1 is fulfilled at any pair $(x, 0)$ with $x \in X^*$. Therefore, in general, neither Assumption 2 nor Assumption 1 guarantees $|\mathcal{M}| < \infty$. □

Example 3.11. For $M(\lambda) := \lambda^2$, we have $\mathrm{SOL} = \mathbb{R}_+ \setminus \{0\} \times \{0\}$. Hence, $|\mathcal{M}| = 1$, while no element of SOL satisfies Assumption 2. Because $S(x) = \{0\}$ for all $x \in X^*$ and since the function $x \mapsto s(x) = 0$ is continuous at any $x \in X^*$, Lemma 3.2 implies that each pair $(x, 0)$ with $x \in X^*$ fulfills Assumption 1. Therefore, in general, Assumption 1 does not imply Assumption 2. □

Example 3.12. For

$$M(\lambda) := \sin(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $0 \in \Lambda^*$. Hence, $X^* = \mathbb{R}^2_+ \setminus \{0\}$ follows. For $\hat{x} := (0,1)$, we find that $(\hat{x}, \pi) \in \mathrm{SOL}$ and

$$\hat{x}^\top M'(\pi)\hat{x} = \cos(\pi) \neq 0.$$

Thus, the pair $(\hat{x}, \pi)$ fulfills Assumption 2 and, interestingly, also the strict complementarity condition $\hat{x}_i + (M(\pi)\hat{x})_i > 0$ for $i = 1, 2$. However, we show now that Assumption 1 is violated there. Let us assume the contrary, i.e., that Assumption 1 is satisfied at the pair $(\hat{x}, \pi)$. For any $\varepsilon \in (0, \pi)$, we can pick $x = (x_1, x_2) \in X^* \setminus \{\hat{x}\}$ sufficiently close to $\hat{x}$ such that

$$x_1 > 0 \quad \text{and} \quad x_2 > 0. \tag{3.20}$$

Then, according to Assumption 1, Lemma 3.2 shows that there is some $c = c(x) \in [\pi - \varepsilon, \pi + \varepsilon]$ satisfying $(x, c) \in \mathrm{SOL}$, or equivalently, the system

$$\min\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} cx_2 \\ cx_1 + \sin(c)x_2 \end{pmatrix} \right\} = 0 \tag{3.21}$$

must be fulfilled. According to (3.20), however, the first equality in (3.21) implies $c = 0$, a contradiction. Therefore, Assumption 1 can not be satisfied at $(\hat{x}, \pi)$ and hence, Assumption 2 does not imply Assumption 1 in general. ☐

3.4. Application to symmetric linear Eigenvalue complementarity problems.

In this subsection, we assume that the EiCP is linear, i.e., mapping M is given by

$$M(\lambda) = \lambda B - A,$$

where $A, B \in \mathbb{R}^{n \times n}$. For the remainder, B is assumed to be positive definite. The latter property is quite common and guarantees that the linear EiCP has a solution [28, Section 1]. When needed, the symmetry of A and B is assumed later on. For such symmetric linear EiCPs, we will show that there is some $c > 0$, so that $\|F(x)\|$ is an upper bound for $c\,\mathrm{dist}[x, X^*]$, if x is sufficiently close to x^* and $x^* \in X^*$ is arbitrary but fixed.

Lemma 3.13. *It holds that* $|\mathcal{M}| < \infty$.

Proof. Similar to [36, Proposition 3], one can show that $|\Lambda^*| < \infty$. Thus, $|\mathcal{M}| < \infty$ follows. ☐

The next result can be found in [31] and elsewhere.

Lemma 3.14. *For all* $(x, \lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}$, *the equation* $x^\top M(\lambda)x = 0$ *is valid if and only if*

$$\lambda = \frac{x^\top A x}{x^\top B x}.$$

With Lemma 3.3 d) in mind, the previous lemma implies that the implicit function f in Sections 3.1–3.2 for the linear EiCP is given by

$$f(x) = \frac{x^\top A x}{x^\top B x} \tag{3.22}$$

for all x near x^*. Thus, Lemma 3.14 also implies that a complementary eigenvalue λ^* relative to the complementary eigenvector x^* is unique with $\lambda^* = f(x^*)$.

Lemma 3.15. *Assumptions 1 and 2 are satisfied at* (x^*, λ^*).

Proof. According to Lemma 3.14, we observe that

$$S(x) = \left\{ \frac{x^\top Ax}{x^\top Bx} \right\} \quad \text{for all } x \in X^*. \tag{3.23}$$

The function $s : X^* \to \Lambda^*$, defined by $s(x) := (x^\top Ax)/(x^\top Bx)$, is continuous at x^* and, by (3.23), fulfills $s(x) \in S(x)$ for all $x \in X^*$ (near x^*). Thus, Lemma 3.2 implies that Assumption 1 is satisfied at $(x^*, \lambda^*) = (x^*, s(x^*))$. As B is positive definite, we obtain $(x^*)^\top M'(\lambda^*)x^* = (x^*)^\top Bx^* > 0$, i.e., (x^*, λ^*) satisfies Assumption 2. $\square$

Lemma 3.16. *The continuous mapping* $H : \mathbb{R}^n \to \mathbb{R}^n$*, defined by*

$$H(x) := \left(\frac{x^\top Ax}{x^\top Bx} B - A \right) x \quad \textit{for } x \in \mathbb{R}^n \setminus \{0\} \quad \textit{and} \quad H(0) := 0, \tag{3.24}$$

is ostensibly affine near x^**, i.e., there is* $\varepsilon > 0$ *with* $H(x) = H(x^*) + H'(x^*)(x - x^*)$ *being satisfied for each* $x \in F^{-1}(0) \cap (x^* + \varepsilon\mathbb{B}) = X^* \cap (x^* + \varepsilon\mathbb{B})$*, where* F *is defined according to* (1.5).

Proof. The continuity of H can be shown in a standard way. Furthermore, Lemma 3.14 implies $F^{-1}(0) = X^* \cup \{0\}$. Thus, the fact that H is ostensibly affine near x^* for some $\varepsilon > 0$ is a consequence of (3.22) and Lemmas 3.7, 3.13, and 3.15. $\square$

In the remainder, the mapping H is defined by (3.24).

Theorem 3.17. *If the matrices* A *and* B *are symmetric, then* $T_{X^*}(x^*) = \mathfrak{D}F(x^*)$.

Proof. We notice that $T_{X^*}(x^*) = T_{F^{-1}(0)}(x^*)$. Then, due to (2.3), it suffices to show that $\mathfrak{D}F(x^*) \subset T_{X^*}(x^*)$. Because of Lemma 3.13, Lemma 3.5, and Lemma 2.4, X^* is T-conical near x^*. Evidently, H is continuously differentiable near $x^* \neq 0$. Therefore, to show $\mathfrak{D}F(x^*) \subset T_{X^*}(x^*)$, we are going to apply Lemma 2.7.

If, for some $d \in \mathfrak{D}F(x^*)$, (2.4) is satisfied for all $i \in \{1, \dots, n\}$ and all $t \geq 0$ sufficiently small, then Lemma 2.7 yields $d \in T_{X^*}(x^*)$. Therefore, let us assume that there is $d \in \mathfrak{D}F(x^*)$ with $\|d\| = 1$ such that

$$J := \big\{ i \mid H_i(x^*) = H_i'(x^*)d = 0 \big\} \neq \varnothing. \tag{3.25}$$

As needed in Lemma 2.7, we will show that $H_i(x^* + td) = 0$ holds for all $i \in J$ and all $t \in [0, \tau]$ with $\tau > 0$ sufficiently small. Then, $d \in T_{X^*}(x^*)$ follows.

As first step, we show that there exists some $\tau \in (0, \|x^*\|)$ such that

$$f(x^* + td) = \lambda^* \quad \text{for all } t \in [0, \tau], \tag{3.26}$$

where the implicit function f (see Lemma 3.3) takes the form (3.22). Therefore, the function $\omega(t) := f(x^* + td)$ can be written as

$$\omega(t) = \frac{a_0 + ta_1 + t^2a_2}{b_0 + tb_1 + t^2b_2} \quad \text{for } t \in [0, \|x^*\|), \tag{3.27}$$

where $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$ are constants satisfying $b_0 > 0$, $b_2 > 0$, and $b_0 + tb_1 + t^2b_2 > 0$ for all $t \in [0, \|x^*\|)$ since B is positive definite. Now, (3.27) yields

$$\nabla f(x^* + td)^\top d = \omega'(t) = \frac{p'(t)q(t) - p(t)q'(t)}{q^2(t)} \tag{3.28}$$

and further

$$d^\top \nabla^2 f(x^* + td)d = \omega''(t) = \frac{p''(t)q(t) - p(t)q''(t)}{q^2(t)} - 2\omega'(t)\frac{q'(t)}{q(t)}, \tag{3.29}$$

where the functions $p, q : [0, \|x^*\|) \to \mathbb{R}$ are defined by

$$p(t) := a_0 + ta_1 + t^2 a_2 \quad \text{and} \quad q(t) := b_0 + tb_1 + t^2 b_2.$$

With the index sets

$$J_0 := \{i \mid H_i(x^*) = x_i^* = 0\}, \quad J_1 := \{i \mid x_i^* > 0\}, \quad \text{and} \quad J_2 := \{i \mid H_i(x^*) > 0\},$$

Lemma 2.6 provides

$$\begin{aligned} \min\{d_i, H_i'(x^*)d\} &= 0 \quad \text{for } i \in J_0, \\ H_i'(x^*)d &= 0 \quad \text{for } i \in J_1, \\ d_i &= 0 \quad \text{for } i \in J_2. \end{aligned} \tag{3.30}$$

Now, taking into account the symmetry of matrices A and B, Lemma 3.3 a) and c), and Theorem 3.4 a), we observe that

$$\nabla f(x^*) = -\frac{2}{(x^*)^\top B x^*}(\lambda^* B - A)x^* = -\frac{2}{(x^*)^\top B x^*} H(x^*). \tag{3.31}$$

Thus, it holds that $(\nabla f(x^*))_i = 0$ for all $i \in J_0 \cup J_1$. Therefore, with (3.30) in mind,

$$\nabla f(x^*)^\top d = \sum_{i \in J_0 \cup J_1} (\nabla f(x^*))_i d_i + \sum_{i \in J_2} (\nabla f(x^*))_i d_i = 0 \tag{3.32}$$

follows. Since

$$\nabla f(x) = -\frac{2}{x^\top B x}(f(x)B - A)x$$

is valid for all x close to x^*, short computations and $\beta := (x^*)^\top B x^*$ give

$$d^\top \nabla^2 f(x^*)d = \frac{2}{\beta^2}\left(2d^\top H(x^*)(Bx^*)^\top d - \beta\, d^\top H'(x^*)d\right). \tag{3.33}$$

From (3.30), we get $d^\top H'(x^*)d = 0$. Therefore, (3.31)–(3.33) yield

$$d^\top \nabla^2 f(x^*)d = \frac{4}{\beta^2}(Bx^*)^\top d d^\top H(x^*) = 0. \tag{3.34}$$

Because $\lambda^* = f(x^*) = a_0/b_0 = p(0)/q(0)$, a combination of (3.28) for $t = 0$ and (3.32) yields $a_1 = \lambda^* b_1$. Moreover, using (3.29) for $t = 0$, (3.32), and (3.34), we get $a_2 = \lambda^* b_2$. Hence, we observe that $p(t) = \lambda^* q(t)$ is valid for all $t > 0$ small enough. Thus, the claim in (3.26) is true for some $\tau \in (0, \|x^*\|)$.

As second step, we show that

$$H_i(x^* + td) = 0 \quad \text{for all } (i, t) \in J \times [0, \tau]. \tag{3.35}$$

For any $i \in J$, let us consider the function $h_i : [0, \tau] \to \mathbb{R}$ with $h_i(t) := H_i(x^* + td)$. Evidently, $i \in J$ yields

$$h_i'(0) = H_i'(x^*)d = 0. \tag{3.36}$$

Moreover, (3.26) implies $h_i(t) = \big((\lambda^* B - A)(x^* + td)\big)_i$ for all $t \in [0, \tau]$. Thus, h_i is affine on $[0, \tau]$. Finally, (3.36) and $i \in J$ give $h_i(t) = h_i(0) = H_i(x^*) = 0$ for all $t \in [0, \tau]$. Therefore, (3.35) is valid and, hence, Lemma 2.7 implies $d \in T_{X^*}(x^*)$. $\square$

Corollary 3.18. *Suppose that the matrices A and B are symmetric. Then, for any $x^* \in X^*$, the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F(0) := 0$ and*

$$F(x) := \min \left\{ x, \frac{x^\top Ax}{x^\top Bx} Bx - Ax \right\} \quad \text{for } x \neq 0$$

provides a local error bound at x^*.

Proof. The assertion follows from Lemmas 3.13–3.16, Theorem 3.17, and Corollary 3.9. □

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