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ON EXISTENCE OF SOLUTIONS FOR VECTOR EQUILIBRIUM PROBLEMS VIA COERCING FAMILY

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Abstract. In this paper, some existence theorem for a solution of vector equilibrium problems in real Hausdorff topological vector spaces by using coercing family are established. The compactness of the solution sets of the vector equilibrium problems under suitable conditions is investigated. Upper semicontinuity of the solution set mapping is shown. Also, the lower semicontinuity of the solution set mapping of the vector equilibrium problem on some residual sets is provided.

1. Introduction

The well known KKM (Knaster-Kuratowski-Mazurkiewicz) theorem was first presented in 1929 in finite-dimensional Euclidean space [22]. This principle is exerted in many theorems with finite intersection property. In 1961, the KKM theorem was extended to an infinite dimensional space by Key Fan [12]. After that, some other generalizations of KKM were proposed by Brézis-Nirenberg-Stampacchia [7] and Dugundji-Granas [11] and many other researchers [3, 10, 17, 21, 23, 26, 30]. The importance of KKM principle and its extensions are determined by wide applications in nonlinear analysis such as variational inequalities, fixed point theory, Nash equilibrium problem, equilibrium problem, optimization and so on. For various applications of KKM principle or of its generalizations, you can see [10, 18, 23, 25]. One of the most important applications of KKM theorem is in equilibrium problems. Equilibrium problems are an inequality that involves a functional and must be solved on a certain set. This type of problems was introduced by Blum and Oettli in 1994 to incorporate some problems in nonlinear analysis such as variational inequalities, fixed point theory, Nash equilibrium problem, optimization and etc [6]. Later on, many authors tried to generalize and improve this developing area of analysis ([1, 4, 5, 8, 9, 13]). Authors in [14] established the lower semicontinuity of the solution mapping of the vector equilibrium problems in the setting of topological vector spaces without using a continuity assumption. In 2013, Mahato and Nahak obtained the existence results for mixed equilibrium problems in a reflexive Banach space [24]. Authors in [27] proved some existence results for combinations of a vector equilibrium problem and a vector variational inequality problem in non-compact setting. Recently, in [20], two shrinking extragradient algorithms are presented to find the solution sets of equilibrium problems for pseudomonotone bifunctions and to

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find the sets of fixed points of quasi nonexpansive mappings in a real Hilbert space. Recently, some authors are interested in studying Hadamard manifolds by using the equilibrium problems, see [28] and the references therein. Some authors provided some existence theorems for a solution set of equilibrium problems via scalarization method, see [15] and the references therein. Moreover, some authors introduced two extragradient algorithms for solving equilibrium problems in Hadamard manifolds. Furthermore, they proved that any sequence generated by the proposed algorithms converges to a solution of the equilibrium problems under suitable assumptions [19]. Authors in [29] studied extended mixed vector equilibrium problems in Hausdorff topological vector spaces and by using generalized KKM-Fan theorem proved some existence results in noncompact domain.

The purpose of this paper is to study vector equilibrium problem in the real Hausdorff topological vector spaces to get some existence results. We first introduce the concept of coercing family and recall the generalized version of KKMF principle. Next, by using this new version of KKMF principle, we investigate some existence results for solution of vector equilibrium problems. The compactness of the solution sets of the vector equilibrium problems under suitable conditions is investigated. Upper semicontinuity of the solution set mapping is shown. Also the lower semicontinuity of the solution set mapping of the vector equilibrium problem on some residual sets is provided.

2. Preliminaries

Throughout the paper, unless otherwise specified, let X and Y be two real Hausdorff topological vector spaces, K a nonempty closed convex subset of X, $\{C(x) : x \in K\}$ a family of proper convex cones with nonempty interior where $C : K \to 2^Y$, 2^Y denotes the set of all subsets of Y and $Y^* = L(Y, \mathbb{R})$ the topological dual of Y. We denote the duality pairing between X^* and X, with $\langle ., . \rangle$ and the open line segment joining between $x, y \in K$ by $]x, y[$. Let A be a nonempty subset of X. The family of all nonempty finite subsets of A is denoted by $\mathcal{F}(A)$. In topological vector space X, let *int*, *cl*, and *co* denote the interior, closure and convex hull, respectively.

A nonempty subset P of Y is called a cone if $\lambda P \subset P$, for all $\lambda \geq 0$, and it is convex if $\lambda P + (1 - \lambda)P \subset P$, for all $\lambda \in [0, 1]$. Moreover, it is a convex cone when (i) $P + P = P$, (ii) $\lambda P \subset P$, for all $\lambda \geq 0$. The cone P is said to be pointed whenever $P \cap -P = \{0\}$. If P is a pointed convex cone of Y, then P induces a partial ordering on Y (in this case the pair (Y, P) is called an ordered topological vector space) as follows:

$$x \leq y \Leftrightarrow y - x \in P.$$

Let K be a nonempty convex subset of X and K_0 a subset of K. A multi-valued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if

$$coA \subseteq \bigcup_{x \in A} \Gamma(x), \quad \forall A \in \mathcal{F}(K_0).$$

Let K be a nonempty subset of X and $f : K \times K \to Y$ a bifunction. The vector equilibrium problem (denoted by VEP) consists in finding $\overline{x} \in K$ such that

$$f(\overline{x}, y) \notin -int\ C(\overline{x}), \quad \forall y \in K.$$

Also the strong vector equilibrium problem (SVEP) consists in finding $\overline{x} \in K$ such that

$$f(\overline{x}, y) \notin -C(\overline{x})\backslash\{0\}, \quad \forall y \in K,$$

It is obvious that the solution set of SVEP is a subset of the solution set of VEP.

Classical KKM (Knaster-Kuratowski-Mazurkiewicz) theorem as follows is well known in nonlinear analysis. It has a crucial role in the proof of many problems with finite intersection property in the setting of finite-dimensional spaces.

Theorem 2.1 ([22]). *Let $S_n = co\{x_1, \ldots, x_{n+1}\}$ be a closed n-simplex, and $F_1, \ldots, F_{n+1}$ be $n+1$ closed subsets of S_n. If for all set $\{i, j, \ldots, l\} \subset \{1, \ldots, n+1\}$, we have $co\{x_i, x_j, \ldots, x_l\} \subset F_i \cup F_j \cup \cdots \cup F_l$, then $\bigcap_{i=1}^{n+1} F_i \neq \emptyset$.*

The generalization of classical KKM theorem to infinite-dimensional spaces is given by Key Fan [12] as follows.

Lemma 2.2 ([12]). *Let K be an arbitrary set in a Hausdorff topological vector space X and $F : K \to 2^X$ a multivalued mapping with closed values. If F is a KKM mapping and $F(x)$ is a compact set for at least one $x \in X$, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

A new version of Ky Fan's lemma is presented by Mechaiekh and his colleagues [3] by using the notion of coercing family instead of the compactness in Lemma 2.2. We apply this consequence in our work to establish some existence theorems for VEP and SVEP. Before stating a new version of Ky Fan's lemma, we need to express the concept of coercing covering.

In this article we are going to work with the compactly closed sets which are weaker notion than the closed sets. We recall that a subset A of a topological space E is compactly closed if for any compact set D of E, the set $A \cap D$ is closed in E.

Definition 2.3 ([3]). Consider a subset A of a topological vector space Y. A family $\{C_i, K_i\}_{i \in I}$ of pairs of sets is said to be coercing for a map $G : A \to 2^Y$ if and only if:

(i) for each $i \in I$, C_i is contained in a compact convex subset of A, and K_i is a compact subset of Y;
(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subset C_k$;
(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} G(x) \subset K_i$.

Here, I is an index set.

Theorem 2.4 ([3]). *Let X be a topological vector space, C a convex subset of X, D a nonempty subset of C and $F : D \to 2^C$ a KKM mapping with compactly closed values in C. If F admits a coercing family, then $\bigcap_{x \in D} F(x) \neq \emptyset$.*

Definition 2.5 ([2]). Let X and Y be topological spaces and $G : X \to 2^Y$ a multivalued mapping.

(i) G is called lower semicontinuous (l.s.c.) at $\bar{w} \in X$ if for any open set $V \subset Y$ with $V \cap G(\bar{w}) \neq \emptyset$, there exists a neighborhood $N(\bar{w})$ of $\bar{w}$ such that $G(w) \cap V \neq \emptyset$, for all $w \in N(\bar{w})$.
(ii) G is called upper semicontinuous (u.s.c.) at $\bar{w} \in X$ if for any open set $V \subset Y$ with $G(\bar{w}) \subset V$, there exists a neighborhood $N(\bar{w})$ of $\bar{w}$ such that $G(w) \subset V$, for all $w \in N(\bar{w})$.

We say that $G(\cdot)$ is l.s.c. (resp. u.s.c.) on $W \subset X$ if and only if it is l.s.c. (resp. u.s.c.) at each $\bar{w} \in W$. $G(\cdot)$ is said to be continuous on W if and only if it is both l.s.c. and u.s.c. on W.

Proposition 2.6 ([2]). (i) *G is l.s.c. at $\bar{w}$ if and only if for any net $\{w_\alpha\} \subset W$ with $w_\alpha \to \bar{w}$ and any $\bar{x} \in G(\bar{w})$, there exists $x_\alpha \in G(w_\alpha)$ such that $x_\alpha \to \bar{x}$.*
(ii) *If G has compact values, then G is u.s.c. at $\bar{w}$ if and only if for any net $\{w_\alpha\} \subset W$ with $w_\alpha \to \bar{w}$ and any $x_\alpha \in G(w_\alpha)$, there exist a point $\bar{x} \in G(\bar{w})$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \to \bar{x}$.*

A topological space X is said to be a Baire space if the following condition holds: given any countable collection $\{A_n\}_{n=1}^{\infty}$ of the closed subsets of X, each of them has empty interior in X, their union $\cup A_n$ also has empty interior in X. A subset G of X is called residual if it contains a countable intersection of open dense subsets of X.

Lemma 2.7 ([16]). *(Baire category theorem) If X is a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Lemma 2.8 ([16]). *Let X be a Baire space, Y a metric space and $G : X \to 2^Y$ an upper semicontinuous mapping with compact values. Then, there exists a dense residual subset Q of X such that G is lower semicontinuous at each $x \in Q$.*

3. Main result

In the next result, an existence theorem for a solution of VEP is established.

Theorem 3.1. *Let K be a nonempty convex set, and $f : K \times K \to Y$ satisfying the following conditions:*

(i) *for any $x \in K$, $f(x, x) = 0$,*
(ii) *for any compact subset W of K and for any $y \in K$, the set $\{x \in W : f(x, y) \notin -intC(x)\}$ is closed in K,*
(iii) *for any $x \in K$, the set $\{y \in K : f(x, y) \in -intC(x)\}$ is convex,*
(iv) *there exist compact subset B and compact convex subset D of K such that $\forall x \in K \backslash B \ \exists y \in D$ where $f(x, y) \in -intC(x)$.*

Then, the set $\{\bar{x} \in K : f(\bar{x}, y) \notin -intC(\bar{x}),\ \forall y \in K\}$ is nonempty and compact in K.

Proof. We define $F : K \to 2^K$ as follows

$$F(y) = \{x \in K : f(x, y) \notin -intC(x)\}.$$

By (ii), F has compactly closed values in K. We claim that F is a KKM mapping. Indeed, if it is false, then there exist elements $y_1, y_2, \dots, y_n$ of K and $z \in co(\{y_1, y_2, \dots, y_n\})$ such that $z \notin \bigcup_{i=1}^{n} F(y_i)$ and so by the definition of F, we get

$$f(z, y_i) \in -intC(z), \text{ for } i = 1, \dots, n.$$

Thus, by (iii), we have $f(z, z) \in -intC(z)$ which is contradicted by (i). By (iv), $\{(D, B)\}$ is a coercing and so by Theorem 2.4, we have $\bigcap_{x\in K} F(x) = \{x \in K : f(x, y) \notin -intC(x),\ \forall y \in K\}$ is nonempty. Moreover, by (iv), $\bigcap_{x\in K} F(x) \subset B$ and hence $\bigcap_{x\in K} F(x) = \bigcap_{x\in K} F(x) \cap B$ which is compact in K, by (ii). This completes the proof. □

The following result is an existence theorem for strong vector equilibrium problem. It can be also viewed as an existence theorem for VEP, because the solution set of strong vector equilibrium problem is a subset of the solution set of vector equilibrium problem.

Theorem 3.2. *Let K be a nonempty convex and close set and $f : K \times K \to Y$ a continuous mapping in the first argument. Suppose for each $x \in K$, there exists a convex cone $\widetilde{C}(x)$ such that $-C(x) \setminus \{0\} \subseteq -int\widetilde{C}(x)$. Assume that f and $\widetilde{C}$ satisfy the conditions (i)-(iv) of Theorem 3.1. Then, the solution set of SVEP, that is the set $\{\bar{x} \in K : f(\bar{x}, y) \notin -C(\bar{x}) \setminus \{0\},\ \forall y \in K\}$ is nonempty, and moreover it is relatively compact set of K if the following condition holds*

(v) *there are compact subset B and compact convex subset D of K such that for all $x \in K\setminus B$ there exists $y \in D$ where $f(x, y) \in -C(x) \setminus \{0\}$.*

Further, if the graph of the multivalued mapping $x \to H(x) = Y \setminus (-C(x) \setminus \{0\})$ is closed, then the solution set of SVEP is a compact subset of K.

Proof. Since f and $\widetilde{C}$ satisfy the assumptions of Theorem 3.1, then there exists $\bar{x} \in K$ such that $f(\bar{x}, y) \notin -int\widetilde{C}(\bar{x})$, for all $y \in K$. Then, it follows from $-C(\bar{x}) \setminus \{0\} \subseteq -int\widetilde{C}(\bar{x})$ that

$$f(\bar{x}, y) \notin -C(\bar{x}) \setminus \{0\}, \quad \forall y \in K.$$

Hence, $\bar{x}$ is a solution of SVEP. It follows from (v) that the solution set of SVEP is a subset of the compact set B of K and so it is relatively compact set of K. Finally, if for all $y \in K$, $f(x_\alpha, y) \notin -C(x_\alpha)\setminus\{0\}$ and $x_\alpha \to x$, then it follows from the closedness of K that $x \in K$ and also $(x_\alpha, f(x_\alpha, y))$ belongs to the graph of the multivalued mapping H and f is continuous in the first variable. Therefore, we get $f(x, y) \in H(x) = Y\setminus(-C(x)\setminus\{0\})$. This means that x is a solution of SVEP and so the solution set of SVEP is a closed subset of B. Consequently, the solution set of SVEP is a compact subset of K and the proof is completed. □

The next result provides an existence theorem for the solutions of SVEP with new hypothesis.

Theorem 3.3. *Let K be a nonempty closed convex set, and $f : K \times K \to Y$ satisfying the following conditions:*

(i) *for any $x \in K$, $f(x, x) = 0$,*

(ii) *f is continuous in the first argument and convex in the second argument,*

(iii) *the multivalued mapping* $x \to Y \setminus (-C(x) \setminus \{0\})$ *is closed,*
(iv) *there exist compact subset* B *and compact convex subset* D *of* K *such that* $\forall x \in K \backslash B,\ \exists y \in D$ *where* $f(x, y) \in -C(x)$.

Then, the solution set of SVEP is nonempty and compact.

Proof. Let $F : K \to 2^K$ be a multivalued mapping defined by

$$F(y) = \{x \in K \ : \ f(x, y) \notin -C(x) \setminus \{0\}\}. \tag{3.1}$$

It is clear that $F(y)$ is closed for any $y \in K$. Because if $\{x_\alpha\}$ is an arbitrary net in $F(y)$ such that $x_\alpha \to x$, then it follows from $x_\alpha \in F(y)$ that $f(x_\alpha, y) \notin -C(x_\alpha) \setminus \{0\}$ and hence $f(x_\alpha, y) \in Y \setminus (-C(x_\alpha) \setminus \{0\})$. Since f is continuous in the first variable, we have $f(x_\alpha, y) \to f(x, y)$. Hence, by (iii), $f(x, y) \notin -C(x) \setminus \{0\}$ which means that $x \in F(y)$ and so $F(y)$ is closed. We claim that F is a KKM mapping. Suppose on the contrary that F is not a KKM mapping. So, there exist $y_1, \ldots, y_n$ in K and $z \in co\{y_i : 1 = 1, \ldots, n\}$ such that $z \notin \cup_{i=1}^n F(y_i)$ which implies that $f(z, y_i) \in -C(x)\backslash\{0\}$. There exists $\lambda_i \geq 0$ with $\sum \lambda_i = 1$ where $v = \sum_{i=1}^n \lambda_i y_i$. Since f is convex in the second argument, we have $\sum_{i=1}^n \lambda_i f(z, y_i) = f(z, z) \in -C(x) \setminus \{0\}$ which is a contradiction to assumption (i). By (iv), $\{(D, B)\}$ is a coercing family for F. Consequently, F satisfies all the assumptions of Theorem 2.4 and so $\bigcap_{x \in K} F(x) \neq \emptyset$. Hence, there exists $\bar{x} \in K$ such that

$$f(\bar{x}, y) \notin -C(\bar{x})\backslash\{0\}, \quad \forall y \in K.$$

Therefore, $\bar{x}$ is a solution of SVEP. Since the solution set of SVEP equals to $\bigcap_{x \in K} F(x)$, by (iv), it is a closed subset of B. Hence, we deduce that the solution set of SVEP is compact. This completes the proof. □

Let M be the collection of all mappings $f : K \times K \to Y$ satisfying the conditions (i)-(iv) of Theorem 3.1.

We say that a net (f_α) in M converges to $f \in M$ in the first variable if for any $y \in K$ and for any convergent net (x_α) to an element $x \in K$, $f_\alpha(x_\alpha, y) \to f(x, y)$. Denote that $S_K(f)$, the solution set of VEP, with respect to f, i.e., $S_K(f) = \{x \in K : f(x, y) \notin -intC(x), \quad \forall y \in K\}$.

In the next result the upper semicontinuity of the solution set mapping of VEP is considered.

Theorem 3.4. *Assume that* K *is a closed convex set in the Hausdorff topological space* X *and* M *is the set of all mappings* $f : K \times K \to Y$ *which satisfies Theorem 3.1. If the multivalued mapping* $W : K \to 2^Y$ *defined by* $W(x) = Y\backslash(-intC(x))$ *is closed, then the solution set mapping* $S_K : M \to 2^X$ *defined by* $f \to S_K(f)$ *is upper semicontinuous with compact values in* K.

Proof. Since, by Theorem 3.1, the values of the solution set mapping are compact, by Proposition 2.6 (ii), it is enough to show that the graph of the solution set mapping is closed. To verify the closedness of the graph of S_K, suppose $(f_\alpha, x_\alpha) \in G_r(S_K)$ and converges to (f, x). Thus, for all $y \in K$, we have $f_\alpha(x_\alpha, y) \to f(x, y)$. It follows from $x_\alpha \in S_K(f_\alpha)$, the closedness of the graph of W and $f_\alpha(x_\alpha, y) \to f(x, y)$ that $f(x, y) \in W(x) = Y\backslash(-intC(x))$. This completes the proof. □

Theorem 3.5. *Let M be a Baire space and X a meterizable topological vector space. If $W : K \to 2^Y$ defined by $W(x) = Y\backslash(-intC(x))$ is closed, then there exists a dense residual subset Q of M such that the solution set mapping $S_K : M \to 2^X$ is lower semicontinuous on Q.*

Proof. By Theorem 3.4, S_K is upper semicontinuous. Now the results follows from Lemma 2.8. □

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