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STRONG BI-METRIC REGULARITY IN AFFINE OPTIMAL CONTROL PROBLEMS

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Abstract. The paper presents new sufficient conditions for the property of strong bi-metric regularity of the optimality map associated with an optimal control problem which is affine with respect to the control variable (*affine problem*). The optimality map represents the system of first order optimality conditions (Pontryagin maximum principle), and its regularity is of key importance for the qualitative and numerical analysis of optimal control problems. The case of affine problems is especially challenging due to the typical discontinuity of the optimal control functions. A remarkable feature of the obtained sufficient conditions is that they do not require convexity of the objective functional. As an application, the result is used for proving uniform convergence of the Euler discretization method for a family of affine optimal control problems.

1. Introduction

Regularity properties of the system of first order necessary optimality conditions for optimization problems play a key role in qualitative analysis and reliable numerical treatment of such problems (see e.g. the books [1, 4, 7, 8]). For optimal control problems, the investigation of regularity properties of the map associated with the Pontryagin maximum principle (called further *optimality map*) was first initiated in [3], which deals with problems that satisfy the so-called coercivity condition. The latter, however is never fulfilled for problems in which both dynamics and cost are affine with respect to the control (further called *affine problems*). Results about strong metric sub-regularity of the optimality map for affine problems were obtained in the recent papers [9, 10]. The property of *strong metric regularity* (see e.g. [4, Chapter 3]) of the optimality map proved to be important for convergence and error estimates of numerical methods (discretizations, gradient projection, Newton method, etc.). However, more suitable for affine problems is a specific extension the strong metric regularity introduced in [12] under the name *Strong bi-Metric Regularity* (Sbi-MR). The present paper investigates this property for Lagrange-type

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affine optimal control problems of the form

$$\min\Big\{J(u) := \int_0^T [w(t,x(t)) + \langle s(t,x(t)),u(t)\rangle]\,\mathrm{d}t\Big\}, \tag{1.1}$$

subject to

$$\dot{x}(t) = a(t,x(t)) + B(t,x(t))u(t), \quad x(0) = x^0, \tag{1.2}$$

$$u(t) \in U, \quad t \in [0,T]. \tag{1.3}$$

Here the state vector $x(t)$ belongs to $\mathbb{R}^n$, the control function u has values $u(t)$ that belong to a given set U in $\mathbb{R}^m$ for almost every (a.e.) $t \in [0,T]$. Correspondingly, w is a scalar function on $[0,T]\times\mathbb{R}^n$, s is an m-dimensional vector function ($\langle\cdot,\cdot\rangle$ denotes the scalar product), a and B are vector-/matrix-valued functions with appropriate dimensions. The initial state x^0 and the final time $T > 0$ are fixed. The set of feasible control functions u, denoted in the sequel by $\mathcal{U}$, consists of all Lebesgue measurable and bounded functions $u : [0,T] \to U$. Accordingly, the state trajectories x, that are solutions of (1.2) for feasible controls, are absolutely continuous functions on $[0,T]$.

It is well known that the Pontryagin (local) maximum principle can be written in the form of a generalized equation

$$0 \in F(y), \tag{1.4}$$

where $y = (x(\cdot),u(\cdot),p(\cdot))$ encapsulates the state function $x(\cdot)$, the control function $u(\cdot) \in \mathcal{U}$, and the adjoint (co-state) function $p(\cdot)$, and the inclusion $0 \in F(y)$ represents the state equation, the co-state equation, and the maximization condition in the maximum principle (the last being the inclusion of the derivative of the associated Hamiltonian with respect to the control in the normal cone to $\mathcal{U}$ at $u(\cdot)$). The detailed definition of the mapping F in (1.4), called further *optimality map* is given in the next section.

In the next paragraphs we remind the definition of Sbi-MR in the form used in [10] and [11]. Let (Y,d_Y), (Z,d_Z), $(\tilde{Z},d_{\tilde{Z}})$ be metric spaces, with $\tilde{Z} \subset Z$ and $d_Z \le d_{\tilde{Z}}$ on $\tilde{Z}$.[1] Denote by $\mathbf{B}_Y(\hat{y};a)$, $\mathbf{B}_Z(\hat{z};b)$ and $\mathbf{B}_{\tilde{Z}}(\hat{z};b)$ the closed balls in the metric spaces (Y,d_Y), (Z,d_Z) and $(\tilde{Z},d_{\tilde{Z}})$ with radius $a>0$ or $b>0$ centered at $\hat{y}$ and $\hat{z}$, respectively.

Given a set-valued map $\Phi : Y \rightrightarrows Z$, $\operatorname{gph}\Phi := \{(y,z) \in Y\times Z :\ z \in \Phi(y)\}$ denotes the graph of Φ. The inverse map, $\Phi^{-1} : Z \rightrightarrows Y$, is the set-valued map defined as $\Phi^{-1}(z) := \{y \in Y : z \in \Phi(y)\}$.

Definition 1.1. The set-valued map $\Phi : Y \rightrightarrows Z$ is *strongly bi-metrically regular* (Sbi-MR) (with disturbance space $\tilde{Z}$) at $\hat{y} \in Y$ for $\hat{z} \in \tilde{Z}$ with constants $\kappa \ge 0$, $a > 0$ and $b > 0$, if $(\hat{y},\hat{z}) \in \operatorname{graph}(\Phi)$ and the following properties are fulfilled:

(i) the map $\mathbf{B}_{\tilde{Z}}(\hat{z};b) \ni z \mapsto \Phi^{-1}(z) \cap \mathbf{B}_Y(\hat{y};a)$ is single-valued;

(ii) for all $z, z' \in \mathbf{B}_{\tilde{Z}}(\hat{z};b)$

$$d_Y(\Phi^{-1}(z) \cap \mathbf{B}_Y(\hat{y};a), \Phi^{-1}(z') \cap \mathbf{B}_Y(\hat{y};a)) \le \kappa d_Z(z,z'). \tag{1.5}$$

[1] This inequality can be understood as $d_Z(z) \le c\, d_{\tilde{Z}}(z)$ for every $z \in \tilde{Z}$, where c is a constant.

We stress that the difference between this notion and the standard notion of strong metric regularity (see e.g. [4, Chapter 3]) is that the "disturbances" z have to belong to the smaller space, $\tilde{Z}$ (with the bigger distance), but the Lipschitz property in (ii) holds with respect to the smaller distance, d_Z, in the right-side of (1.5). A detailed explanation of the reasons for the appropriateness of this definition is given in [11, Introduction].

Sufficient conditions for more specific problems and some applications of the Sbi-MR property are presented in [10] and [11]. The main aim of the present paper is to obtain new, more general, sufficient conditions for Strong bi-Metric Regularity (Sbi-MR) of the optimality map F in an appropriate space setting. A new feature of these conditions is that they involve not only the second derivative of the associated Hamiltonian with respect to the control, but also its first derivative. Thanks to that, they may be also fulfilled for problems with a non-convex objective functional, which is a new founding in the optimal control context, in general.

We present the sufficient conditions for Sbi-MR in Section 2 and give a detailed proof in Section 3. In Section 4 we specialize these conditions to the case of affine problems with bang-bang solutions and give an example where they apply to a non-convex problem. As an application, in Section 5 we prove that the obtained sufficient conditions imply uniform first order convergence of the Euler discretization scheme when applied to affine problems that are close enough to a reference one. This result is of importance, for example, for the justification of Model Predictive Control methods applied to affine problems.

2. Sufficient conditions for strong bi-metric regularity

We will use the following standard notations. The euclidean norm and the scalar product in $\mathbb{R}^n$ (the elements of which are regarded as column-vectors) are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. The transpose of a matrix (or vector) E is denoted by $E^\top$. For a function $\psi:\mathbb{R}^p\to\mathbb{R}^r$ of the variable z we denote by $\psi_z(z)$ its derivative (Jacobian), represented by an $(r\times p)$-matrix. If $r=1$, $\nabla_z\psi(z)=\psi_z(z)^\top$ denotes its gradient (a vector-column of dimension p). Also for $r=1$, $\psi_{zz}(z)$ denotes the second derivative (Hessian), represented by a $(p\times p)$-matrix. For a function $\psi:\mathbb{R}^{p+q}\to\mathbb{R}$ of the variables (z,v), $\psi_{zv}(z,v)$ denotes its mixed second derivative, represented by a $(p\times q)$-matrix. The space $L^k([0,T],\mathbb{R}^r)$, with $k=1,2$ or $k=\infty$, consists of all (classes of equivalent) Lebesgue measurable r-dimensional vector-functions defined on the interval $[0,T]$, for which the standard norm $\|\cdot\|_k$ is finite. Often the specification $([0,T],\mathbb{R}^r)$ will be omitted in the notations. As usual, $W^{1,k}=W^{1,k}([0,T],\mathbb{R}^r)$ denotes the space of absolutely continuous functions $x:[0,T]\to\mathbb{R}^r$ for which the first derivative belongs to L^k. The norm in $W^{1,k}$ is defined as $\|x\|_{1,k}:=\|x\|_k+\|\dot{x}\|_k$. Moreover, $\mathbf{B}_X(x;r)$ will denote the ball of radius r centered at x in a metric space X.

Allover the paper we use the abbreviation

$$f(t,x,u)=a(t,x)+B(t,x)u,\qquad g(t,x,u)=w(t,x)+\langle s(t,x),u\rangle. \tag{2.1}$$

For problem (1.1)–(1.3) we make the following assumption.

Assumption (A1). The set U is convex and compact; the functions $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ have the form as in (2.1) and are two times differentiable in (t, x), and the derivatives are Lipschitz continuous[2].

Define the Hamiltonian associated with problem (1.1)–(1.3) as usual:

$$H(t, x, p, u) := g(t, x, u) + \langle p, f(t, x, u)\rangle, \quad p \in \mathbb{R}^n.$$

The local form of the Pontryagin maximum (here minimum) principle for problem (1.1)-(1.3) can be represented by the following optimality system for (x, u) and an absolutely continuous (here Lipschitz) function $p : [0, T] \to \mathbb{R}^n$: for a.e. $t \in [0, T]$

$$0 = -\dot{x}(t) + f(t, x(t), u(t)), \quad x(0) - x^0 = 0, \tag{2.2}$$

$$0 = \dot{p}(t) + \nabla_x H(t, x(t), p(t), u(t)), \quad p(T) = 0, \tag{2.3}$$

$$0 \in \nabla_u H(t, x(t), p(t), u(t)) + N_U(u(t)), \tag{2.4}$$

where the normal cone $N_U(u)$ to the set U at $u \in \mathbb{R}^m$ is defined in the usual way,

$$N_U(u) = \begin{cases} \{y \in \mathbb{R}^n \mid \langle y, v - u\rangle \le 0 \text{ for all } v \in U\} & \text{if } u \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

Assumption (A1) implies that there exists a number $M > 0$ such that for any $u \in \mathcal{U}$ the corresponding solution x of (2.2) and also the solution p of (2.3) exist on $[0, T]$ and

$$\max\{|x(t)|, |\dot{x}(t)|, |p(t)|, |\dot{p}(t)|\} \le M \quad \text{for a.e. } t \in [0, T]. \tag{2.5}$$

In what follows, $\bar{M}$ will be any number larger that M.

Let us introduce the metric spaces

$$Y := \{(x, p, u) \in W^{1,1} \times W^{1,1} \times L^1 : x(0) = x^0, \, p(T) = 0\} \cap (\mathbf{B}_{W^{1,\infty}}(0; \bar{M}))^2 \times \mathcal{U}.$$

and

$$Z := L^\infty \times L^\infty \times L^\infty \text{ and } \quad \tilde{Z} := L^\infty \times L^\infty \times W^{1,\infty} \subset Z.$$

The distances in these spaces are induced by norms, therefore we keep the norm-notations: for $y = (x, p, u) \in Y$

$$\|y\| := \|x\|_{1,1} + \|p\|_{1,1} + \|u\|_1$$

and for $z = (\xi, \pi, \rho)$ in Z or in $\tilde{Z}$, respectively,

$$\|z\|_Z := \|\xi\|_1 + \|\pi\|_1 + \|\rho\|_\infty, \qquad \|z\|_\sim := \|\xi\|_\infty + \|\pi\|_\infty + \|\rho\|_{1,\infty}.$$

Notice that Y is a complete metric space, thanks to the compactness of the set U.

Now, we define the set-valued mapping $F : Y \rightrightarrows Z$ as

$$F(y) := \begin{pmatrix} -\dot{x} + f(\cdot, x, u) \\ \dot{p} + \nabla_x H(\cdot, y) \\ \nabla_u H(\cdot, y) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ N_{\mathcal{U}}(u) \end{pmatrix}, \tag{2.6}$$

[2]The assumption of *global* Lipschitz continuity is made for convenience. Since the analysis in this paper is local (in a neighborhood of a reference trajectory $\hat{x}(\cdot)$ in the uniform metric), it can be replaced with *local* Lipschitz continuity.

where $N_{\mathcal{U}}(u)$ is the normal cone to the set $\mathcal{U}$ of admissible controls at u, considered as a subset of L^∞:

$$N_{\mathcal{U}}(u) := \begin{cases} \emptyset & \text{if } u \notin \mathcal{U} \\ \{v \in L^\infty \,:\, v(t) \in N_U(u(t)) \text{ for a.e. } t \in [0,T]\} & \text{if } u \in \mathcal{U}. \end{cases}$$

Notice that $F(Y) \subset Z$, and $\nabla_u H(\cdot, y) \in \tilde{Z}$ thanks to the affine structure of the problem, namely, the independence of $\nabla_u H(\cdot, y)$ of u.

With these definitions, the necessary optimality conditions (2.2)–(2.4) take the form

$$F(y) \ni 0, \tag{2.7}$$

therefore F is called *optimality map* associated with problem (1.1)–(1.3). The main result in this paper is a sufficient condition for Sbi-MR of the optimality mapping $F : Y \rightrightarrows Z$ with perturbation space $\tilde{Z}$. To do this we fix a reference solution $\hat{y} = (\hat{x}, \hat{p}, \hat{u})$. We mention that such always exists since on assumption (A1) problem (1.1)–(1.3) has a solution. To shorten the notations we skip arguments with "hat" in functions, shifting the "hat" on the top of the notation of the function, so that $\hat{f}(t) := f(t, \hat{x}(t), \hat{u}(t))$, $\hat{s}(t) = s(t, \hat{x}(t))$, $\hat{H}(t) := H(t, \hat{x}(t), \hat{u}(t), \hat{p}(t))$, $\hat{H}(t, u) := H(t, \hat{x}(t), u, \hat{p}(t))$, etc. Moreover, denote

$$\hat{A}(t) := f_x(t, \hat{x}(t), \hat{u}(t)), \quad \hat{B}(t) := f_u(t, \hat{x}(t), \hat{u}(t)) = B(t, \hat{x}(t))$$
$$\hat{\sigma}(t) := \nabla_u \hat{H}(t) = \hat{B}(t)^\top \hat{p}(t) + \hat{s}(t).$$

Let us introduce the following functional of $L^1 \ni \delta u \mapsto \Gamma(\delta u) \in \mathbb{R}$:

$$\Gamma(\delta u) := \int_0^T \left[\langle \hat{H}_{xx}(t)\delta x(t), \delta x(t)\rangle + 2\langle \hat{H}_{ux}(t)\delta x(t), \delta u(t)\rangle \right] \mathrm{d}t, \tag{2.8}$$

where δx is the solution of the equation $\dot{\delta x} = \hat{A}\delta x + \hat{B}\delta u$ with initial condition $\delta x(0) = 0$.

Assumption (A2). There exist numbers c_0, $\alpha_0 > 0$ and $\gamma_0 > 0$ such that

$$\int_0^T \langle \sigma(t), \delta u(t)\rangle \,\mathrm{d}t + \Gamma(\delta u) \geq c_0 \|\delta u\|_1^2,$$

for every $\delta u = u' - u$ with $u', u \in \mathcal{U} \cap B_{L^1}(\hat{u}; \alpha_0)$, and for every function $\sigma \in \mathbf{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma_0) \cap (-N_{\mathcal{U}}(u))$.

Assumption (A2) will be analyzed and discussed in details in Section 4. Now we formulate the main theorem.

Theorem 2.1. *Let Assumption (A1) be fulfilled for problem (1.1)–(1.3) and let $\hat{y} = (\hat{x}, \hat{p}, \hat{u})$ be a solution of the optimality system (2.7) (with F defined in (2.6)) for which Assumption (A2) is fulfilled. Let, in addition, the matrix $\hat{H}_{ux}(t)\hat{B}(t)$ be symmetric for a.e. $t \in [0, T]$. Then the optimality map $F : Y \rightrightarrows Z$ is strongly bi-metrically regular at $\hat{y}$ for zero with disturbance space $\tilde{Z} \subset Z$.*

3. Proof of the main result

The proof of Theorem 2.1 consists of several steps.

Step 1. The following result (adapted to the present problem formulation, assumptions, and notations) was proved in [11, Theorem 3.1].[3]

Theorem 3.1. *Let the assumptions in Theorem 2.1 be satisfied. Then strong bi-metric regularity of the set-valued map $y \mapsto F(y)$ at $\hat{y}$ for 0 (in the spaces as in Theorem 2.1) is equivalent to the strong bi-metric regularity of the map $y \mapsto L(y)$, at $\hat{y}$ for 0, where*

$$L(y) = \begin{pmatrix} -\dot{x} + \hat{f} + \hat{A}(x - \hat{x}) + \hat{B}(u - \hat{u}) \\ \dot{p} + \nabla_x \hat{H} + \hat{H}_{xy}(y - \hat{y}) \\ \nabla_u \hat{H} + \hat{H}_{uy}(y - \hat{y}) + N_{\mathcal{U}}(u) \end{pmatrix}.$$

The map L represents the partial linearization of F around $\hat{y} = (\hat{x}, \hat{p}, \hat{u})$. Thanks to the identity $\hat{H}_{uu} = 0$, L maps Y to Z, and moreover, $\hat{y}$ solves the inclusion $L(\hat{y}) \ni 0$.

To shorten the notations, we set for this section (skipping the dependence on t)

$$W := \hat{H}_{xx}, \;\; S := \hat{H}_{ux}, \;\; A := \hat{A} = \hat{f}_x, \;\; B := \hat{B} = B(\hat{x}).$$

We remind the already introduced notation $\hat{\sigma} = \nabla_u \hat{H}$. Then, also having in mind the identity $\hat{H}_{uu} = 0$, we can recast the definition of $L(y)$ as

$$L(y) = \begin{pmatrix} -\dot{x} + \dot{\hat{x}} + A(x - \hat{x}) + B(u - \hat{u}) \\ \dot{p} - \dot{\hat{p}} + W(x - \hat{x}) + S^\top(u - \hat{u}) + A^\top(p - \hat{p}) \\ \hat{\sigma} + S(x - \hat{x}) + B^\top(p - \hat{p}) + N_{\mathcal{U}}(u) \end{pmatrix}.$$

Due to Assumption (A1), we have that $\dot{\hat{x}}$, A, $\dot{\hat{p}}$, W, $\hat{\sigma} \in L^\infty$, and B, $S \in W^{1,\infty}$. We remind that according to (2.7) and (2.6), $\hat{u}$ satisfies the inclusion $\hat{\sigma} + N_{\mathcal{U}}(\hat{u}) \ni 0$.

Step 2. Define the map $\Lambda : L^1 \times \tilde{Z} \to L^\infty$ in the following way: for $u \in L^1$ and $z = (\xi, \pi, \rho) \in \tilde{Z}$,

$$\Lambda(u, z) := \hat{\sigma} + S(x[u, z] - \hat{x}) + B^\top(p[u, z] - \hat{p}) - \rho, \tag{3.1}$$

where $(x[u, z], p[u, z])$ is the solution of the system

$$\dot{x} = \dot{\hat{x}} + A(x - \hat{x}) + B(u - \hat{u}) - \xi, \quad x(0) = x^0, \tag{3.2}$$

$$-\dot{p} = -\dot{\hat{p}} + W(x - \hat{x}) + S^\top(u - \hat{u}) + A^\top(p - \hat{p}) - \pi, \quad p(T) = 0. \tag{3.3}$$

Further we skip the argument z if $z = 0$, so that $x[u] := x[u, 0]$, $p[u] := p[u, 0]$, $\Lambda(u) := \Lambda(u, 0)$.

Lemma 3.2. *Strong bi-metric regularity of the set-valued map L at $\hat{y}$ for 0 (in the spaces as in Theorem 2.1) is equivalent to strong bi-metric regularity of the map $\Lambda(\cdot, 0) + N_{\mathcal{U}}(\cdot) : \mathcal{U} \rightrightarrows L^\infty$ at $\hat{u}$ for zero, with disturbance space $W^{1,\infty} \subset L^\infty$.*

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[3] A Mayer problem is considered in [11], but the result also applies to Lagrange problems after a standard transformation. Moreover, the assumptions in [11] are somewhat weaker than (A1).

Proof. We shall prove that the bi-metric regularity of the map $\Lambda(\cdot,0)+N_{\mathcal{U}}(\cdot)$ implies that of L, which will actually be used later. The proof of the converse is similar and simpler.

For any $z=(\xi,\pi,\rho)\in\tilde{Z}$ and $u\in L^\infty$, we have from (3.2) that

$$\|l^x(\xi)\|_{1,\infty}\le c_1\|\xi\|_\infty,\quad \|l^x(\xi)\|_{1,1}\le c_1'\|z\|_Z, \tag{3.4}$$

where $l^x : L^\infty \to W^{1,\infty}$ is the linear map given by $l^x(\xi) := x[u,z]-x[u,0]$, and c_1 and c_1' are independent of u and z. Using this and (3.3), we obtain (also in a standard way) that

$$\|l^p(\xi,\pi)\|_{1,\infty}\le c_2(\|\xi\|_\infty+\|\pi\|_\infty),\quad \|l^p(\xi,\pi)\|_{1,1}\le c_2'\|z\|_Z, \tag{3.5}$$

where $l^p : L^\infty\times L^\infty \to W^{1,\infty}$ is the linear map given by $l^p(\xi,\pi) := p[u,z]-p[u,0]$, and c_2 and c_2' are constants such as c_1 and c_1'. Notice that the second inequalities in (3.4) and (3.5) imply that for a.e. $t\in[0,T]$

$$\max\{|x[u,z](t)|,\, |\dot{x}[u,z](t)|,\, |p[u,z](t)|,\, |\dot{p}[u,z](t)|\}\le M+c''(\|\xi\|_\infty+\|\pi\|_\infty),$$

where c'' is a constant. This will be used later to ensure that the appearing triples $(u,x[u,z],p[u,z])$ belong to the space Y.

We may represent

$$\Lambda(u,z)=\Lambda(u)+Q(z),$$

where

$$Q(z)=Sl^x(\xi)+B^\top l^p(\xi,\pi)-\rho,\quad \|Q(z)\|_{1,\infty}\le c_3\|z\|_\sim$$

is a linear map and c_3 is a constant.

The inclusion $L(y)\ni z$ can be equivalently reformulated as

$$x=x[u,z],\quad p=p[u,z],\quad \Lambda(u,z)+N_{\mathcal{U}}(u)\ni 0. \tag{3.6}$$

In view of the obtained representations, the last relations are equivalent to

$$x=x[u]+l^x(\xi),\quad p=p[u]+l^p(\xi,\eta),\quad \Lambda(u)+Q(z)+N_{\mathcal{U}}(u)\ni 0.$$

Having in mind the estimations for $\|l^x(\xi)\|_{1,\infty}$, $\|l^p(\xi,\pi)\|_{1,\infty}$ and $\|Q(z)\|_{1,\infty}$, obtaining Sbi-MR of L from that of $\Lambda+N_{\mathcal{U}}$ becomes a routine task. We will sketch the rest of the proof for completeness.

First we observe that there is a constant c_4 such that $\|Q(z)\|_\infty\le c_4\|z\|_Z$. Let κ, α and β be the constants in the definition of the Sbi-MR of the map $\Lambda+N_{\mathcal{U}}$. Fix

$$\bar{\alpha}=(c_1'+c_2')\bar{\beta}+\alpha,\qquad \bar{\beta}=\min\left\{\frac{\beta}{c_3},\frac{\bar{M}-M}{c''}\right\},\qquad \bar{\kappa}=c_1'+c_2'+c_4\kappa.$$

For any $z\in\tilde{Z}$ with $\|z\|_\sim\le\bar{\beta}$ we have $\|Q(z)\|_{1,\infty}\le\beta$. Then there exists a unique solution $u(z)\in\mathbf{B}_{L^1}(\hat{u};\alpha)$ of the inclusion $\Lambda(u,z)+N_{\mathcal{U}}(u)\ni 0$. Moreover, for z_1, $z_2\in\tilde{Z}$ with $\|z_i\|_\sim\le\bar{\beta}$ we have

$$\|u(z_1)-u(z_2)\|_1\le\kappa\|Q(z_1-z_2)\|_\infty\le c_4\kappa\|z_1-z_2\|_Z.$$

From the first two relations in (3.6) we have for $x(z_i)=x[u(z_i),z_i]$ and $p(z_i)=p[u(z_i),z_i]$

$$\|x(z_1)-x(z_2)\|_{1,1}+\|p(z_1)-p(z_2)\|_{1,1}\le c_1'\|z_1-z_2\|_Z+c_2'\|z_1-z_2\|_Z.$$

Thus L is Sbi-MR at $\hat{y}$ for zero with constants $\bar{\kappa}$, $\bar{\alpha}$, $\bar{\beta}$. ☐

Step 3. According to Lemma 3.2, it is enough to prove Sbi-MR of $\Lambda + N_{\mathcal{U}}$ in the spaces specified in the formulation of the lemma. It is convenient to use the notation

$$\langle v, u\rangle_1 := \int_0^T \langle v(t), u(t)\rangle \, \mathrm{d}t$$

for the duality pairing of L^1 and L^∞, where $v \in L^\infty$ and $u \in L^1$. The map $\Lambda : L^1 \to L^\infty$ is linear and continuous, and we shall show that its derivative, Λ', satisfies the equality

$$\langle \Lambda'\delta u, \delta u\rangle_1 = \Gamma(\delta u), \quad \forall\, \delta u \in L^1, \tag{3.7}$$

where the mapping $\Gamma : L^1 \to \mathbb{R}$ is defined in (2.8).[4] In the notations introduced in this section the definition of Γ reads as

$$\Gamma(\delta u) = \langle W\delta x, \delta x\rangle_1 + 2\langle S\delta x, \delta u\rangle_1, \tag{3.8}$$

where δx is the solution of $\dot{\delta x} = A\delta x + B\delta u$ with $\delta x(0) = 0$. Let δp be the solution of the equation

$$-\dot{\delta p} = A^\top \delta p + W\delta x + S^\top \delta u, \quad \delta p(T) = 0.$$

Since $u \mapsto \Lambda(u) := \hat{\sigma} + S(x[u] - \hat{x}) + B^\top(p[u] - \hat{p})$ is linear, we deduce

$$\Lambda'(u)\delta u = S\delta x + B^\top \delta p. \tag{3.9}$$

Integrating by parts the expression $\langle \delta p, \dot{\delta x}\rangle_1$ we obtain the equality

$$\langle \delta p, A\delta x + B\delta u\rangle_1 = \langle \delta p, \dot{\delta x}\rangle_1 = -\langle \delta x, \dot{\delta p}\rangle_1 = \langle \delta x, A^\top\delta p + W\delta x + S^\top\delta u\rangle_1.$$

Hence,

$$\langle \delta p, B\delta u\rangle_1 = \langle \delta x, W\delta x + S^\top \delta u\rangle_1,$$
$$\langle B^\top \delta p, \delta u\rangle_1 = \langle W\delta x, \delta x\rangle_1 + \langle S\delta x, \delta u\rangle_1,$$
$$\langle S\delta x, \delta u\rangle_1 + \langle B^\top\delta p, \delta u\rangle_1 = \langle W\delta x, \delta x\rangle_1 + 2\langle S\delta x, \delta u\rangle_1 = \Gamma(\delta u),$$

which implies (3.7) in view of (3.9).

Equality (3.7) allows to reformulate the inequality in Assumption (A2) as

$$\int_0^T \langle \sigma(t), \delta u(t)\rangle \, \mathrm{d}t + \langle \Lambda'\delta u, \delta u\rangle_1 \geq c_0\|\delta u\|_1^2 \tag{3.10}$$

with σ and δu as in (A2).

Step 4. Next, we will prove that for every $\alpha \in (0, \alpha_0)$ (see Assumption (A2)) and for every $\Delta \in W^{1,\infty}$ with $\|\Delta\|_{1,\infty} < c_0\alpha$ the inclusion

$$\Lambda(u) + N_{\mathcal{U}}(u) \ni \Delta \tag{3.11}$$

has a solution $\tilde{u} \in L^1$ satisfying $\|\tilde{u} - \hat{u}\|_1 < \alpha$. For this, we consider the inclusion

$$\Lambda(u) + N_{\mathcal{U}\cap \mathbf{B}_{L^1}(\hat{u};\alpha)}(u) \ni \Delta. \tag{3.12}$$

This inclusion represents the standard necessary optimality condition for the problem

$$\min\left\{ J_0(u) := \int_0^T \left[\frac{1}{2}\langle Wx[u], x[u]\rangle + \langle Sx[u], u\rangle + \langle \Delta, u\rangle\right]\right\},$$

[4] Similar representations are known, see e.g. in [6], but in the space L^2. Here the space setting is different and the specificity of the affine problem is essential.

where $x[u]$ is defined around (3.2), with the control constraints $u \in \mathcal{U}$ and $u \in \mathbf{B}_{L^1}(\hat{u};\alpha)$. This is due to the well-known fact that $\Lambda(u)$ is the derivative of J_0 at u in L^1 (the proof of this fact uses a similar argument as the proof of the relation (3.7)). Due to the weak compactness of $\mathcal{U} \cap \mathbf{B}_{L^1}(\hat{u};\alpha)$ in L^1, this problem has a solution $\tilde{u}$, which then is a solution of (3.12).

Now we use the relation

$$N_{\mathcal{U}\cap\mathbf{B}_{L^1}(\hat{u};\alpha)}(u) = N_{\mathcal{U}}(u) + N_{\mathbf{B}_{L^1}(\hat{u};\alpha)}(u). \tag{3.13}$$

It follows from [2, Theorem 3.1], which, formulated for the particular space setting and sets, $\mathcal{U} \subset L^1$ and $\mathcal{V} := \mathbf{B}_{L^1}(\hat{u};\alpha) \subset L^1$, reads as follows: the equality (3.13) holds, provided that the set $\operatorname{Epi} s_{\mathcal{U}} + \operatorname{Epi} s_{\mathcal{V}}$ is weak* closed, where $\operatorname{Epi} s_{\mathcal{W}}$ is the epigraph of $s_{\mathcal{W}}$ and $s_{\mathcal{W}} : L^\infty \to \mathbb{R}$ is the support function to the set $\mathcal{W} \subset L^1$, that is, $s_{\mathcal{W}}(l) := \sup_{w\in\mathcal{W}}\langle l, w\rangle_1$. The weak* closedness of this set is proved in Proposition 3.1, case (i), in [2], which requires (in our case) that $\mathcal{U}$ and the interior of $\mathbf{B}_{L^1}(\hat{u};\alpha)$ have a nonempty intersection, which is obviously fulfilled.

Due to (3.13) and (3.12), there exists $\nu \in N_{\mathbf{B}_{L^1}(\hat{u};\alpha)}(\tilde{u})$ such that

$$\nu + \Lambda(\tilde{u}) - \Delta \in -N_{\mathcal{U}}(\tilde{u}),$$

hence,

$$\langle \nu, \hat{u} - \tilde{u}\rangle_1 + \langle \Lambda(\tilde{u}) - \Delta, \hat{u} - \tilde{u}\rangle_1 \geq 0.$$

We have $\langle \nu, \hat{u} - \tilde{u}\rangle_1 \leq 0$ since $\hat{u} \in \mathbf{B}_{L^1}(\hat{u};\alpha)$. Thus

$$\langle \Lambda(\tilde{u}), \hat{u} - \tilde{u}\rangle_1 - \langle \Delta, \hat{u} - \tilde{u}\rangle_1 \geq 0.$$

Since Λ is linear and satisfies (3.7), and since $\Lambda(\hat{u}) = \hat{\sigma}$ in view of (3.1), we obtain that

$$\begin{aligned} 0 &\geq \langle \Lambda(\hat{u}), \tilde{u} - \hat{u}\rangle_1 + \langle \Lambda(\tilde{u}) - \Lambda(\hat{u}), \tilde{u} - \hat{u}\rangle_1 + \langle \Delta, \hat{u} - \tilde{u}\rangle_1 \\ &= \langle \Lambda(\hat{u}), \tilde{u} - \hat{u}\rangle_1 + \langle \Lambda'(\tilde{u} - \hat{u}), \tilde{u} - \hat{u}\rangle_1 + \langle \Delta, \hat{u} - \tilde{u}\rangle_1 \\ &= \langle \hat{\sigma}, \tilde{u} - \hat{u}\rangle_1 + \Gamma(\tilde{u} - \hat{u}) + \langle \Delta, \hat{u} - \tilde{u}\rangle_1. \end{aligned}$$

Moreover, we have $\hat{\sigma} \in -N_{\mathcal{U}}(\hat{u})$. Then Assumption (A2) in the form of (3.10) applied for $\delta u = \tilde{u} - \hat{u}$ and $\sigma = \hat{\sigma}$ implies that

$$0 \geq c_0\|\tilde{u} - \hat{u}\|_1^2 + \langle \Delta, \hat{u} - \tilde{u}\rangle_1.$$

Hence,

$$\|\tilde{u} - \hat{u}\|_1 \leq \frac{\|\Delta\|_\infty}{c_0} < \alpha.$$

Since $\tilde{u}$ belongs to the interior of $\mathbf{B}_{L^1}(\hat{u};\alpha)$, thus $N_{\mathbf{B}_{L^1}(\hat{u};\alpha)}(\tilde{u}) = \{0\}$, we obtain that $\nu = 0$, therefore $\tilde{u}$ is a solution of the inclusion (3.11).

Step 5. First, we shall estimate $\|\Lambda(u_1) - \Lambda(u_2)\|_{1,\infty}$ for two functions u_1, $u_2 \in L^1$. Denote $\delta u = u_1 - u_2$, $\delta x = x[u_1] - x[u_2]$, $\delta p = p[u_1] - p[u_2]$. Then there is a constant c_1 independent of u_1 and u_2 such that

$$\|\delta x\|_\infty \leq c_1\|\delta u\|_1, \qquad \|\delta p\|_\infty \leq c_1\|\delta u\|_1.$$

Using the definition of Λ and Assumption (A1) we can estimate

$$\|\Lambda(u_1) - \Lambda(u_2)\|_\infty \leq c_2\|\delta u\|_1$$

with some constant c_2. Then

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}t}(\Lambda(u_1) - \Lambda(u_2)) \Big\|_\infty \le \|S(A\delta x + B\delta u) - B^\top(W\delta x + S^\top \delta u + A^\top \delta p)\|_\infty + \|\dot S \delta x + \dot B^\top \delta p\|_\infty \le c_3 \|\delta u\|_1,$$

where c_3 is another constant and in the last estimate we use the assumed symmetry of $SB = \hat H_{ux}\hat B$. Thus

$$\|\Lambda(u_1) - \Lambda(u_2)\|_{1,\infty} \le (c_2 + c_3)\|u_1 - u_2\|_1 =: c_4\|u_1 - u_2\|_1. \tag{3.14}$$

Now we choose the number α in such a way that

$$0 < \alpha \le \alpha_0, \qquad c_0\alpha \le \alpha_0, \qquad (c_0 + c_4)\alpha \le \gamma_0.$$

Consider two disturbances Δ_1, $\Delta_2 \in W^{1,\infty}$ with $\|\Delta_i\|_{1,\infty} < c_0\alpha$, and two solutions u_1, $u_2 \in \mathbf{B}_{L^1}(\hat u; \alpha)$ of (3.11) corresponding to Δ_1 and Δ_2, respectively.

Let $\sigma := \Lambda(u_2) - \Delta_2$, by (3.14) we have

$$\begin{aligned} \|\sigma - \hat\sigma\|_{1,\infty} &\le \|\Lambda(u_2) - \Lambda(\hat u)\|_{1,\infty} + \|\Delta_2\|_{1,\infty} \\ &\le c_4\|u_2 - \hat u\|_1 + c_0\alpha \\ &< c_4\alpha + c_0\alpha \\ &\le \gamma_0. \end{aligned}$$

Moreover, we have $\sigma = \Lambda(u_2) - \Delta_2 \in -N_{\mathcal U}(u_2)$ because u_2 solves the variational inequality (3.11) with $\Delta = \Delta_2$. Similarly as in Step 4 we obtain the following chain of inequalities:

$$\begin{aligned} 0 &\ge \langle \Lambda(u_1) - \Delta_1, u_1 - u_2\rangle_1 \\ &= \langle \Lambda(u_2) - \Delta_2, u_1 - u_2\rangle_1 + \langle \Lambda(u_1) - \Lambda(u_2), u_1 - u_2\rangle_1 + \langle \Delta_2 - \Delta_1, u_1 - u_2\rangle_1 \\ &= \langle \sigma, u_1 - u_2\rangle_1 + \langle \Lambda'(u_1 - u_2), u_1 - u_2\rangle_1 + \langle \Delta_2 - \Delta_1, u_1 - u_2\rangle_1 \\ &= \langle \sigma, u_1 - u_2\rangle_1 + \Gamma(u_1 - u_2) + \langle \Delta_2 - \Delta_1, u_1 - u_2\rangle_1, \end{aligned}$$

Having in mind also that $\|u_2 - \hat u\|_1 < \alpha \le \alpha_0$, we can apply Assumption (A2) (in the form as in (3.10)) to the latter inequality. We obtain

$$0 \ge c_0\|u_1 - u_2\|_1^2 + \langle \Delta_2 - \Delta_1, u_1 - u_2\rangle_1,$$

which implies that $\|u_1 - u_2\|_1 \le \frac{1}{c_0}\|\Delta_1 - \Delta_2\|_\infty$. This proves the Sbi-MR property of $\Lambda + N_{\mathcal U}$ with constants $\kappa = (c_0)^{-1}$, α, and $\beta = c_0\alpha$. The proof of Theorem 2.1 is complete.

4. Some special cases

We begin with few comments. Assumption (A2) with the particular choice $\sigma = \hat\sigma$, reads as

$$\int_0^T \langle \hat\sigma(t), \delta u(t)\rangle \,\mathrm{d}t + \Gamma(\delta u) \ge c_0\|\delta u\|_1^2. \tag{4.1}$$

This inequality, required for all $\delta u \in \mathcal{U} - \hat{u}$, is shown in [9] to be sufficient for the property of *strong metric sub-regularity*, which is substantially weaker than Sbi-MR. Moreover, the condition[5]

$$\int_0^T \langle \hat{\sigma}(t), \delta u(t) \rangle \,\mathrm{d}t + \frac{1}{2}\Gamma(\delta u) \geq c_0 \|\delta u\|_1^2, \quad \forall\, \delta u \in \mathcal{U} - \hat{u}, \quad \|\delta u\|_1 \text{ – small enough,}$$

is sufficient for strict local optimality of $\hat{u}$ in an L^1-neighborhood. This last condition is weaker than (4.1), as shown in [9].

Assumption (A2) is fulfillled on the following (more compact) one.

Assumption (A2'). There exist numbers c_0, $\alpha_0 > 0$ and $\gamma_0 > 0$ such that

$$\int_0^T |\langle \sigma(t), \delta u(t) \rangle| \,\mathrm{d}t + \Gamma(\delta u) \geq c_0 \|\delta u\|_1^2, \tag{4.2}$$

for every function $\sigma \in \mathbf{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma_0)$ and for every $\delta u \in \mathcal{U} - \mathcal{U}$ with $\|\delta u\|_1 \leq \alpha_0$.

Obviously (A2') implies (A2), since for $\sigma \in -N_{\mathcal{U}}(u)$ and $u' \in \mathcal{U}$ it holds that $\langle \sigma(t), u'(t) - u(t) \rangle \geq 0$.

Now we focus on the first-order term in (4.2) under an additional condition introduced in [5] in a somewhat stronger form and for box-like sets U.

Assumption (B). The set U is a convex and compact polyhedron. Moreover, there exist numbers $\kappa > 0$ and $\tau > 0$ such that for every unit vector e parallel to some edge of U and for every $s \in [0, T]$ for which $\langle \hat{\sigma}(s), e \rangle = 0$ it holds that

$$|\langle \hat{\sigma}(t), e \rangle| \geq \kappa |t - s| \qquad t \in [s - \tau, s + \tau] \cap [0, T].$$

The next lemma claims that Assumption (B) remains valid for all functions σ close enough to $\hat{\sigma}$ in $W^{1,\infty}$.

Lemma 4.1. *Let assumptions (A1) and (B) be fulfilled. Then there exist numbers $\kappa' > 0$, $\tau' > 0$ and $\gamma' > 0$ such that for every function $\sigma \in \mathbf{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma')$, for every unit vector e parallel to some edge of U and for every $s \in [0, T]$ for which $\langle \sigma(s), e \rangle = 0$, it holds that*

$$|\langle \sigma(t), e \rangle| \geq \kappa' |t - s| \qquad t \in [s - \tau', s + \tau'] \cap [0, T].$$

Proof. The proof combines arguments from the proof of Proposition 3.4 in [11] and the proof of Proposition 4.1, therefore we only sketch it focusing on the differences with the proofs mentioned above.

First of all, Assumption (B) implies that the reference control $\hat{u}$ is piece-wise constant. This follows from the fact that $\langle \hat{\sigma}(t), e \rangle$ has not more than $T/\tau + 1$ zeros in $[0, T]$ and U has a finite number of edges. More details are given in the proof of Proposition 4.1 in [9].

From the definition of $\hat{\sigma}$, (A1) and the fact that $\hat{u}$ is a piece-wise constant function we obtain that $\hat{\sigma}$ has a piece-wise continuous derivative. Let us fix e as in

[5] The left-hand side in the next inequality is just the second order Taylor expansion of the objective functional $J(u)$ in (1.1).

Assumption (B), and denote $\hat{\sigma}_e := \langle \hat{\sigma}(t), e \rangle$. Let $\hat{s}_1, \ldots, \hat{s}_k$ be the zeros of $\hat{\sigma}_e$ in $[0, T]$. For $\delta > 0$ define

$$\Omega(\delta) := \cup_{i=1}^k [\hat{s}_i - \delta, \hat{s}_i + \delta].$$

Choose $\delta > 0$ so small that $\delta < \tau$ and there are no other points of discontinuity of $\dot{\hat{\sigma}}$ in $\Omega(\delta)$ except possibly $\hat{s}_1, \ldots, \hat{s}_k$. Denote

$$\dot{\hat{\sigma}}_e^-(\hat{s}_i) := \lim_{t \to \hat{s}_i - 0} \dot{\hat{\sigma}}(t), \quad \dot{\hat{\sigma}}_e^+(\hat{s}_i) := \lim_{t \to \hat{s}_i + 0} \dot{\hat{\sigma}}(t), \quad i = 1, \ldots, k.$$

By choosing $\delta > 0$ smaller, if needed, we may ensure that

$$|\dot{\hat{\sigma}}(t) - \dot{\hat{\sigma}}_e^-(\hat{s}_i)| \le \frac{\kappa}{4} \text{ for } t \in [\hat{s}_i - \delta, \hat{s}_i], \quad |\dot{\hat{\sigma}}(t) - \dot{\hat{\sigma}}_e^+(\hat{s}_i)| \le \frac{\kappa}{4} \text{ for } t \in [\hat{s}_i, \hat{s}_i + \delta].$$

Then for every i and $t \in [\hat{s}_i - \delta, \hat{s}_i]$ we have from Assumption (B) that

$$\begin{aligned} \kappa|t - \hat{s}_i| &\le |\hat{\sigma}_e(t) - \hat{\sigma}_e(\hat{s}_i)| \\ &= \Big| \int_{\hat{s}_i}^t \dot{\hat{\sigma}}_e(\theta) \, \mathrm{d}\theta \Big| \\ &\le \int_{\hat{s}_i}^t |\dot{\hat{\sigma}}_e^-(\hat{s}_i)| \, \mathrm{d}\theta + \int_{\hat{s}_i}^t |\dot{\hat{\sigma}}_e^-(\hat{s}_i) - \dot{\hat{\sigma}}_e(\theta)| \, \mathrm{d}\theta \\ &\le |t - \hat{s}_i| \, |\dot{\hat{\sigma}}_e^-(\hat{s}_i)| + \frac{\kappa}{4}|t - \hat{s}_i| \end{aligned}$$

Hence,

$$|\dot{\hat{\sigma}}_e^-(\hat{s}_i)| \ge \frac{3\kappa}{4}.$$

Analogously we obtain the same estimate for $|\dot{\hat{\sigma}}_e^+(\hat{s}_i)|$.

Obviously there exists $\eta > 0$ such that $|\hat{\sigma}_e(t)| \ge \eta$ for every $t \in [0, T] \setminus \Omega(\delta/2)$. By choosing the number $\gamma \in (0, \kappa/4]$ sufficiently small we have that for every $\sigma \in \mathbf{B}_{W^{1,\infty}}(\hat{\sigma}; \gamma)$ the function $\sigma_e = \langle \sigma, e \rangle$ has no zeros in $[0, T] \setminus \Omega(\delta/2)$. Now let us take an arbitrary σ as in the last sentence. Let s be an arbitrary zero of σ_e in $[0, T]$. Then there exists $\hat{s}_i$ such that $|s - \hat{s}_i| \le \delta/2$. For $t \in [s - \delta/2, s + \delta/2]$ we can estimate

$$\begin{aligned} |\sigma_e(t)| &= \Big| \int_s^t \dot{\sigma}_e(\theta) \, \mathrm{d}\theta \Big| \\ &\ge \Big| \int_s^t \dot{\hat{\sigma}}_e(\theta) \, \mathrm{d}\theta \Big| - \int_s^t |\dot{\sigma}_e(\theta) - \dot{\hat{\sigma}}_e(\theta)| \, \mathrm{d}\theta \Big| \\ &\ge \Big| \int_s^t \dot{\hat{\sigma}}_e(\theta) \, \mathrm{d}\theta \Big| - \gamma|t - s|. \end{aligned}$$

For the last integral we have

$$\Big| \int_s^t \dot{\hat{\sigma}}_e(\theta) \, \mathrm{d}\theta \Big| \ge \Big| \int_s^t \zeta(\theta) \, \mathrm{d}\theta \Big| - \int_s^t |\dot{\hat{\sigma}}_e(\theta) - \zeta(\theta)| \, \mathrm{d}\theta,$$

where $\zeta(\theta)$ is either $\dot{\hat{\sigma}}_e^-(\hat{s}_i)$ or $\dot{\hat{\sigma}}_e^+(\hat{s}_i)$ depending on whether $\theta < \hat{s}_i$ or $\theta > \hat{s}_i$. Thus we can estimate

$$|\sigma_e(t)| \ge \frac{3\kappa}{4}|t - s| - \frac{\kappa}{4}|t - s| - \gamma|t - s| \ge \frac{\kappa}{4}|t - s|.$$

Thus we obtain the claim of the lemma with $\kappa' = \kappa/4$, $\tau' = \delta/2$ and $\gamma' = \gamma$. $\square$

Proposition 4.2. *Let assumptions (A1) and (B) be fulfilled. Then there exist numbers c_0, $\alpha_0 > 0$ and $\gamma_0 > 0$ such that*

$$\int_0^T |\langle \sigma(t), \delta u(t)\rangle| \, \mathrm{d}t \geq c_0 \|\delta u\|_1^2, \tag{4.3}$$

for every function $\sigma \in \mathbf{B}_{W^{1,\infty}}(\hat{\sigma};\gamma_0)$ and for every $\delta u \in \mathcal{U} - \mathcal{U}$ with $\|\delta u\|_1 \leq \alpha_0$.

Having at hand Lemma 4.1, the proof repeats that of Proposition 4.1 in [9].

Remark 4.3. *A more slightly precise modification of the proof of Lemma 4.1 shows that the number κ' can be taken as any number smaller than κ (from Assumption (B)). Moreover, the constant c_0 in Proposition 4.2 is directly related with number κ' (thus with κ). In the simplest case of scalar control and $U = [u_1, u_2]$ is straightforward. As obtained in the proof of Lemma 4.1, Assumption (B) implies in this case that $\hat{\sigma}$ has finite number of zeros, $\hat{s}_1, \dots, \hat{s}_k$, and $\dot{\hat{\sigma}}$ is piece-wise continuous. If the number Q satisfies*

$$\liminf_{t \to \hat{s}_i} |\dot{\hat{\sigma}}(t)| \geq Q \quad 1 = 1, \dots, k,$$

(the liminf *is taken over all t at which the derivative exists) then a simple calculation shows that the claim of Proposition 4.2 holds with any number $c_0 \leq Q/(8k(u_2 - u_1))$.*

Example 1. This example shows that Sbi-MR of the optimality mapping may hold even for problems that are non-convex, namely, the objective functional J in (1.1) is even directionally non-convex at the optimal control $\hat{u}$. Consider the problem

$$\min \Big\{ J(u) := \int_0^1 \Big[-\frac{\alpha}{2}(x(t))^2 - \beta x(t) + u(t) \Big] \, \mathrm{d}t \Big\},$$

subject to

$$\dot{x} = u, \quad x(0) = 0, \quad u(t) \in [0,1].$$

Here α and β are positive parameters satisfying $\beta > 1$, $2\alpha \leq \beta$.

The solution of the adjoint equation $\dot{p} = \alpha x + \beta$, $p(1) = 0$ is strictly monotone increasing and the switching function, $\sigma(t) = p(t) + 1$, is positive at $t = 1$. This implies that only optimal control has the structure

$$\hat{u}(t) = \begin{cases} 1 & \text{for } t \in [0, \tau], \\ 0 & \text{for } t \in (\tau, 1]. \end{cases}$$

The corresponding solutions of the primal and the adjoint equations are

$$\hat{x}(t) = \begin{cases} t & \text{for } t \in [0, \tau], \\ \tau & \text{for } t \in (\tau, 1], \end{cases}$$

and

$$\hat{p}(t) = \begin{cases} \frac{\alpha}{2}(\tau^2 + t^2) + \beta t - \alpha\tau - \beta & \text{for } t \in [0, \tau], \\ t(\alpha\tau + \beta) - \alpha\tau - \beta & \text{for } t \in (\tau, 1]. \end{cases}$$

A simple calculation shows that for $\beta > 1$, τ is given by

$$\tau = \frac{-(\beta - \alpha) + \sqrt{(\beta - \alpha)^2 + 4\alpha(\beta - 1)}}{2\alpha} \in (0,1).$$

For the corresponding switching function $\hat\sigma = \hat p + 1$ we have $\dot{\hat\sigma}(\tau) = \alpha\tau + \beta > \beta$. Then Assumption (B) is fulfilled with $\kappa < \beta$. According to Remark 4.3, we have

$$\int_0^1 |\hat\sigma(t)\delta u(t)|\,\mathrm{d}t \ge \frac{\beta}{2}\|\delta u\|_1^2 \qquad \forall\, \delta u \in \mathcal{U} - \mathcal{U} \ \text{ with a sufficiently small } \|\delta u\|_1.$$

Moreover,

$$\Gamma(\delta u) = -\int_0^1 \alpha(\delta x(t))^2\,\mathrm{d}t = -\alpha\int_0^1 \Big(\int_0^t \delta u(s)\,\mathrm{d}s\Big)^2\,\mathrm{d}t \ge -\alpha\|\delta u\|_1^2.$$

Thus for $2\alpha < \beta$ Assumption (A2') is fulfilled and the optimality mapping for the considered problem is Sbi-MR at $(\hat x, \hat u, \hat p)$ for zero. On the other hand, considering again the expression for the second variation Γ, we see that $\Gamma(\delta u) < 0$, except some specially constructed control variations δu. Thus the objective functional $J(u)$ in this example is not convex even directionally at the solution point $\hat u$.

5. An application: uniform convergence of the Euler discretization

In this section we prove that the sufficient conditions for Sbi-MR given in Theorem 2.1 imply a property that can be called *uniform strong sub-regularity* concerning a family of optimal control problems "neighboring" a given reference problem. This property is shown to imply a *uniform* error estimate for the accuracy of the Euler discretization scheme, applied to any of the problems of the family.

We consider again the reference problem (1.1)-(1.3) together with the fixed solution $(\hat x, \hat p, \hat u)$ of its optimality system (2.2)–(2.4). The assumptions in Theorem 2.1 will hold in this section, with the additional assumption that f and g are time-invariant.

Together with the reference problem, we consider a family of problems of the same kind, each defined by a pair of time-invariant functions $\pi := (\tilde f, \tilde g)$ satisfying Assumption (A1) (with f and g replaced with $\tilde f$ and $\tilde g$). Any such pair will be called admissible, and $(\mathcal{P}_\pi)$ will denote the problem corresponding to the pair π, that is, the problem

$$\min_{u\in\mathcal{U}} \left\{ \int_0^T \tilde g(x(t), u(t))\,\mathrm{d}t \right\} \tag{5.1}$$

subject to

$$\dot x(t) = \tilde f(x(t), u(t)), \quad x(0) = x^0. \tag{5.2}$$

Due to relation (2.5), we restrict our consideration to admissible pairs π defined on the set $D := \mathbf{B}_{\mathbb{R}^n}(0, \bar M) \times U$. Given a positive number ρ, we denote by $\mathcal{H}_\rho$ the set of all admissible pairs $\pi = (\tilde f, \tilde g)$ such that

$$\|\tilde f - f\|_{1,\infty} + \|\tilde g - g\|_{1,\infty} \le \rho, \tag{5.3}$$

where the $W^{1,\infty}$-norms are taken for functions defined on the set D.

For a given $\pi = (\tilde f, \tilde g) \in \mathcal{H}_\rho$, we consider the mapping $\Phi_\pi : Y \to Z$ defined by

$$\Phi_\pi(x, p, u) = \begin{pmatrix} \dot x - \tilde f(x, u) \\ \dot p + \nabla_x \tilde H(x, p, u) \\ \nabla_u \tilde H(x, p, u) + N_{\mathcal{U}}(u) \end{pmatrix} \tag{5.4}$$

where $\tilde{H}$ is the Hamiltonian corresponding to the pair π, and where as before $N_{\mathcal{U}}(u) \subset L^\infty$ is the normal cone to the set $\mathcal{U}$ of admissible controls at u. The following lemma is technical.

Lemma 5.1. *Let $\pi = (\tilde{f}, \tilde{g})$ belong to $\mathcal{H}_\rho$ and $\varphi_\pi : Y \to Z$ be defined as*

$$\varphi_\pi(x,p,u) = \begin{pmatrix} \varphi_\pi^1(x,p,u) \\ \varphi_\pi^2(x,p,u) \\ \varphi_\pi^3(x,p,u) \end{pmatrix} := \begin{pmatrix} f(x,u) - \tilde{f}(x,u) \\ \nabla_x \tilde{H}(x,p,u) - \nabla_x H(x,p,u) \\ \nabla_u \tilde{H}(x,p,u) - \nabla_u H(x,p,u) \end{pmatrix}. \tag{5.5}$$

There exists a positive constant c such that

$$d_Z(\varphi_\pi(y), 0) \le c\rho \quad \forall y \in Y. \tag{5.6}$$

Proof. Let $y = (x,p,u) \in Y$. We estimate each one of the components of $\varphi_\pi(y)$. First,

$$\|\varphi_\pi^1(y)\|_1 = \|f(x,u) - \tilde{f}(x,u)\|_1 \le T\rho.$$

In a similar way,

$$\begin{aligned} \|\varphi_\pi^2(y)\|_1 &= \|\nabla_x \tilde{H}(x,p,u) - \nabla_x H(x,p,u)\|_1 \\ &\le \|\tilde{f}_x - f_x\|_1 \|p\|_\infty + \|\tilde{g}_x - g_x\|_1 \\ &\le (\bar{M}+1)T\rho. \end{aligned}$$

Analogously,

$$\|\varphi_\pi^3(y)\|_\infty \le (\bar{M}+1)\rho.$$

The result follows. ☐

We remind the notion of Strong Metric sub-Regularity (SMsR) for a set-valued mapping $\Phi : Y \to Z$. We make use of this notion in the following results.

Definition 5.2. A set valued mapping $\Phi : Y \to Z$ is Strongly Metrically sub-Regular (SMsR) at y^* for zero if $0 \in \Phi(y^*)$ and there exist $a, b > 0$ and $\kappa > 0$ such that for any $z \in B_Z(0,b)$ and any solution $y \in B_Y(y^*, a)$ of the inclusion $z \in \Phi(y)$ it holds that $d_Y(y, y^*) \le \kappa d_Z(z, 0)$. We call a, b and κ the parameters of SMsR.

According to Theorem 3.1 in [9], Assumption (A2) implies that the optimality map F in (2.6) is SMsR at $\hat{y}$ for zero (see Section 4). We fix its parameters $a, b > 0$ and $\kappa > 0$ of SMsR.

Proposition 5.3. *Let π belong to $\mathcal{H}_\rho$ and $y^* \in B_Y(\hat{y}, a)$ be a solution of problem $(\mathcal{P}_\pi)$. There exists a positive constant κ' such that*

$$d_Y(\hat{y}, y^*) \le \kappa' \rho, \tag{5.7}$$

for all sufficiently small ρ.

Proof. We can write $\Phi_\pi = \varphi_\pi + F$, where φ_π is the map (5.5) in Lemma 5.1 and F is the optimality mapping (2.6). Let $c > 0$ be the constant in that lemma, so that $d_Z(\varphi_\pi(y), 0) \le c\rho$ for all $y \in Y$. We can choose ρ small enough to ensure $\varphi_\pi(y) \in B_Z(0,b)$ for all $y \in Y$. Since y^* is a solution of problem $(\mathcal{P}_\pi)$, the inclusion $0 \in \varphi_\pi(y^*) + F(y^*)$ is satisfied. By SMsR, we have the desired inequality with $\kappa' := c\kappa$. ☐

Analogously as we defined the functional Γ; given a $\pi \in \mathcal{H}$ and a reference solution y^* of problem $(\mathcal{P}_\pi)$, we consider the functional $\Gamma_\pi : L^1 \to \mathbb{R}$ defined in terms of π and y^* as in (2.8). Explicitly,

$$\Gamma_\pi(\delta u) = \int_0^T \Big[\langle \tilde{H}_{xx}(y^*(t))\delta x(t), \delta x(t)\rangle + 2\langle\, \tilde{H}_{ux}(y^*(t))\delta x(t), \delta u(t)\rangle\Big]\, \mathrm{d}t,$$

where δx is the solution of the equation $\dot{\delta x}(t) = \tilde{f}_x(x^*(t), u^*(t))\delta x(t) + \tilde{f}_u(x^*(t), u^*(t))\delta u(t)$ with initial condition $\delta x(0) = 0$.

The following lemma establishes an estimation involving the functionals Γ_π and Γ.

Lemma 5.4. *Let π belong to $\mathcal{H}_\rho$ and $y^* \in B_Y(\hat{y}, a)$ be a solution of problem $(\mathcal{P}_\pi)$. There exists a constant $\eta > 0$ such that*

$$|\Gamma(v - u^*) - \Gamma_\pi(v - u^*)| \le \eta\,\rho \|v - u^*\|_1^2 \quad \forall v \in \mathcal{U},$$

for all sufficiently small ρ.

Proof. Using Proposition 5.3 and the Lipschitz continuity of the functions involved, we can find positive constants c_w and c_s such that

$$\|\hat{H}_{xx} - \tilde{H}^*_{xx}\|_1 \le c_w \rho, \tag{5.8}$$

and

$$\|\hat{H}_{ux} - \tilde{H}^*_{ux}\|_\infty \le c_s \rho. \tag{5.9}$$

Let $v \in \mathcal{U}$ and $v' = v - u^*$, we denote by $\delta\hat{x}$ and δx^* the solutions of

$$\dot{x} = \hat{A}x + \hat{B}v', \;\; x(0) = 0, \qquad \dot{x} = \tilde{A}^* x + \tilde{B}^* v', \;\; x(0) = 0, \tag{5.10}$$

respectively. There exist positive constants d_1 and d_2 such that

$$\max\{\|\delta\hat{x}\|_\infty, \|\delta x^*\|_\infty\} \le d_1 \|v'\|_1, \tag{5.11}$$

and

$$\|\delta\hat{x} - \delta x^*\|_\infty \le d_2 \rho \|v'\|_1. \tag{5.12}$$

Now,

$$\begin{aligned}
|\Gamma(v') - \Gamma_\pi(v')| &\le \left| \int_0^T \Big[\langle\, \hat{H}_{xx}\delta\hat{x}, \delta\hat{x}\rangle - \langle\, \tilde{H}^*_{xx}\delta x^*, \delta x^*\rangle\Big] \right| \\
&\quad + 2\left| \int_0^T \Big[\langle\, \hat{H}_{ux}\delta\hat{x} - \tilde{H}^*_{ux}\delta x^*, v'\rangle\Big] \right| \\
&\le \int_0^T |\langle\, \hat{H}_{xx}\delta\hat{x}, \delta\hat{x} - \delta x^*\rangle| + \int_0^T |\langle\, \hat{H}_{xx}\delta\hat{x} - \tilde{H}^*_{xx}\delta x^*, \delta x^*\rangle| \\
&\quad + 2\int_0^T |\langle\, \hat{H}_{ux}\delta\hat{x} - \tilde{H}^*_{ux}\delta x^*, v'\rangle| \\
&\le \|\hat{H}_{xx}\delta\hat{x}\|_1 \|\delta\hat{x} - \delta x^*\|_\infty + \big(\|\hat{H}_{xx}(\delta\hat{x} - \delta x^*)\|_1 \\
&\quad + \|(\hat{H}_{xx} - \tilde{H}^*_{xx})\delta x^*\|_1\big)\|\delta x^*\|_\infty \\
&\quad + \big(\|\hat{H}_{ux}(\delta\hat{x} - \delta x^*)\|_\infty + \|(\hat{H}_{ux} - \tilde{H}^*_{ux})\delta x^*\|_\infty\big)\|v'\|_1.
\end{aligned}$$

Taking (5.8)-(5.12) into account, the result follows. □

Theorem 5.5. *There exist $\zeta, \tilde{a}, \tilde{b} > 0$ and $\tilde{\kappa} > 0$ such that if $\pi \in \mathcal{H}_\zeta$ and $y^* \in B_Y(\hat{y}, a)$ is a solution for problem $(\mathcal{P}_\pi)$, then the map Φ_π is SMsR at y^* for zero with parameters $\tilde{a}, \tilde{b}, \tilde{\kappa}$.*

Proof. Let c_0, $\alpha_0 > 0$ and $\gamma_0 > 0$ be the numbers in Assumption (A2). If y^* is a solution for problem $(\mathcal{P}_\pi)$, we have $0 \in \Phi_\pi(y^*)$. By Proposition 5.3, there exists $\zeta > 0$ such that for any $\pi \in \mathcal{H}_\zeta$, $d_Y(\hat{y}, y^*) < \kappa'\zeta$ for some constant $\kappa' > 0$. We consider ζ small enough to guarantee $\|\hat{\sigma} - \tilde{\sigma}^*\|_{1,\infty} \le \gamma_0$ (in the estimation of $\|\dot{\hat{\sigma}} - \dot{\tilde{\sigma}}^*\|_\infty$ we use the symmetry of $\hat{H}_{ux}\hat{B}$ similarly as in the estimation before (3.14)) and $\|\hat{u} - u^*\|_1 < \alpha_0/2$. Let $\tilde{\alpha}_0 := \alpha_0/2$, so $B_{L^1}(u^*; \tilde{\alpha}_0) \subset B_{L^1}(\hat{u}; \alpha_0)$.

By Assumption (A2),

$$\int_0^T \langle \tilde{\sigma}^*, v - u^* \rangle + \Gamma(v - u^*) \ge c_0 \|v - u^*\|_1^2 \quad \forall v \in \mathcal{U} \cap B_{L^1}(u^*; \tilde{\alpha}_0), \tag{5.13}$$

or

$$\int_0^T \langle \tilde{\sigma}^*, v - u^* \rangle + \Gamma_\pi(v - u^*) \ge c_0 \|v - u^*\|_1^2 + \Big[\Gamma_\pi(v - u^*) - \Gamma(v - u^*)\Big],$$

for all $v \in \mathcal{U} \cap B_{L^1}(u^*; \tilde{\alpha}_0)$. Taking into account Lemma 5.4, we can choose ζ smaller if needed to ensure

$$\Gamma_\pi(v - u^*) - \Gamma(v - u^*) \ge -\frac{c_0}{2} \|v - u^*\|_1^2 \quad \forall v \in \mathcal{U} \cap B_{L^1}(u^*; \tilde{\alpha}_0).$$

Thus,

$$\int_0^T \langle \sigma^*, v - u^* \rangle + \Gamma_\pi(v - u^*) \ge \frac{c_0}{2} \|v - u^*\|_1^2 \quad \forall v \in \mathcal{U} \cap B_{L^1}(u^*; \tilde{\alpha}_0). \tag{5.14}$$

Let L be a bound for the Lipschitz constants of f, g and H their first derivatives in x, and H_{xu}, H_{up}. It is easy to see that $\tilde{L} := L + 2(1 + \bar{M})\zeta$ is a bound for the Lipschitz constants of $\tilde{f}, \tilde{g}$ and $\tilde{H}$, their first derivatives in x, and $\tilde{H}_{xu}, \tilde{H}_{up}$ for all $\pi \in \mathcal{H}_\zeta$. Analogously, we can find a bound $\tilde{M}$, depending on ζ and $\bar{M}$, for the functions $\tilde{f}, \tilde{H}$ and their derivatives (see Remark 2.1 in [9]). Finally, (5.14) implies that the hypotheses of Theorem 3.1 in [9] are fulfilled. We conclude that Φ_π is SMsR at y^* for zero. According to that theorem, the parameters of SMsR can be chosen as depending only on $\tilde{M}, T, c_0$ and $\tilde{L}$. This completes the proof. □

From now on, we only consider elements $\pi \in \mathcal{H}_\zeta$, since the latter theorem ensures that each map Φ_π is SMsR at a solution $y^* \in B_Y(\hat{y}, a)$ for zero. This automatically ensures that y^* is the unique local solution in $B_Y(\hat{y}, a)$ of the inclusion $0 \in \Phi_\pi(y)$.

Let $\{t_n\}_{n=0}^N$ be a grid on $[0, T]$ with equally spaced nodes and a step size h, that is, $t_k = kT/N$ for $i = 0, \ldots, N$. Given a $\pi \in \mathcal{H}_\zeta$, the discrete time problem $(\mathcal{P}_\pi^h)$ obtained by the Euler discretization is

$$\min_{u \in U^N} \left[h \sum_{i=0}^{N-1} \tilde{g}(x_i, u_i) \right] \tag{5.15}$$

subject to

$$x_{i+1} = x_i + h\tilde{f}(x_i, u_i), \quad x_0 = x^0. \tag{5.16}$$

The local form of the discrete time minimum principle implies that for any locally optimal solution (x, u) of problem $(\mathcal{P}^h_\pi)$ there exists a vector $p = (p_0, \ldots, p_N)$ such that

$$x_{i+1} = x_i + h\tilde{f}(x_i, u_i),\ x_0 = x^0, \tag{5.17}$$

$$\lambda_i = \lambda_{i+1} + h\nabla_x \tilde{H}(x_i, u_i, p_{i+1}),\ p_N = 0, \tag{5.18}$$

$$0 \in \nabla_u \tilde{H}(x_i, u_i, p_{i+1}) + N_U(u_i), \tag{5.19}$$

where i runs between 0 and $N-1$. Let (x^h, u^h) be a solution of problem $(\mathcal{P}^h_\pi)$ and p^h the corresponding co-state vector, so that $y^h = (x^h, p^h, u^h)$ satisfies (5.17)-(5.19). In order to compare this solution with the reference solution of $y^* = (x^*, p^*, u^*)$ of the continuous-time problem $(\mathcal{P}_\pi)$, we embed the sequence (x^h, p^h, u^h) into the space $W^{1,1} \times W^{1,1} \times L^1$ considering $y_h = (x_h, p_h, u_h)$ defined by

$$x_h(t) := x^h_i + \frac{t-t_i}{h}(x^h_{i+1} - x^h_i), \quad u_h(t) := u^h_i, \quad p_h(t) := p^h_i + \frac{t-t_i}{h}(p^h_{i+1} - p^h_i), \tag{5.20}$$

for $t \in [t_i, t_{i+1})$, $i = 0, \ldots, N-1$.

We need the following technical assumption to apply results in [9]. It is a crucial assumption, at least because it may happen that y_h is close to some other local solution of the continuous-time problem, and we have to eliminate this possibility.

Assumption (C1). Let $\pi \in \mathcal{H}_\zeta$. We assume that problem $(\mathcal{P}_\pi)$ has a solution y^* in $B_Y(\hat{y}, a)$. Moreover, the embedded solution y_h in (5.20) of problem $(\mathcal{P}^h_\pi)$ belongs to $B_Y(y^*, \tilde{a})$ for all sufficiently small h.

The following theorem is a direct consequence of Theorem 5.5 and Theorem 5.1 in [9].

Theorem 5.6. *There exists a positive constant C such that for all $\pi \in \mathcal{H}_\zeta$ for which Assumption (C1) holds, the estimate*

$$\|x_h - x^*\|_{1,1} + \|p_h - p^*\|_{1,1} + \|u_h - u^*\|_1 \le Ch \tag{5.21}$$

holds for all sufficiently small h.

Proof. By Theorem 5.5, the parameters $\tilde{a}, \tilde{b}, \tilde{\kappa}$ of SMsR of Φ_π at y^* for zero are the same for all $\pi \in \mathcal{H}_\zeta$ satisfying Assumption (C1).

Let $\pi \in \mathcal{H}_\zeta$. In order to make use of the SMsR property of the map Φ_π, we have to estimate the residuals

$$\Delta_1 := \dot{x}_h - \tilde{f}(x_h, u_h),$$

$$\Delta_2 := \dot{p}_h + \nabla_x \tilde{H}(x_h, p_h, u_h),$$

$$\Delta_3 := \nabla_u \tilde{H}(x^h_i, p^h_i, u^h_i) - \nabla_u \tilde{H}(x_h, p_h, u_h),\ t \in [t_i, t_{i+1}),\ i = 0, \ldots, N-1.$$

Repeating the calculations in the proof of Theorem [9, Theorem 5.1], we obtain

$$\max\{\|\Delta_1\|_1, \|\Delta_2\|_1, \|\Delta_3\|_\infty\} \le \max\{1, T\}\, \tilde{L}(1 + 2\tilde{M})h, \tag{5.22}$$

where $\tilde{L}, \tilde{M}$ are the numbers in the proof of Theorem 5.5. We can choose $h_0 > 0$ depending on $\tilde{L}, \tilde{M}, T$ and b so that $\|\Delta_1\|_1 + \|\Delta_2\|_1 + \|\Delta_3\|_\infty \leq b$ for all $h \leq h_0$. The claim follows from the SMsR property of Φ_π with $C := 3\kappa(1+2\tilde{M})\tilde{L} \max\{1, T\}$. The proof is complete since this holds for any arbitrary $\pi \in \mathcal{H}_\zeta$ satisfying Assumption (C1). □

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