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FIXED POINTS OF COMPOSITIONS OF NONEXPANSIVE MAPPINGS: FINITELY MANY LINEAR REFLECTORS

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ABSTRACT. Nonexpansive mappings play a central role in modern optimization and monotone operator theory because their fixed points can describe solutions to optimization or critical point problems. It is known that when the mappings are sufficiently "nice", then the fixed point set of the composition coincides with the intersection of the individual fixed point sets.

In this paper, we explore the situation for compositions of linear reflectors. We provide positive results, upper bounds, and limiting examples. We also discuss classical reflectors in the Euclidean plane.

1. INTRODUCTION

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}, \tag{1.1}$$

and induced norm $\|\cdot\|\colon X \to \mathbb{R}\colon x \mapsto \sqrt{\langle x, x\rangle}$. A mapping $R\colon X \to X$ is *nonexpansive* if $(\forall x \in X)(\forall y \in X)\ \|Rx - Ry\| \leq \|x - y\|$. Nonexpansive operators play a central role in modern optimization because the set of fixed points $\operatorname{Fix} R := \{x \in X \mid x = Rx\}$ often represents solutions to optimization or inclusion problems (see, e.g., [3]). A central question is the following

Given nonexpansive maps $R_1, R_2, \ldots, R_{m-1}, R_m$ on X with $\bigcap_{i=1}^m \operatorname{Fix} R_i \neq \varnothing$, what can we say about $\operatorname{Fix}(R_m R_{m-1} \cdots R_1)$?

Clearly,

$$\bigcap_{i=1}^m \operatorname{Fix} R_i \subseteq \operatorname{Fix}(R_m R_{m-1} \cdots R_1); \tag{1.2}$$

see also [6, Proposition 4.3] for a local version. Note that one cannot expect equality to hold in (1.2):

Example 1.1. Suppose that $X \neq \{0\}$. Then $\operatorname{Fix}(-\operatorname{Id}) \cap \operatorname{Fix}(-\operatorname{Id}) = \{0\}$ while $\operatorname{Fix}(-\operatorname{Id})(-\operatorname{Id}) = \operatorname{Fix}(\operatorname{Id}) = X$.

However, equality in (1.2) does hold for "nice" nonexpansive maps such as averaged mappings (see, e.g., [3, Corollary 4.51]) or even strongly nonexpansive mappings (see [5, Lemma 2.1]).

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In this note, we aim to study $\operatorname{Fix}(R_m R_{m-1} \cdots R_1)$ for certain mappings that are not nice but that do have some additional structure. To describe this, let us denote the *projector* (or nearest point mapping) associated with a nonempty closed convex subset C of X by P_C. The corresponding *reflector*

$$\mathrm{R}_C := 2\mathrm{P}_C - \mathrm{Id} \tag{1.3}$$

is known to be nonexpansive (see, e.g., [3, Corollary 4.18]). Note that

$$\operatorname{Fix} \mathrm{R}_C = C \tag{1.4}$$

and that $\mathrm{R}_{\{0\}} = -\,\mathrm{Id}$, so the class of reflectors is "bad" (see Example 1.1). We also have

$$\mathrm{R}_C = \mathrm{P}_C - \mathrm{P}_{C^\perp} \quad \text{provided that } C \text{ is a closed linear subspace of } X, \tag{1.5}$$

in which case $\mathrm{P}_C, \mathrm{P}_{C^\perp}, \mathrm{R}_C$ are all linear operators (see, e.g., [3, Corollary 3.24]). When C is a hyperplane containing the origin, then we shall refer to R_C as a *classical* reflector. Classical reflectors are basic building blocks: indeed, the *Cartan-Dieudonné Theorem* (see, e.g., [7, Theorem 8.1] or [11, Section 2.4]) states that every linear isometry on $\mathbb{R}^n$ is the composition of at most n classical reflectors.

A very satisfying result is available for two general linear reflectors:

Fact 1.2 ([2, Proposition 3.6])**.** *Let U_1 and U_2 be closed linear subspaces of X. Then*

$$\operatorname{Fix}(\mathrm{R}_{U_2} \mathrm{R}_{U_1}) = (U_1 \cap U_2) \oplus (U_1^\perp \cap U_2^\perp). \tag{1.6}$$

and

$$\mathrm{P}_{U_1} \operatorname{Fix}(\mathrm{R}_{U_2} \mathrm{R}_{U_1}) = U_1 \cap U_2. \tag{1.7}$$

Fact 1.2 was used in [2] to analyze the Douglas–Rachford operator $T := \frac{1}{2}(\mathrm{Id} + \mathrm{R}_{U_2} \mathrm{R}_{U_1})$. Note that $\operatorname{Fix} T = \operatorname{Fix}(\mathrm{R}_{U_2} \mathrm{R}_{U_1})$! It was shown that $\mathrm{P}_{U_1} T^n \to \mathrm{P}_{U_1 \cap U_2}$ pointwise. Iterating T is actually a special case of employing Rockafellar's proximal point algorithm [10].

We also note that Fact 1.2 provides an alternative explanation of Example 1.1: indeed, set $U_1 = U_2 = \{0\}$ in Fact 1.2. Then $\mathrm{R}_{U_1} = \mathrm{R}_{U_2} = -\,\mathrm{Id}$, $U_1^\perp = U_2^\perp = X$, and $\operatorname{Fix}(\mathrm{R}_{U_2} \mathrm{R}_{U_1}) = (U_1 \cap U_2) \oplus (U_1^\perp \cap U_2^\perp) = X$.

Fact 1.2 nurtures the hope that there might exist a nice formula for $\operatorname{Fix}(\mathrm{R}_{U_3} \mathrm{R}_{U_2} \mathrm{R}_{U_1})$ and that there might be a way to recover $U_1 \cap U_2 \cap U_3$ by projecting $\operatorname{Fix}(\mathrm{R}_{U_3} \mathrm{R}_{U_2} \mathrm{R}_{U_1})$ suitably. Unfortunately, this hope was crushed with the following example:

Example 1.3 ([1, Example 2.1])**.** Suppose that $X = \mathbb{R}^2$, $U_1 = \mathbb{R}(0,1)$, $U_2 = \mathbb{R}(\sqrt{3},1)$, and $U_3 = \mathbb{R}(-\sqrt{3},1)$. Then $U_1 \cap U_2 \cap U_3 = \{0\}$, $x := (-\sqrt{3},-1) \in \operatorname{Fix}(\mathrm{R}_{U_3} \mathrm{R}_{U_2} \mathrm{R}_{U_1})$ yet $\{\mathrm{P}_{U_1}x, \mathrm{P}_{U_2}x, \mathrm{P}_{U_3}x\} \cap U_1 \cap U_2 \cap U_3 = \varnothing$. The fixed point sets for all six permutations of the reflectors are depicted in Figure 1.

We are now in a position to describe precisely our aim.

The goal of this note is to study the fixed point set of the composition of finitely many reflectors associated with closed linear subspaces.

In Section 2, we obtain several positive results (see Lemma 2.3 and Theorem 2.6), an upper bound (see Theorem 2.7) as well as limiting examples. Section 3 focusses

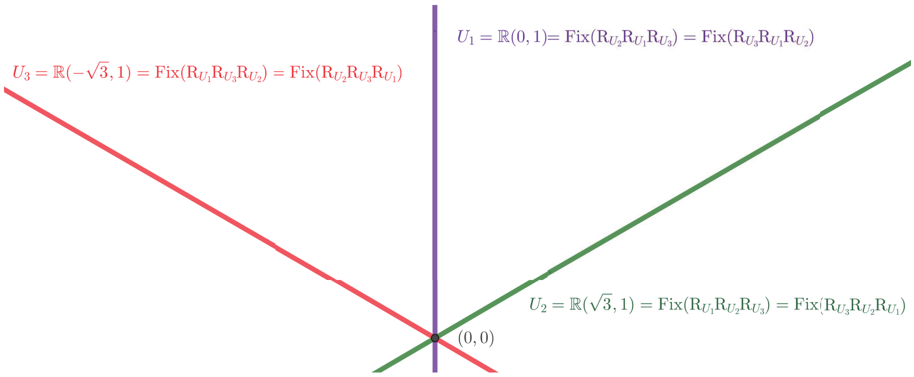

FIGURE 1. The fixed point sets for Example 1.3

mainly on classical reflectors in the Euclidean plane for which precise information is available. The notation employed is standard and follows largely [3].

2. General results

We start with a simple observation.

Lemma 2.1. *Let U be a closed linear subspace of X. Then the following hold:*

(i) $\mathrm{R}_{U^\perp}\mathrm{R}_U = \mathrm{R}_U\mathrm{R}_{U^\perp} = -\mathrm{Id}$.
(ii) $-\mathrm{R}_U = \mathrm{R}_U \circ (-\mathrm{Id}) = \mathrm{R}_{U^\perp}$.
(iii) $\mathrm{Fix}(-\mathrm{R}_U) = \mathrm{Fix}\,\mathrm{R}_{U^\perp} = U^\perp$.

Proof. We shall employ (1.5) repeatedly. (i): $\mathrm{R}_{U^\perp}\mathrm{R}_U = (\mathrm{P}_{U^\perp} - \mathrm{P}_U)(\mathrm{P}_U - \mathrm{P}_{U^\perp}) = 0 - \mathrm{P}_U - \mathrm{P}_{U^\perp} + 0 = -\mathrm{Id}$. (ii): $-\mathrm{R}_U = -(\mathrm{P}_U - \mathrm{P}_{U^\perp}) = (\mathrm{P}_{U^\perp} - \mathrm{P}_{U^{\perp\perp}}) = \mathrm{R}_{U^\perp}$. (iii): Combine (ii) with (1.4). ☐

The following result, which is a consequence of Lemma 2.1, provides a case when we have precise knowledge of the fixed point set of the composition of three reflectors:

Proposition 2.2. *Let U and V be closed linear subspaces of X. Then* $\mathrm{R}_V\mathrm{R}_{U^\perp}\mathrm{R}_U = \mathrm{R}_V\mathrm{R}_U\mathrm{R}_{U^\perp} = \mathrm{R}_{U^\perp}\mathrm{R}_U\mathrm{R}_V = \mathrm{R}_U\mathrm{R}_{U^\perp}\mathrm{R}_V = -\mathrm{R}_V = \mathrm{R}_{V^\perp}$ *and thus* $\mathrm{Fix}(\mathrm{R}_V\mathrm{R}_{U^\perp}\mathrm{R}_U) = \mathrm{Fix}(\mathrm{R}_V\mathrm{R}_U\mathrm{R}_{U^\perp}) = \mathrm{Fix}(\mathrm{R}_{U^\perp}\mathrm{R}_U\mathrm{R}_V) = \mathrm{Fix}(\mathrm{R}_U\,\mathrm{R}_{U^\perp}\mathrm{R}_V) = V^\perp$.

We now turn to m operators and obtain a very general result which clearly shows the effect of cyclically shifting a composition:

Lemma 2.3. *Let $R_1, \dots, R_m$ be arbitrary maps from X to X. Then*

$$\begin{aligned}
&\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1) && \text{(2.1a)}\\
&= (R_m R_{m-1} \cdots R_3 R_2)\big(\mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2)\big) && \text{(2.1b)}\\
&= (R_m R_{m-1} \cdots R_3)\big(\mathrm{Fix}(R_2 R_1 R_m R_{m-1} \cdots R_4 R_3)\big) && \text{(2.1c)}\\
&\ \vdots && \text{(2.1d)}\\
&= R_m R_{m-1}\big(\mathrm{Fix}(R_{m-2} R_{m-3} \cdots R_2 R_1 R_m R_{m-1})\big) && \text{(2.1e)}\\
&= R_m\big(\mathrm{Fix}(R_{m-1} R_{m-2} \cdots R_2 R_1 R_m)\big). && \text{(2.1f)}
\end{aligned}$$

Proof. Let $x \in \mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1)$. Then

$$R_1 x = R_1(R_m R_{m-1} \cdots R_2 R_1)x = (R_1 R_m R_{m-1} \cdots R_3 R_2)(R_1 x)$$

and so $R_1 x \in \mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2)$. It follows that

$$R_1\big(\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1)\big) \subseteq \mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2). \tag{2.2}$$

The same reasoning gives

$$\begin{aligned}
(R_2 R_1)\big(\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1)\big) &\subseteq R_2\big(\mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2)\big) && \text{(2.3a)}\\
&\subseteq \mathrm{Fix}(R_2 R_1 R_m R_{m-1} \cdots R_4 R_3), && \text{(2.3b)}
\end{aligned}$$

hence

$$\begin{aligned}
(R_3 R_2 R_1)\big(\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1)\big) &\subseteq (R_3 R_2)\big(\mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2)\big) && \text{(2.4a)}\\
&\subseteq R_3\big(\mathrm{Fix}(R_2 R_1 R_m R_{m-1} \cdots R_4 R_3)\big) && \text{(2.4b)}\\
&\subseteq \mathrm{Fix}(R_3 R_2 R_1 R_m R_{m-1} \cdots R_5 R_4) && \text{(2.4c)}
\end{aligned}$$

until finally

$$\begin{aligned}
\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1) &= (R_m R_{m-1} \cdots R_2 R_1)\big(\mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1)\big) && \text{(2.5a)}\\
&\subseteq (R_m R_{m-1} \cdots R_2)\big(\mathrm{Fix}(R_1 R_m R_{m-1} \cdots R_3 R_2)\big) && \text{(2.5b)}\\
&\ \ \vdots && \text{(2.5c)}\\
&\subseteq R_m\big(\mathrm{Fix}(R_{m-1} R_{m-2} \cdots R_2 R_1 R_m)\big) && \text{(2.5d)}\\
&\subseteq \mathrm{Fix}(R_m R_{m-1} \cdots R_2 R_1). && \text{(2.5e)}
\end{aligned}$$

Hence equality holds throughout (2.5) and we are done. ☐

Lemma 2.3 allows us to derive a result complementary to Proposition 2.2:

Proposition 2.4. *Let U and V be closed linear subspaces of X. Then*

$$\mathrm{Fix}(\mathrm{R}_{U^\perp}\, \mathrm{R}_V\, \mathrm{R}_U) = \mathrm{Fix}(\mathrm{R}_U\, \mathrm{R}_V\, \mathrm{R}_{U^\perp}) = \mathrm{R}_U(V^\perp). \tag{2.6}$$

Proof. Using Lemma 2.3 and Proposition 2.2, we obtain

$$\mathrm{Fix}(\mathrm{R}_U\, \mathrm{R}_V\, \mathrm{R}_{U^\perp}) = \mathrm{R}_U(\mathrm{Fix}(\mathrm{R}_V\, \mathrm{R}_U\, \mathrm{R}_{U^\perp})) = \mathrm{R}_U(V^\perp). \tag{2.7}$$

Now $\mathrm{R}_U(V^\perp)$ is a subspace and thus $\mathrm{R}_U(V^\perp) = (-\mathrm{R}_U)(V^\perp) = \mathrm{R}_{U^\perp}(V^\perp) = \mathrm{Fix}(\mathrm{R}_{U^\perp}\, \mathrm{R}_V\, \mathrm{R}_U)$ by the first part of the proof. ☐

Remark 2.5. Comparing Proposition 2.2 and Proposition 2.4, we note that it is *not necessarily true* that $\mathrm{R}_U(V^\perp) = V^\perp$; indeed, see Example 3.3 below for a concrete instance. Hence, unlike the case of just two linear reflectors (see Fact 1.2), *the order of the operators does influence the fixed point set!*

While Remark 2.5 points out the importance of the order of the operators, there does exist a nice permutation of the reflectors yielding the same fixed point set. To describe this result, observe first $(\mathrm{R}_{U_m} \cdots \mathrm{R}_{U_1})^* = \mathrm{R}^*_{U_1} \cdots \mathrm{R}^*_{U_m} = \mathrm{R}_{U_1} \cdots \mathrm{R}_{U_m}$ because linear projectors and (hence) reflectors are self-adjoint. Combining this with an old result by Riesz and Sz.-Nagy which states that $\operatorname{Fix} T = \operatorname{Fix} T^*$ for any nonexpansive linear operator $T\colon X \to X$ (see [9, page 408 in Section X.144] or [8]), we obtain the following positive result:

Theorem 2.6. *Let $U_1, \ldots, U_m$ be closed linear subspaces of X. Then*

$$\operatorname{Fix}(\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_2} \mathrm{R}_{U_1}) = \operatorname{Fix}(\mathrm{R}_{U_1} \mathrm{R}_{U_2} \cdots \mathrm{R}_{U_{m-1}} \mathrm{R}_{U_m}). \tag{2.8}$$

We now turn to three linear subspaces. The next result narrows down the location of fixed points.

Theorem 2.7. *Let U, V, W be closed linear subspaces of X. Then*

$$\operatorname{Fix}(\mathrm{R}_W \mathrm{R}_V \mathrm{R}_U) = \operatorname{Fix}\big(4\mathrm{P}_W\mathrm{P}_V\mathrm{P}_U - 2(\mathrm{P}_W\mathrm{P}_V + \mathrm{P}_W\mathrm{P}_U + \mathrm{P}_V\mathrm{P}_U) + \mathrm{P}_W + \mathrm{P}_V + \mathrm{P}_U\big) \tag{2.9}$$

and

$$\operatorname{Fix}(\mathrm{R}_W \mathrm{R}_V \mathrm{R}_U) \subseteq U + V + W. \tag{2.10}$$

Proof. Let $x \in X$. Then $x \in \operatorname{Fix}(\mathrm{R}_W \mathrm{R}_V \mathrm{R}_U) \Leftrightarrow x = (2\mathrm{P}_W - \mathrm{Id})(2\mathrm{P}_V - \mathrm{Id})(2\mathrm{P}_U - \mathrm{Id})x$ and (2.9) follows by expanding and simplifying. In turn, $\operatorname{Fix}(\mathrm{R}_W \mathrm{R}_V \mathrm{R}_U) \subseteq U+V+W$ because $\operatorname{Fix}(4\mathrm{P}_W\mathrm{P}_V\mathrm{P}_U - 2(\mathrm{P}_W\mathrm{P}_V + \mathrm{P}_W\mathrm{P}_U + \mathrm{P}_V\mathrm{P}_U) + \mathrm{P}_W + \mathrm{P}_V + \mathrm{P}_U) \subseteq \operatorname{ran}(4\mathrm{P}_W\mathrm{P}_V\mathrm{P}_U - 2(\mathrm{P}_W\mathrm{P}_V + \mathrm{P}_W\mathrm{P}_U + \mathrm{P}_V\mathrm{P}_U) + \mathrm{P}_W + \mathrm{P}_V + \mathrm{P}_U) \subseteq W + V + U$. ☐

The approach utilized in the proof of Theorem 2.7 to derive the description of the fixed point set also works for any *odd* number of reflectors; however, the resulting algebraic expressions don't seem to provide further insights. The superset obtained; however, will easily generalize to an *odd* number of reflectors:

Theorem 2.8. *Let $U_1, \ldots, U_m$ be an* odd *number of closed linear subspaces of X. Then*

$$\operatorname{Fix}(\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1}) \subseteq U_1 + U_2 + \cdots + U_m. \tag{2.11}$$

Proof. Let $x \in X$. Then $x \in \operatorname{Fix}(\mathrm{R}_{U_m} \cdots \mathrm{R}_{U_1}) \Leftrightarrow x = \mathrm{R}_{U_m} \cdots \mathrm{R}_{U_1} x \Leftrightarrow x = (2\mathrm{P}_{U_m} - \mathrm{Id}) \cdots (2\mathrm{P}_{U_1} - \mathrm{Id})x \Rightarrow x \in (-1)^m x + \operatorname{ran}(\sum_i \mathrm{P}_{U_i}) \Rightarrow 2x \in \operatorname{ran}(\sum_i \mathrm{P}_{U_i}) \subseteq \sum_i U_i$. ☐

Remark 2.9. Theorem 2.8 is false when m is assumed to be even: indeed, assume that U is a proper closed linear subspace of X, and set $U_1 := U_2 := U$. Then $\mathrm{R}_{U_2} \mathrm{R}_{U_1} = \mathrm{Id}$ and hence

$$\operatorname{Fix}(\mathrm{R}_{U_2} \mathrm{R}_{U_1}) = X \supsetneqq U = U_1 + U_2. \tag{2.12}$$

The next example shows that the upper bound provided in Theorem 2.8 is sometimes sharp:

Example 2.10. Let $U_1, \ldots, U_m$ be closed linear subspaces of X which are assumed to be *pairwise orthogonal*: $U_i \perp U_j$ whenever $i \neq j$. Then either

$$m \text{ is odd and } \operatorname{Fix}(\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1}) = U_1 + U_2 + \cdots + U_m \tag{2.13}$$

or

$$m \text{ is even and } \operatorname{Fix}(-\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1}) = U_1 + U_2 + \cdots + U_m. \tag{2.14}$$

Proof. Assume first that m is odd. Let $(x_1,\ldots,x_m) \in U_1 \times \cdots \times U_m$ and set $x = x_1 + \cdots + x_m$. Write $\mathrm{R}_{U_i} = \mathrm{P}_{U_i} - \mathrm{P}_{U_i^\perp}$ for each i (see (1.5)). Then

$$\mathrm{R}_{U_1} x = (\mathrm{P}_{U_1} - \mathrm{P}_{U_1^\perp})(x_1 + \cdots + x_m) = x_1 - x_2 - \cdots - x_m. \tag{2.15}$$

Hence

$$\mathrm{R}_{U_2} \mathrm{R}_{U_1} x = (\mathrm{P}_{U_2} - \mathrm{P}_{U_2^\perp})(x_1 - x_2 \cdots - x_m) = -x_1 - x_2 + x_3 \cdots + x_m. \tag{2.16}$$

and further

$$\begin{aligned} \mathrm{R}_{U_3} \mathrm{R}_{U_2} \mathrm{R}_{U_1} x &= (\mathrm{P}_{U_3} - \mathrm{P}_{U_3^\perp})(-x_1 - x_2 + x_3 \cdots + x_m) \\ &= x_1 + x_2 + x_3 - x_4 - \cdots - x_m. \end{aligned} \tag{2.17}$$

In general, for $1 \leq k \leq m$, we have

$$(-1)^{k-1} \mathrm{R}_{U_k} \mathrm{R}_{U_{k-1}} \cdots \mathrm{R}_{U_1} x = (x_1 + \cdots + x_k) - (x_{k+1} + \cdots + x_m). \tag{2.18}$$

In particular, because $m-1$ is even, we obtain $\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} = x$. This completes the proof of (2.13).

Now assume that m is even. Set $U_{m+1} := \{0\}$. Then $\mathrm{R}_{U_{m+1}} = -\operatorname{Id}$ and $m+1$ is odd. Therefore, we obtain (2.14) from the odd case we just proved. ☐

In contrast to Example 2.10, we conclude this section with another example which will illustrate that the upper bound in Theorem 2.8 is not always attained:

Example 2.11. Assume that U is a closed linear subspace of X such that $\{0\} \subsetneqq U$. Let m be an odd positive integer, and let $i \in \{1,\ldots,m\}$. Then set $U_i := \{0\}$ and $U_j := U$ for every $j \neq i$. Then $m-1$ is even and $\mathrm{R}_{\{0\}} = -\operatorname{Id}$. Hence $\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1} = -\mathrm{R}_U^{m-1} = -\operatorname{Id}$ and therefore

$$\operatorname{Fix}(\mathrm{R}_{U_m} \mathrm{R}_{U_{m-1}} \cdots \mathrm{R}_{U_1}) = \operatorname{Fix}(-\operatorname{Id}) = \{0\} \subsetneqq U = U_1 + U_2 + \cdots + U_m. \tag{2.19}$$

3. The Euclidean plane $\mathbb{R}^2$

Let us now specialize the general result of the last section to the Euclidean plane and classical reflectors. We start with some well known results whose statements can be found, e.g., in [12].

Set

$$\operatorname{Refl}\colon \mathbb{R} \to \mathbb{R}^{2\times 2}\colon \alpha \mapsto \begin{pmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{pmatrix}. \tag{3.1}$$

It is clear that Refl is periodic, with minimal period π. The importance of Refl stems from the fact that it describes all classical reflectors on $\mathbb{R}^2$; indeed,

$$\mathrm{R}_{\mathbb{R}\cdot(\cos(\alpha),\sin(\alpha))} = \operatorname{Refl}(\alpha) \tag{3.2}$$

for every $\alpha \in \mathbb{R}$. It is convenient to also define

$$\operatorname{Rot}\colon \mathbb{R} \to \mathbb{R}^{2\times 2}\colon \alpha \mapsto \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}. \tag{3.3}$$

Note that for every $\alpha \in \mathbb{R}$, $\operatorname{Rot}(\alpha)$ describes the counterclockwise rotation by α; the operator Rot is periodic with minimal period 2π.

The following result provides "calculus rules" for the composition of reflectors and rotators. It can be verified using matrix multiplication and addition theorems for sine and cosine.

Fact 3.1. *Let α and β be in $\mathbb{R}$. Then the following hold:*

(i) $\mathrm{Rot}(\beta)\mathrm{Rot}(\alpha) = \mathrm{Rot}(\alpha+\beta)$.

(ii) $\mathrm{Refl}(\beta)\mathrm{Refl}(\alpha) = \mathrm{Rot}(2(\beta-\alpha))$.

(iii) $\mathrm{Rot}(\beta)\mathrm{Refl}(\alpha) = \mathrm{Refl}(\alpha+\frac{1}{2}\beta)$.

(iv) $\mathrm{Refl}(\beta)\mathrm{Rot}(\alpha) = \mathrm{Refl}(\beta-\frac{1}{2}\alpha)$.

We are now in a position to classify the fixed point sets of compositions of classical reflectors on $\mathbb{R}^2$:

Theorem 3.2. *Let $\alpha_1,\ldots,\alpha_m$ be in $\mathbb{R}$. Consider the composition of m classical reflectors,*

$$S_m := \mathrm{Refl}(\alpha_m)\cdots\mathrm{Refl}(\alpha_1)\mathrm{Refl}(\alpha_1), \tag{3.4}$$

and set $\beta_m := \alpha_m - \alpha_{m-1} + \cdots - (-1)^m\alpha_1$. Then exactly one of the following holds:

(i) *m is odd, $S_m = \mathrm{Refl}(\beta_m)$, and $\mathrm{Fix}\, S_m = \mathbb{R}(\cos(\beta_m),\sin(\beta_m))$.*

(ii) *m is even, $S_m = \mathrm{Rot}(2\beta_m)$, and $\mathrm{Fix}\, S_m = \begin{cases}\mathbb{R}^2, & \text{if } \beta_m \in \mathbf{Z}\pi;\\ \{0\}, & \text{otherwise.}\end{cases}$*

Proof. We proceed by induction on m, discussing the odd and even cases separately.

Base case: Case 1: Assume that $m = 1$. Then, we have $\beta_1 = \alpha_1$ and $S_1 = \mathrm{Refl}(\alpha_1) = \mathrm{Refl}(\beta_1)$ so $\mathrm{Fix}\, S_1 = \mathrm{Fix}\,\mathrm{Refl}(\alpha_1) = \mathbb{R}(\cos(\alpha_1),\sin(\alpha_1)) = \mathbb{R}(\cos(\beta_1),\sin(\beta_1))$ by (3.2) as announced.

Case 2: Now assume that $m = 2$. Then $\beta_2 = \alpha_2 - \alpha_1$. Using Fact 3.1(ii), we obtain $S_2 = \mathrm{Refl}(\alpha_2)\mathrm{Refl}(\alpha_1) = \mathrm{Rot}(2(\alpha_2-\alpha_1)) = \mathrm{Rot}(2\beta_2)$ and the claim follows.

Inductive step: We assume that the result is true for some integer $m \geq 2$. Then

$$S_{m+1} = \mathrm{Refl}(\alpha_{m+1})\mathrm{Refl}(\alpha_m)\cdots\mathrm{Refl}(\alpha_2)\mathrm{Refl}(\alpha_1) = \mathrm{Refl}(\alpha_{m+1})S_m \tag{3.5}$$

and

$$\beta_{m+1} = \alpha_{m+1} - \beta_m. \tag{3.6}$$

Case 1: $m+1$ is odd; equivalently, m is even. Using the inductive hypothesis, Fact 3.1(iv), and (3.6), we obtain

$$S_{m+1} = \mathrm{Refl}(\alpha_{m+1})S_m = \mathrm{Refl}(\alpha_{m+1})\mathrm{Rot}(2\beta_m) \tag{3.7a}$$

$$= \mathrm{Refl}(\alpha_{m+1} - \tfrac{1}{2}(2\beta_m)) = \mathrm{Refl}(\beta_{m+1}) \tag{3.7b}$$

and the result follows.

Case 2: $m+1$ is even; equivalently, m is odd. Using the inductive hypothesis, Fact 3.1(ii), and (3.6), we obtain

$$S_{m+1} = \mathrm{Refl}(\alpha_{m+1})S_m = \mathrm{Refl}(\alpha_{m+1})\mathrm{Refl}(\beta_m) \tag{3.8a}$$

$$= \mathrm{Rot}(2(\alpha_{m+1}-\beta_m)) = \mathrm{Rot}(2\beta_{m+1}) \tag{3.8b}$$

and the result follows. $\square$

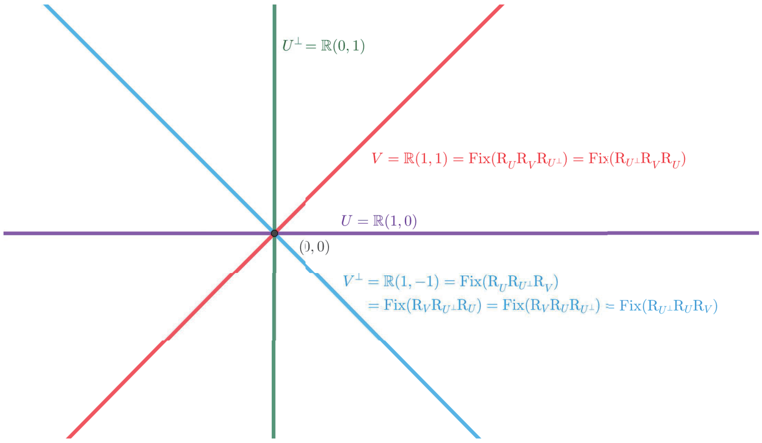

FIGURE 2. The fixed point sets for Example 3.3

The next example, which was used in an algorithmic context in [4, Example 2.30], illustrates Proposition 2.2, Proposition 2.4, and Theorem 3.2.

Example 3.3 ([4, Example 2.30])**.** Set $U := \mathbb{R}(1,0)$ so that $U^\perp = \mathbb{R}(0,1)$, and $V := \mathbb{R}(1,1)$. Then $\mathrm{R}_U = \mathrm{Refl}(0)$, $\mathrm{R}_{U^\perp} = \mathrm{Refl}(\pi/2)$, and $\mathrm{R}_V = \mathrm{Refl}(\pi/4)$. Moreover, the following hold:

(i) $\mathrm{Fix}(\mathrm{R}_{U^\perp}\,\mathrm{R}_V\,\mathrm{R}_U) = \mathrm{Fix}(\mathrm{Refl}(\pi/2)\mathrm{Refl}(\pi/4)\mathrm{Refl}(0))$
$= \mathbb{R}(\cos(\pi/4),\,\sin(\pi/4)) = \mathbb{R}(1,1) = V = \mathrm{R}_U(V^\perp)$.

(ii) $\mathrm{Fix}(\mathrm{R}_U\,\mathrm{R}_V\,\mathrm{R}_{U^\perp}) = \mathrm{Fix}(\mathrm{Refl}(0)\mathrm{Refl}(\pi/4)\mathrm{Refl}(\pi/2))$
$= \mathbb{R}(\cos(\pi/4),\,\sin(\pi/4)) = \mathbb{R}(1,1) = V = \mathrm{R}_U(V^\perp)$.

(iii) $\mathrm{Fix}(\mathrm{R}_V\,\mathrm{R}_{U^\perp}\,\mathrm{R}_U) = \mathrm{Fix}(\mathrm{Refl}(\pi/4)\mathrm{Refl}(\pi/2)\mathrm{Refl}(0))$
$= \mathbb{R}(\cos(-\pi/4),\,\sin(-\pi/4)) = \mathbb{R}(1,-1) = V^\perp$.

(iv) $\mathrm{Fix}(\mathrm{R}_V\,\mathrm{R}_U\,\mathrm{R}_{U^\perp}) = \mathrm{Fix}(\mathrm{Refl}(\pi/4)\mathrm{Refl}(0)\mathrm{Refl}(\pi/2))$
$= \mathbb{R}(\cos(3\pi/4),\,\sin(3\pi/4)) = \mathbb{R}(-1,1) = V^\perp$.

(v) $\mathrm{Fix}(\mathrm{R}_{U^\perp}\,\mathrm{R}_U\,\mathrm{R}_V) = \mathrm{Fix}(\mathrm{Refl}(\pi/2)\mathrm{Refl}(0)\mathrm{Refl}(\pi/4))$
$= \mathbb{R}(\cos(3\pi/4),\,\sin(3\pi/4)) = \mathbb{R}(-1,1) = V^\perp$.

(vi) $\mathrm{Fix}(\mathrm{R}_U\,\mathrm{R}_{U^\perp}\,\mathrm{R}_V) = \mathrm{Fix}(\mathrm{Refl}(0)\mathrm{Refl}(\pi/2)\mathrm{Refl}(\pi/4))$
$= \mathbb{R}(\cos(-\pi/4),\,\sin(-\pi/4)) = \mathbb{R}(1,-1) = V^\perp$.

Remark 3.4. Example 3.3 clearly shows that the order of the reflectors influences the fixed point set. See also Figure 2 for a visualization.

Example 3.5. Let $\gamma \in \mathbb{R}$, and let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ all be small in absolute value. Set $\alpha_1 := \gamma + \pi/6 + \varepsilon_1$, $\alpha_2 := \gamma + \varepsilon_2$, $\alpha_3 := \gamma - \pi/6 + \varepsilon_3$, and $\varepsilon := \varepsilon_1 - \varepsilon_2 + \varepsilon_3$, and suppose that $U_i = \mathbb{R}(\cos(\alpha_i), \sin(\alpha_i))$ for $i \in \{1,2,3\}$. Then it follows from Theorem 3.2 that

$$\mathrm{R}_{U_3}\,\mathrm{R}_{U_2}\,\mathrm{R}_{U_1} = \mathrm{Refl}(\alpha_3)\mathrm{Refl}(\alpha_2)\mathrm{Refl}(\alpha_1) = \mathrm{Refl}(\alpha_3 - \alpha_2 + \alpha_1) \tag{3.9a}$$

$$= \mathrm{Refl}(\gamma + \varepsilon) \tag{3.9b}$$

and

$$\mathrm{Fix}(\mathrm{R}_{U_3}\,\mathrm{R}_{U_2}\,\mathrm{R}_{U_1}) = \mathbb{R}\big(\cos(\gamma+\varepsilon), \sin(\gamma+\varepsilon)\big). \tag{3.10}$$

However, $U_1 \cap U_2 \cap U_3 = \{0\}$.

Remark 3.6. Consider the setting of Example 3.5.

(i) No matter which of the operators in $\{\mathrm{P}_{U_1}, \mathrm{P}_{U_2}, \mathrm{P}_{U_3}, \mathrm{P}_{U_1^\perp}, \mathrm{P}_{U_2^\perp}, \mathrm{P}_{U_3^\perp}\}$ we apply to $\mathrm{Fix}(\mathrm{R}_{U_3}\,\mathrm{R}_{U_2}\,\mathrm{R}_{U_1})$, we always obtain a line and never the singleton $U_1 \cap U_2 \cap U_3 = \{0\}$.

(ii) If each $\varepsilon_i = 0$, then $\varepsilon = 0$ and $\mathrm{Fix}(\mathrm{R}_{U_3}\,\mathrm{R}_{U_2}\,\mathrm{R}_{U_1}) = U_2$. If additionally $\gamma = \pi/6$, then $U_2 = \mathbb{R}(\cos(\pi/6), \sin(\pi/6)) = \mathbb{R}(\sqrt{3}/2, 1/2)$ and we recover precisely Example 1.3.

We conclude with a comment on higher-dimensional Euclidean space.

Remark 3.7 ($\mathbb{R}^3$ and beyond)**.** Considering reflectors and rotations in $\mathbb{R}^3$ (see, e.g., [13]) or even $\mathbb{R}^n$ is more complicated because there is no "easy" counterpart of Fact 3.1. However, using the fact that eigenvalues of isometries are always drawn from ± 1 or from nonreal complex conjugate pairs of magnitude 1, one obtains at least the parity result that

$$m + n \equiv \dim(\mathrm{Fix}(R_m R_{m-1} \cdots R_1)) \mod 2 \tag{3.11}$$

for m classical reflectors $R_1, \ldots, R_m$ on $\mathbb{R}^n$.

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