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OPTIMAL DYNAMIC PERIMETER CONTROL WITH VIRTUAL QUEUE

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Abstract. The problem of perimeter dynamical control with virtual queue is analysed and solved by the use of the optimal control theory. All possible eight different cases of solutions are presented and analysed depending on parameters and input data of the system. The results are shown to be optimal with sufficiency.

1. Introduction

The problem of an urban perimeter control in the urban area to improve mobility and prevent congestions and traffic jams has been intensively investigated from the different points of view, see e.g. [2–4]. The problem of perimeter dynamic control with virtual queue for so-called "single reservoir system" was considered in a well known paper [1] where the different policies were suggested by direct investigation of the control variations and different properties of optimality were shown. That paper describes an adaptive control approach to improve urban mobility and relieve congestion. The basic idea consists in monitoring and controlling aggregate vehicular accumulations at the neighborhood level. Evidently the processes of vehicles in the zone and waiting at the perimeter border vehicles are random processes. Thus the considered processes of current accumulation of vehicles in the zone $n(t)[veh]$ and the waiting at the perimeter border vehicles so-called " virtual queue " $VQ(t)[veh]$ should be considered as the mathematical expectations of the real random processes.

Here we take another approach in respect to optimization and present the full solution for all feasible cases by the application of the modern, straightforward, and appropriate approach of optimal control theory, [7], to get optimal solutions in a simple and general way. The aggregated model formulation was taken from [6] and somehow modified to avoid the state discontinuous right-hand side of ODE.

2. The model

The inflow demand $I(t)$ $[veh/s]$ to an urban area is restricted in order to minimize the total delay both for vehicles inside the city area and vehicles that are waiting at the "virtual queue" on the border of the perimeter. The internal demand is assumed to be relatively small and is neglected. The control variable (dimensionless) $u(t)$ is acting as the virtual green split of signalizing green-red lights on the boundary

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of the area. The current accumulation of vehicles in the zone is $n(t)$ $[veh]$. The waiting at the perimeter border vehicles are stored in a so-called "virtual queue" $VQ(t)$ $[veh]$. The boundary transfer capacity is denoted as c $[veh/s]$. The output flow in the area (trip completion rate) is presented by the unimodal MFD function $O(n)$ $[veh/s]$. $O(0) = 0$, and it can exhibit a range of multiple maxima with equal flow O^* at some horizontal interval in n. The output exit function according to [1] presents a macroscopic fundamental diagram, MFD.

We shall distinguish Case 1 when $0^* \le c$ and Case 2 when $c \le O^*$.

The dynamic model has the following form

$$\begin{aligned} \frac{dn}{dt} &= u(t)c - O(n), \\ \frac{dVQ(t)}{dt} &= I(t) - u(t)c, \\ J = \int_0^T (n(t) + VQ(t))dt &\to \min, \\ 0 \le u \le 1,\ 0 \le VQ(t),\ n(0) &= n_0,\ VQ(0) = VQ_0. \end{aligned} \tag{2.1}$$

We shall denote

$$O^* = \max_n O(n),$$

and

$$n^* = arg \min_n \ \forall n : \{O(n) = O^*.\}$$

Thus we have

$$O(n^*) = O^*.$$

For the right end of the possible interval with $O = O^*$ we shall denote

$$n^{**} = arg \max_n \ \forall n : \{O(n) = O^*.\}$$

Note that

$$\frac{dO(n)}{dn} > 0 \ \forall n : \ \{n < n^*\}$$

and

$$\frac{dO(n)}{dn} < 0 \ \forall n : \ \{n > n^{**}\}.$$

We also denote

$$t^* = arg \min_t \ \forall t : \{n(t) = n^*\},$$

thus

$$n(t^*) = n^*,$$
$$t^* = arg \min_t \ \forall t : \{n(t) = n^{**}\},$$

thus

$$n(t^{**}) = n^{**},$$

and

$$n_c = arg \min_n (O(n) = c),$$

thus

$$O(n_c) = c.$$

Other notation is

$$t_c = arg(n(t) = n_c),$$

thus

$$n(t_c) = n_c.$$

Also

$$\bar{n}_c = arg\max_n(O(n) = c),$$

thus

$$O(\bar{n}_c) = c.$$

Other notation is

$$\bar{t}_c = arg(n(t) = \bar{n}_c),$$

thus

$$n(\bar{t}_c) = \bar{n}_c.$$

The schematic plot of the output exit function $O(n)$ is shown in Fig. 1.

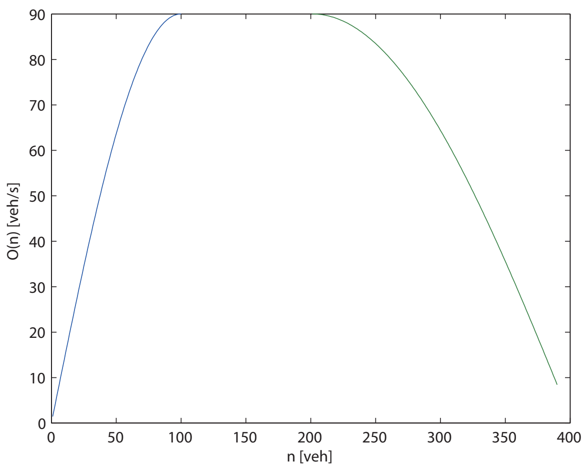

FIGURE 1. Output exit function $O(n)$

3. Conditions of optimality

Following Pontryagin Maximal Principle (PMP), [7], we consider the differential system with state constraint and free end condition as

$$\begin{aligned} \frac{dx}{dt} &= f(x,u), \\ J = \int_{t_0}^{T} f_0(x,u)dt &\to \min, \\ x(t_0) &= x_0,\ u \in U,\ 0 \leq g(x), \end{aligned} \tag{3.1}$$

and form the augmented Hamiltonian as

$$H = p^T f(x,u) - f_0(x,u) - \lambda g(x) \tag{3.2}$$

The conditions of optimality are:

$$\begin{aligned} u &= arg \max_u H(p,u,x) \\ \frac{dp}{dt} &= -\frac{\partial H}{dx}, \\ p(T) &= 0,\ 0 \le \lambda, \end{aligned} \tag{3.3}$$

and the transversality conditions for the free state variables at the end $t = T$ as

$$p(T) = 0.$$

In respect to the model (2.1) we have according to PMP the following:

$$\begin{aligned} H &= p_n(u(t)c - O(n)) + p_{VQ}(I(t) - u(t)c) - n - VQ - \lambda(-VQ), \\ u &= sign(p_n - p_{VQ}), \\ \frac{dp_n}{dt} &= 1 + \frac{dO}{dn}p_n,\ \frac{dp_{VQ}}{dt} = 1 - \lambda, \\ p_n(T) &= 0,\ p_{VQ}(T) = 0, \\ 0 &\le \lambda,\ p_n(T) = p_{VQ}(T) = 0. \end{aligned} \tag{3.4}$$

When the switching function $S = p_n - p_{VQ} = 0$ we have a so-called '*singular arc*' and the control u is not determined from the maximization of the augmented Hamiltonian.

Intuitively it is clear that the optimal policy is to keep variable n as close to the interval $[n^*, n^{**}]$ as possible. If the lower constraint for the virtual queue is active, $VQ = 0$, then the control is fully determined by this constraint, $u = I/c$. However to strongly prove the optimality we need to construct the corresponding trajectories for costates which satisfies the differential equations and transversality conditions from (3.4).

4. Classification

There are the following different cases and subcases to be considered:

(1) $0^* \le c$ and $0^* \ge c$
(2) $0^* \ge c$ and $0^* \le c$
(3) $n(t_0) < n^*$ and $n(t_0) > n^{**}$

Thus altogether there are eight cases and subcases to be considered.

5. Optimization - Case 1

5.1. **Subcase 1.1.** Here we have $0^* \le c$, $VQ > 0$. We also assume that $n(t_0) < n^*$. We propose the following scenario and show that it is the optimal solution. The optimal control is $u = 1$ while n increases to the value $n = n^*$. Then the control can take any value while the condition $n^* \le n \le n^{**}$ is satisfied.

To fit this scenario the costate trajectories must satisfy the conditions

(1)
$$p_{VQ}(t) \le p_n(t) < 0,\ \lambda = 0\ \forall t : \{t_0 \le t < t^* < T, \}$$

(2)
$$p_{VQ}(t) = p_n(t) \le 0,\ \lambda = 0,\ \forall t : \{t^* \le t < T\},$$

(3)
$$p_n(T) = p_{VQ}(T) = 0.$$
Note that $dO(n)/dn > 0$ when $n < n^*$, and thus $dp_n/dt < dp_{VQ}/dt$ when $n < n^*$. These conditions easily may be satisfied according to the costate differential equations from (3.4). The optimal trajectories of costates are schematically shown on Fig. 2. Note that at $t = t^*$ we have $n = n^*$ and for $t^* \leq t,\ n^* \leq n \leq n^{**}$ we

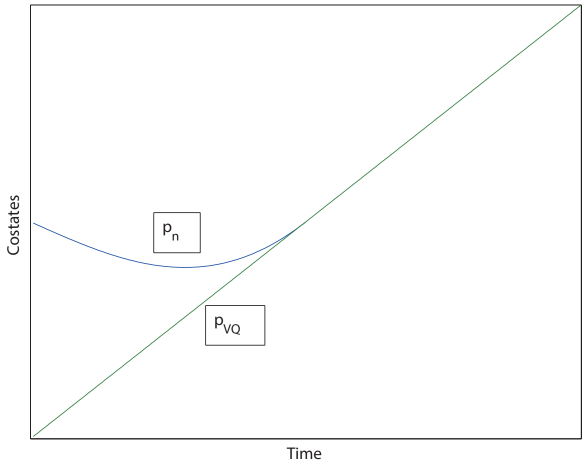

FIGURE 2. Schematic optimal trajectories of costates - case 1, subcase 1

have $dO(n)/dn = 0$ and thus $dp_n/dt = dp_{VQ}/dt = 1$. The optimal policy according to PMP is as follows:

(1)
$$u = 1\ \forall t : \{t_0 \leq t < t^*\},\ n : \{n_0 \leq n < n^*,\}$$
(2)
$$0 \leq u \leq 1,\ \forall t : \{t^* \leq t \leq T\},\ n : \{n^* \leq n \leq n^{**}\}.$$
At the interval 2 we have the singular arc with $S = dp_n/dt - dp_{VQ}/dt = 0,\ dS/dt = 0$.

5.2. **Subcase 1.2.** Here we have $0^* \leq c$. We also assume that $n(t_0) < n^*$. Starting from some point $t = t_Q < t^*$ we have $VQ = 0$. This means that at $t \leq t_Q$ we have $I < cu$ and $dVQ/dt < 0$. We propose the following scenario:

(1)
$$u = 1\ \forall t : \{t_0 \leq t < t_Q,\ VQ > 0,\ dVQ/dt < 0,\ 0 < S,\ n : \{n_0 \leq n < n(t_Q),\}$$
(2)
$$u = I/c\ \forall t : \{t_Q \leq t \leq T,\ VQ = 0,\}$$
$$dVQ/dt = I - cu = 0,\ S = 0,\}$$
Accordingly n increased if (a) $I > O(n)$ and decreased if (b) $I < O(n)$. When we have subcase (a) and $n = n^*,\ S = 0$ then the control u can take any value $O^* \leq u \leq I/c$. If $u < I/c$ then the constraint $VQ = 0$ will became inactive.

The behaviour of costates according to the costate equations from (3.4) will be as follows:

(1)
$$0 \geq p_n > p_{VQ},\ \lambda = 0\ \forall t : \{t_0 \leq t < t_Q,\ VQ > 0,\ dVQ/dt < 0,\ 0 < S\}$$

(2)
$$0 \geq p_n = p_{VQ},\ \ \lambda = -(dO/dn)p_n \geq 0\ \forall t : \{t_Q \leq t \leq T,$$
$$VQ = 0,\ dVQ/dt = I - cu = 0,\ S = 0,\}$$
$$p_n(T) = p_{VQ}(T) = 0.$$

If we have subcase (a) and $t^{**} < T$ then we take

$$u = O^*/c < I/c,\ \forall\, t : \{t^{**} \leq t \leq T,\}$$

and the constraint will be inactive. The schematic costate trajectories for subcase 1.2 are shown in Fig.3.

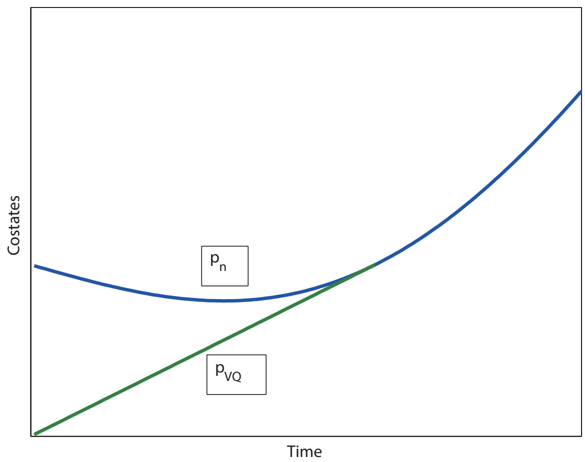

FIGURE 3. Schematic optimal trajectories of costates - case 1, subcase 2

5.3. **Subcase 1.3.** Here we have $0^* \leq c$, $VQ > 0$. We also assume that $n(t_0) > n^{**}$. We propose the following scenario and show that it is the optimal solution. The optimal control is $u = 0$ while n decreases to the value $n = n^{**}$. Then the control can take any value while the condition $n^* \leq n \leq n^{**}$ is satisfied.

To fit this scenario the costate trajectories must satisfy the conditions

(1)
$$p_{VQ}(t) \geq p_n(t) < 0\ \lambda = 0,\ \forall t : \{t_0 \leq t < t^{**} \leq T\},$$

(2)
$$p_{VQ}(t) = p_n(t) \leq 0,\ \lambda = 0,\ \forall t : \{t^{**} \leq t \leq T\},$$

(3)
$$p_n(T) = p_{VQ}(T) = 0.$$

Note that we have $\frac{dO(n)}{dn} < 0$ when $n > n^{**}$, and thus $\frac{dp_n}{dt} > \frac{dp_{VQ}}{dt}$ when $n > n^{**}$. These conditions easily may be satisfied according to the costate differential equations from (3.4). The optimal trajectories of costates are schematically shown on Fig. 4.

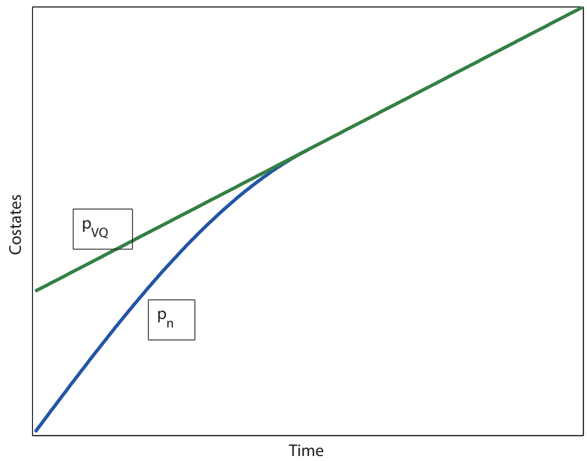

FIGURE 4. Schematic optimal trajectories of costates - case 1, subcase 3

5.4. **Subcase 1.4.** Here we should have $c \geq O^*$, and $n > n^{**}$, $VQ = 0$. This case is not feasible because with the optimal policy $u = 0$ it follows that $dVQ/dt = I > 0$ and thus the active constraint $VQ = 0$ can not be kept.

6. Optimization - Case 2

6.1. **Subcase 2.1.** Here we have $0^* \geq c$. We also assume that $VQ > 0$, and $n(t_0) < n^*$.

We propose the following scenario

$$u = 1 \; \forall t : \{t_0 \leq t < T, \; VQ > 0, \; 0 < S.\}$$

Accordingly n increased to the value $n = n_c$ where $O(n_c) = c$.

The behaviour of costates according to the costate equations from (3.4) will be as follows:

$$p_n > p_{VQ}, \; \lambda = 0 \; \forall t : \{t_0 \leq t < T, \; VQ > 0, \;\; 0 < S\}$$

$$p_n(T) = p_{VQ}(T) = 0, \;\; \lambda = 0.$$

The value n increases to $n = n_c$ and remains at this steady-state value.

The schematic costate trajectories for subcase 2.1 are shown in Fig.5.

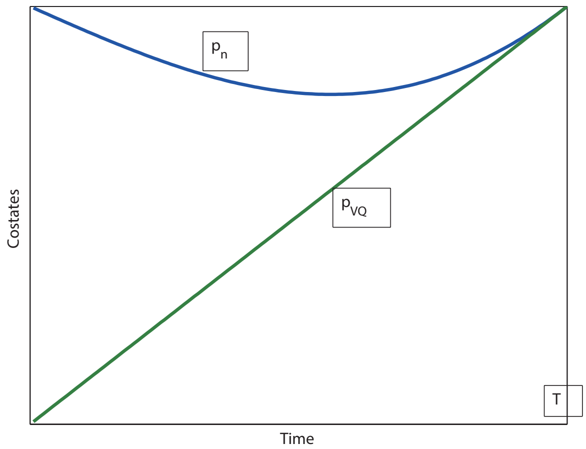

FIGURE 5. Schematic optimal trajectories of costates - case 2, subcase 1

6.2. **Subcase 2.2.** Here we have $O^* \geq c$. We also assume that $n(t_0) < n^*$ and $VQ = 0$ starting from the time moment $t = t_Q$. We propose the following scenario

$$u = 1\ \forall t : \{t_0 \leq t < t_Q,\ VQ > 0,\ 0 < S.\}$$

At the moment $t = t_Q$ the constraint $VQ \leq 0$ become active and we have the singular arc with the singular control

$$u = I/c,\ \forall t : \{t_Q \leq t \leq T,\ VQ = 0,\ S = 0.\}$$

Accordingly n achieved the value $n = n_I$ where $O(n_I) = I$ and then remained at that point. The costates behave like this

$$p_n > p_{VQ}, \lambda = 0,\ \forall\, t : \{t_0 \leq t < t_Q\ \ S > 0\},$$
$$p_n = p_{VQ}, \lambda = -(dO/dn)p_n \geq 0,\ \forall\, t : \{t_Q \leq t \leq T,\ S = 0,\}$$
$$p_n(T) = p_{VQ}(T) = 0.$$

Schematic optimal trajectories of costates are similar to the shown in Fig.3.

6.3. **Subcase 2.3.** Here we have $O^* \geq c$. We also assume that $n(t_0) > n^{**}$ and $VQ > 0$.

We propose the following scenario

$$u = 0\ \forall\, t : \{t_o \leq t \leq t_s \leq t^{**} < T.\},$$
$$u = 1\ \forall\, t : \{t_s \leq t \leq T.\}$$

Here $t = t_s$ is a switching point for the control in time. It follows that $t_s > \bar{t}_c$ because otherwise the value n^* will never be reached with $dn/dt = c - O(n) > 0$. The corresponding costate trajectories will be as follows

$$p_n < p_{VQ},\ S < 0,\ \forall\, t : \{t_0 < t_s,\}$$
$$p_n > p_{VQ},\ S > 0,\ \forall\, t : \{t_s \leq T.\}$$

Note that for $t^* \leq t \leq t^{**}$ we have $dp_n/dt = 1$ and thus $dS/dt = 0$. At the interval $t^* < t \leq T$ we have $dp_n/dt < 1$ and thus $dS/dt < 0$. At the point $t = T$ we have

$p_n(T) = p_{VQ}(T) = 0$. Schematic optimal trajectories of costates are shown in Fig. 6.

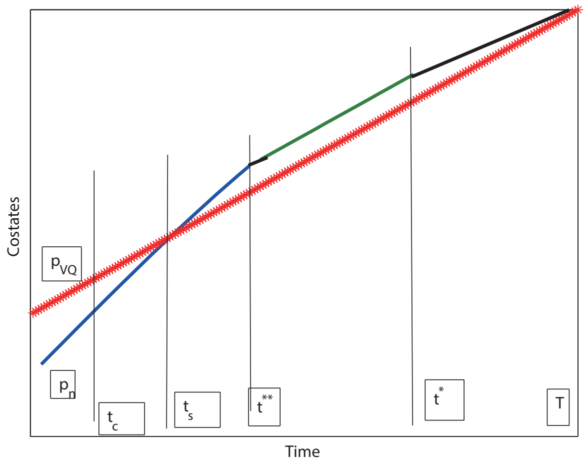

FIGURE 6. Schematic optimal trajectories of costates - case 2, subcase 3

6.4. **Subcase 2.4.** Here we have $O^* \geq c$. We also assume that $n(t_0) > n^{**}$ and $VQ = 0$ starting from the time moment $t = t_Q$. We propose the following scenario

$$u = 0,\ \forall\, t : \{t_0 \leq t \leq t^{**},\ VQ > 0, S < 0,\}$$

$$0 \leq u \leq 1\, \forall\, t : \{t^{**} \leq t_Q,\ VQ > 0,\ S = 0,\}$$

$$u = I/c,\ VQ = 0\, \forall\, t : \{t_Q \leq t \leq T.\}$$

Note that if $t^* < T$ then

$$S = 0,\ \lambda = -\frac{dO}{dn} p_n \geq 0,\ \forall\, t : \{t_* \leq t \leq T.\}$$

At the moment $t = t_Q$ the constraint $VQ \leq 0$ become active and we have the singular arc with the singular control $u = I/c$.

The costates behave like this

$$p_n < p_{VQ}, \lambda = 0,\ \forall\, t : \{t_0 \leq t < t^{**},\ VQ > 0S < 0\},$$

$$p_n = p_{VQ},\ S = 0,\ \lambda = 0,\ \forall\, t : \{t^* \leq t < t^{**},\}$$

$$p_n = p_{VQ},\ S = 0,\ \lambda = -(dO/dn)p_n > 0,\ \forall\, t : \{t^* \leq t \leq T,\}$$

$$p_n(T) = p_{VQ}(T) = 0.$$

Schematic costate trajectories are similar to shown in Fig. 4.

7. Discussion

Theorem: The problem has linear objective integrand and convex in state variables right-hand side of the differential equations for dynamics. The costates in the optimal solutions are negative and equal to zero at the end. Thus the scalar product of the costate vector and the right-hand side of ODE is the concave function of the state variables. The time is fixed. It follows that any solution satisfying the PMP is sufficiently optimal.

Proof. According to the Krotov's identity, [5], we have for the smooth with smooth partial derivatives function $V(t, x)$ and the problem (3.1) that

$$R(t, x, u) = \frac{\partial V}{\partial x} f(x, u) - f_0(x) + \frac{\partial V}{\partial t},$$

$$J = V(T, x_T) - V(t_0, x_0) - \int_{t_0}^{T} R(t, x, u)dt.$$

As far as

$$V(T, x_T) - V(t_0, x_0) = Const,$$

the sufficient condition of optimality is

$$sup_u R(t, x, u) = \mu(t),$$

when the optimal u transfers the state vector from given initial to the given end position. Taking $V(t, x) = p(t)x$, and calculating $\frac{\partial R}{\partial x} = 0$, we see that the solution corresponds to the absolute $max_x R(t, x, u)$ if the vector $p(t)$ satisfied equations for costates and the transversality condition $p(T) = 0$. So the PMP is a sufficient condition of optimality in this case. □

8. Conclusions

We have demonstrated the optimal solutions for all possible cases for both state and costate variables that are satisfied the PMP. Note that according to [5] the PMP presents here the sufficient conditions of optimality.

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