Pure and Applied Functional Analysis
Volume 6, Number 5, 2021, 939–947

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

FORMULA OF THE OVERTAKING OPTIMAL SOLUTION FOR A GROWTH MODEL WITH AK TECHNOLOGY

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Abstract. We consider the AK-type capital accumulation model with continuous time and present a general formula of the overtaking optimal solution. This general formula can be used for models with not only twice continuously differentiable utility but also continuously differentiable utility. Moreover, we derive that under a mild requirement, Euler equation and transversality condition are necessary and sufficient for the overtaking optimality, although the integrant is not necessarily finite. Our result can be applied to a CRRA utility function, and in this case, we can derive a simple necessary and sufficient condition for the existence of the overtaking optimal solution.

1. Introduction

This paper treats an extension of the classical capital accumulation model with linear technology. Such a model is usually called an AK model. In an AK model, there is no steady state for the simultaneous differential equation of the capital accumulation equation and Euler equation, and thus the solution can diverge. This feature makes the AK model difficult to analyze mathematically. That is, there may be an admissible consumption path such that the utility of this path is infinite. If so, Euler equation is no longer necessary for optimality.

To solve this problem, we can use the notion of the **overtaking criterion**, which can compare two admissible consumption paths whose utilities are infinite. Adopting this criterion, we can derive a **general formula of the overtaking optimal solution** for the AK model. This formula is expressed by equation (3.2) in this paper (Theorem 3.2).

Several notes on the optimization technique are needed. Typical macroeconomic textbooks that treat the continuous time growth model (e.g., Romer [10], Blanchard and Fischer [3], Barro and Sara-i-Martin [2], Acemoglu [1]) argue both Euler equation and transversality condition. To our knowledge, almost all these textbooks use the Hamiltonian and Pontryagin's maximal principle in deriving Euler equation. However, Pontryagin's maximal principle cannot be used in the present paper because overtaking optimality admits paths whose utilities may be infinite. This difficulty was solved by Hosoya [6].

2020 *Mathematics Subject Classification.* 91B55, 49K15.
Key words and phrases. capital accumulation, AK model, overtaking optimality, Euler equation, transversality condition, CRRA utility function.

If a path satisfies both Euler equation and transversality condition, then it is overtaking optimal. This can easily be shown using the usual method. However, the necessity of the transversality condition is difficult to verify. In usual model, we can verify it by using a perturbation technique to the inner solution. (See Kamihigashi [7].) This logic is no longer used if the existence of the inner solution of the model is hard to verify directly. However, we can show the necessity of the transversality condition for an overtaking optimal solution of the AK model under mild conditions. (Theorem 3.3)

There is another merit in treating the overtaking optimality. That is, we can treat a model without time-discounting. The time-discount rate was introduced in the literature by Cass [5] and Koopmans [8] independently. The existence of a positive time-discount rate makes the problem much easier to solve. However, such a discount rate does not appear in the work of Ramsey [9], which is the origin of the capital accumulation model. To our understanding, Ramsey considered that the capital accumulation model belongs to not positive theory but normative theory, and from the normative viewpoint, the ideal social planner must not discriminate against future people. Fortunately, we can treat Ramsey's original problem using the overtaking optimality.

We stress that our assumption of the model is weak. For example, we assume that the instantaneous utility is continuously differentiable but do not assume that it is twice differentiable. Additionally, we do not assume the boundedness of the utility function, and thus we treat all constant relative risk aversion (CRRA) utility functions. If the instantaneous utility function is a CRRA function, then we can derive a simple formula for the solution. (Theorem 4.1) Although the same formula was presented by Barro and Sara-i-Martin [2], we additionally give a necessary and sufficient condition for the existence of the solution, and our condition is wider than their requirement.

Section 2 presents our model and prepares the necessary notions and definitions. The main formula and general results are given in section 3. Section 4 considers the case in which the instantaneous utility is a CRRA function.

2. Model and Basic Notions

2.1. Model. The classical AK model is written as

$$\begin{aligned} \max \quad & \int_0^\infty e^{-\rho t} u(c(t))dt \\ \text{subject to. } & k(0) = \bar{k} > 0,\ k(t) \ge 0,\ c(t) \ge 0, \\ & \dot{k}(t) = \gamma k(t) - c(t) \text{ a.e.}, \\ & c(\cdot) \in W, \end{aligned}$$

where k is the amount of capital, c is the amount of private consumption, and ρ is the time discount rate. The set W is some functional space. The equation

$$\dot{k}(t) = \gamma k(t) - c(t)$$

represents two economic relationships. The first relationship is the equality between production and consumption. That is,

$$\kappa k(t) = c(t) + i(t),$$

where the positive value κ represents the linear production technology and i denotes the amount of investment. The second relationship is between the capital stock and investment. That is,

$$\dot{k}(t) = i(t) - dk(t),$$

where $d \geq 0$ is the capital wastage ratio. Combining these two equations, we have

$$\dot{k}(t) = \kappa k(t) - dk(t) - c(t) \equiv \gamma k(t) - c(t),$$

where $\gamma = \kappa - d$, as desired.

In this paper, however, u is not necessarily bounded and ρ is not necessarily positive, and thus, $\int_0^\infty e^{-\rho t} u(c(t))dt$ may not be definable. Therefore, we modify the above problem as:

$$\begin{aligned} &\max \quad lim_{T\to\infty} \int_0^T e^{-\rho t} u(c(t))dt \\ &\text{subject to. } k(0) = \bar{k} > 0,\ k(t) \geq 0,\ c(t) \geq 0, \\ &\qquad \dot{k}(t) = \gamma k(t) - c(t) \text{ a.e.}, \\ &\qquad c(\cdot) \in W, \end{aligned} \tag{2.1}$$

where lim means either lim sup or lim inf.

We make the following three assumptions in this paper.

Assumption 1. The function u is a continuous, strictly concave, and increasing function defined on $\mathbb{R}_+$ and continuously differentiable on $\mathbb{R}_{++}$.

Assumption 2. The value γ is positive.

Assumption 3. The set W denotes the set of all locally integrable functions on $\mathbb{R}_+$.

Note that, under Assumption 3, k must be absolutely continuous on every compact set in $\mathbb{R}_+$.

2.2. **Overtaking Optimality.** We have converted the classical AK problem to problem (2.1). However, even for this model, some problematic cases exist. If there exist $(k_1(t), c_1(t))$ and $(k_2(t), c_2(t))$ such that

$$lim_{T\to\infty} \int_0^T e^{-\rho t} u(c_i(t))dt = +\infty$$

for each i, then these must be equivalent, even though $c_1(t) > c_2(t)$ for every t. This is problematic, and thus we need a criterion that can compare these processes appropriately.

First, we define the notion of admissibility.

Definition 2.1. A pair of functions $(k(t), c(t))$ defined on $\mathbb{R}_+$ is called **admissible** if the following properties hold.

1. $c(t)$ is integrable and $k(t)$ is absolutely continuous on any compact interval in $\mathbb{R}_+$.[1]
2. $k(t) \geq 0$ and $c(t) \geq 0$ for all t.
3. The following differential equation

$$\dot{k}(t) = \gamma k(t) - c(t)$$

holds for almost all $t \in \mathbb{R}_+$.

Let A be the set of all admissible pairs, and $A_{\bar{k}}$ be the set of all admissible pairs such that $k(0) = \bar{k}$.

Define a binary relation $\succ^*$ such that for any $(k_1(t), c_1(t)), (k_2(t), c_2(t)) \in A_{\bar{k}}$,

$$(k_1(t), c_1(t)) \succ^* (k_2(t), c_2(t)) \Leftrightarrow lim_{T\to\infty} \int_0^T e^{-\rho t}[u(c_1(t)) - u(c_2(t))]dt > 0,$$

and define $(k_1(t), c_1(t)) \succsim^* (k_2(t), c_2(t))$ if and only if $(k_2(t), c_2(t)) \not\succ^* (k_1(t), c_1(t))$. If *lim* is lim sup, the maximal element of $\succsim^*$ is called the **overtaking optimal solution** of problem (2.1), and if *lim* is lim inf, the maximal element of $\succsim^*$ is called the **weak overtaking optimal solution** of problem (2.1). See Carlson, Haurie, and Leizarowitz [4] for more detailed arguments.[2]

We mention that for $(k^*(t), c^*(t)) \in A_{\bar{k}}$, if

$$\int_0^\infty e^{-\rho t} u(c^*(t))dt$$

is defined and finite, then it is a solution of (2.1) if and only if it is a weakly overtaking optimal solution of (2.1), if and only if it is an overtaking optimal solution of (2.1). Therefore, the notion of overtaking optimality is an extension of usual optimality.

2.3. **Euler Equation and Transversality Condition.** Let $(k(t), c(t))$ be admissible. This pair is said to satisfy the **Euler equation** on an interval I if and only if $c(t)$ is continuous, $u \circ c$ is continuously differentiable, and

$$\frac{d}{dt}(u \circ c)(t) = (\rho - \gamma)(u \circ c)(t) \tag{2.2}$$

holds for all t.

In the case of the traditional model, it is said that $(k^*(t), c^*(t))$ is a solution only if it satisfies Euler equation on $\mathbb{R}_+$. However, in our model, $c^*(t)$ is not necessarily continuous, and thus this statement is a little incorrect. Therefore, we extend the above definition. An admissible pair $(k(t), c(t))$ satisfies the Euler equation **a.e.** on an interval I if and only if there exists a continuous function $\tilde{c}(t)$ such that $(k(t), \tilde{c}(t))$ is also admissible and satisfies Euler equation on I. Then, we can modify the above claim to the following one: "$(k^*(t), c^*(t))$ is a solution only if it satisfies the Euler equation a.e. on $\mathbb{R}_+$".

[1]In this paper, the term 'interval' means a convex set in $\mathbb{R}$ that includes at least two different points.

[2]In economics, our overtaking optimality is sometimes called **catching-up optimality**, and in this case, our weak overtaking optimality is called overtaking optimality.

However, in our model, $\int_0^\infty e^{-\rho t}u(c^*(t))dt$ may be not defined, and thus this statement is not obvious. Additionally, Euler equation is only a necessary condition of an **inner solution**. This problem was solved by Hosoya [6], although the proof is very long.

Next, again let $(k(t), c(t))$ be admissible. This pair is said to satisfy **transversality condition** if and only if

$$\lim_{t\to\infty} e^{-\rho t}u'(c(t))k(t) = 0. \tag{2.3}$$

It is also said that usually $(k^*(t), c^*(t))$ is a solution if and only if it satisfies Euler equation and transversality condition. The necessity of transversality condition for optimality cannot easily be derived. However, in this paper, the necessity of this condition can be derived.

2.4. **Linear Differential Equation.** Consider the following linear differential equation:

$$\dot{x}(t) = a(t)x(t) + b(t),$$

where $a(t), b(t)$ are locally integrable functions. The general solution of the above equation is

$$x(t) = e^{\int_0^t a(s)ds}\left[x(0) + \int_0^t e^{-\int_0^s a(\tau)d\tau}b(s)ds\right].$$

In the AK model, both the capital accumulation equation and Euler equation are linear. Thus, this formula is frequently used in the proof of our main result. This is the key idea of this paper.

3. Results

3.1. **The Known Facts.** In this subsection, we present results of Hosoya [6]. These results are heavily used in this paper.

First, we define a notion of the inner solution.

Definition 3.1. Suppose that $(k(t), c(t))$ is admissible. We say that this pair is **positive on** $[0, T]$ if and only if the following two statements hold.

1) $k(t) > 0$ for all $t \in [0, T]$.
2) There exists $c > 0$ such that the Lebesgue measure of the set $\Delta(c) = \{t \in [0, T] | c(t) < c\}$ is zero.

If $(k^*(t), c^*(t))$ is a (weak, or strong) overtaking optimal solution, then it is called an **inner solution** if and only if it is positive on $[0, T]$ for all $T > 0$.

Then, the following results hold.

Fact 1. Let $T > 0$, and suppose that $(k^*(t), c^*(t))$ is a weak overtaking optimal solution of (2.1) and positive on $[0, T]$. Then, $(k^*(t), c^*(t))$ satisfies Euler equation a.e. on $[0, T]$.

Fact 2. Suppose that

$$\lim_{c\to 0} u'(c) = +\infty.$$

If $(k^*(t), c^*(t))$ is a weak overtaking optimal solution of (2.1), then $(k^*(t), c^*(t))$ is an inner solution, and thus Euler equation holds a.e. on $\mathbb{R}_+$.

Fact 3. Suppose that $(k^*(t), c^*(t))$ is admissible, $c^*(t)$ is a continuous function, and both Euler equation on $\mathbb{R}_+$ and transversality condition hold. Then, $(k^*(t), c^*(t))$ is an overtaking optimal solution of (2.1).

3.2. **Main Results.** Hereafter, we assume the following.

Assumption 4. $\lim_{c\to 0} u'(c) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$.

Note that, by Assumption 4, we can define $(u')^{-1} : \mathbb{R}_{++} \to \mathbb{R}_{++}$, and $(u')^{-1}$ is surjective and decreasing.

For $C > 0$, define

$$\varphi(C) = \int_0^\infty e^{-\gamma t}(u')^{-1}(Ce^{(\rho-\gamma)t})dt \tag{3.1}$$

and

$$B_1 = \{C > 0 | \varphi(C) \le \bar{k}\},\ B_2 = \{C > 0 | \varphi(C) \ge \bar{k}\}.$$

Note that φ is a positive and nonincreasing function of C. We can in fact show that if both B_1 and B_2 are nonempty, then $\inf B_1 \in B_1$.

Theorem 3.2. *If there exists a weak overtaking optimal solution of* (2.1)*, then both* B_1, B_2 *are nonempty and* $C^* \equiv \inf B_1 > 0$. *Moreover, the pair* $(k^*(t), c^*(t))$ *defined by*

$$c^*(t) = (u')^{-1}(C^*e^{(\rho-\gamma)t}),\ k^*(t) = e^{\gamma t}\left[\bar{k} - \int_0^t e^{-\gamma s}c^*(s)ds\right] \tag{3.2}$$

is a weak overtaking optimal solution of (2.1)*. Furthermore, every weakly overtaking optimal solution of* (2.1) *is the same as the above solution a.e..*

Proof. By Facts 1-2, if $(k^*(t), c^*(t))$ is a weak overtaking optimal solution of (2.1), then $c^*(t)$ satisfies Euler equation a.e., and thus we can assume that $c^*(t)$ is continuous, and

$$\frac{d}{dt}(u' \circ c^*)(t) = (\rho - \gamma)(u' \circ c^*)(t).$$

This implies that $u' \circ c^*$ is a solution of the following linear differential equation

$$\dot{x}(t) = (\rho - \gamma)x(t),$$

and thus,

$$c^*(t) = (u')^{-1}(u'(c^*(0))e^{(\rho-\gamma)t}).$$

Moreover,

$$\dot{k}^*(t) = \gamma k^*(t) - c^*(t),\ k^*(0) = \bar{k},$$

and thus, we have

$$k^*(t) = e^{\gamma t}\left[\bar{k} - \int_0^t e^{-\gamma s}c^*(s)ds\right].$$

Because $k^*(t) \ge 0$ for all $t \ge 0$, we have

$$\bar{k} \ge \int_0^\infty e^{-\gamma t}c^*(t)dt = \varphi(u'(c^*(0))),$$

which implies that $u'(c^*(0)) \in B_1$. Therefore, B_1 is nonempty. If $C < u'(c^*(0))$ and $C \in B_1$, define

$$c^+(t) = (u')^{-1}(Ce^{(\rho-\gamma)t}),\ k^+(t) = e^{\gamma t}\left[\bar{k} - \int_0^t e^{-\gamma s}c^+(s)ds\right].$$

Then, $(k^+(t), c^+(t))$ is admissible and $c^+(t) > c^*(t)$ for all $t \ge 0$. This implies that $(k^*(t), c^*(t))$ is not weakly overtaking optimal, which is a contradiction. Therefore, we have that $C \in B_2$ for every $C < u'(c^*(0))$, and hence, B_2 is nonempty and $C^* = \inf B_1 = u'(c^*(0))$. This completes the proof. □

The following result presents the necessary and sufficient condition for the existence of the solution.

Theorem 3.3. *Suppose that either φ is continuous or there exists $C > 0$ such that $\bar{k} \le \varphi(C) < +\infty$. Then, the following three requirements are equivalent.*

1) *There exists a weak overtaking optimal solution of* (2.1).
2) *There exists an overtaking optimal solution of* (2.1).
3) *There exists $C^* > 0$ such that $\varphi(C^*) = \bar{k}$.*

Moreover, the overtaking optimal solution is given by equation (3.2) *in this case.*

Note that, this is a kind of results that proves equivalence between weak overtaking optimality and overtaking optimality. A similar result is found in Zaslavski [11], where the setup and assumptions of model is different from ours.[3]

We will show in the proof of this theorem that 3) implies that $(k^*(t), c^*(t))$ defined in (3.2) satisfies the transversality condition. Therefore, we can say that, under Assumptions 1, 2, 3, and 4, and either the continuity of φ or the existence of $C > 0$ such that $\bar{k} \le \varphi(C) < +\infty$, the Euler equation and the transversality condition are the necessary and sufficient condition for the overtaking optimality.

Proof. The fact that 2) implies 1) is obvious.

Suppose that 1) holds. By Theorem 3.2, we have that both B_1 and B_2 are nonempty, and $C^* = \inf B_1 > 0$. If φ is continuous, then we have that $\varphi(C^*) = \bar{k}$, and thus 3) holds. Next, suppose that there exists $C > 0$ such that $\bar{k} \le \varphi(C) < +\infty$. If $\varphi(C) = \bar{k}$, then 3) holds. Otherwise, we have that $\bar{k} < \varphi(C) < +\infty$. Choose $C_n = \frac{n}{n+1}C^*$ and $C_n' = \frac{n+1}{n}C^*$. Then, $C_n \in B_2$ and $C_n' \in B_1$. Because of the monotone convergence theorem, we have that $\lim_{n\to\infty}\varphi(C_n') = \varphi(C^*)$, which implies that $\varphi(C^*) \le \bar{k}$ and $C < C^*$. Therefore, for sufficiently large n, $C \le C_n$ and $\bar{k} \le \varphi(C_n) \le \varphi(C) < +\infty$. Again by the monotone convergence theorem, we have that $\lim_{n\to\infty}\varphi(C_n) = \varphi(C^*)$, which implies that $\varphi(C^*) \ge \bar{k}$, and thus $\varphi(C^*) = \bar{k}$ and 3) holds.

Suppose that 3) holds. Let $(k^*(t), c^*(t))$ be defined by (3.2). Then,

$$\begin{aligned} e^{-\rho t}u'(c^*(t))k^*(t) &= C^*\left[\bar{k} - \int_0^t e^{-\gamma s}c^*(s)ds\right] \\ &\to C^*(\bar{k} - \varphi(C^*)) = 0, \end{aligned}$$

[3]See ch.6 of Zaslavski [11]. Note that, because our model includes $e^{-\rho t}$, it is nonautonomous, and thus, in this monograph, not Theorem 5.20 but Theorem 6.29 corresponds to our Theorem 3.3.

which asserts that $(k^*(t), c^*(t))$ satisfies both Euler equation and transversality condition. By Fact 3, we must have that this pair is an overtaking optimal solution, and 2) holds. This completes the proof. □

4. Example

For $\theta > 0$, the following function u_θ is called the CRRA utility function.

$$u_\theta(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1, \\ \log c & \text{if } \theta = 1. \end{cases}$$

Barro and Sara-i-Martin [2] showed that under a CRRA utility function, if γ, θ, ρ satisfy several inequalities, then the admissible pair defined by (3.2) is a (not overtaking optimal, but usual) solution. We can extend their results. That is, there exists an inequality that holds if and only if there exists a weak overtaking optimal solution of (2.1), and in this case, this solution is given by (3.2).

Theorem 4.1. *Suppose that $u = u_\theta$. Then, there exists a weak overtaking optimal solution of* (2.1) *if and only if*

$$\rho - (1-\theta)\gamma > 0, \tag{4.1}$$

and if so,

$$c^*(t) = \frac{(\rho - (1-\theta)\gamma)\bar{k}}{\theta} e^{\frac{\gamma-\rho}{\theta}t}, \ k^*(t) = \bar{k} e^{\frac{\gamma-\rho}{\theta}t} \tag{4.2}$$

is overtaking optimal.[4]

Proof. In this case,

$$u'(c) = c^{-\theta}, \ (u')^{-1}(x) = x^{-\frac{1}{\theta}}.$$

Therefore,

$$\varphi(C) = \begin{cases} C^{-\frac{1}{\theta}} \frac{\theta}{\rho-(1-\theta)\gamma} & \text{if } \rho - (1-\theta)\gamma > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Thus, by Theorems 3.2 and 3.3, there exists a weak overtaking optimal solution of (2.1) if and only if (4.1) holds. In this case,

$$\varphi(C^*) = \bar{k} \Leftrightarrow C^* = \left(\frac{\theta}{(\rho - (1-\theta)\gamma)\bar{k}} \right)^\theta,$$

and thus,

$$(u')^{-1}(C^* e^{(\rho-\gamma)t}) = \frac{(\rho - (1-\theta)\gamma)\bar{k}}{\theta} e^{\frac{\gamma-\rho}{\theta}t},$$

which implies that $(c^*(t), k^*(t))$ defined by (4.2) coincides with that defined by (3.2), and thus this is an overtaking optimal solution of (2.1). This completes the proof. □

[4] Check that this solution is the same as that given by (3.2).

Acknowledgement

The research agenda pursued in this paper was proposed by Hiroyuki Ozaki. We are grateful to him for pointing out an important issue in economic dynamics. We also appreciate the helpful comments from an anonymous reviewer.

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Manuscript received May 29 2020
revised July 2 2020

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