Pure and Applied Functional Analysis
Volume 6, Number 5, 2021, 917–938

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

OPTIMAL GROWTH IN THE TWO-SECTOR RSL MODEL WITH CAPITAL-INTENSIVE CONSUMPTION GOODS: A DYNAMIC PROGRAMMING APPROACH

LIUCHUN DENG AND MINAKO FUJIO

Abstract. We study the two-sector Robinson-Shinkai-Leontief (RSL) model of optimal economic growth with discounting for the case of capital-intensive consumption goods. We identify explicitly a lower bound for the discount factor above which the optimal growth path for any initial stock converges to the modified golden rule stock in finite periods, following the prediction of the undiscounted RSL model. This result echoes the earlier finding in Khan-Mitra [12]. We also provide conditions under which the straight-down-to-the-turnpike policy is optimal.

1. Introduction

Since Khan-Mitra's finding of topological chaos in the Robinson-Solow-Srinivasan (RSS) model [9], a distinct strand of literature carries out an extensive examination of the dynamic behavior of this model of economic growth with Leontief production technology. The RSS model, being "a specific instance of the general theory of intertemporal resource allocation",[1] has been deployed as a technical device to interrogate May's claim of "simple mathematical models with very complicated dynamics"[2] in an intertemporal optimization framework. Deliberately chosen to be simple, the RSS model testifies how complicated economic dynamics can be understood by geometric construction [10,11,14], delineated by the dynamic programming approach [12,13,15], and to a large extent, further consolidated and re-understood by circling back to geometry [7].

One of the key assumptions in the RSS model is that production of investment goods only uses labor. This assumption significantly simplifies the analysis but is nevertheless restrictive and rules out what is so called upward inertia, an important source of optimal chaos as identified by Mitra-Nishimura-Sorger [17], in the first place. Relaxing this assumption, Fujio [4], [5], and [6] study the so-called Robinson-Shinkai-Leontief (RSL) model, which also nests Nishimura-Yano [18] as a special case, and offer a comprehensive characterization of optimal growth without

2020 *Mathematics Subject Classification.* 91B55, 37E05.

Key words and phrases. RSL model, RSS model, dynamic programming, discounted, undiscounted.

Liuchun Deng acknowledges the support of the Start-up Grant from Yale-NUS College.

[1]It is quoted from the abstract of Khan-Mitra [8].

[2]See his 1976 article in Nature [16].

discounting. In a recent paper, Deng-Fujio-Khan [2] makes a first attempt to investigate the discounted RSL model. A key parameter ζ has been identified for the characterization of optimal policy correspondence. [2] demonstrates that the system always converges to the golden rule stock or a two-period cycle when $\zeta \leq 1$.

In this paper, we turn to the case of $\zeta > 1$. In particular, we prove that there is a uniform lower bound, $1/\zeta$, for the discount factor above which the optimal program converges to the modified golden-rule stock in finite periods. This result echoes and extends earlier finding as in Khan-Mitra [12] and Fujio [6]: the dynamic properties of the optimal policy of a model with sufficiently patient agents is qualitatively the same as those of a model without discounting.[3] Our proof relies heavily on the modified guess-and-verify approach which exploits convexity of the optimization problem as in Khan-Mitra [12].

The finding of this threshold discount factor adds to the understanding of optimal chaos for sufficiently patient agents. In their celebrated example, Nishimura-Yano [18] uses a special case of the RSL model with full depreciation of capital ($d = 1$) to demonstrate that, for any discount factor that is arbitrarily close to unity, it is possible to construct a growth model by carefully picking the technological parameters[4] such that the optimal policy leads to chaos. However, our results, specialized to the Nishimura-Yano example,[5] suggest that, once the technological parameters of the model are chosen, there is no optimal chaos for the discount factor above $1/\zeta$. Our result complements the Nishimura-Yano example and calls for a comprehensive bifurcation analysis of how the optimal policy of the RSL model changes with respect to the discount factor.

Moreover, our analysis further identifies the conditions under which the straight-down-to-the-turnpike policy is optimal. Interestingly, under certain parameter restrictions, the discount factor has to be strictly below one for the straight-down-to-the-turnpike policy to emerge, which stands in sharp contrast to the case of the RSS model. We provide further characterization of the optimal policy correspondence when the discount factor is above $1/\zeta$. Like what is obtained in Deng-Fujio-Khan [2] for $\zeta \leq 1$, the optimal policy hinges on the technological parameters and their complex interplay.

The rest of this paper is organized as follows. In the next section, we present the setup of the RSL model and the existing results upon which our analysis is based. In Section 3, we characterize the optimal policy for each parameter specification when the agents are sufficiently patient. We offer concluding remarks in the last section. All the proofs are collected in the Appendix.

2. The RSL model of optimal growth

2.1. **The Setup.** We consider the two-sector discrete-time Robinson-Shinkai-Leontief model of optimal economic growth with discounting, which nests the technological

---

[3]For the "folk theorem" on the discounted versus undiscounted growth models, see the first paragraph of the introduction of Khan-Mitra [12] and references therein.

[4]As will be made clear, the technological parameters are the coefficients in the Leontief production function.

[5]Note that in what follows we report our results for $0 < d < 1$ for notational simplicity, but our results can be readily extended to the case of $d = 1$.

specification of the RSS model [9,12,13] as a special case. There are two sectors. In the consumption good sector, it requires one unit of labor and a_C units of capital to produce one unit of good. In the investment good sector, it requires one unit of labor and a_I units of capital to produce b units of good. Throughout our discussion, we focus on the case of capital-intensive consumption goods:

$$a_C > a_I. \tag{2.1}$$

Labor supply is fixed and normalized to be unity in each time period t. Denote the capital stock in the current period by x, the capital stock in the next period by x', and the depreciation rate of capital by $d \in (0,1)$. The *transition possibility set* illustrated as in Figure **??** is formally defined as

$$\Omega = \{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-d)x \geq 0, x' - (1-d)x \leq b \min\{1, x/a_I\}\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. Denote by y output of consumption goods. For any $(x, x') \in \Omega$, we define a correspondence

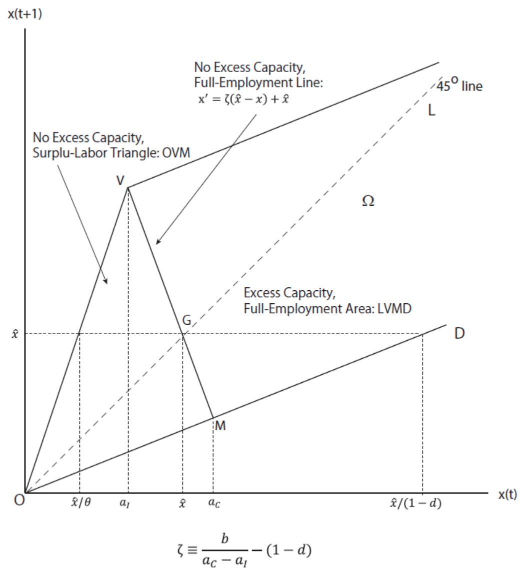

FIGURE 1. Transition Possibility Set Ω

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq (1/a_C)(x - (a_I/b)(x' - (1-d)x)) \\ \text{and } \ 0 \leq y \leq 1 - (1/b)(x' - (1-d)x)\}.$$

A felicity function, $w : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is linear and given by $w(y) = y$. The reduced form utility function, $u : \Omega \longrightarrow \mathbb{R}_+$, is defined on Ω such that

$$u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}.$$

The future utility is discounted with a discount factor $\rho \in (0,1)$.

An *economy* E consists of a triplet (Ω, u, ρ). A *program from* x_0 is a sequence $\{x_t, y_t\}$ such that for all $t \in \mathbb{N}$, $(x_t, x_{t+1}) \in \Omega$ and $y_t = \max \Lambda(x_t, x_{t+1})$. A program

$\{x_t, y_t\}$ is called *stationary* if for all $t \in \mathbb{N}, (x_t, y_t) = (x_{t+1}, y_{t+1})$. For all $0 < \rho < 1$, a program $\{x_t^*, y_t^*\}$ from x_0 is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t [u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)] \leq 0$$

for every program $\{x_t, y_t\}$ from x_0.

To ensure the existence of a stock expansible by the factor ρ^{-1}, we assume

$$\theta \equiv b/a_I + (1-d) > 1/\rho. \tag{2.2}$$

2.2. **Preliminaries.** Define the marginal rate of transformation of capital between today and tomorrow under full utilization of capital as

$$\zeta \equiv b/(a_C - a_I) - (1-d).$$

It is a key parameter for our analysis. In Khan-Mitra's work on the RSS model, the counterpart of ζ is their ξ.

According to Deng-Fujio-Khan [2], we know there exists a modified golden rule stock, which is the stationary optimal capital stock, for the RSL model. The modified golden rule stock, is given by

$$\hat{x} = \frac{a_C(\zeta + 1 - d)}{\zeta + 1} = \frac{a_C b}{b + d(a_C - a_I)}.$$

Deng-Fujio-Khan [2] has provided a complete characterization for $\zeta \leq 1$. The focus of our analysis in this paper is on the case of $\zeta > 1$. To proceed, we write explicitly the reduced-form utility function

$$u(x, x') = \begin{cases} \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x', & \text{for } x' \leq \zeta(\hat{x} - x) + \hat{x} \\ \\ \frac{1-d}{b} x - \frac{1}{b} x' + 1, & \text{for } x' > \zeta(\hat{x} - x) + \hat{x} \end{cases}$$

where the first line stands for the case of full utilization of capital while the second line stands for the case of full utilization of labor.

Just to recap, we impose three parametric assumptions on the analysis in what follows: (1) $a_C > a_I$; (2) $\rho\theta > 1$; (3) $\zeta > 1$.

3. The results

3.1. **The Dynamic Programming Approach.** Define the value function $V : \mathbb{R}_+ \to \mathbb{R}$ as

$$V(x) = \sum_{t=0}^{\infty} \rho^t \left[u(x(t), x(t+1)) - u(\hat{x}, \hat{x}) \right]$$

where $\{x(t), y(t)\}$ is an optimal program starting from $x(0) = x$. By construction, we have $V(\hat{x}) = 0$. The value function is continuous. We know from [2] that the value function V is also concave and strictly increasing. For each $x \in \mathbb{R}_+$, the Bellman equation[6]

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') - u(\hat{x}, \hat{x}) + \rho V(x')\}$$

[6] The *Principle of Optimality* was originally stated in Bellman [1].

holds where $\Gamma(x) = \{x' : (x, x') \in \Omega\}$. For each $x \in \mathbb{R}_+$, Define the optimal policy correspondence

$$h(x) = \arg \max_{x' \in \Gamma(x)} \{u(x, x') - u(\hat{x}, \hat{x}) + \rho V(x')\}.$$

A program $\{x(t), y(t)\}$ from $x(0)$ is optimal if and only if it satisfies the equation: $V(x(t)) = u(x(t), x(t+1)) - u(\hat{x}, \hat{x}) + \rho V(x(t+1))$ for $t \geq 0$.[7]

The following proposition adapted from [2] provides a partial characterization for $\zeta > 1$. Figure 2 illustrates this partial characterization of the optimal policy correspondence.

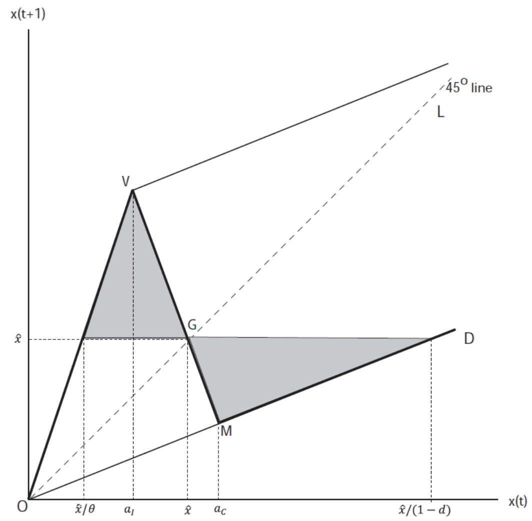

FIGURE 2. Optimal Policy Correspondence for $\zeta > 1$

Proposition 3.1. *Let $\zeta > 1$. The optimal policy correspondence h satisfies*

$$h(x) \subset \begin{cases} \{\theta x\} & \text{for } x \in (0, \hat{x}/\theta] \\ [\hat{x}, \theta x] & \text{for } x \in (\hat{x}/\theta, a_I] \\ [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] & \text{for } x \in (a_I, \hat{x}] \\ [\zeta(\hat{x} - x) + \hat{x}, \hat{x}] & \text{for } x \in (\hat{x}, a_C] \\ [(1-d)x, \hat{x}] & \text{for } x \in (a_C, \hat{x}/(1-d)). \\ \{(1-d)x\} & \text{for } x \in [\hat{x}/(1-d), \infty) \end{cases}$$

.

3.2. **The Optimal Policy Correspondence.** In this subsection, we will characterize the optimal policy correspondence when the discount factor ρ is sufficiently close to one. Since we can rewrite $\zeta > 1$ as $(\theta a_I - a_C) + (a_I - (1-d)a_C) > 0$, it is useful to consider three cases for $\zeta > 1$, in which the RSS model is covered by Case II:

(I) $\theta a_I > a_C$ and $a_I \geq (1-d)a_C$

[7]This equivalence result is well known in the literature. We refer the interested reader to Footnote 22 in [2].

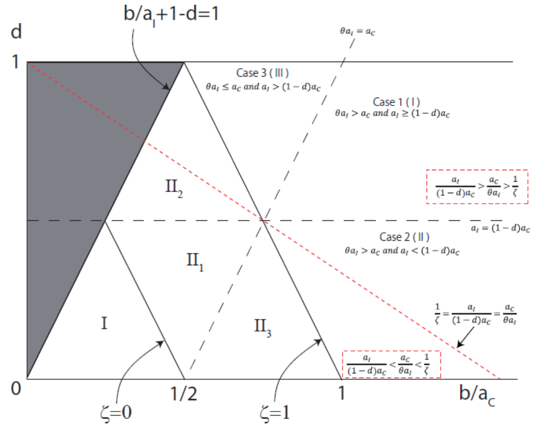

FIGURE 3. Parameter Specification for $a_I/a_C = 1/2$

(II) $\theta a_I > a_C$ and $a_I < (1-d)a_C$
(III) $\theta a_I \le a_C$ and $a_I > (1-d)a_C$.

Figure 3 provides a delineation of the parameter space. It can be seen from the figure that the parameter space for $\zeta > 1$ are divided into three regions.

Proposition 3.2. *The optimal policy correspondence is characterized by a straight-down-to-the-turnpike policy:*

$$h(x) = \begin{cases} \{\theta x\} & \text{for } x \in (0, \hat{x}/\theta] \\ \{\hat{x}\} & \text{for } x \in (\hat{x}/\theta, \hat{x}/(1-d)) \\ \{(1-d)x\} & \text{for } x \in [\hat{x}/(1-d), \infty) \end{cases}$$

if the discount factor satisfies

$$\frac{a_I}{(1-d)a_C} > \rho > \frac{a_C}{\theta a_I}$$

This proposition establishes the sufficient condition under which the straight-down-to-the-turnpike policy is optimal. We notice from the proof that this condition is not necessary under certain parameter restrictions. Moreover, it should be noted that there may simply not exist ρ that satisfies the restrictions on the discount factor in the proposition above. Consider, for example, the RSS model with $a_I = 0$.

The result can be further specialized to Case I and II.

Corollary 3.3 (Case I)**.** *Let $\theta a_I > a_C$ and $a_I \ge (1-d)a_C$. If $\rho > a_C/(\theta a_I)$, the optimal policy correspondence is characterized by a straight-down-to-the-turnpike policy.*

Corollary 3.4 (Case II)**.** *Let $\theta a_I > a_C$ and $a_I < (1-d)a_C$. If $\frac{a_C}{\theta a_I} < \rho < \frac{a_I}{(1-d)a_C} < 1$, the optimal policy correspondence is characterized by a straight-down-to-the-turnpike policy.*

Figure 4 illustrates the straight-down-to-the-turnpike policy for Case I. In contrast to the result for the RSS model, Corollary 3.4 suggests that the discount factor

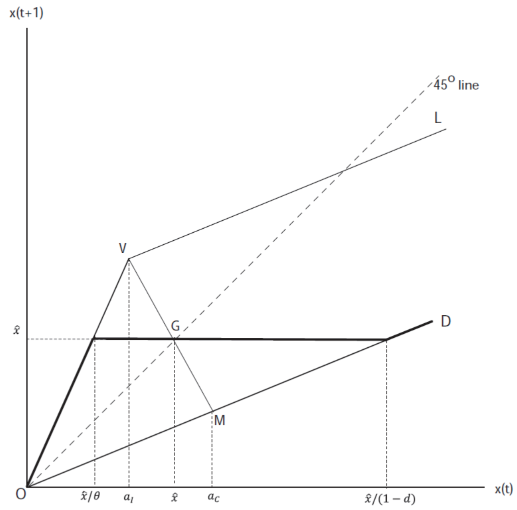

FIGURE 4. Case I: $\theta a_I > a_C$ and $a_I \geq (1-d)a_C$

has to be in an intermediate range for the straight-down-to-the-turnpike policy to become optimal.

Before turning to identifying the common cutoff for the discount factor above which the optimal dynamics converges to the golden rule stock in finite periods, we provide the following ordering that turns out to be quite useful for further characterization of the optimal policy correspondence. Consider three ratios: $\frac{1}{\zeta}$, $\frac{a_I}{(1-d)a_C}$, and $\frac{a_C}{\theta a_I}$. We have

$$\left(\frac{1}{\zeta} - \frac{a_C}{\theta a_I}\right)\left(\frac{a_C}{\theta a_I} - \frac{a_I}{(1-d)a_C}\right) \geq 0, \tag{3.1}$$

where the equality holds if and only if $\frac{1}{\zeta} = \frac{a_I}{(1-d)a_C} = \frac{a_C}{\theta a_I}$. The case of equality is illustrated as the red dash line in Figure 3 which further divides Case II into two subregions. Inequality 3.1 implies that $\frac{a_C}{\theta a_I}$ is always in between $\frac{a_I}{(1-d)a_C}$ and $\frac{1}{\zeta}$.

Following closely the modified guess-and-verify approach taken by [12], we now demonstrate that there exists an interval over which the optimal policy is the modified golden rule if the discount factor is sufficiently high.

Proposition 3.5. *If $\rho > \max\{\frac{1}{\zeta}, \frac{a_C}{\theta a_I}\}$, $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}, \hat{x}/(1-d))$.*

Proposition 3.6. *If $a_C/(\theta a_I) \geq \rho > \frac{1}{\zeta}$, $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}/\theta, \hat{x})$.*

Proposition 3.5 and 3.6 are direct generalizations of Lemma 7 and Proposition 4 in [12]. The basic proof idea is that we guess and verify the optimal policy and the value function for the optimization problem restricted to a relatively small interval. Given that the value function is concave, we can then prove the policy is also optimal for the original optimization problem. We obtain the following corollaries from Propositions 3.2 – 3.6 for each case.

Corollary 3.7 (Case I). *Let $\theta a_I > a_C$ and $a_I \geq (1-d)a_C$. If $\rho > 1/\zeta$, $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}/\theta, \hat{x})$.*

Corollary 3.8 (Case II)**.** *Let $\theta a_I > a_C$ and $a_I < (1-d)a_C$. Let $\rho > 1/\zeta$. Consider two subcases:*

(1) $1/\zeta \geq a_C/(\theta a_I)$.
$h(x) = \{\hat{x}\}$ *for* $x \in (\hat{x}, \hat{x}/(1-d))$;

(2) $1/\zeta < a_C/(\theta a_I)$.
If $\rho < a_I/((1-d)a_C)$, $h(x) = \{\hat{x}\}$ *for* $x \in (\hat{x}/\theta, \hat{x})$.
If $\rho > a_C/(\theta a_I)$, $h(x) = \{\hat{x}\}$ *for* $x \in (\hat{x}, \hat{x}/(1-d))$.

Corollary 3.9 (Case III)**.** *Let $\theta a_I \leq a_C$ and $a_I > (1-d)a_C$. If $\rho > 1/\zeta$, $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}/\theta, \hat{x})$.*

Proposition 3.3 and Corollaries 3.7 – 3.9 are counterpart of Proposition 9 in [6] for the RSL model of optimal growth without discounting. Notably, a common cutoff $1/\zeta$ emerges from the above results.

Theorem 3.10. *If $\rho > 1/\zeta$, the optimal dynamics converges to the modified golden rule stock $\hat{x}$ in finite periods.*

In the RSS model of optimal growth, $1/\xi$ has been identified as the threshold for the discount factor above which the optimal dynamics leads to global convergence. Our theorem generalizes the finding in [12] by showing that the cutoff $1/\zeta$, which is the counterpart of $1/\xi$, applies to the general setting of the RSL model.

Based on Corollaries 3.8 and 3.9, with additional parameter restrictions, we can fully characterize the optimal policy correspondence for Case II and III provided that the discount factor is sufficiently high.

Proposition 3.11 (Case II)**.** *Let $\theta a_I > a_C$ and $a_I < (1-d)a_C$. There exists an integer $n \geq 0$ such that $\theta a_I \in (\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}}]$. If $\rho > 1/\zeta$ and $\rho^{n+1}(1-d)^{n+1} > a_I/a_C$, the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{\theta x\} & \text{for } x \in (0, a_I] \\ \{\zeta(\hat{x} - x) + \hat{x}\} & \text{for } x \in [a_I, \hat{x}) \\ \{\hat{x}\} & \text{for } x \in [\hat{x}, \hat{x}/(1-d)) \\ \{(1-d)x\} & \text{for } x \in [\hat{x}/(1-d), \infty) \end{cases}.$$

If $\theta a_I \leq \hat{x}/(1-d)$, then $n = 0$ and thus, under Case II, we can always pick ρ sufficiently close to 1 such that $\rho^{n+1}(1-d)^{n+1} > a_I/a_C$ and $\rho > 1/\zeta$. The policy function is illustrated in Figure 5. If $\rho > \max\{\frac{1}{\zeta}, \frac{a_C}{\theta a_I}\}$ but $\rho^{n+1}(1-d)^{n+1} < a_I/a_C$, the optimal policy will have a flat top over the internal that covers a_I.

Proposition 3.12 (Case III)**.** *Let $\theta a_I \leq a_C$ and $a_I > (1-d)a_C$. There exists an integer $n \geq 0$ such that $(1-d)a_C \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n})$. If $\rho > \frac{1}{\zeta}$ and $(\rho\theta)^{n+1}a_I < a_C$, the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{\theta x\} & \text{for } x \in (0, \hat{x}/\theta] \\ \{\hat{x}\} & \text{for } x \in (\hat{x}/\theta, \hat{x}] \\ \{\zeta(\hat{x} - x) + \hat{x}\} & \text{for } x \in (\hat{x}, a_C) \\ \{(1-d)x\} & \text{for } x \in [a_C, \infty) \end{cases}.$$

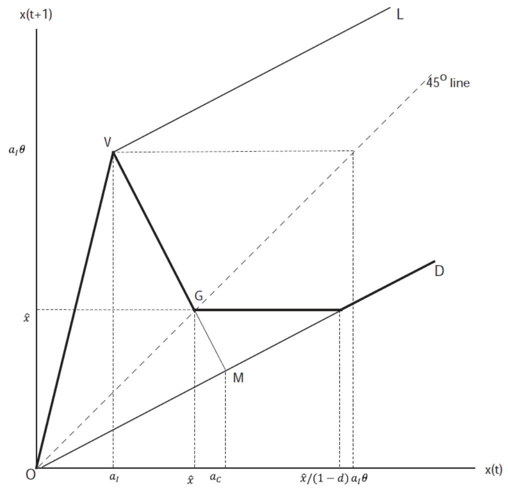

FIGURE 5. Case II: $\theta a_I > a_C$ and $a_I < (1-d)a_C$

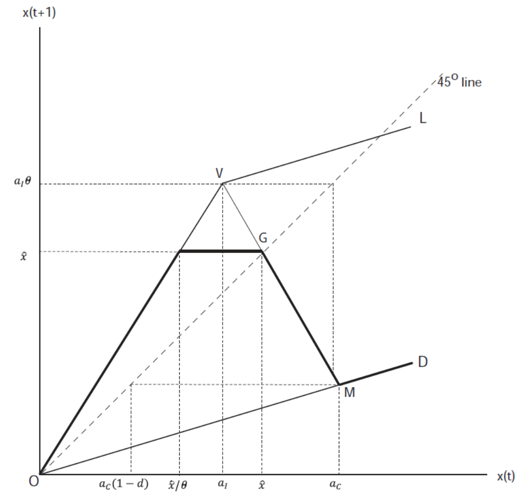

FIGURE 6. Case III: $\theta a_I \leq a_C$ and $a_I > (1-d)a_C$

If $a_C(1-d) \geq \hat{x}/\theta$, then $n = 0$ and thus, under Case III, we can always pick ρ sufficiently close to 1 such that $(\rho\theta)^{n+1}a_I < a_C$ and $\rho > 1/\zeta$. The policy function is illustrated in Figure 6. If $\rho > \frac{1}{\zeta}$ but $(\rho\theta)^{n+1}a_I > a_C$, the optimal policy will have a flat bottom over the interval that covers a_C.

To avoid repetition, we report our results above for $d < 1$. The results carry through for $d = 1$, the setting of the Nishimura-Yano example. When $d = 1$, there are two possible cases: Case I and III. In both cases, optimal policy leads to global convergence for $\rho > 1/\zeta$.

4. Concluding remarks

In this paper, we have shown that for $\zeta > 1$ the optimal dynamics of the RSL model converges in finite periods to the modified golden rule stock when the discount factor is above $1/\zeta$. A natural question arises: what if the discount factor is below that threshold? Would the system undergo a qualitative change as shown in the RSS model by Khan-Mitra [12]. More importantly, since the RSL model nests both the RSS model and the Nishimura-Yano example [18] as special cases, it would be interesting to see if alternative forms of complicated dynamics, which do not belong to the class of check or tent maps, will emerge. For example, Deng-Fujio-Khan [3] has shown an intriguing Z-shaped map characterizes the equilibrium dynamics of the RSL model. Even though it has been proven that the optimal dynamics cannot be presented by the same Z-shaped map, it remains open whether and how complicated dynamics arise in the general setting of the two-sector RSL model . Furthermore, the literature on the discounted RSL model of optimal growth has not yet brought geometry, the very signature of the Khan-Mitra work, into play. We envisage a fruitful geometric investigation that synthesizes the existing findings and points to avenues for future research.

5. Appendix: Proofs of the results

5.1. Proof of Proposition 3.2.

Proof. We solve for the optimal policy function by the standard guess-and-verify approach. Postulate a candidate value function based on the straight-down-to-turnpike policy:

$$W(x) = \begin{cases} \frac{a_I\theta}{a_C b}\rho^n(\theta^n x - \hat{x}) - \frac{1-\rho^n}{1-\rho}u(\hat{x},\hat{x}) & \text{for } x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n}) \\ \frac{1-d}{b}\rho^n\left[(1-d)^n x - \hat{x}\right] + \frac{1-\rho^n}{1-\rho}\left[1 - u(\hat{x},\hat{x})\right] & \text{for } x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}}) \end{cases}$$

where $n = 0, 1, 2...$

We now verify if $W(x)$ satisfies the Bellman equation

$$W(x) = \max_{x' \in \Gamma(x)} \{u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x')\}$$

and we consider four cases: (a) $x \in [\hat{x}/\theta, \hat{x})$; (b) $x \in [\hat{x}, \hat{x}/(1-d))$; (c) $x \in [\hat{x}/\theta^{n+1}, \hat{x}/\theta^n)$ for $n \geq 1$; (d) $x \in [\hat{x}/(1-d)^n, \hat{x}/(1-d)^{n+1})$ for $n \geq 1$.

Case (a): We have $W(x) = u(x, \hat{x}) - u(\hat{x}, \hat{x}) + \rho W(\hat{x})$. Pick $x' > \hat{x}$ such that $(x, x') \in \Omega$. There exists $n_0 \geq 0$ such that $x' \in [\hat{x}/(1-d)^{n_0}, \hat{x}/(1-d)^{n_0+1})$. For

$x' \le \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned}
& u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x') \\
= & \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x' - u(\hat{x}, \hat{x}) \\
& + \frac{1-d}{b} \rho^{n_0+1} \left[(1-d)^{n_0} x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1}}{1-\rho} \left[1 - u(\hat{x}, \hat{x})\right] \\
= & \frac{1}{a_C b} \left[(1-d) a_C \rho^{n_0+1} (1-d)^{n_0} - a_I\right] x' + \frac{a_I \theta}{a_C b} x \\
& - u(\hat{x}, \hat{x}) - \frac{1-d}{b} \rho^{n_0+1} \hat{x} + \frac{\rho - \rho^{n_0+1}}{1-\rho} \left[1 - u(\hat{x}, \hat{x})\right]
\end{aligned}$$

which strictly decreases with x' for any given n_0 because $a_C(1-d)\rho < a_I$, $\rho < 1$, and $(1-d) < 1$. For $x' > \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned}
& u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x') \\
= & \frac{1-d}{b} x - \frac{1}{b} x' + 1 - u(\hat{x}, \hat{x}) \\
& + \frac{1-d}{b} \rho^{n_0+1} \left[(1-d)^{n_0} x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1}}{1-\rho} \left[1 - u(\hat{x}, \hat{x})\right] \\
= & \frac{1}{b} \left[\rho^{n_0+1} (1-d)^{n_0+1} - 1\right] x' + \frac{1-d}{b} x + 1 \\
& - u(\hat{x}, \hat{x}) - \frac{1-d}{b} \rho^{n_0+1} \hat{x} + \frac{\rho - \rho^{n_0+1}}{1-\rho} \left[1 - u(\hat{x}, \hat{x})\right]
\end{aligned}$$

which strictly decreases with x' for any given n_0 for $\rho < 1$ and $(1-d) < 1$.

For any $x' > \hat{x}$, we have shown that $[u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x')]$ strictly decreases with x', so we have

$$u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x') < u(x, \hat{x}) - u(\hat{x}, \hat{x}) + \rho W(\hat{x}) \tag{5.1}$$

for any $x' > \hat{x}$. On the other hand, for $x' < \hat{x}$ such that $(x, x') \in \Omega$, there exists $n_0 \ge 0$ such that $x' \in [\hat{x}/\theta^{n_0+1}, \hat{x}/\theta^{n_0})$. We have

$$\begin{aligned}
& u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x') \\
= & \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x' - u(\hat{x}, \hat{x}) + \frac{a_I \theta}{a_C b} \rho^{n_0+1} (\theta^{n_0} x' - \hat{x}) - \frac{\rho - \rho^{n_0+1}}{1-\rho} u(\hat{x}, \hat{x}) \\
= & \frac{a_I}{a_C b} \left[(\rho\theta)^{n_0+1} - 1\right] x' + \frac{a_I \theta}{a_C b} x - u(\hat{x}, \hat{x}) - \frac{a_I \theta}{a_C b} \rho^{n_0+1} \hat{x} - \frac{\rho - \rho^{n_0+1}}{1-\rho} u(\hat{x}, \hat{x})
\end{aligned}$$

which strictly increases with x' for any given n_0 because $\rho\theta > 1$. It implies Inequality [5.1] for any $x' < \hat{x}$. Therefore, $\hat{x}$ is optimal for $x \in [\hat{x}/\theta, \hat{x})$.

Case (b): We have $W(x) = u(x, \hat{x}) - u(\hat{x}, \hat{x}) + \rho W(\hat{x})$. Pick $x' < \hat{x}$ such that $(x, x') \in \Omega$. There exists $n_0 \ge 0$ such that $x' \in [\hat{x}/\theta^{n_0+1}, \hat{x}/\theta^{n_0})$. For $x' > \zeta(\hat{x} -$

$x) + \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = & \frac{1-d}{b}x - \frac{1}{b}x' + 1 - u(\hat{x},\hat{x}) + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \\ = & \frac{1}{b}\left[\frac{a_I\theta}{a_C}\rho(\rho\theta)^{n_0} - 1\right]x' + \frac{1-d}{b}x + 1 \\ & -u(\hat{x},\hat{x}) - \frac{a_I\theta}{a_C b}\rho^{n_0+1}\hat{x} - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \end{aligned}$$

which strictly increases with x' for any given n_0 for $\rho > a_C/(\theta a_I)$ and $\rho\theta > 1$. For $x' \leq \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \\ = & \frac{a_I}{a_C b}\left[(\rho\theta)^{n_0+1} - 1\right]x' + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \frac{a_I\theta}{a_C b}\rho^{n_0+1}\hat{x} - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \end{aligned}$$

which strictly increases with x' for any given n_0 because $\rho\theta > 1$.

We have shown that $[u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')]$ strictly increases with x' for any $x' < \hat{x}$, which implies Inequality [5.1] for any $x' < \hat{x}$. On the other hand, for $x' > \hat{x}$ such that $(x,x') \in \Omega$, there exists $n_0 \geq 0$ such that $x' \in [\hat{x}/(1-d)^{n_0}, \hat{x}/(1-d)^{n_0+1})$. We have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = & \frac{1-d}{b}x - \frac{1}{b}x' + 1 - u(\hat{x},\hat{x}) + \\ & + \frac{1-d}{b}\rho^{n_0+1}\left[(1-d)^{n_0}x' - \hat{x}\right] + \frac{\rho-\rho^{n_0+1}}{1-\rho}\left[1 - u(\hat{x},\hat{x})\right] \\ = & \frac{1}{b}\left[(\rho(1-d))^{n_0+1} - 1\right]x' + \frac{1-d}{b}x + 1 \\ & -u(\hat{x},\hat{x}) - \frac{1-d}{b}\rho^{n_0+1}\hat{x} - \frac{\rho-\rho^{n_0+1}}{1-\rho}[1 - u(\hat{x},\hat{x})]. \end{aligned}$$

which strictly decreases with x' for a given n_0 because $\rho < 1$ and $(1-d) < 1$. It again implies Inequality [5.1] for any $x' > \hat{x}$. Therefore, $\hat{x}$ is optimal for $x \in [\hat{x}, \hat{x}/(1-d))$.

Case (c): We have $W(x) = u(x,\theta x) - u(\hat{x},\hat{x}) + \rho W(\theta x)$. For $x' < \theta x$,

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \\ = & \frac{a_I}{a_C b}\left[(\rho\theta)^{n_0+1} - 1\right]x' + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \frac{a_I\theta}{a_C b}\rho^{n_0+1}\hat{x} - \frac{\rho-\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}) \end{aligned}$$

which strictly increases with x' because $\rho\theta > 1$. Therefore, θx is optimal for $x \in [\hat{x}/\theta^{n+1}, \hat{x}/\theta^n)$ for $n \geq 1$.

Case (d): We have $W(x) = u(x,(1-d)x) - u(\hat{x},\hat{x}) + \rho W((1-d)x)$. For $x' > (1-d)x$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \; & \frac{1-d}{b}x - \frac{1}{b}x' + 1 - u(\hat{x},\hat{x}) + \\
& + \frac{1-d}{b}\rho^{n_0+1}\left[(1-d)^{n_0}x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1}}{1-\rho}\left[1 - u(\hat{x},\hat{x})\right] \\
= \; & \frac{1}{b}\left[(\rho(1-d))^{n_0+1} - 1\right]x' + \frac{1-d}{b}x + 1 \\
& - u(\hat{x},\hat{x}) - \frac{1-d}{b}\rho^{n_0+1}\hat{x} - \frac{\rho-\rho^{n_0+1}}{1-\rho}[1 - u(\hat{x},\hat{x})]
\end{aligned}$$

which strictly decreases with x' because $\rho(1-d) < 1$. Therefore, $(1-d)x$ is optimal for $x \in [\hat{x}/(1-d)^n, \hat{x}/(1-d)^{n+1})$ for $n \geq 1$.

In sum, we have verified that the postulated value function W satisfies the Bellman equation and the straight-down-to-the-turnpike policy is optimal. □

5.2. **Proof of Proposition 3.5.**

Proof. Consider two cases: (A) $\frac{1}{\zeta} \geq \frac{a_C}{\theta a_I}$ and (B) $\frac{1}{\zeta} < \frac{a_C}{\theta a_I}$.

Case (A): Given Inequality [3.1], $\frac{1}{\zeta} \geq \frac{a_C}{\theta a_I}$ implies $\frac{1}{\zeta} \geq \frac{a_C}{\theta a_I} \geq \frac{a_I}{(1-d)a_C}$. Pick $k \in (a_I, \hat{x})$ such that $\zeta(\hat{x} - k) + \hat{x} < \hat{x}/(1-d)$. We proceed in two steps: (1) We first guess and verify a value function $W(\cdot)$ for the transition correspondence $\Gamma(\cdot)$ being restricted to a subset $[k, \hat{x}/(1-d)]$. (2) Based on the results from Step (1) and given the concavity of the utility function, we then prove $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}, \hat{x}/(1-d))$.

Step (1): We postulate the following value function

$$W(x) = \begin{cases} \left(\frac{1}{a_C - a_I} - \rho\zeta\frac{1-d}{b}\right)(x - \hat{x}) & \text{for } x \in [k, \hat{x}) \\ \frac{1-d}{b}(x - \hat{x}) & \text{for } x \in [\hat{x}, \hat{x}/(1-d)] \end{cases}.$$

Our claim is that W solves the following functional equation

$$W(x) = \max_{x' \in \Gamma(x) \cap [k, \hat{x}/(1-d)]} \{u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')\}$$

with the policy function solving the functional equation given by

$$g(x) = \begin{cases} \zeta(\hat{x} - x) + \hat{x} & \text{for } x \in [k, \hat{x}) \\ \hat{x} & \text{for } x \in [\hat{x}, \hat{x}/(1-d)] \end{cases}.$$

We consider two cases: (a) $x \in [k, \hat{x})$; (b) $x \in [\hat{x}, \hat{x}/(1-d)]$.

Case (a): For $\hat{x} \leq x' < \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \; & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x' - \hat{x}) \\
= \; & \frac{1-d}{b}\left(\rho - \frac{a_I}{a_C(1-d)}\right)x' + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\frac{1-d}{b}\hat{x} \\
< \; & u(x, \zeta(\hat{x} - x) + \hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x} - x) + \hat{x}),
\end{aligned}$$

where the inequality follows from $\rho > \frac{1}{\zeta} \geq \frac{a_I}{(1-d)a_C}$ and $x' < \zeta(\hat{x}-x)+\hat{x}$. For $x' < \hat{x} < \zeta(\hat{x}-x)+\hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\left(\frac{1}{a_C - a_I} - \rho\zeta\frac{1-d}{b}\right)(x'-\hat{x}) \\
= \ & \frac{1}{b}\left(-\frac{a_I}{a_C} + \frac{b\rho}{a_C - a_I} - \zeta(1-d)\rho^2\right)x' \\
& + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C - a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
= \ & \frac{1}{b}\left(-\frac{a_I}{a_C} + 1 + (\zeta\rho - 1)(1-(1-d)\rho)\right)x' \\
& + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C - a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
< \ & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho > 1/\zeta$, $(1-d)\rho < 1$, and $a_I < a_C$. For $x' > \zeta(\hat{x}-x)+\hat{x} > \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x'-\hat{x}) \\
= \ & \frac{1}{b}\left(\rho(1-d)-1\right)x' + 1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \rho\frac{1-d}{b}\hat{x} \\
< \ & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho(1-d) < 1$ and $x' > \zeta(\hat{x}-x)+\hat{x}$.

Thus, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' = \zeta(\hat{x}-x)+\hat{x}$ and we have

$$\begin{aligned}
& u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}) \\
= \ & \frac{1-d}{b}\left(\rho - \frac{a_I}{a_C(1-d)}\right)\zeta(\hat{x}-x) + \frac{a_I\theta}{a_C b}(x-\hat{x}) \\
= \ & \left(\rho\zeta\frac{1-d}{b} - \frac{a_I\zeta}{a_C b} - \frac{a_I\theta}{a_C b}\right)(\hat{x}-x) = W(x).
\end{aligned}$$

Case (b): For $x' < \zeta(\hat{x}-x)+\hat{x} \le \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \; & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)(x'-\hat{x}) \\
= \; & \left(-\frac{a_I}{a_C b} + \frac{\rho}{a_C-a_I} - \rho^2\zeta\frac{1-d}{b}\right)x' \\
& +\frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
= \; & \left[(-\rho\zeta+1)\left(\rho\frac{1-d}{b} - \frac{1}{b}\right) + \frac{a_C-a_I}{a_C b}\right]x' \\
& +\frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
< \; & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho > 1/\zeta$, $\rho(1-d) < 1$, and $a_C > a_I$. For $\zeta(\hat{x}-x)+\hat{x} \le x' < \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \; & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)(x'-\hat{x}) \\
= \; & \left(-\frac{1}{b} + \frac{\rho}{a_C-a_I} - \rho^2\zeta\frac{1-d}{b}\right)x' \\
& +1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
= \; & (-\rho\zeta+1)\left(\rho\frac{1-d}{b} - \frac{1}{b}\right)x' \\
& +1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \rho\left(\frac{1}{a_C-a_I} - \rho\zeta\frac{1-d}{b}\right)\hat{x} \\
< \; & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho > 1/\zeta$ and $\rho(1-d) < 1$. For $x' > \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \; & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x'-\hat{x}) \\
< \; & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho(1-d) < 1$.

Therefore, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' = \hat{x}$ and we have $u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}) = \frac{1-d}{b}(x-\hat{x}) = W(x)$.

We have shown so far that $W(\cdot)$ is indeed the value function with the postulated policy being the policy function for the optimization problem restricted to the interval $[k, \hat{x}/(1-d)]$.

Step (2): Suppose on the contrary there exists $x \in (\hat{x}, \hat{x}/(1-d))$ such that there exists $x' \in h(x)$ such that $x' \neq \hat{x}$. Consider an optimal program $\{x(t), y(t)\}$ such that $x(0) = x$ and $x(1) = x'$. Consider an alternative program $\{\bar{x}(t), \bar{y}(t)\}$ starting from x such that $\bar{x}(t) = \hat{x}$ for any $t \geq 1$. We have

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \geq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

Consider a program that is a convex combination of the two programs: $\tilde{x}(t) = \lambda x(t) + (1-\lambda)\bar{x}(t)$ for $\lambda \in (0,1)$. This problem is well-defined because of convexity of Ω. We have

$$\begin{aligned}
& \sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) \\
\geq \;& \lambda \sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) + (1-\lambda) \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)) \\
\geq \;& \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))
\end{aligned}$$

where the first inequality follows from concavity of the utility function. Since we know from Proposition 3.1 that $x(t) \in [a_C(1-d), a_I\theta]$ for any $t \geq 1$, we can pick λ sufficiently close to zero such that $\tilde{x}(t) \in [k, \hat{x}/(1-d)]$. From Step (1) we know

$$\sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) < \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

which leads to the desired contradiction.

Case (B): Given Inequality [3.1], $\frac{1}{\zeta} < \frac{a_C}{\theta a_I}$ implies $\frac{1}{\zeta} < \frac{a_C}{\theta a_I} < \frac{a_I}{(1-d)a_C}$. Our argument above carries through for $\rho > \frac{a_I}{(1-d)a_C}$. For $\frac{a_C}{\theta a_I} < \rho < \frac{a_I}{(1-d)a_C}$, Proposition 3.2 applies. We only need to consider $\rho = \frac{a_I}{(1-d)a_C}$. Following the same steps as above, pick $k \in (a_I, \hat{x})$ such that $\zeta(\hat{x} - k) + \hat{x} < \hat{x}/(1-d)$. We postulate the following value function

$$W(x) = \begin{cases} \frac{a_I\theta}{a_C b}(x - \hat{x}) & \text{for } x \in [k, \hat{x}) \\ \\ \frac{1-d}{b}(x - \hat{x}) & \text{for } x \in [\hat{x}, \hat{x}/(1-d)] \end{cases}.$$

Our claim is that W solves the following functional equation

$$W(x) = \max_{x' \in \Gamma(x) \cap [k, \hat{x}/(1-d)]} \{u(x, x') - u(\hat{x}, \hat{x}) + \rho W(x')\}$$

with the policy function given by

$$g(x) = \begin{cases} [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] & \text{for } x \in [k, \hat{x}) \\ \hat{x} & \text{for } x \in [\hat{x}, \hat{x}/(1-d)] \end{cases}.$$

We consider two cases: (a) $x \in [k, \hat{x})$; (b) $x \in [\hat{x}, \hat{x}/(1-d)]$.

Case (a): For $\hat{x} \leq x' \leq \zeta(\hat{x}-x)+\hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x'-\hat{x}) \\
= \ & \frac{1-d}{b}\left(\rho - \frac{a_I}{a_C(1-d)}\right)x' + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \rho\frac{1-d}{b}\hat{x}
\end{aligned}$$

which stays constant for any $\hat{x} \leq x' \leq \zeta(\hat{x}-x)+\hat{x}$ given $\rho = \frac{a_I}{(1-d)a_C}$. For $x' < \hat{x} < \zeta(\hat{x}-x)+\hat{x}$, we have

$$\begin{aligned}
u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') &= \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \frac{\rho a_I\theta}{a_C b}(x'-\hat{x}) \\
&< u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho\theta > 1$. For $x' > \zeta(\hat{x}-x)+\hat{x} > \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x'-\hat{x}) \\
= \ & \frac{1}{b}\left(\rho(1-d)-1\right)x' + 1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \rho\frac{1-d}{b}\hat{x} \\
< \ & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho(1-d) < 1$ and $x' > \zeta(\hat{x}-x)+\hat{x}$.

Therefore, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' \in [\hat{x}, \zeta(\hat{x}-x)+\hat{x}]$ and we have $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') = W(x)$.

Case (b): For $x' < \zeta(\hat{x}-x)+\hat{x} \leq \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \frac{\rho a_I\theta}{a_C b}(x'-\hat{x}) \\
< \ & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho\theta > 1$. For $\zeta(\hat{x}-x)+\hat{x} \leq x' < \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \frac{\rho a_I\theta}{a_C b}(x'-\hat{x}) \\
< \ & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho > a_C/(a_I\theta)$. For $x' > \hat{x}$, we have

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\
= \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\frac{1-d}{b}(x'-\hat{x}) \\
< \ & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}),
\end{aligned}$$

where the inequality follows from $\rho(1-d) < 1$.

Therefore, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' = \hat{x}$ and we have $u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}) = \frac{1-d}{b}(x-\hat{x}) = W(x)$.

We have shown so far that $W(\cdot)$ is indeed the value function with the postulated policy being the policy function for the optimization problem restricted to the interval $[k, \hat{x}/(1-d)]$. The second step follows closely from that for Case (A) and thus is omitted here. Therefore, we have obtained the desired conclusion. $\square$

5.3. **Proof of Proposition 3.6.**

Proof. We first consider $a_C/(\theta a_I) > \rho > \frac{1}{\zeta}$ Pick $K \in (\hat{x}, a_C)$ such that $\zeta(\hat{x}-K)+\hat{x} > \hat{x}/\theta$. We proceed in two steps: (1) We first guess and verify a value function $W(\cdot)$ for the transition correspondence $\Gamma(\cdot)$ being restricted to a subset $[\hat{x}/\theta, K]$. (2) Based on the results in Step (1) and given the concavity of the utility function, we then prove $h(x) = \{\hat{x}\}$ for $x \in (\hat{x}/\theta, \hat{x})$.

Step (1): We postulate the following value function

$$W(x) = \begin{cases} \frac{a_I\theta}{a_C b}(x-\hat{x}) & \text{for } x \in [\hat{x}/\theta, \hat{x}] \\ \\ \frac{a_I}{a_C b}\left(\theta+\zeta-\rho\theta\zeta\right)(x-\hat{x}) & \text{for } x \in (\hat{x}, K] \end{cases}.$$

Our claim is that W solves the following functional equation

$$W(x) = \max_{x' \in \Gamma(x) \cap [\hat{x}/\theta, K]} \{u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')\}$$

with the policy function given by

$$g(x) = \begin{cases} \hat{x} & \text{for } x \in [\hat{x}/\theta, \hat{x}] \\ \zeta(\hat{x}-x)+\hat{x} & \text{for } x \in (\hat{x}, K] \end{cases}.$$

We consider two cases: (a) $x \in [\hat{x}/\theta, \hat{x}]$; (b) $x \in (\hat{x}, K]$.

Case (a): For $x' < \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \;& \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\frac{a_I\theta}{a_C b}(x'-\hat{x}) \\ < \;& u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}), \end{aligned}$$

where the inequality follows from $\rho\theta > 1$. For $x' > \zeta(\hat{x}-x)+\hat{x} \geq \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \;& 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \frac{\rho a_I}{a_C b}\left(\theta+\zeta-\rho\theta\zeta\right)(x'-\hat{x}) \\ = \;& \frac{a_I}{b a_C}\left(-\frac{a_C}{a_I} + \rho\theta + \rho\zeta - \zeta\theta\rho^2\right)x' \\ & +1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \frac{\rho a_I}{a_C b}\left(\theta+\zeta-\rho\theta\zeta\right)\hat{x} \\ = \;& \frac{a_I}{b a_C}\left((\rho\theta-1)(1-\rho\zeta) - \frac{a_C}{a_I} + 1\right)x' \\ & +1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \frac{\rho a_I}{a_C b}\left(\theta+\zeta-\rho\theta\zeta\right)\hat{x} \\ < \;& u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}), \end{aligned}$$

where the inequality follows from $\rho > 1/\zeta$, $\rho\theta > 1$, and $a_C > a_I$. For $\zeta(\hat{x}-x)+\hat{x} \geq x' > \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \ & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \frac{\rho a_I}{a_C b}\left(\theta + \zeta - \rho\theta\zeta\right)(x' - \hat{x}) \\ = \ & \frac{a_I}{a_C b}(\rho\theta - 1)(1-\rho\zeta)x' + \frac{a_I\theta}{a_C b}x - u(\hat{x},\hat{x}) - \frac{\rho a_I}{a_C b}\left(\theta + \zeta - \rho\theta\zeta\right)\hat{x} \\ < \ & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}), \end{aligned}$$

where the inequality follows from $\rho > 1/\zeta$ and $\rho\theta > 1$.

Therefore, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' = \hat{x}$ and we have $u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}) = \frac{a_I\theta}{a_C b}(x-\hat{x}) = W(x)$.

Case (b): For $x' < \zeta(\hat{x}-x)+\hat{x} < \hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \ & \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' - u(\hat{x},\hat{x}) + \rho\frac{a_I\theta}{a_C b}(x' - \hat{x}) \\ < \ & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}), \end{aligned}$$

where the inequality follows from $\rho\theta > 1$. For $\hat{x} \geq x' > \zeta(\hat{x}-x)+\hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \rho\frac{a_I\theta}{a_C b}(x' - \hat{x}) \\ < \ & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}), \end{aligned}$$

where the inequality follows from $\rho < a_C/(\theta a_I)$. For $x' > \hat{x} \geq \zeta(\hat{x}-x)+\hat{x}$, we have

$$\begin{aligned} & u(x,x') - u(\hat{x},\hat{x}) + \rho W(x') \\ = \ & 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) + \frac{\rho a_I}{a_C b}\left(\theta + \zeta - \rho\theta\zeta\right)(x' - \hat{x}) \\ = \ & \frac{a_I}{a_C b}\left((\rho\theta - 1)(1-\rho\zeta) - \frac{a_C}{a_I} + 1\right)x' \\ & +1 + \frac{1-d}{b}x - u(\hat{x},\hat{x}) - \frac{\rho a_I}{a_C b}\left(\theta + \zeta - \rho\theta\zeta\right)\hat{x} \\ < \ & u(x,\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\hat{x}), \end{aligned}$$

where the inequality follows from $\rho\zeta > 1$, $\rho\theta > 1$ and $a_I < a_C$.

Therefore, $u(x,x') - u(\hat{x},\hat{x}) + \rho W(x')$ is maximized when $x' = \zeta(\hat{x}-x)+\hat{x}$ and we have

$$\begin{aligned} & u(x,\zeta(\hat{x}-x)+\hat{x}) - u(\hat{x},\hat{x}) + \rho W(\zeta(\hat{x}-x)+\hat{x}) \\ = \ & \frac{1}{b}\left(\frac{a_I}{a_C}\rho\theta - 1\right)\zeta(\hat{x}-x) + \frac{1-d}{b}(x-\hat{x}) \\ = \ & \frac{a_I}{a_C b}\left(\theta + \zeta - \rho\theta\zeta\right)(x - \hat{x}) = W(x). \end{aligned}$$

We have shown so far that $W(\cdot)$ is indeed the value function for the optimization problem restricted to the interval $[\hat{x}/\theta, K]$.

Step (2): Suppose on the contrary there exists $x \in (\hat{x}/\theta, \hat{x})$ such that there exists $x' \in h(x)$ such that $x' \neq \hat{x}$. Consider an optimal program $\{x(t), y(t)\}$ such that $x(0) = x$ and $x(1) = x'$. Consider an alternative program $\{\bar{x}(t), \bar{y}(t)\}$ starting from x such that $\bar{x}(t) = \hat{x}$ for any $t \geq 1$. We have

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \geq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

Consider a program that is a convex combination of the two programs: $\tilde{x}(t) = \lambda x(t) + (1-\lambda)\bar{x}(t)$ for $\lambda \in (0,1)$. This problem is well-defined because of convexity of Ω. We have

$$\begin{aligned} & \sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) \\ \geq \quad & \lambda \sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) + (1-\lambda) \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)) \\ \geq \quad & \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)) \end{aligned}$$

where the first inequality follows from concavity of the utility function. Since we know from Proposition 3.1 that $x(t) \in [a_C(1-d), a_I\theta]$ for any $t \geq 1$, we can pick λ sufficiently close to zero such that $\tilde{x}(t) \in [\hat{x}/\theta, K]$. From Step (1) we know

$$\sum_{t=0}^{\infty} \rho^t u(\tilde{x}(t), \tilde{x}(t+1)) < \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

which leads to the desired contradiction.

Now consider $\rho = a_C/(\theta a_I)$. The proof above carries through with one modification: the policy function is given by

$$\begin{cases} \hat{x} & \text{for } x \in [\hat{x}/\theta, \hat{x}] \\ [\zeta(\hat{x} - x) + \hat{x}, \hat{x}] & \text{for } x \in (\hat{x}, K] \end{cases}.$$

We have now obtained the desired conclusion. □

5.4. Proof of Proposition 3.11.

Proof. Since $\rho^{n+1}(1-d)^{n+1} > a_I/a_C$, $\rho(1-d) > a_I/a_C$. Since $\rho(1-d) > a_I/a_C$ and $\rho > 1/\zeta$, from Inequality [3.1] we know $\rho > a_C/(\theta a_I)$. Given Proposition 3.1 and 3.5, we only need to verify for $x \in (\hat{x}/\theta, \hat{x})$. Pick $x' \in (\hat{x}, \min\{\theta x, \zeta(x - \hat{x}) + \hat{x}\})$. There exists an integer $n_0 \geq 0$ such that $x' \in (\frac{\hat{x}}{\theta^{n_0}}, \frac{\hat{x}}{\theta^{n_0+1}}]$. By construction, $n_0 \leq n$. We know from Proposition 3.1 and 3.5,

$$\begin{aligned} & u(x, x') - u(\hat{x}, \hat{x}) + \rho V(x') \\ = \quad & \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x' - u(\hat{x}, \hat{x}) + \frac{\rho - \rho^{n_0+1}}{1-\rho}(1 - u(\hat{x}, \hat{x})) \\ & + \frac{1-d}{b} \rho^{n_0+1}((1-d)^{n_0} x' - \hat{x}) \\ < \quad & u(x, \min\{\theta x, \zeta(x - \hat{x}) + \hat{x}\}) - u(\hat{x}, \hat{x}) + \rho V(\min\{\theta x, \zeta(x - \hat{x}) + \hat{x}\}) \end{aligned}$$

where the inequality follows from $\rho^{n+1}(1-d)^{n+1} > a_I/a_C$ and $n_0 \leq n$. Therefore, $h(x) = \{\min\{\theta x, \zeta(x-\hat{x})+\hat{x}\}\}$ for $x \in (\hat{x}/\theta, \hat{x})$. □

5.5. **Proof of Proposition 3.12.**

Proof. Given Proposition 3.1 and 3.6, we only need to verify for $x \in (\hat{x}, \hat{x}/(1-d))$. Pick $x' \in (\max\{(1-d)x, \zeta(x-\hat{x})+\hat{x}\}, \hat{x})$. There exists an integer $n_0 \geq 0$ such that $x' \in [\frac{\hat{x}}{(1-d)^{n_0+1}}, \frac{\hat{x}}{(1-d)^{n_0}})$. By construction, $n_0 \leq n$. We know from Proposition 3.1 and 3.6,

$$\begin{aligned}
& u(x,x') - u(\hat{x},\hat{x}) + \rho V(x') \\
= \;& 1 - \frac{1}{b}x' + \frac{1-d}{b}x - u(\hat{x},\hat{x}) \\
& + \frac{\rho - \rho^{n_0+1}}{1-\rho}(1-u(\hat{x},\hat{x})) + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) \\
< \;& u(x, \max\{(1-d)x, \zeta(x-\hat{x})+\hat{x}\}) - u(\hat{x},\hat{x}) \\
& + \rho V(\max\{(1-d)x, \zeta(x-\hat{x})+\hat{x}\})
\end{aligned}$$

where the inequality follows from $(\rho\theta)^{n+1}a_I < a_C$ and $n_0 \leq n$. Therefore, $h(x) = \{\max\{(1-d)x, \zeta(x-\hat{x})+\hat{x}\}\}\}$ for $x \in (\hat{x}, \hat{x}/(1-d))$. □

References

[1] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.

[2] L. Deng, M. Fujio, and M. Ali Khan, *Eventual periodicity in the two-sector RSL model: equilibrium vis-à-vis optimum growth*, Working Paper (2019).

[3] L. Deng, M. Fujio, and M. Ali Khan, *Optimal growth in the Robinson-Shinkai-Leontief model: the case of capital-intensive consumption goods*, Stud. in Nonln. Dyn. & Econom. **23** (2019), 20190032.

[4] M. Fujio, *The Leontief two-sector model and undiscounted optimal growth with irreversible investment: the case of labor-intensive consumption goods*, J. of Econ. **86** (2005), 145–159.

[5] M. Fujio, *Undiscounted optimal growth in a Leontief two-sector model with circulating capital: the case of a capital-intensive consumption good*, J. of Econ. Behav. & Org. **66** (2008), 420–436.

[6] M. Fujio, *Optimal transition dynamics in the Leontief two-sector growth model with durable capital: the case of capital intensive consumption goods*, Jap. Econ. Rev. **60** (2009), 490–511.

[7] M. Ali Khan and A. Piazza, *Optimal cyclicity and chaos in the 2-sector RSS model: an anything-goes construction*, J. Econ. Behav. & Org. **80** (2011), 397–417.

[8] M. Ali Khan and T. Mitra, *On choice of technique in the Robinson-Solow-Srinivasan model*, Intl. J. Econ. Th. **1** (2005), 83–110.

[9] M. Ali Khan and T. Mitra, *On topological chaos in the Robinson-Solow-Srinivasan model*, Econ. Lett. **88** (2005), 127–133.

[10] M. Ali Khan and T. Mitra, *Discounted optimal growth in the two-sector RSS model: a geometric investigation*, Adv. Math. Econ. **8** (2006), 349–381.

[11] M. Ali Khan and T. Mitra, *Optimal growth in a two-sector RSS model without discounting: a geometric investigation*, Jap. Econ. Rev. **58** (2007), 191–225.

[12] M. Ali Khan and T. Mitra, *Optimal growth under discounting in the two-sector Robinson-Solow-Srinivasan model: a dynamic programming approach*, J. Diff. Eq. App. **12** (2007), 151–168.

[13] M. Ali Khan and T. Mitra, *Impatience and dynamic optimal behavior: a bifurcation analysis of the Robinson-Solow-Srinivasan model*, Nonln. Anal.: Th., Meh. & App. **75** (2012), 1400–1418.

[14] M. Ali Khan and T. Mitra, *Optimal growth in a two-sector RSS model with discounting: a further geometric investigation*, Adv. Math. Econ. **17** (2013), 1–33.

[15] M. Ali Khan and T. Mitra, *Complicated dynamics and parametric restrictions in the Robinson-Solow-Srinivasan (RSS) model*, Adv. Math. Econ., forthcoming.

[16] R. May, *Simple mathematical models with very complicated dynamics*, Nature **261** (1976), 459–467.

[17] T. Mitra, K. Nishimura, and G. Sorger, *Optimal cycles and chaos*, in R.-A. Dana, C. Le Van, T. Mitra and K. Nishimura (eds) Handb. Opm. Gr. **1** (2006), 141–170.

[18] K. Nishimura and M. Yano, *Nonlinear dynamics and chaos in optimal growth: an example*, Econometrica **63** (1995), 981–1001.

Manuscript received December 29 2019
revised March 3 2020

L. Deng
Division of Social Sciences, Yale-NUS College, Singapore
E-mail address: `liuchun.deng@yale-nus.edu.sg`

M. Fujio
Department of Economics, Yokohama National University, Japan
E-mail address: `fujio-minako-rm@ynu.ac.jp`