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FATOU'S LEMMA FOR UNBOUNDED GELFAND INTEGRABLE MAPPINGS

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Abstract. The objective of this paper is to provide Fatou-type results for sequences of Gelfand integrable mappings defined on a measure space Ω with values in the topological dual $E^\star$ of a separable Banach space E. We introduce the assumption of mean weak boundedness that encompasses stronger conditions considered previously. The proof of our Fatou-type results consists in applying successively the scalar version of Komlós' theorem.

Since Schmeidler [45], Fatou's lemma is one of the essential tool in proving the existence of Walrasian equilibria in economic models with a measure space of economic agents as initiated by Aumann [4]. With infinitely many commodities, with commodity space being the topological dual of a separable Banach space, the version of Fatou's lemma with the Gelfand integral applies directly to models of spacial economies (Cornet and Médecin [16]) and models with differentiated commodities (Ostroy and Zame [41] and Martins-da-Rocha [36]).

1. Introduction

Since the seminal contribution of Aumann [4], economists represent perfect competition by modeling the set of agents as a non-atomic finite positive measure space. In these economic models, Fatou's lemma for vector-valued mappings is essential in proving the existence of a Walrasian competitive equilibrium. When economic agents trade finitely many commodities, the range space is the standard finite dimensional Euclidean space $\mathbb{R}^n$ where n is the number of commodities. In this finite dimensional environment, Schmeidler [45] provided a version of Fatou's lemma that was used by Hildenbrand [24, 25] to prove the existence of a Walrasian competitive equilibrium. In Schmeidler [45]'s version of Fatou's lemma, mappings take values in the standard positive orthant $\mathbb{R}^n_+$. An extension to mappings with values in some abstract cone of $\mathbb{R}^n$ has been proposed by Balder and Hess [9]. This more general version of Fatou's lemma was first applied by Cornet, Topuzu and Yildiz [18] to allow for satiated consumers, and then by Angeloni and Martins-da-Rocha [2] to incorporate differential information.

In a model with infinitely many commodities, economic agents (consumers, producers) make their optimal choice of actions in an infinite dimensional commodity space. We refer to Mas Colell and Zame [38], for instance, for a detailed discussion on the model and to Khan and Yannelis [32], Bewley [13], Noguchi [40],

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Podczeck [43], Martins-da-Rocha [35] and Lee [34] for the existence of a Walrasian competitive equilibrium using the Bochner integral. This lead various authors to establish versions of Fatou's lemma for mappings with values in a Banach space, using either the Bochner integral or the Gelfand integral. We refer to Khan and Majumdar [27], Balder [5], Yannelis [47–49], Rustichini [44], Balder and Hess [9], Khan and Sagara [28] for Fatou-type results dealing with Bochner integrable functions.

When the commodity space $E^\star$ is the topological dual of a separable Banach space E, aggregation is usually represented by the Gelfand integral. This is the case for models of spatial economies (Cornet and Médecin [16]) and models with differentiated commodities (Ostroy and Zame [41] and Martins-da-Rocha [36]). We refer to Martins-da-Rocha [36] and Khan and Sagara [29] for the existence of a Walrasian competitive equilibrium using the Gelfand integral. For the Gelfand integral, versions of Fatou's lemma have been proposed by Cornet and Médecin [17], Balder [8], a previous version of the current paper (Cornet and Martins-da-Rocha [15]), Balder and Sambucini [11], Khan and Sagara [31] and Greinecker and Podczeck [22]. Our paper contributes to this literature and introduces the assumption of mean weak boundedness that encompasses stronger conditions considered previously in approximate versions of Fatou's lemma with Gelfand integral.

In order to present our contribution, we introduce informally the following notations, with full detail given in Section 2.1. The commodity space is $E^\star$, the topological dual space of a separable Banach space $(E, \|\cdot\|)$, the set of agents is a finite positive measure space $(\Omega, \mathcal{A}, \mu)$, and an allocation is a mapping that assigns to each agent $a \in \Omega$ a bundle $f(a) \in X \subseteq E^\star$, i.e., the commodity set X is a given subset of the commodity space $E^\star$. We restrict attention to allocations f that are feasible in the sense that: (i) $f(a) \in X$ for all a, (ii) $f : \Omega \to E^\star$ is Gelfand integrable, and (iii) f satisfies

$$\int_\Omega f d\mu = \int_\Omega e d\mu$$

for some exogenously given Gelfand integrable mapping $e : \Omega \to E^\star$ that represents the agents' initial endowments. The main approach to prove the existence of a Walrasian competitive equilibrium involves the construction of a sequence (f_n) of Gelfand integrable feasible allocations to which will be applied a version of Fatou's lemma. Gelfand integrability then implies that

$$\forall x \in E, \quad \int_\Omega \langle x, f_n(a)\rangle d\mu(a) = \int_\Omega \langle x, e(a)\rangle d\mu(a)$$

where $\langle\cdot,\cdot\rangle : E \times E^\star \to \mathbb{R}$ is the canonical duality function. In most economic models, the space E is a Banach lattice, with E_+ denoting its nonnegative cone, and the commodity set X is a subset of the dual nonnegative cone $E^\star_+$ (see Aliprantis and Border [1] for formal definitions). Since each f_n is feasible, we deduce the following *mean scalar boundedness* property:

$$\forall x \in E, \quad \sup_n \int_\Omega |\langle x, f_n\rangle| d\mu < \infty.^1 \tag{1.1}$$

[1]Indeed, for every $x \in E$, $f_n(a) \in X \subseteq E^\star_+$ implies $\int_\Omega |\langle x, f_n\rangle| d\mu \leqslant \int_\Omega \langle x^+, f_n\rangle d\mu + \int_\Omega \langle x^-, f_n\rangle d\mu = \int_\Omega \langle x^+, e\rangle d\mu + \int_\Omega \langle x^-, e\rangle d\mu$.

In a previous version of this paper (Cornet and Martins-da-Rocha [15]), we showed that the *uniform integrability* condition of Cornet and Médecin [17] and Balder [8]

$$\lim_{\alpha\to\infty}\sup_n \int_{\|f_n\|^\star \geqslant \alpha} \|f_n\|^\star \, d\mu = 0$$

can be replaced by the following *mean norm boundedness* condition:

$$\sup_n \int_\Omega \|f_n\|^\star \, d\mu < \infty.$$

This last condition is particularly useful when the commodity space $E^\star = M(K)$, the space of signed Borel measures on a compact space K, since models with differentiated commodities consider K as the set of commodity characteristics (see Jones [26], Mas Colell [37], Ostroy and Zame [41], Martins-da-Rocha [36] and Greinecker and Podczeck [21]). Indeed, choosing the vector x to be any constant function, we get that the mean norm boundedness property follows from the mean scalar property.

Consider now an economic environment in which agents share risks by trading their initial endowments in an uncertain environment represented by a countably generated probability space $(S, \Sigma, \mathbb{P})$. In the line of Bewley [12], agents choose contingent consumption bundles in $L^\infty_+(\mathbb{P})$, that is, formally $E = L^1(\mathbb{P})$ endowed with the L^1-norm and $E^\star = L^\infty(\mathbb{P})$ where $\|\cdot\|^\star$ is the (essential) sup-norm. We point out a fundamental difference between this model and the previous model with differentiated commodities (for which $E^\star = M(K)$), since now there does not always exist a vector $x \in E_+ = L^1_+(\mathbb{P})$ such that $\|y\|^\star = \langle x, y\rangle$ for every $y \in L^\infty_+(\mathbb{P})$. In other words, the mean norm boundedness condition does not necessarily follow from the mean scalar boundedness. Thus, an additional boundedness assumption needs to be imposed. For instance, Khan and Sagara [30, Section 7.2] assume that there exists a uniform bound $M > 0$ on consumption sets; in other words, the sequence (f_n) of Gelfand integrable mappings in Fatou's lemma is assumed to satisfy $\|f_n(a)\|^\star \leqslant M$ for each agent a.

The main contribution of this paper is to provide a new approximate version of Fatou's lemma when the sequence (f_n) of Gelfand integrable mappings satisfies a general condition that encompasses the two following important cases of (i) mean norm boundedness and (ii) pointwise boundedness, i.e., $\sup_n \|f_n(a)\|^\star < \infty$ for a.e. $a \in \Omega$. In order to do it, the mean norm boundedness condition is replaced by the following strictly weaker requirement:

$$\sup_n \int_{\|f_n\|^\star > \rho} \|f_n\|^\star \, d\mu < \infty$$

where $\rho : \Omega \to (0, \infty)$ is a given measurable function that is neither assumed to be bounded nor to be integrable. If the sequence (f_n) is mean norm bounded, then ρ can be chosen to be a constant function and if the sequence (f_n) is pointwise bounded, then ρ can be chosen as follows $\rho(a) := \sup_n \|f_n(a)\|^\star$ for every $a \in \Omega$.

2. Statement of results

2.1. **Gelfand integrable mappings.** In the whole paper we assume that $(\Omega, \mathcal{A}, \mu)$ is a finite complete positive measure space, $(E, \|.\|)$ is a separable Banach space, with topological dual space $E^\star$. Let $\mathcal{L}^0(\mathbb{R})$ denote the set of $\mathcal{A}$-measurable functions $f : \Omega \to \mathbb{R}$, and let $\mathcal{L}^1(\mathbb{R})$ be the space of μ-integrable real-valued functions defined on Ω, and denote $\|\cdot\|_1$ its Lebesgue semi-norm defined by $\|f\|_1 := \int_\Omega |f(a)| d\mu(a)$. We let $L^1(\mathbb{R})$ denote the corresponding quotient space. We shall mainly consider on the space $E^\star$ the weak-star topology $\sigma(E^\star, E)$, denoted $w^\star$, and the limit, the closure of a set (etc..) for this topology will be denoted $\lim$, cl (etc..). For $x \in E$ and $f \in E^\star$, we denote by $\langle x, f\rangle := f(x)$ the canonical dual product, and by $\|.\|^\star$ the dual norm on $E^\star$, i.e., $\|f\|^\star := \sup_{x\neq 0} |\langle f, x\rangle| / \|x\|$. We denote by B and $B^\star$, the closed unit balls in $(E, \|.\|)$ and $(E^\star, \|.\|^\star)$, respectively. If (x_k) is a sequence in $E^\star$, then we denote by $\mathrm{Ls}\{x_k\}$ the set of $w^\star$-limit points of the set $\{x_k : k \in \mathbb{N}\}$. If $C \subseteq E$, then $C^\circ := \{x^\star \in E^\star : \forall x \in C, \ \langle x, x^\star\rangle \leq 0\}$ denotes the negative polar cone of C and if $D \subseteq E^\star$, then $D^\circ := \{x \in E : \forall x^\star \in D, \ \langle x, x^\star\rangle \leq 0\}$ denotes the negative polar cone of D. If $F \subseteq E$ is a linear subspace of E, then the negative polar F° coincides with the orthogonal $F^\perp := \{x^\star \in E^\star : \forall x \in F, \ \langle x, x^\star\rangle = 0\}$. Note that if X is a subset of $E^\star$, then

$$X \subseteq \bigcap_{F\in\mathcal{F}} [X + F^\perp] \subseteq \mathrm{cl}\, X,$$

where $\mathcal{F}$ is the collection of all finite dimensional linear subspaces of E. In particular if E is finite dimensional, then $X = \bigcap_{F\in\mathcal{F}} [X + F^\perp]$.

A mapping f from Ω to $E^\star$ is said to be **Gelfand measurable**, if for every $x \in E$, the real-valued function $\langle x, f\rangle : a \mapsto \langle x, f(a)\rangle$ from Ω to $\mathbb{R}$ is measurable, which is equivalent to saying that for each Borelian $B \subseteq E^\star$, $f^{-1}(B) := \{a \in \Omega : f(a) \in B\}$ belongs to $\mathcal{A}$ (see the Appendix), and f is said to be **Gelfand integrable**, if for every $x \in E$, the function $\langle x, f\rangle$ is integrable. If f is Gelfand integrable, it can be shown (see Diestel and Uhl [19, pp. 52-53]) that for each $A \in \mathcal{A}$, there exists a unique $x^\star_A \in E^\star$ such that

$$\forall x \in E, \quad \langle x, x^\star_A\rangle = \int_\Omega \langle x, f(a)\rangle \mathbf{1}_A(a) d\mu(a) = \int_\Omega \langle x, f\rangle \mathbf{1}_A d\mu^2$$

thus $x^\star_A$ is simply denoted by $\int_A f d\mu$.

Proposition 2.1. *Let $f : \Omega \to E^\star$ be Gelfand integrable, then*

$$\|f\|_{\mathcal{G}} := \sup_{x\in B} \int_\Omega |\langle x, f(a)\rangle| d\mu(a) < \infty.$$

Proof. Let f be a Gelfand integrable mapping. We let T_f be the mapping from E to $L^1(\mathbb{R})$ defined by

$$\forall x \in E, \quad T_f(x) := [a \mapsto \langle x, f(a)\rangle].$$

[2] We denote by $\mathbf{1}_A$ the indicator function of the set A, i.e., $\mathbf{1}_A(a) = 1$ if $a \in A$ and $\mathbf{1}_A(a) = 0$ otherwise.

The linear mapping T_f from $(E, \|.\|)$ to $(L^1(\mathbb{R}), \|.\|_1)$ has a closed graph (Diestel and Uhl [19, pp. 52–53]). From Banach's closed graph theorem, T_f is continuous. Thus $\|f\|_{\mathcal{G}} < \infty$. □

Note that if f is a Gelfand measurable mapping, then the function $a \mapsto \|f(a)\|^\star$ is measurable (see Proposition A.1 in the appendix). A Gelfand measurable mapping f is said to be **norm integrable** if $a \mapsto \|f(a)\|^\star$ is integrable. Obviously, a norm integrable mapping is Gelfand integrable but the converse is not true in general; see Counterexample A.11 in the appendix. We recall the following notions about sequences of integrable mappings.

Definition 2.2. A sequence (f_n) of Gelfand integrable mappings from Ω to $E^\star$ is said to be

(1) **integrably bounded** if there exists a real-valued integrable function $\varphi \in \mathcal{L}^1(\mathbb{R})$ such that

$$\sup_n \|f_n(a)\|^\star \leqslant \varphi(a) \quad \text{a.e.},$$

(2) **uniformly norm integrable** if the sequence of real-valued functions $(\|f_n\|^\star)$ is uniformly integrable, i.e.

$$\lim_{\alpha \to \infty} \sup_n \int_{\{\|f_n\|^\star \geqslant \alpha\}} \|f_n(a)\|^\star \, d\mu(a) = 0,$$

(3) **mean norm bounded** if

$$\sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a) < \infty,$$

(4) **mean weakly bounded** if the two following conditions hold:
   (i) for all $x \in E$, the sequence of real-valued functions $(\langle x, f_n \rangle)$ is mean bounded, i.e.,

$$\forall x \in E, \quad \sup_n \int_\Omega |\langle x, f_n(a) \rangle| d\mu(a) < \infty,$$

   (ii) there exists a measurable function $\rho \in \mathcal{L}^0(\mathbb{R})$ such that

$$\sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star \, d\mu(a) < \infty.$$

We note that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) and the converse implications are not true in general. Indeed, (1) $\Longrightarrow$ (2) is obvious, (2) $\Longrightarrow$ (3) holds since uniform norm integrability is equivalent to mean norm boundednes and equicontinuity (see Neveu [39]), and choosing $\rho = 0$ we get (3) $\Longrightarrow$ (4). To see that the implication (4) $\Longrightarrow$ (3) is not valid, we refer to Counterexample A.12 in the appendix.

Remark 2.3. When the sequence (f_n) is norm bounded, in the sense that $\sup_n \|f_n\|^\star < \infty$ almost everywhere, then (4ii) is automatically satisfied if we choose $\rho := \sup_n \|f_n\|^\star$. Even if (f_n) satisfies condition (4i), this does not necessarily imply that (f_n) is mean norm bounded. We refer to Counterexample A.12 in the appendix.

2.2. **Fatou's lemma.** Our main contribution is to provide Fatou-type results for sequences of Gelfand integrable mappings satisfying the mean weakly boundedness conditions and we recall that this condition has been previously proved to be a strict weakening of the mean boundedness condition.

Before presenting our main results, we associate to every given sequence (f_n) of Gelfand integrable mappings $f_n : \Omega \to E^\star$ the cone $C_{(f_n)}$ of all vectors $x \in E$ such that the sequence of real-valued functions $(\max\{0, -\langle x, f_n\rangle\})$ is uniformly integrable, i.e.,

$$C_{(f_n)} := \{x \in E : \text{ the sequence } (\max\{0, -\langle x, f_n\rangle\}) \text{ is uniformly integrable}\}.$$

Our first result provides a convex version of Fatou's lemma for Gelfand integrable mappings.

Theorem 2.4 (Convex Fatou's lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$, which is mean weakly bounded and such that* $\lim \int_\Omega f_n d\mu$ *exists in $E^\star$.*

Then there exists a Gelfand integrable mapping f from Ω to $E^\star$ such that

$$\int_\Omega f d\mu - \lim \int_\Omega f_n d\mu \in \left[C_{(f_n)}\right]^\circ, \tag{2.1}$$

and

$$f(a) \in \overline{\text{co}}\, \text{Ls}\{f_n(a)\} \quad \text{a.e.} \tag{2.2}$$

Moreover, f satisfies

$$\int_\Omega \|f(a)\|^\star d\mu(a) \leqslant \int_\Omega \rho d\mu + \sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star d\mu(a), \tag{2.3}$$

where $\rho \in \mathcal{L}^0(\mathbb{R})$ is the measurable function associated to the mean weakly boundedness condition.

The proof of Theorem 2.4 is given in Section 3 as a direct consequence of Theorem 3.2 also of interest for itself. Our second result formulates an approximate version of Fatou's lemma.

Theorem 2.5 (Approximate Fatou's lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$, which is mean weakly bounded and such that* $\lim \int_\Omega f_n d\mu$ *exists in $E^\star$.*

Then, there exist a Gelfand integrable mapping f from Ω to $E^\star$ satisfying the properties (2.1), (2.2) and (2.3) of Theorem 2.4 such that for each finite dimensional subspace F of E, there exists a Gelfand measurable mapping f_F from Ω to $E^\star$ satisfying

$$\forall x \in F, \quad \int_\Omega \langle x, f(a)\rangle d\mu(a) = \int_\Omega \langle x, f_F(a)\rangle d\mu(a)$$

together with

$$f_F(a) \in \text{Ls}\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f_F(a)\|^\star \, d\mu(a) \leqslant \int_\Omega \rho d\mu + \sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star \, d\mu(a),$$

where $\rho \in \mathcal{L}^0(\mathbb{R})$ *is the measurable function associated to the mean weakly bounded condition.*

Theorem 2.5 will be proved in Section 4 as a consequence of Theorem 3.2. When the sequence (f_n) is mean norm bounded (i.e., when we can choose $\rho = 0$), we get the following result that corresponds to Theorem 2.2 in Cornet and Martins-da-Rocha [15] and Corollary 4.1 in Balder and Sambucini [11].

Corollary 2.6. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a finite positive complete measure space, let* $(E, \|.\|)$ *be a separable Banach space, let* (f_n) *be a sequence of Gelfand integrable mappings from* Ω *to* $E^\star$*, which is mean norm bounded and such that* $\lim \int_\Omega f_n d\mu$ *exists in* $E^\star$*.*

(1) [Convex Fatou's lemma]. *There exists a Gelfand integrable mapping* f *from* Ω *to* $E^\star$ *such that*

$$\int_\Omega f d\mu - \lim \int_\Omega f_n d\mu \in \left[C_{(f_n)}\right]^\circ$$

together with

$$f(a) \in \overline{\text{co}}\, \text{Ls}\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

(2) [Approximate Fatou's lemma]. *For each finite dimensional subspace* F *of* E*, there exist a Gelfand integrable mapping* f_F *from* Ω *to* $E^\star$ *such that*

$$\int_\Omega f_F d\mu - \lim \int_\Omega f_n d\mu \in \left[C_{(f_n)}\right]^\circ + F^\perp,$$

together with

$$f_F(a) \in \text{Ls}\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f_F(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

(3) [Finite Dimensional Fatou's lemma]. *If* E *is finite dimensional, then there exists a Gelfand integrable mapping* f_E *from* Ω *to* $E^\star$ *such that*

$$\int_\Omega f_E d\mu - \lim \int_\Omega f_n d\mu \in \left[C_{(f_n)}\right]^\circ$$

together with

$$f_E(a) \in \text{Ls}\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f_E(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

We provide hereafter a sufficient condition for a sequence of mappings to satisfy the assumptions of Corollary 2.6. The result first appeared in Cornet and Martins-da-Rocha [15]; see also Greinecker and Podczeck [22].

Lemma 2.7. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$ such that*

$$\forall n \in \mathbb{N}, \quad f_n(a) \in C^\star + \varphi_n(a) B^\star \quad \text{a.e.},$$

where $C^\star \subseteq E^\star$ is a closed convex cone, (φ_n) is a sequence of uniformly integrable positive real-valued functions defined on Ω.

(a) $-(C^\star)^\circ \subseteq C_{(f_n)}$.

(b) *If* $\lim \int_\Omega f_n d\mu$ *exists in* $E^\star$ *and* $C^\star$ *has a* $w^\star$*-compact sole,*[3] *then the sequence* (f_n) *is mean norm bounded.*

Proof. Part (a) is obvious. We now prove Part (b). Let $C^\star$ be a closed convex cone with a $w^\star$-compact sole. There exists $e \in E$ such that for every $x^\star \in C^\star \setminus \{0\}$, $\langle e, x^\star \rangle > 0$ and such that the following set $S := \{x^\star \in C^\star : \langle e, x^\star \rangle = 1\}$ is $w^\star$-compact. It follows that S is $\|.\|^\star$-bounded by some $m > 0$. In particular, for every $x^\star \in C^\star$, $\langle e, x^\star \rangle \geqslant m \|x^\star\|^\star$. For each $n \in \mathbb{N}$, we consider the following correspondence $F_n : a \mapsto C^\star \cap [\{f_n(a)\} - \varphi_n(a) B^\star]$. Applying Theorem A.2, there exists two measurable mappings $c_n : \Omega \mapsto C^\star$ and $b_n : \Omega \mapsto B^\star$ such that for every $n \in \mathbb{N}$,

$$\forall a \in \Omega, \quad f_n(a) = c_n(a) + \varphi_n(a) b_n(a).$$

Since the sequence $(\int_\Omega f_n d\mu)$ converges, we can suppose (passing to a subsequence if necessary) that the sequences $(\int_\Omega c_n d\mu)$ and $(\int_\Omega \varphi_n b_n d\mu)$ converges in $E^\star$. Now, let $v^\star := \lim \int_\Omega c_n d\mu$, then

$$\limsup_n \int_\Omega \|c_n(a)\|^\star \, d\mu(a) \leqslant \frac{1}{m} \langle e, v^\star \rangle \mu(\Omega)$$

and the sequence (c_n) is mean norm bounded. It follows that the sequence (f_n) is also mean norm bounded. □

Applying Lemma 2.7, we present hereafter a consequence of Corollary 2.6.

Corollary 2.8. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$ such that*

$$\forall n \in \mathbb{N}, \quad f_n(a) \in C^\star + \varphi_n(a) B^\star \quad \text{a.e.},$$

where $C^\star$ is closed convex cone in $E^\star$ with a $w^\star$-compact sole, and (φ_n) is a sequence of uniformly integrable positive functions. Suppose that $\lim \int_\Omega f_n d\mu$ *exists in* $E^\star$.

(1) [Convex Fatou's lemma]. *There exists a Gelfand integrable mapping f from Ω to $E^\star$ such that*

$$\int_\Omega f d\mu - \lim \int_\Omega f_n d\mu \in -C^\star$$

together with

$$f(a) \in \overline{\text{co}}\, \text{Ls}\{f_n(a)\} \quad \text{a.e.}$$

[3]That is, there exists $e \in E$, such that for each $c^\star \in C^\star \setminus \{0\}$, $\langle e, c^\star \rangle > 0$ and $S := \{c^\star \in C^\star : \langle e, c^\star \rangle = 1\}$ is $w^\star$-compact.

and

$$\int_\Omega \|f(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

(2) [Approximate Fatou's lemma]. *For every finite dimensional subspace F of E, there exists a Gelfand integrable mapping f_F from Ω to $E^\star$ such that*

$$\int_\Omega f_F d\mu - \lim \int_\Omega f_n d\mu \in F^\perp - C^\star$$

together with

$$f_F(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f_F(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

(3) [Finite Dimensional Fatou's lemma]. *If E is finite dimensional, then there exists a Gelfand integrable mapping f_E from Ω to $E^\star$ such that*

$$\int_\Omega f_E d\mu - \lim \int_\Omega f_n d\mu \in -C^\star$$

together with

$$f_E(a) \in \mathrm{Ls}\{f_n(a)\} \quad \text{a.e.}$$

and

$$\int_\Omega \|f_E(a)\|^\star \, d\mu(a) \leqslant \sup_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

Remark 2.9. If E is finite dimensional, then every pointed closed convex cone has a compact sole and Corollary 2.8 generalizes the version of Fatou's lemma proved in Cornet, Topuzu and Yildiz [18].

Finally, the following remark describes another framework of general equilibrium theory with a continuum of consumers and differentiated commodities for which Corollary 2.8 can be used to prove the existence of Walras equilibria (see Martins-da-Rocha [36]). It should however be noticed that the convexity of preferences is required to apply Corollary 2.8 to the commodity space $M(K)$.

Remark 2.10. Let K be a compact metric space and let $E = C(K)$ be the separable Banach space of continuous real-valued functions endowed with the supremum norm. The topological dual space $E^\star$ is then $M(K)$, the space of finite Radon measures on K. Let $C(K)_+ := \{x \in C(K) : \forall t \in K,\ x(t) \geqslant 0\}$ and $M(K)_+ := \{f \in M(K) : \forall x \in C(K)_+,\ \langle x, f\rangle \geqslant 0\}$ be the positive cones of $C(K)$ and $M(K)$, respectively. Then the set $M(K)_+$ is a closed convex cone with a $w^\star$-compact sole. Take e in $C(K)$ defined by $e(t) = 1$ for each $t \in K$; then for each m in $M(K)_+$, $\langle e, m\rangle = \|m\|^\star$.

2.3. **The link with other results.** In Cornet and Médecin [17], the sequence (f_n) is supposed to be integrably bounded. This implies that the sequence (f_n) is mean norm bounded. Moreover, the set $C_{(f_n)}$ coincides with the whole space E in that case. Hence Theorems 2.4 and 2.5 generalize Theorem 1 in Cornet and Médecin [17].

If a sequence (f_n) is uniformly norm integrable, then it is mean norm bounded and the set $C_{(f_n)}$ coincides with the whole space E. Hence, Theorems 2.4 and 2.5 generalize Theorems 1 and 2 in Balder [8]. More precisely, in Balder [8] it is proved that if a sequence (f_n) of Gelfand integrable mappings is supposed to be uniformly norm integrable, then for each open neighborhood W of zero, there exists a Gelfand integrable mapping f_W from Ω to $E^\star$ such that

$$\int_\Omega f_W d\mu - \lim_n \int_\Omega f_n d\mu \in W \quad \text{and} \quad f_W(a) \in \operatorname{cl} \operatorname{Ls}_n\{f_n(a)\} \ a.e.$$

Since uniform norm integrability of the sequence (f_n) implies that the sequence (f_n) is mean norm bounded and $C_{(f_n)} = E$, we can apply Theorem 2.5 to a finite dimensional subspace F of E such that $F^\perp \subseteq W$. Thus there exists a Gelfand integrable mapping f_F from Ω to $E^\star$ such that

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in F^\perp \subseteq W \quad \text{and} \quad f_F(a) \in \operatorname{Ls}_n\{f_n(a)\} \ a.e.$$

Note that the above proof leads to the statement that $f_F(a) \in \operatorname{Ls}_n\{f_n(a)\}$ which is more precise than the one in Balder [8], ti.e., $f_W(a) \in \operatorname{cl} \operatorname{Ls}_n\{f_n(a)\}$.

3. A more general version of Theorem 2.4 and its proof

Our proof of Fatou's lemma relies on a straightforward extension of the important result by Komlós (Theorem A.8 in Appendix). We first recall the following definition of Komlós convergence, simply called K-convergence in the following.

Definition 3.1. Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and let $(E, \|.\|)$ be a separable Banach space. A sequence (f_n) of mappings from Ω to $E^\star$ is said to be K-convergent to a mapping $f : \Omega \to E^\star$, denoted

$$f_n \xrightarrow{\text{K}} f,$$

if for every strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, there exists a null set $N \in \mathcal{A}$ (i.e., $\mu(N) = 0$) such that

$$\forall a \in \Omega \setminus N, \quad (1/n) \sum_{k=1}^{n} f_{\varphi(k)}(a) \xrightarrow{w^\star} f(a).$$

We now state the following theorem, which is more general than Theorem 2.4 and is also of interest for itself.

Theorem 3.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$ which is mean weakly bounded.*

Then there exists a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ and a Gelfand integrable mapping f from Ω to $E^\star$ such that

(a) *the sequence* $(f_{\theta(n)})$ K-*converges to* f*, and*

$$\int_\Omega \|f(a)\|^\star \, d\mu(a) \leqslant \int_\Omega \rho d\mu + \liminf_n \int_{\{\|f_{\theta(n)}\|^\star > \rho\}} \left\|f_{\theta(n)}(a)\right\|^\star d\mu(a)$$

where $\rho \in \mathcal{L}^0(\mathbb{R})$ *is the measurable function associated to mean weakly boundedness;*

(b) *We have*

$$\forall A \in \mathcal{A}, \quad \forall x \in C_{(f_n)}, \quad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_n \int_A \langle x, f_{\theta(n)}(a)\rangle d\mu(a);$$ [4]

(c) *there exists* $\xi \in \mathcal{L}^0(\mathbb{R})$ *such that for every finite dimensional subspace* F *of* E*,*

$$f(a) \in \mathrm{coLs}\{f_{\theta(n)}(a)\} + \xi(a)B^\star \cap F^\perp \quad \text{a.e..}$$

In particular $f(a) \in \overline{\mathrm{co}}\,\mathrm{Ls}\{f_{\theta(n)}(a)\}$ *a.e.*

We recall that a sequence (f_n) of Gelfand integrable mappings from Ω to $E^\star$ is said to be **weakly convergent** to a Gelfand integrable mapping f, if for every $x \in E$, the sequence of real-valued functions $\langle x, f_n\rangle : a \mapsto \langle x, f_n(a)\rangle$ converges to the function $\langle x, f\rangle : a \mapsto \langle x, f(a)\rangle$ for the weak topology $\sigma(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$. A family $\mathcal{H}$ of Gelfand integrable mappings from Ω to $E^\star$ is said to be **uniformly weak integrable** if for every $x \in E$, the family $(\langle x, f\rangle)_{f\in\mathcal{H}}$ of real-valued functions is uniformly integrable.

A direct consequence of Theorem 3.2 is the following weak sequential compactness criteria.

Corollary 3.3. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a finite positive complete measure space and let* $(E, \|.\|)$ *be a separable Banach space. If* $\mathcal{H}$ *is a family of Gelfand integrable mappings from* Ω *to* $E^\star$ *which is uniformly weak integrable and there exists a measurable function* $\rho \in \mathcal{L}^0(\mathbb{R})$ *such that*

$$\sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n\|^\star \, d\mu < \infty$$

then $\mathcal{H}$ *is weakly sequentially compact.*

Proof. Consider a sequence (f_n) of mappings in $\mathcal{H}$. Since (f_n) is uniformly weak integrable, for every $x \in E$, the sequence $(\langle x, f_n\rangle)$ of real-valued functions is mean bounded in the sense of property (4i) of Definition 2.2. Moreover, the set $C_{(f_n)}$ coincides with the whole space E. Since (f_n) also satisfies property (4ii), we deduce that (f_n) is mean weakly bounded. We can then apply part (a) of Theorem 3.2 and get the existence of a Gelfand integrable mapping f together with a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that $(f_{\theta(n)})$ K-converges to f. Moreover since $C_{(f_n)} = E$, from part (b) of Theorem 3.2, we get that

$$\int_\Omega \langle x, f(a)\rangle d\mu = \lim_n \int_\Omega \langle x, f_{\theta(n)}(a)\rangle d\mu.$$

[4]Recall that $C_{(f_n)}$ is the set of all vectors $x \in E$ such that the sequence $(\max\{0, -\langle x, f_n\rangle\})$ is uniformly integrable.

Fix an arbitrary $h \in \mathcal{L}^\infty(\mathbb{R})$. Observe that the sequence (hf_n) satisfied the same properties as the sequence (f_n). Applying the above argument, we deduce that

$$\int_\Omega h(a)\langle x, f(a)\rangle d\mu = \lim_n \int_\Omega h(a)\langle x, f_{\theta(n)}(a)\rangle d\mu.$$

In particular $(f_{\theta(n)})$ weakly converges to f. □

Remark 3.4. If a sequence (f_n) of Gelfand integrable mappings is uniformly norm integrable, then (f_n) is uniformly weak integrable and $C_{(f_n)} = E$. In particular, if $\mathcal{H}$ is a family of uniformly norm integrable mappings, then $\mathcal{H}$ is weakly sequentially compact.

The proof of Theorem 3.2 will be given in three steps corresponding to part (a), (b) and (c).

3.1. **Proof of Part (a).** The following proposition is an extension to vector-valued mappings, of the important result by Komlós (Theorem A.8 in Appendix).

Proposition 3.5. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space, let $(E, \|.\|)$ be a separable Banach space, and let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$, which is mean weakly bounded. Then, passing to a subsequence if necessary, we can assume that (f_n)* K*-converges to a Gelfand integrable mapping f from Ω to $E^\star$. Moreover, we have*

$$\forall x \in E, \quad \int_\Omega |\langle x, f(a)\rangle| d\mu(a) \leqslant \sup_n \int_\Omega |\langle x, f_n(a)\rangle| d\mu(a)$$

and

$$\int_\Omega \|f(a)\|^\star \, d\mu(a) \leqslant \int_\Omega \rho d\mu + \liminf_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star \, d\mu(a)$$

where $\rho \in \mathcal{L}^0(\mathbb{R})$ is the measurable function associated to the mean weakly boundedness condition.

Remark 3.6. Observe that if the sequence (f_n) is mean norm bounded, then we can choose $\rho := 0$ and get that

$$\int_\Omega \|f(a)\|^\star \, d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^\star \, d\mu(a).$$

Proof of Proposition 3.5. Since the sequence (f_k) satisfies property (4ii), there exists a measurable function $\rho \in \mathcal{L}^0(\mathbb{R})$ such that

$$\sup_k \int_{\{\|f_k\|^\star > \rho\}} \|f_k(a)\|^\star \, d\mu(a) < \infty.$$

Passing to a subsequence if necessary, we can assume that

$$\liminf_k \int_{\{\|f_k\|^\star > \rho\}} \|f_k(a)\|^\star \, d\mu(a) = \lim_k \int_{\{\|f_k\|^\star > \rho\}} \|f_k(a)\|^\star \, d\mu(a).$$

For each $k \in \mathbb{N}$, we let $\psi_k : \Omega \to \mathbb{R}$ be defined by

$$\forall a \in \Omega, \quad \psi_k(a) := \begin{cases} \|f_k(a)\|^\star, & \text{if } \|f_k(a)\|^\star > \rho(a) \\ 0, & \text{otherwise.} \end{cases}$$

Let (x_j) be a $\|.\|$-dense sequence in E. We define for each $j,k \in \mathbb{N}$,

$$\varphi_{j,k}(a) := \langle x_j, f_k(a) \rangle \quad \text{and} \quad \varphi_{\infty,k} := \psi_k.$$

Since the sequence (f_k) is mean weakly bounded and the sequence (ψ_k) is mean bounded, then, for every $j \in \mathbb{N} \cup \{\infty\}$,

$$\sup_k \int_\Omega |\varphi_{j,k}(a)| d\mu(a) < \infty.$$

It is now possible to apply Komlós' Theorem (Theorem A.8 in Appendix) repeatedly in a diagonal procedure. Passing to a subsequence if necessary, this yields a family $(\varphi_j)_{j \in \mathbb{N} \cup \{\infty\}}$ of integrable real-valued functions such that for every $j \in \mathbb{N} \cup \{\infty\}$ and every strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$

$$\frac{1}{n} \sum_{k=1}^{n} \varphi_{j,\theta(k)}(a) \longrightarrow \varphi_j(a) \quad \text{a.e.}$$

Choosing $\theta : k \mapsto k$, we get that for every $j \in \mathbb{N}$,

$$\langle x_j, \frac{1}{n} \sum_{k=1}^{n} f_k(a) \rangle \longrightarrow \varphi_j(a) \quad \text{a.e.} \tag{3.1}$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \psi_k(a) \longrightarrow \varphi_\infty(a) \quad \text{a.e.} \tag{3.2}$$

Fix $a \in \Omega$ outside the exceptional null-set and for each $n \in \mathbb{N}$, define

$$g_n(a) := \frac{1}{n} \sum_{k=1}^{n} f_k(a).$$

Then applying (3.2), $\limsup_n \|g_n(a)\|^\star \leqslant \max\{\rho(a), \varphi_\infty(a)\} < \infty$. Now, from Banach-Alaoglu's Theorem and passing to a subsequence if necessary, we get that $(g_n(a))$ converges to some $f(a) \in E^\star$ (for the $w^\star$-topology). Applying (3.1), for every $j \in \mathbb{N}$,

$$\langle x_j, f(a) \rangle = \varphi_j(a).$$

We have thus proved that for every $j \in \mathbb{N}$ and every strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$,

$$\langle x_j, \frac{1}{n} \sum_{k=1}^{n} f_{\theta(k)}(a) \rangle \longrightarrow \langle x_j, f(a) \rangle \quad \text{a.e.}$$

Since the sequence (x_j) is $\|.\|$-dense in E, it follows that for every strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$

$$\frac{1}{n} \sum_{k=1}^{n} f_{\theta(k)}(a) \xrightarrow{w^\star} f(a) \quad \text{a.e.} \tag{3.3}$$

i.e., the sequence (f_n) K-converges to f, in particular, the mapping f is Gelfand measurable.

For every $x \in E$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\langle x, f_k(a)\rangle \longrightarrow \langle x, f(a)\rangle \quad \text{a.e.}$$

Applying Fatou's lemma for positive real-valued functions,

$$\begin{aligned}\int_\Omega |\langle x, f(a)\rangle| d\mu(a) &\leqslant \liminf_n \int_\Omega \frac{1}{n}\sum_{k=1}^{n}|\langle x, f_k(a)\rangle| d\mu(a)\\ &\leqslant \sup_n \int_\Omega |\langle x, f_n(a)\rangle| d\mu(a).\end{aligned}$$

Hence the mapping f is Gelfand integrable and

$$\|f\|_{\mathcal{G}} \leqslant \sup_n \|f_n\|_{\mathcal{G}}.$$

Moreover, $\|f(a)\|^\star \leqslant \liminf_n \|g_n(a)\|^\star$ for almost every a in Ω. Applying Fatou's lemma for positive functions, we deduce that

$$\begin{aligned}\int_\Omega \|f(a)\|^\star d\mu(a) &\leqslant \int_\Omega \rho d\mu + \liminf_n \frac{1}{n}\sum_{k=1}^{n}\int_{\{\|f_k\|^\star>\rho\}} \|f_k(a)\|^\star d\mu(a)\\ &\leqslant \int_\Omega \rho d\mu + \liminf_k \int_{\{\|f_k\|^\star>\rho\}} \|f_k(a)\|^\star d\mu(a).\end{aligned}$$

□

Remark 3.7. We refer to Balder [6] and Balder and Hess [10] for other extensions of Komlós' result, mainly in the case of Bochner integration.

3.2. **Proof of Part (b).** The proof of Part (b) follows from the following proposition.

Proposition 3.8. *Let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$,* K*-converging to a Gelfand integrable mapping $f : \Omega \to E^\star$. Then*

$$\forall x \in C_{(f_n)}, \quad \forall A \in \mathcal{A}, \quad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_n \int_A \langle x, f_n(a)\rangle d\mu(a).$$

Proof. Let $A \in \mathcal{A}$ and let $x \in C_{(f_n)}$, passing to a subsequence if necessary, we can assume that

$$\alpha := \liminf_k \int_A \langle x, f_k\rangle d\mu = \lim_k \int_A \langle x, f_k\rangle d\mu.$$

We define, for each $k \in \mathbb{N}$, the function $\varphi_k : \Omega \to \mathbb{R}$, by:

$$\varphi_k(a) := -\langle x, f_k(a)\rangle^- = -\max\{0, -\langle x, f_k(a)\rangle\}.$$

Note that the sequence of real-valued functions (φ_k) is uniformly integrable over A. Moreover, for every $k \in \mathbb{N}$, we have $\langle x, f_k(a)\rangle \geqslant \varphi_k(a)$ for each $a \in A$. Applying Komlós' Theorem (Theorem A.8 in Appendix), we can pass to a subsequence if necessary and deduce the existence of an integrable real-valued function $\varphi \in \mathcal{L}^1(\mathbb{R})$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\varphi_k(a) \longrightarrow \varphi(a) \quad \text{for a.e. } a \in A.$$

From the uniform integrability (which follows from the definition of the set $C_{(f_n)}$), we get

$$\frac{1}{n}\sum_{k=1}^{n}\int_A \varphi_k d\mu \longrightarrow \int_A \varphi d\mu,$$

so it follows that

$$\alpha - \int_A \varphi d\mu = \lim_n \left[\frac{1}{n}\sum_{k=1}^{n}\int_A (\langle x, f_k(a)\rangle - \varphi_k(a))d\mu(a)\right].$$

Since the sequence (f_k) is K-convergent to f,

$$\frac{1}{n}\sum_{k=1}^{n}\langle x, f_k(a)\rangle \longrightarrow \langle x, f(a)\rangle \quad \text{for a.e. } a \in A.$$

Since $\langle x, f_k(a)\rangle \geqslant \varphi_k(a)$ for each $k \in \mathbb{N}$, we apply Fatou's lemma (for real-valued functions)

$$\alpha - \int_A \varphi d\mu \geqslant \int_A [\langle x, f\rangle - \varphi]\, d\mu = \int_A \langle x, f\rangle d\mu - \int_A \varphi d\mu.$$

Consequently

$$\alpha := \liminf_n \textstyle\int_A \langle x, f_n(a)\rangle d\mu(a) \geq \int_A \langle x, f(a)\rangle d\mu(a). \qquad \square$$

3.3. **Proof of Part (c).** We now prove a lower closure result as a consequence of the Komlós convergence of a sequence of mappings. The proof will be given in two steps; first we consider the finite dimensional case, and second the general case of a separable Banach space.

The finite dimensional version is based on the following result by Page [42]. For the sake of completeness, we propose a simple and direct proof.

Proposition 3.9 (Page [42]). *Let E be a finite dimensional vector space and let (f_n) be a sequence of integrable mappings from Ω to $E^\star$ that is mean norm bounded and* K*-converges to an integrable mapping $f : \Omega \to E^\star$. Then*

$$f(a) \in \text{coLs}_n\{f_n(a)\} \quad \text{a.e.}$$

Remark 3.10. When E is finite dimensional, a sequence of functions is mean norm bounded if, and only if, it is mean weakly bounded.

Proof of Proposition 3.9. Let (f_n) be a sequence of mean norm bounded mappings from Ω to $E^\star$, K-converging to $f : \Omega \to E^\star$. Following Gaposhkin's lemma A.9, there exists a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that for each n, $f_{\theta(n)} = g_n + h_n$, where the sequence (g_n) is uniformly integrable and the sequence (h_n) converges almost everywhere to 0. Since $(f_{\theta(n)})$ K-converges to f, it follows that (g_n) K-converges to f. From Proposition 3.8, the sequence (g_n) weakly converges to f. Now applying Proposition C in Artstein [3], we get that $f(a) \in \text{coLs}\{g_n(a)\}$ almost everywhere. Since $\text{Ls}\{g_n(a)\} \subseteq \text{Ls}\{f_n(a)\}$, it follows that $f(a) \in \text{coLs}\{f_n(a)\}$ a.e. $\square$

Remark 3.11. The proof of Proposition 3.9 is based on Proposition C in Artstein [3]. However, Proposition C in Artstein [3] can be seen as a corollary of Propositions 3.5, 3.8 and 3.9. Indeed, let (f_n) be a sequence of integrable mappings from Ω to $E^\star$ (E is finite dimensional) such that (f_n) weakly converges to an integrable mapping f. The sequence (f_n) is then mean norm bounded. Applying Propositions 3.5 and 3.9, there exists a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ and an integrable mapping g such that $(f_{\theta(n)})$ K-converges to g and $g(a) \in \mathrm{coLs}\{f_{\theta(n)}(a)\}$ almost everywhere. Since (f_n) weakly converges, it follows from Proposition IV.2.3 in Neveu [39] that (f_n) is uniformly integrable. Applying Proposition 3.8, the sequence $(f_{\theta(n)})$ weakly converges to g. Hence $g = f$ almost everywhere and $f(a) \in \mathrm{coLs}\{f_n(a)\}$ almost everywhere.

Applying Proposition 3.9, we now provide a proof of the lower closure result in the general setting.

Proposition 3.12. *Let (f_n) be a sequence of Gelfand integrable mappings from Ω to $E^\star$ that is mean weakly bounded and* K*-converges to a Gelfand integrable mapping $f : \Omega \to E^\star$. Then there exists $\xi \in \mathcal{L}^\infty(\mathbb{R}_+)$ such that for every finite dimensional subspace F of E,*

$$f(a) \in \mathrm{coLs}\{f_n(a)\} + \xi(a)B^\star \cap F^\perp \quad \text{a.e.}$$

Proof. Since the sequence (f_n) is mean weakly bounded, there exists a measurable function $\rho \in \mathcal{L}^0(\mathbb{R}_+)$ such that

$$\sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star \, d\mu(a) < \infty.$$

We let for each $n \in \mathbb{N}$,

$$\forall a \in \Omega, \quad \psi_n(a) := \begin{cases} \|f_n(a)\|^\star & \text{if } \|f_n(a)\|^\star > \rho(a) \, , \\ 0 & \text{elsewhere.} \end{cases}$$

The sequence (ψ_n) is mean norm bounded. Applying Komlós' Theorem (Theorem A.8 in Appendix) and passing to a subsequence if necessary, we can suppose that the sequence (ψ_n) is K-convergent to an integrable function $\psi \in \mathcal{L}^1(\mathbb{R})$. Let F be a finite dimensional subspace of E. We consider π the projection mapping from $E^\star$ to $F^\star$, defined by

$$\forall x^\star \in E^\star, \quad \pi(x^\star) = [x \in F \mapsto \langle x, x^\star \rangle].$$

Observe that the sequence $(\psi_n, \pi(f_n))$ is K-convergent to $(\psi, \pi(f))$. Applying Proposition 3.9,

$$(\psi(a), \pi(f(a))) \in \mathrm{coLs}\left\{(\psi_n(a), \pi(f_n(a)))\right\} \quad \text{a.e.}$$

Let $a \in \Omega$ outside the exceptional null set. There exists a finite set I, a finite family $(\lambda_i)_{i\in I} \in [0,1]^I$ such that $\sum_{i\in I} \lambda_i = 1$, and there exists a finite family $(\theta_i)_{i\in I}$ of strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$(\psi(a), \pi(f(a))) = \sum_{i\in I} \lambda_i \lim_n \left(\psi_{\theta_i(n)}(a), \pi(f_{\theta_{i(n)}}(a))\right).$$

Recall that $\|f_n(a)\|^\star \leqslant \max\{\rho(a), \psi_n(a)\}$. Let $i \in I$, since the sequence $(\psi_{\varphi_i(n)}(a))$ converges, passing to a subsequence if necessary, we can suppose that the sequence $(f_{\varphi_i(n)}(a))$ $w^\star$-converges to some $h_i(a) \in \mathrm{Ls}\{f_n(a)\} \subseteq E^\star$ with $\|h_i(a)\|^\star \leqslant \max\{\rho(a), \psi(a)\}$. It follows that

$$\pi(f(a)) = \sum_{i\in I} \lambda_i \pi(h_i(a)) \in \pi\left(\mathrm{coLs}\{f_n(a)\}\right).$$

Note that $\left\|\sum_{i\in I} \lambda_i h_i(a)\right\|^\star \leqslant \sum_{i\in I} \lambda_i \|h_i(a)\|^\star \leqslant \max\{\rho(a), \psi(a)\}$, hence

$$f(a) \in \mathrm{coLs}\{f_n(a)\} + \rho(a)B^\star \cap F^\perp,$$

where $\xi(a) := \max\{\rho(a), \psi(a)\} + \|f(a)\|^\star$. □

The proof of Part (c) of Theorem 3.2 follows from Proposition 3.12 and the following proposition.

Proposition 3.13. *Let L be a multifunction from Ω to $E^\star$, let f be a mapping from Ω to $E^\star$ and let ξ be a positive real-valued function such that for every finite dimensional subspace F of E,*

$$f(a) \in L(a) + \xi(a)B^\star \cap F^\perp \quad \text{a.e.}$$

Then

$$f(a) \in \mathrm{cl}\, L(a) \quad \text{a.e.}$$

Proof. Let (e_i) be a dense sequence in E, and for each $n \in \mathbb{N}$, let F_n be the vector subspace of E generated by $\{e_0, e_1, \dots, e_n\}$. Clearly there exists $\Omega' \subseteq \Omega$ such that $\mu(\Omega \setminus \Omega') = 0$ and

$$\forall a \in \Omega', \quad f(a) \in \bigcap_{n\in\mathbb{N}} \left(L(a) + \rho(a)B^\star \cap F_n^\perp\right).$$

Let $a \in \Omega'$, there exists a sequence $(z_n(a))$ in $E^\star$ satisfying $f(a) - z_n(a) \in L(a)$ and $z_n(a) \in \rho(a)B^\star \cap F_n^\perp$. Passing to a subsequence if necessary, we can suppose that $(z_n(a))$ $w^\star$-converges to $z(a)$. It follows that $f(a) - z(a) \in \mathrm{cl}\, L(a)$. Moreover, since $z_n(a) \in F_n^\perp$, we deduce that for every i, $\langle e_i, z(a)\rangle = 0$. In particular $z(a) = 0$, hence $f(a)$ belongs to $\mathrm{cl}\, L(a)$. □

4. Proof of Theorem 2.5

We start by proving Theorem 2.5 when $(\Omega, \mathcal{A}, \mu)$ is non-atomic and then we provide the proof in the general case.

4.1. **The case $(\Omega, \mathcal{A}, \mu)$ is non-atomic.** Since the sequence (f_n) is mean weakly bounded, there exists a measurable function $\rho \in \mathcal{L}^0(\mathbb{R}_+)$ such that

$$\sup_n \int_{\{\|f_n\|^\star > \rho\}} \|f_n(a)\|^\star \, d\mu(a) < \infty.$$

We let for each $n \in \mathbb{N}$,

$$\forall a \in \Omega, \quad \psi_n(a) := \begin{cases} \|f_n(a)\|^\star & \text{if } \|f_n(a)\|^\star > \rho(a), \\ 0 & \text{elsewhere.} \end{cases}$$

Then the sequence (ψ_n) is clearly mean bounded.

Let F be a finite dimensional linear subspace of E. Let π be the projection mapping from $E^\star$ to $F^\star$, the algebraic dual of F, defined by

$$\forall x^\star \in E^\star, \quad \pi(x^\star) := [x \in F \mapsto \langle x, x^\star \rangle].$$

Let $\|\cdot\|_F^\star$ be the norm on $F^\star$ defined by

$$\forall x^\star \in E^\star, \quad \|\pi(x^\star)\|_F^\star := \sum_{x \in B_F} |\langle x, x^\star \rangle|$$

where B_F is a finite basis of F. We define $\xi_n : \Omega \to \mathbb{R}$ $(n \in \mathbb{N})$ be defined by

$$\xi_n(a) := \|\pi(f_n(a))\|_F^\star.$$

Since the sequence (f_n) is mean weakly bounded, the sequence (ξ_n) is mean bounded.

Applying Theorem 3.2 to the sequence (ψ_n, ξ_n, f_n), we can suppose, passing to a subsequence if necessary, that there exists a Gelfand integrable mapping f from Ω to $E^\star$ and integrable functions ψ, ξ from Ω to $[0, +\infty)$ such that

$$(\psi_n, \xi_n, f_n) \xrightarrow{\mathrm{K}} (\psi, \xi, f) \quad \text{a.e.}$$

In particular, we have

$$\forall x \in C_{(f_n)}, \quad \int_\Omega \langle x, f \rangle d\mu \leqslant \liminf \int_\Omega \langle x, f_n \rangle d\mu$$

and

$$(\psi(a), \xi(a), f(a)) \in \mathrm{coLs}\{(\psi_n(a), \xi_n(a), f_n(a))\} + (\mathbb{R} \times \mathbb{R} \times F)^\perp \quad \text{a.e.}$$

It follows that

$$(\psi(a), \xi(a), \pi[f(a)]) \in \mathrm{coLs}\{(\psi_n(a), \xi_n(a), \pi[f_n(a)])\} \quad \text{a.e.}$$

Following Carathéodory's theorem, we let $I := \{1, \ldots, \ell + 3\}$, where ℓ is the dimension of F. Then, for almost every $a \in \Omega$, there exists $(\lambda_i(a))_{i \in I} \in [0,1]^I$ such that $\sum_{i \in I} \lambda_i(a) = 1$ and $(\theta_i)_{i \in I}$ strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$(\psi(a), \xi(a), \pi[f(a)]) = \sum_{i \in I} \lambda_i(a) \lim_n (\psi_{\theta_i(n)}(a), \xi_{\theta_i(n)}(a), \pi[f_{\theta_i(n)}(a)]).$$

In particular, for each $i \in I$, $\psi_i(a) := \lim_n \psi_{\theta_i(n)}(a) < \infty$. Recall that

$$\left\| f_{\theta_i(n)}(a) \right\|^\star \leqslant \max\{\rho(a), \psi_{\theta_i(n)}(a)\}.$$

It follows that there exists $s_i(a) \in \mathrm{Ls}\{f_n(a)\}$ such that a subsequence of $(f_{\theta_i(n)}(a))$ $w^\star$-converges to $s_i(a)$. In particular, we have

$$(\psi(a), \xi(a), \pi[f(a)]) = \sum_{i \in I} \lambda_i(a) (\psi_i(a), \xi_i(a), \pi[s_i(a)]) \quad \text{a.e.}$$

and for each $i \in I$,

$$\|s_i(a)\|^\star \leqslant \max\{\rho(a), \psi_i(a)\} \quad \text{and} \quad \|\pi(s_i(a))\|_F^\star \leqslant \xi_i(a).$$

Applying Theorem A.2, Proposition A.6 and Corollary A.3 in the Appendix, we can suppose that for each $i \in I$, the functions λ_i, ψ_i and ξ_i are measurable and the mappings s_i are Gelfand measurable selections of $\mathrm{Ls}\{f_n(\cdot)\}$. Note that

$$\int_\Omega \sum_{i\in I} \lambda_i(a) \left\|\pi(s_i(a))\right\|_F^\star d\mu(a) \leqslant \int_\Omega \sum_{i\in I} \lambda_i(a)\xi_i d\mu \leqslant \int_\Omega \xi d\mu < \infty.$$

Applying the Extended Lyapunov Theorem A.10 in the Appendix, there exists a measurable partition $(B_i)_{i\in I}$ of Ω such that the mapping $\Phi := (\psi_i, \xi_i, \pi(s_i(.)))$ is integrable over B_i and such that

$$\int_\Omega \sum_{i\in I} \lambda_i(a)\Phi(a)d\mu(a) = \sum_{i\in I} \int_{B_i} \Phi(a)d\mu(a).$$

Then $f_F := \sum_{i\in I} \mathbf{1}_{B_i} s_i$ is a Gelfand measurable selection of the correspondence $a \longmapsto \mathrm{Ls}\{f_n(a)\}$, and moreover

$$\int_\Omega \left\|\pi(f_F(a))\right\|_F^\star d\mu(a) \leqslant \sum_{i\in I} \int_{B_i} \left\|\pi(s_i(a))\right\|_F^\star d\mu(a) \leqslant \int_\Omega \xi d\mu < \infty.$$

It follows that $\pi(f_F)$ is integrable. Now

$$\begin{aligned} \pi(\int_\Omega f_F d\mu) &= \sum_{i\in I} \int_{B_i} \pi[s_i(a)]d\mu(a) \\ &= \int_\Omega \sum_i \lambda_i(a)\pi[s_i(a)]d\mu(a) = \int_\Omega \pi(f)d\mu. \end{aligned}$$

Hence

$$\int_\Omega \pi(f)d\mu = \int_\Omega \pi(f_F)d\mu.$$

4.2. **The general case.** We now provide the proof of Theorem 2.5 in the general case, i.e., without assuming anymore that $(\Omega, \mathcal{A}, \mu)$ is non-atomic. Using a classical argument, the set Ω can be partitioned into a non-atomic part $\Omega^{na} \in \mathcal{A}$ and a purely atomic part $\Omega^{pa} \in \mathcal{A}$, so that the set Ω^{pa} can be written as the disjoint union of at most countably many measurable atoms $(A^i)_{i\in I}$ $(I \subseteq \mathbb{N})$. Furthermore, for every $i \in I$ and every $n \in N$, the measurable mapping $f_n : \Omega \to E^\star$ takes a constant value $f_n^i \in E^\star$ for a.e. $a \in A^i$. Since the sequence (f_n) is mean weakly bounded, for each $i \in I$, the sequence (f_n^i) is norm bounded, and thus remains in a $w^\star$-compact subset of $E^\star$ by Alaoglu's theorem. Consequently, by a diagonal extraction argument, we can pass to a sequence and get that for every $i \in I$, the sequence (f_n^i) $w^\star$-converges to some element $f^i \in E^\star$. We let $f^{pa} : \Omega^{pa} \to E^\star$ be defined by $f^{pa}(a) = f^i$ if $a \in A^i$.

We prove that f^{pa} satisfies conditions similar to (2.1), (2.2) and (2.3) of Theorem 2.4 when Ω is replaced by Ω^{pa}. Since $(f_n(a))$ $w^\star$-converges to $f^{pa}(a)$ for a.e. $a \in \Omega^{pa}$, we have

$$f^{pa}(a) \in \mathrm{Ls}\{f_n(a)\} \quad \text{a.e. in } \Omega^{pa}. \tag{4.1}$$

Moreover, we also have $\|f^{pa}(a)\|^\star \leqslant \liminf \|f_n(a)\|^\star$ for a.e. $a \in \Omega^{pa}$ which implies

$$\int_{\Omega^{pa}} \|f^{pa}(a)\|^\star \, d\mu \leqslant \int_{\Omega^{pa}} \rho d\mu + \sup \int_{\Omega^{pa}} \mathbf{1}_{\{\|f_n\|^\star > \rho\}} \|f_n\|^\star \, d\mu. \tag{4.2}$$

Fix an arbitrary $x \in E$. By condition (4i), we have

$$\sup_n \int_{\Omega^{pa}} |\langle x, f_n \rangle| d\mu < \infty.$$

We can then apply Fatou's lemma to deduce that

$$\int_{\Omega^{pa}} |\langle x, f^{pa} \rangle| d\mu \leqslant \liminf \int_{\Omega^{pa}} |\langle x, f_n \rangle| d\mu.$$

We have thus proved that $f^{pa} : \Omega^{pa} \to E^\star$ is Gelfand integrable. Now fix $x \in C_{(f_n)}$. Recall that the sequence $(\langle x, f_n \rangle^-)$ is uniformly integrable. Since $(f_n(a))$ converges to $f^{pa}(a)$ for almost every $a \in \Omega^{pa}$, we deduce that

$$\int_{\Omega^{pa}} \langle x, f^{pa} \rangle d\mu \leqslant \liminf \int_{\Omega^{pa}} \langle x, f_n \rangle d\mu. \tag{4.3}$$

We now consider the non-atomic part Ω^{na}. Applying the version of Fatou's lemma proved previously to the non-atomic part Ω^{na}, we get the existence of a Gelfand integrable function f^{na} from Ω^{na} to $E^\star$ satisfying

$$\forall x \in C_{(f_n)}, \quad \int_{\Omega^{na}} \langle x, f^{na} \rangle d\mu \leqslant \liminf \int_{\Omega^{na}} \langle x, f_n \rangle d\mu, \tag{4.4}$$

$$f^{na}(a) \in \overline{\mathrm{co}}\, \mathrm{Ls}\{f_n(a)\} \quad \text{a.e. in } \Omega^{na}, \tag{4.5}$$

$$\int_{\Omega^{na}} \|f^{na}\|^\star \, d\mu \leqslant \int_{\Omega^{na}} \rho d\mu + \sup_n \int_{\Omega^{na}} \mathbf{1}_{\{\|f_n\|^\star > \rho\}} \|f_n\|^\star \, d\mu. \tag{4.6}$$

Moreover, for every finite dimensional subspace F of E, there exists $f_F^{na} : \Omega^{na} \to E^\star$ such that

$$f_F^{na}(a) \in \mathrm{Ls}\{f_n(a)\} \quad \text{a.e. in } \Omega^{na}$$

and

$$\forall x \in F, \quad \int_{\Omega^{na}} \langle x, f^{na} \rangle d\mu = \int_{\Omega} \langle x, f_F^{na} \rangle d\mu.$$

We now define the mapping $f : \Omega \to E^\star$ by $f(a) := f^{pa}(a)$ if $a \in \Omega^{pa}$ and $f(a) := f^{na}(a)$ if $a \in \Omega^{na}$. Combining (4.1) and (4.5), we deduce that f satisfies (2.2). Combining (4.2) and (4.6), we deduce that f satisfies (2.1). Combining (4.3) and (4.4), for every $x \in C_{(f_n)}$

$$\begin{aligned}\int_{\Omega} \langle x, f \rangle \mu &\leqslant \liminf \int_{\Omega^{pa}} \langle x, f_n \rangle d\mu + \liminf \int_{\Omega^{na}} \langle x, f_n \rangle d\mu \\ &\leqslant \liminf \int_{\Omega} \langle x, f_n \rangle d\mu = \lim \int_{\Omega} \langle x, f_n \rangle d\mu.\end{aligned}$$

This proves that f satisfies (2.1). We now define the mapping $f_F : \Omega \to E^\star$ by $f_F(a) := f^{pa}(a)$ if $a \in \Omega^{pa}$ and $f_F(a) := f_F^{na}(a)$ if $a \in \Omega^{na}$. One checks that the mapping f_F satisfies the conditions of Theorem 2.5. □

Appendix A. Appendix

A.1. **Measurable mappings.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and let $(E, \|.\|)$ be a separable Banach space. We note $\mathcal{B}$ the Borel σ-algebra of $(E^\star, w^\star)$. We recall that a mapping f from Ω to $E^\star$ is said to be Gelfand measurable if, for every $x \in E$, the function $a \mapsto \langle x, f(a)\rangle$ is measurable, and f is said to be measurable, if for every $B \in \mathcal{B}$, $f^{-1}(B)$ belongs to $\mathcal{A}$.

Proposition A.1. *Let f be a mapping from Ω to $E^\star$. Then f is Gelfand measurable if, and only if, it is measurable. Moreover, if f is measurable, then the function $a \mapsto \|f(a)\|^\star$ is measurable.*

Proof. Let (x_i) a norm dense sequence in B the unit ball of E. For each $i \in \mathbb{N}$ and each $\alpha > 0$, we let $V_{i,\alpha} := \{x^\star \in E^\star \ : \ |\langle x_i, x^\star\rangle| < \alpha\}$. We note $\mathcal{D}$ the σ-algebra generated by the family of all $V_{i,\alpha}$. Since $V_{i,\alpha}$ is open in $(E^\star, w^\star)$, we have $\mathcal{D} \subseteq \mathcal{B}$. It follows that if f is measurable, then f is Gelfand measurable. Note that

$$\bigcup_{i\in\mathbb{N}} \bigcap_{n>0} V_{i,\alpha+1/n} = \alpha B^\star = \{x^\star \in E^\star \ : \ \|x^\star\|^\star \leqslant \alpha\} \in \mathcal{D}.$$

Consequently, if f is Gelfand measurable, then the mapping $a \mapsto \|f(a)\|^\star$ is measurable.

Let d be the following distance defined on $E^\star$,

$$\forall (x^\star, y^\star) \in E^\star \times E^\star, \quad d(x^\star, y^\star) = \sum_{i\geqslant 0} \frac{|\langle x_i, x^\star - y^\star\rangle|}{2^i}.$$

Let $\mathcal{B}_d$ be the Borel σ-algebra defined by d. Note that $\mathcal{B}_d \subseteq \mathcal{D}$. The topology defined by the distance d coincide with the $w^\star$-topology on closed bounded subsets of $E^\star$. It follows that if W is a $w^\star$-open subset of $E^\star$, then for each $k \in \mathbb{N}$, $W \cap kB^\star$ is d-open, in particular, $W \cap kB^\star \in \mathcal{D}$. Since $W = \bigcup_k W \cap kB^\star$, it follows that $W \in \mathcal{D}$, and then $\mathcal{B} \subseteq \mathcal{D}$. Hence $\mathcal{B} = \mathcal{D}$ and the result follows. □

A.2. **Measurable selections.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and let $(E, \|.\|)$ be a separable Banach space. A multifunction F from Ω into $E^\star$ is said to be **graph measurable** if the graph G_F of F belongs to the σ-algebra $\mathcal{A} \otimes \mathcal{B}$, where

$$G_F := \{(a, x^\star) \in \Omega \times E^\star \ : \ x^\star \in F(a)\}.$$

A mapping f from Ω to $E^\star$ is a selection of F if $f(a) \in F(a)$ for almost every $a \in \Omega$. We recall the following classical result providing the existence of a measurable selection.

Theorem A.2 (Aumann Selection Theorem)**.** *Let $(E, \|.\|)$ be a separable Banach space, let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space, and let F be a graph measurable multifunction from Ω to $E^\star$ with nonempty values. Then there exists a measurable selection f of F.*

The proof of this theorem is given in Castaing–Valadier [14, Theorem III.22, p.74]. We provide hereafter a direct application of this theorem.

Corollary A.3. *Let $(E, \|.\|)$ be a separable Banach space, let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space, let F be a graph measurable multifunction from Ω to $E^\star$ with non-empty values, let I be a finite set, and let f be a measurable selection of F. Suppose that for almost every $a \in \Omega$, for each $i \in I$, there exist $\lambda_i(a) \in [0,1]$ and $f_i(a) \in F(a)$ such that*

$$f(a) = \sum_{i \in I} \lambda_i(a) f_i(a) \quad \text{and} \quad \sum_{i \in I} \lambda_i(a) = 1.$$

Then for each $i \in I$, λ_i may be chosen as a measurable function from Ω to $[0,1]$ and f_i may be chosen as a measurable selection of F.

Proof. We let $\Sigma(I)$ be the set of all $(\alpha_i) \in [0,1]^I$ such that $\sum_i \alpha_i = 1$. Let π be the linear mapping from $\Sigma(I) \times (E^\star)^I$ to $E^\star$ defined by

$$\forall ((\alpha_i), (x_i^\star)) \in \Sigma(I) \times (E^\star)^I, \quad \pi((\alpha_i), (x_i^\star)) := \sum_{i \in I} \alpha_i x_i^\star.$$

For each $a \in \Omega$, we let

$$H(a) := \pi^{-1}(\{f(a)\}) \cap \left(\Sigma(I) \times F(a)^I\right).$$

The multifunction H is graph measurable with non empty values. The end proof of the proof then follows from the application of Theorem A.2 to the multifunction H. □

A.3. **Measurability of limes superior.** We consider $(E, \|.\|)$ a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a (possibly not complete) finite measure space. A multifunction F from Ω into $E^\star$ is said to be measurable if for each $w^\star$-open subset V of $E^\star$, the set $F^-(V) := \{a \in \Omega : F(a) \cap V \neq \emptyset\}$ belongs to $\mathcal{A}$.

Proposition A.4. *Let F be a multifunction from Ω to $E^\star$.*

(1) *Suppose that $(\Omega, \mathcal{A}, \mu)$ is complete. If F is graph measurable, then F is measurable.*
(2) *Suppose that F is closed valued. If F is measurable, then F is graph measurable.*

Proof. Part (1) follows directly from the Projection Theorem in Castaing–Valadier [14, Theorem III.23]. Now we prove Part (2) of the proposition. Since $(E, \|.\|)$ is a separable Banach space, $E^\star$ is the countable union of $w^\star$-compact metrizable subsets. It follows from Schwartz [46] that $E^\star$ is a Lusin space, in particular, there exists a separable and completely metrizable topology τ, stronger than the $w^\star$ topology, but generating the same Borel sets. Since F is $w^\star$-closed valued, it is τ-closed valued. Applying Proposition III.13 in Castaing–Valadier [14], the graph of F is measurable. □

Proposition A.5. *Let F and F_n, $n \in \mathbb{N}$ be graph measurable multifunctions from Ω into $E^\star$.*

(1) *If $(\Omega, \mathcal{A}, \mu)$ is complete, then the multifunction $\operatorname{cl} F$ defined by $a \mapsto \operatorname{cl} F(a)$ is graph measurable.*
(2) *The multifunctions $\bigcup_n F_n$ and $\bigcap_n F_n$ are graph measurable.*

Proof. We start by proving Part (1). Since the multifunction F is graph measurable, by Proposition A.4, F is measurable. Let V be a $w^\star$-open subset of $E^\star$. For each $a \in A$,

$$F(a) \cap V \neq \emptyset \Longleftrightarrow [\mathrm{cl}\, F(a)] \cap V \neq \emptyset.$$

It follows that if F is measurable, then cl$F$ is measurable. Once again applying Proposition A.4, the multifunction cl$F$ is graph measurable.

Part (2) is an immediate consequence of the following equalities:

$$\mathrm{Graph}\,(\cup_n F_n) = \cup_n \mathrm{Graph}\,(F_n) \;\text{ and }\; \mathrm{Graph}\,(\cap_n F_n) = \cap_n \mathrm{Graph}\,(F_n)\,. \qquad \square$$

If (C_n) is a sequence of subsets of $E^\star$, we denote by LsC_n the sequential limes superior of (C_n) relative to $w^\star$, i.e., a vector $x \in E^\star$ belongs to LsC_n if there exists a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ and a sequence (x_n) such that $\lim x_n = x$ and $x_n \in C_{\theta(n)}$ for each $n \in \mathbb{N}$.

Proposition A.6. *Let (F_n) be a sequence of graph measurable multifunctions from Ω into $E^\star$. The multifunction $a \mapsto \mathrm{Ls}F_n(a)$ is graph measurable. In particular, if (f_n) is a sequence of measurable mappings from Ω to $E^\star$, then the multifunction $a \mapsto \mathrm{Ls}\{f_n(a)\}$ is graph measurable.*

Proof. Note that if (C_n) is a sequence of non-empty subsets of $E^\star$, then

$$\mathrm{Ls}C_n = \bigcup_{p \in \mathbb{N}} \mathrm{Ls}(C_n \cap pB^\star).$$

Indeed, let $x \in \mathrm{Ls}C_n$. There exists a sequence (x_n) and a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that $x_n \in C_{\theta(n)}$ for each $n \in \mathbb{N}$ and

$$x_n \xrightarrow{w^\star} x.$$

It follows that the sequence (x_n) is $\|.\|^\star$-bounded. Hence following Proposition A.5, without any loss of generality, we can suppose that there exists a $w^\star$-compact convex and metrizable subset K of $E^\star$, such that

$$\forall a \in \Omega, \quad \bigcup_n F_n(a) \subseteq K.$$

This implies that

$$\mathrm{Ls}F_n(a) = \bigcap_n \mathrm{cl} \bigcup_{p \geqslant n} F_p(a).$$

Following Proposition A.5, the multifunction $a \mapsto \mathrm{Ls}F_n(a)$ is graph measurable. This ends the proof of Proposition A.6. $\square$

Remark A.7. We refer to Hess [23] for related results on the measurability of the limes superior.

A.4. **Komlós limits.** Let $(E, \|.\|)$ be a separable Banach space and let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. A sequence (f_n) of mappings from Ω to $E^\star$ is said to be K-convergent to a mapping $f : \Omega \to E^\star$, if for every strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$,

$$\frac{1}{n} \sum_{i=1}^{n} f_{\theta(i)}(a) \xrightarrow{w^\star} f(a) \quad \text{a.e.}$$

We now state the following fundamental result for real-valued functions.

Theorem A.8 (Komlós [33]). *Suppose that (f_n) is a sequence of integrable real-valued functions such that*

$$\sup_n \int_\Omega |f_n| d\mu < +\infty.$$

Then there exist an integrable real-valued function $f : \Omega \to \mathbb{R}$ and a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that $(f_{\theta(n)})$ is K*-convergent to f.*

A.5. **Gaposhkin.** Gaposhkin [20, Lemma C.I] states:

Lemma A.9 (Gaposhkin's lemma). *Let E be a finite dimensional vector space, let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, and let (f_n) be a mean norm bounded sequence of integrable mappings from Ω to $E^\star$. Then there exists a strictly increasing function $\theta : \mathbb{N} \to \mathbb{N}$ such that for each $n \in \mathbb{N}$, $f_{\theta(n)} = g_n + h_n$, where the sequence (g_n) is uniformly integrable and the sequence (h_n) converges almost everywhere to 0.*

A.6. **Lyapunov.** The following theorem, due to Balder [7], is a consequence of the classical Lyapunov theorem.

Theorem A.10 (Extended Lyapunov). *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic finite measure space, let I be a finite set, let $\ell \in \mathbb{N}$, let $(f_i)_{i \in I}$ be a family of measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $\mathbb{R}^\ell$, and let $(\lambda_i)_{i \in I}$ be measurable functions from Ω to $[0, 1]$ satisfying $\sum_{i \in I} \lambda_i(a) = 1$ together with*

$$\int_\Omega \sum_{i \in I} \lambda_i(a) |f_i(a)| d\mu(a) < +\infty.$$

Then there exists a measurable partition $(B_i)_{i \in I}$ of Ω such that for each $i \in I$, the function f_i is integrable over B_i and

$$\int_\Omega \sum_{i \in I} \lambda_i(a) f_i(a) d\mu(a) = \sum_{i \in I} \int_{B_i} f_i d\mu.$$

A.7. **Counterexamples.** We first illustrate the fact that a Gelfand integrable mapping may not be norm integrable.

Counterexample A.11. Let $E = L^1(T, \mathcal{B}(T), \lambda)$ be the space of real-valued functions defined on $T := [0, +\infty)$ and integrable with respect to the Lebesgue measure λ. Then $(E, \|.\|_1)$ is a separable Banach space whose topological dual is $E^\star = L^\infty(T, \mathcal{B}(T), \lambda)$. We let $\Omega := [0, 1]$, $\mathcal{A} := \mathcal{B}([0, 1])$ and μ the Lebesgue measure on $[0, 1]$. For each $a \in \Omega$, we let

$$f(a) := t \mapsto (1/a) \exp(-|t + \ln a|).$$

Then the mapping f is Gelfand measurable but not norm integrable. Indeed, for each $t \in T$:

$$\int_\Omega |f(a, t)| d\mu(a) = \int_{[0, +\infty[} \exp(-|t - x|) d\lambda(x) = 2 - \exp(-t).$$

It follows from Fubini-Lebesgue theorem, that for all $x \in E$,

$$\int_\Omega |\langle x, f(a) \rangle| d\mu(a) = \int_T |x(t)| \int_\Omega |f(a, t)| d\mu(a) d\mu(t) \leqslant 2 \|x\|_1 < +\infty.$$

Hence, the mapping f is Gelfand integrable. However, $\|f(a)\|^{\star} = 1/a$ for a.e. $a \in \Omega := [0,1]$ and the function $a \mapsto \|f(a)\|^{\star}$ is not integrable.

We provide hereafter a sequence of Gelfand integrable mappings which is mean weakly bounded and almost everywhere bounded, but which is not mean bounded.

Counterexample A.12. Let $E = L^1(T, \mathcal{B}(T), \lambda)$ be the space of real-valued functions defined on $T := [0, +\infty)$ and integrable with respect to the Lebesgue measure λ. Then $(E, \|.\|_1)$ is a separable Banach space whose topological dual is $E^{\star} = L^{\infty}(T, \mathcal{B}(T), \lambda)$. We let $\Omega := [0,1]$, $\mathcal{A} := \mathcal{B}([0,1])$ and μ be the Lebesgue measure on $[0,1]$. For each $n \in \mathbb{N}$, for each $a \in \Omega$, we let

$$f_n(a) := t \mapsto \frac{1}{a^{1-\frac{1}{n}}} \exp(-|t + \ln a|).$$

For each $n \in \mathbb{N}$, f_n is Gelfand measurable but the sequence (f_n) is not mean norm bounded. Indeed, for each $t \in T$,

$$\int_{\Omega} |f_n(a,t)| d\mu(a) \leqslant \int_{\Omega} \frac{1}{a} \exp(-|t + \ln a|) d\mu(a) \leqslant 2 - \exp(-t).$$

It follows from Fubini-Lebesgue theorem, that for all $x \in E$,

$$\int_{\Omega} |\langle x, f_n(a)\rangle| d\mu(a) = \int_T |x(t)| \int_{\Omega} |f_n(a,t)| d\mu(a) d\mu(t) \leqslant 2 \|x\|_1 < \infty.$$

Hence the sequence (f_n) is mean weakly bounded. However,

$$\|f_n(a)\|^{\star} = \frac{1}{a^{1-\frac{1}{n}}} \quad \text{a.e.}$$

Thus the sequence (f_n) is a.e. bounded. However, for each $n \in \mathbb{N}$,

$$\int_{\Omega} \|f_n(a)\|^{\star} d\mu(a) = \int_{\Omega} a^{-1+\frac{1}{n}} d\mu(a) = n,$$

hence the sequence (f_n) is not mean norm bounded.

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