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CONTROL THEORY IN INFINITE DIMENSION FOR THE OPTIMAL LOCATION OF ECONOMIC ACTIVITY: THE ROLE OF THE SOCIAL WELFARE FUNCTION

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Abstract. In this paper we deal with a family of optimal control problems in infinite dimension with state constraints. We approach such problems with the dynamic programming approach identifying (in cases not yet known in the literature) a closed-form solution v of the associated Hamilton-Jacobi-Bellman (HJB) equation, which is a PDE in a suitable Hilbert space. Consequently we are able prove a verification theorem, to show that v is indeed the value function, and to provide the optimal control in closed-loop form. The abstract problem is motivated by an economic application in the context of continuous spatiotemporal growth models with capital diffusion, where a social planner chooses the optimal location of economic activity across space by maximization of an utilitarian social welfare function.

1. Introduction

1.1. **The mathematical problem.** In this paper we consider the following linear controlled system in a Hilbert space H, which is both the state and the control space:

$$x'(t) = Lx(t) - Nc(t), \quad t \geq 0.$$

Here $x(\cdot) : \mathbb{R}_+ \to H$ is the state trajectory, $c(\cdot) : \mathbb{R}_+ \to H$ is the control strategy, $L, N : H \to H$ are suitable linear operators. We require that the state variable satisfies the constraint

$$\langle x(t), \varphi \rangle > 0, \qquad \forall t \geq 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H and $\varphi \in H$ is a given vector. The aim is to maximize, over the associated set of admissible control strategies, i.e. locally integrable $c(\cdot)$ such that the above state constraint is satisfied, a functional of the form

$$\int_0^\infty e^{-\rho t} \mathcal{U}(c(t)) \mathrm{d}t,$$

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where $\rho > 0$ and $\mathcal{U}$ is a "utility-like" function. To solve the problem we use the dynamic programming technique in infinite dimension (see, e.g., Bensoussan et al. [2] and Li and Yong [16]) introducing the value function V and the associated Hamilton-Jacobi-Bellman (HJB) equation.

The infinite dimensional dynamic programming approach in economic models was first implemented by Fabbri and Gozzi [10] (in a different context) and then employed, among others, in a previous work by Boucekkine et al. [3] in a problem similar to the one presented as an application here. The novelty, with respect to the previous contribution [3], is that here we abstract from the specific form of the operators L, N, and of the function $\mathcal{U}$. Indeed, here L, N are abstract linear operators and we specify the properties of them (in particular the spectral properties of L) and of the function $\mathcal{U}$ that enable us to prove our main results: the verification theorem (Theorem 3.2); the explicit solution v of the HJB equation (Proposition 4.2); the existence and explicit expression of the optimal feedback control together with the identification $v = V$ (Theorem 5.1); the long-run behavior of the optimal state trajectory (Theorem 5.3).

1.2. **The economic motivation.** The problem is motivated by economic growth models with space dimension. There is indeed a growing interest in the economic literature in the role of space in decision-making. While the economic analysis has incorporated the spatial dimension for quite a long time (Von Thunen [21]), it is the rise of the so-called *New Economic Geography* (NEG) which did induce such a recent boom in this literature stream (see Krugman [13] for an insightful review of the NEG, and Fujita and Thisse [11], for a master textbook in this area). An overwhelming part of the latter stream has been concerned with the identification and characterization of spatial externalities and the inherent agglomeration mechanisms. Concretely, researchers in the area target either first nature causes for agglomeration for given technology and demographic spatial distributions (Krugman [14]) or second nature causes through the identification of mechanisms (typically, economies of scale or spillovers) leading to agglomeration for example when labor is mobile (Krugman [15]). An overwhelming majority of this paper is purely static: individuals do not save over time and therefore no capital accumulation is considered. Following an earlier contributions in mathematical geography by Isard and Liossatos [12], Brito [7] is to our knowledge the first who attempted to insert space in otherwise standard neoclassical growth models, giving rise to a bunch of papers on optimal growth within spatiotemporal frames (in particular, Boucekkine et al. [3, 5, 6], and Fabbri [9]).

From the economic point of view, this paper builds on Boucekkine et al. [3]. We focus on the problem faced by a central planner who has to choose the optimal distribution of economic activity (say, investment and production) over space (here the unit circle for simplicity). We assume that technology is homogeneous across space, which amounts to assuming that technological spillovers are quick enough for all the locations to use the same technology. In contrast, as it is most of the time the case, individuals are harder to move. We do not justify such an immobility but just assume it. In other words, the central planner can move capital but not people. Accordingly, our problem investigates a first nature cause of (potential) spatial externalities. Moreover, we generalize the social welfare function considered by Boucekkine et al. [3] by introducing a form of imperfect altruism in the social preferences. While only the Benthamite social welfare function is considered in the latter (that is, the planner consider the sum of utilities of all the individuals present in the economy), we consider a continuum of social welfare functions ranging from the Benthamite to the Millian form (*i.e* only the average consumer is considered). To our knowledge, this is the first time this is done within a spatial setting. Clearly, this would allow to handle a set of new population ethics and political economy issues, for example concerning regional particularism. Earlier papers using this continuum of social welfare functions in non-spatial settings have been typically devoted to study the normative implications of 1984 Parfit's [20] population ethics theory in different economic contexts (see Nerlove et al. [18], Palivos and Yip [19], and more recently, Boucekkine and Fabbri, [4]).

Importantly enough, in the induced control problem, the capital spatiotemporal dynamics follows a parabolic PDE in the tradition opened by Isard and Liossatos [12], and can be lifted into our abstract setting. Hence, it is possible, at the end of the day, to individuate the optimal spatiotemporal control and dynamics.

1.3. **Plan of the paper.** The paper is organized as follows. Sections 2 to 5 give all the mathematical steps needed to solve explicitly the optimal control problem. Section 6 describes the economic application and apply the preceding mathematical theory. Section 7 displays some complementary numerical exercises. Section 8 concludes.

2. The optimal control problem

Let $(D, \mathcal{D}, \mu)$ be a countably generated measure space and let us consider the separable Hilbert space $H := \mathbf{L}^2(D, \mu; \mathbb{R})$, with its usual norm $|\cdot|$ and scalar product $\langle \cdot, \cdot \rangle$. We identify, by the usual Riesz identification, H with its topological dual. We shall identify the elements of H, which are equivalence classes of functions coinciding μ−a.e., with (one of) their representative functions. So, the pointwise relationship must be intended μ−a.e.. We consider the positive and strictly positive orthants of H, i.e. the sets

$$H_+ := \{f \in H : \ f \geq 0\}, \qquad H_{++} := \{f \in H : \ f > 0\}.$$

Finally, we denote

$$H_+^0 := H_+ \setminus \{f \equiv 0\}.$$

In the following we set $\mathbb{R}_+ := [0, +\infty)$.

Given this setting, we are concerned with the following optimal control problem in the space H. Let $L : D(L) \subseteq H \to H$ be a (possibly) unbounded linear operator and let $N : H \to H$ a linear bounded operator. Given $x_0 \in H$ and a control function $c \in L^1_{loc}(\mathbb{R}_+; H)$, we consider the following abstract *state equation* in H:

$$x'(t) = Lx(t) - Nc(t), \qquad x(0) = x_0. \tag{2.1}$$

We introduce the following assumption

Assumption 2.1. *L is a closed and densely defined operator generating a C_0-semigroup in H.*

We use the notaton e^{tL} for the semigroup generated by L and, according to Bensoussan et al. [2], we define the *mild solution* to (2.1) as the function

$$x(t) := e^{tL}x_0 - \int_0^t e^{(t-s)L}Nc(s)\mathrm{d}s. \tag{2.2}$$

To stress the dependence of x on x_0, c, we write $x^{x_0,c}$. (2.2) is also a *weak solution* to (2.1), i.e.

(2.3)

$$\langle x(t), \psi\rangle = \langle x_0, \psi\rangle + \int_0^t \langle x(s), L^*\psi\rangle \mathrm{d}s - \int_0^t \langle Nc(s), \psi\rangle \mathrm{d}s, \quad \forall \psi \in D(L^*), \ \forall t \geq 0,$$

where $L^* : D(L^*) \subseteq H \to H$ denotes the adjoint of L.

Assumption 2.2. *$u : D \times \mathbb{R}_+ \to \mathbb{R}_+$ is such that $u(\theta, \cdot)$ is a utility function for each $\theta \in D$ — i.e. increasing and concave.*

Next, given $u : D \times \mathbb{R}_+ \to \mathbb{R}_+$ as above $u(\theta, \cdot)$, we consider the functional

$$\mathcal{U}(z) := \int_D u(\theta, z(\theta))\mu(\mathrm{d}\theta), \qquad z \in H_+,$$

and the functional on $L^1_{loc}(\mathbb{R}_+; H_+)$

$$c \mapsto \mathcal{J}(c) := \int_0^\infty e^{-\rho t}\mathcal{U}(c(t))\mathrm{d}t, \tag{2.4}$$

where $\rho > 0$ is a given discount factor. Notice that both $\mathcal{U}$ and $\mathcal{J}$ inherit from u the concavity.

Remark 2.3. The choice of working only with nonnegative (or, equivalently, bounded from below) utility function is done here just for the sake of brevity, in order to avoid technical complications that would arise in treating the case of utility functions not bounded from below. The latter case might be considered and treated as well at the price of technical complications.

Assumptions 2.1–2.2 will be standing from now on.

Let $\varphi \in H_{++}$. The aim is to maximize $\mathcal{U}$ when c ranges over the convex set

$$\mathcal{A}^{\varphi}_{++}(x_0) = \{c \in L^1_{loc}(\mathbb{R}_+; H_+) : \ x^{x_0,c}(t) \in H^{\varphi}_{++} \quad \text{for a.e. } t \geq 0\},$$

where H^{φ}_{++} is the open set

$$H^{\varphi}_{++} := \{x \in H : \ \langle x, \varphi\rangle > 0\} \subset H.$$

Notice that $H^0_+ \subset H^{\varphi}_{++}$. Hence, if the semigroup e^{tL} preserves H^0_+, i.e. it maps H^0_+ into itself, then $\mathcal{A}^{\varphi}_{++}(x_0)$ is not empty when $x_0 \in H^0_+$, as the null control $c \equiv 0$ belongs to it. Moreover, if also N is positivity preserving, we have the following monotonicity property:

$$c_1 \leq c_2 \implies x^{x_0,c_1} \geq x^{x_0,c_2}.$$

In particular

$$c_1 \leq c_2, \quad c_2 \in \mathcal{A}^{\varphi}_{++}(x_0) \implies c_1 \in \mathcal{A}^{\varphi}_{++}(x_0). \tag{2.5}$$

So, given $x_0 \in H^{\varphi}_{++}$, we are interested in the following optimal control problem:

$$(P^{\varphi}) \qquad \text{Maximize } \mathcal{J} \text{ over the set } \mathcal{A}^{\varphi}_{++}(x_0),$$

whose value function is

$$V^{\varphi}(x_0) := \sup_{c \in \mathcal{A}^{\varphi}_{++}(x_0)} \mathcal{J}(c).$$

Note that we cannot say ex ante that V^{φ} is finite. Sufficient conditions for finiteness will be provided later.

Remark 2.4. Problem (P^{φ}) is the most natural one from the mathematical point of view when φ is chosen suitably, in the sense that it admits an explicit solution. However, we anticipate that the meaningful problem from the economic point of view would be the (more difficult) one:

$$(P_+) \qquad \text{Maximize } \mathcal{J} \text{ over the set } \mathcal{A}_+(x_0)$$

whose value function is

$$V_+(x_0) := \sup_{c \in \mathcal{A}_+(x_0)} \mathcal{J}(c).$$

where

$$\mathcal{A}_+(x_0) := \{c \in L^1_{loc}(\mathbb{R}_+; H_+) : x^{x_0,c}(t) \in H^0_+ \quad \text{for a.e. } t \geq 0\}.$$

Note that $H^0_+ \subset H^{\varphi}_{++}$: hence, for $x_0 \in H^0_+$, $\mathcal{A}_+(x_0) \subseteq \mathcal{A}^{\varphi}_{++}(x_0)$ and, consequently $V^{\varphi}(x_0) \geq V_+(x_0)$. Moreover, if $\hat{c} \in \mathcal{A}^{\varphi}_{++}(x_0)$ is optimal for (P^{φ}) and belongs to $\mathcal{A}_+(x_0)$, then it is also clearly optimal for (P_+). In the illustration of the results, we will just test numerically, ex post, that the optimal control $\hat{c} \in \mathcal{A}^{\varphi}_{++}(x_0)$ also belongs to $\mathcal{A}_+(x_0)$.

3. The HJB equation and the verification theorem

In this section we provide a verification theorem for the problem and, accordingly, the solution in a special case. The results will be used in the next section to treat our motivating economic application.

The Hamilton-Jacobi-Bellman (HJB) equation associated to the optimal control problem (P^{φ}) is

$$\rho v(x) = \langle Lx, \nabla v(x) \rangle + \mathcal{H}(\nabla v(x)), \qquad x \in H^{\varphi}_{++}, \tag{3.1}$$

where

$$\mathcal{H}(q) := \sup_{z \in H_+} \mathcal{H}_{CV}(q; z), \quad q \in H,$$

and

$$\mathcal{H}_{CV}(q; z) := \mathcal{U}(z) - \langle Nz, q \rangle, \quad z \in H_+, \quad q \in H.$$

Notice that the above supremum may not be finite in general. A sufficient condition for the finiteness is: $\mathcal{U}(z) \leq C|z|^{\alpha}$ for some $C > 0$ and $\alpha \in (0,1)$, N^* preserving H_{++}, and $q \in H_{++}$. These will be used later.

Definition 3.1. We call *classical solution* to (3.1) (on H^{φ}_{++}) a function $v \in C^1(H^{\varphi}_{++};\mathbb{R})$ such that $\nabla v \in C(H^{\varphi}_{++}; D(L^*))$ and such that[1]

$$\rho v(x) = \langle x, L^*\nabla v(x)\rangle + \mathcal{H}(\nabla v(x)), \qquad \forall x \in H^{\varphi}_{++}.$$

Now we turn to state our Verification Theorem. Typically, to prove such theorem for infinite horizon problems, a condition on the solution v computed on the admissible trajectories when $t \to +\infty$ is needed. This is exactly the analogous of the so-called transversality condition arising in the maximum principle approach. The condition that we shall use is

$$\lim_{t\to+\infty} e^{-\rho t} v(x^{x_0,c}(t)) = 0, \qquad \forall c \in \mathcal{A}^{\varphi}_{++}(x_0). \tag{3.2}$$

Theorem 3.2 (Verification)**.** *Let v be a classical solution to* (3.1) *in H^{φ}_{++}, let $x_0 \in H^{\varphi}_{++}$ and let* (3.2) *hold. Then:*

(i) *$v(x_0) \geq V^{\varphi}(x_0)$;*

(ii) *if, moreover, there exists $\hat{c} \in \mathcal{A}^{\varphi}_{++}(x_0)$ such that* (3.2) *holds and*

$$N^*\nabla v(x^{x_0,\hat{c}}(s)) \in D^+\mathcal{U}(\hat{c}(s)) \quad \textit{for a.e. } s \geq 0, \tag{3.3}$$

where $D^+\mathcal{U}$ denotes the superdifferential of $\mathcal{U}$, then $v(x_0) = V^{\varphi}(x_0)$ and $\hat{c}$ is optimal for (P^{φ}) starting at x_0, i.e. $\mathcal{J}(\hat{c}) = V^{\varphi}(x_0)$.

Proof. (i) Let $c \in \mathcal{A}^{\varphi}_{++}(x_0)$. By chain's rule in infinite dimension, we have, for every $t \geq 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[e^{-\rho t} v(x^{x_0,c}(t)))]\\ &= e^{-\rho t}\Big(- \rho v(x^{x_0,c}(t)) + \langle x^{x_0,c}(t), L^*\nabla v(x^{x_0,c}(t))\rangle - \langle Nc(t), \nabla v(x^{x_0,c}(t))\rangle\Big).\end{aligned}$$

Now we add and subtract $e^{-\rho t}\mathcal{U}(c(t))$ to the right hand side, use the fact that v solves HJB, and integrate over $[0,t]$. We get, for every $t \geq 0$,

$$\begin{aligned}&e^{-\rho t} v(x^{x_0,c}(t)) + \int_0^t e^{-\rho s}\mathcal{U}(c(s))\mathrm{d}s =\\ &v(x_0) + \int_0^t e^{-\rho s}\Big(- \mathcal{H}(\nabla v(x^{x_0,c}(s)) + \mathcal{H}_{CV}(\nabla v(x^{x_0,c}(s); c(s))\Big)\mathrm{d}s,\end{aligned}$$

Observe that, since $\mathcal{U}$ is concave (and hence sublinear from above), and $c(\cdot) \in \mathbf{L}^1_{loc}(\mathbb{R}_+; H_+)$ then both sides of the above inequality are finite for every $t \geq 0$. Now, rearranging the terms and taking into account the definition of $\mathcal{H}$, we get, for every $t \geq 0$,

$$v(x_0) \geq e^{-\rho t} v(x^{x_0,c}(t)) + \int_0^t e^{-\rho s}\mathcal{U}(c(s))\mathrm{d}s \tag{3.4}$$

— with equality if $c = \hat{c}$ verifies (3.3) (equivalently, (3.6)) s-a.e. on $[0,t]$. Since $\mathcal{U}$ is nonnegative (Assumption 2.2), by monotone convergence theorem w have then it

[1] This equality, in particular, implies that $\mathcal{H}(\nabla v(x))$ is finite for every $x \in H^{\varphi}_{++}$.

must be

$$\lim_{t\to+\infty}\int_0^t e^{-\rho s}\mathcal{U}(c(s))\mathrm{d}s = \int_0^\infty e^{-\rho s}\mathcal{U}(c(s))\mathrm{d}s =: \mathcal{J}(c). \tag{3.5}$$

Hence, passing (3.4) to the $\lim_{t\to+\infty}$ and using (3.2), we conclude

$$v(x_0) \geq \mathcal{J}(c)$$

Then, by definition of V^φ and since $c \in \mathcal{A}^\varphi_{++}(x_0)$ was arbitrary, we immediately get the claim.

(ii) Notice that, by concavity of $\mathcal{U}$, (3.3) is equivalent to

$$\hat{c}(s) \in \operatorname{argmax}_{z\in H^\varphi_{++}}\{\mathcal{U}(z) - \langle Nz, \nabla v(x^{x_0,\hat{c}}(s))\rangle\}, \quad \forall s \geq 0, \tag{3.6}$$

the usual closed loop condition for optimality. Hence, for $c = \hat{c}$ we have equality in (3.4). Hence, passing to the $\lim_{t\to+\infty}$ and using (3.2) and (3.5), we get the equality

$$v(x_0) = \mathcal{J}(\hat{c})$$

Since $\mathcal{J}(\hat{c}) \leq V^\varphi(x_0)$, combining with part (i), the claim follows. □

4. A solution of the HJB equation

In this section we further specify the model to deal with an explicit solution. We consider the following assumption.

Assumption 4.1.

(i) There exists an eigenvector $b_0 \in H_{++}$ for $L^* : D(L^*) \subseteq H \to H$ with eigenvalue $\lambda_0 \in \mathbb{R}$. Without loss of generality we assume that $|b_0|_H = 1$.

(ii) $\rho > \lambda_0(1-\gamma)$.

(iii) $[Nz](\theta) = n(\theta)z(\theta)$, where $n \in L^\infty(D, \mathcal{D}, \mu; (0, +\infty))$.

(iv) $u(\theta,\xi) = \dfrac{\xi^{1-\gamma}}{1-\gamma}f(\theta)$, where $\gamma \in (0,1)\cup(1,\infty)$ and $f \in L^\infty(D,\mathcal{D},\mu;\mathbb{R}_+)$.

(v) $\displaystyle\int_D f(\theta)^{\frac{1}{\gamma}}(n(\theta)b_0(\theta))^{\frac{\gamma-1}{\gamma}}\mu(\mathrm{d}\theta) < \infty$, $\displaystyle\int_D \left(\frac{f(\theta)}{n(\theta)b_0(\theta)}\right)^{p/\gamma}\mu(\mathrm{d}\theta) < \infty$.

Notice that Assumptions 4.1(iv) implies Assumption 2.2. Also notice that, in general, b_0 may be not unique. A sufficient condition for the uniqueness of b_0 is that L^* is a diagonal operator with respect to a given orthonormal basis in H. This will be the case in Section 6.

Proposition 4.2. *Let Assumption 4.1 hold. Then*

$$v(x) = \alpha\frac{\langle x, b_0\rangle^{1-\gamma}}{1-\gamma}, \qquad x \in H^{b_0}_{++}, \tag{4.1}$$

where

$$\alpha = \gamma^\gamma\left(\frac{\int_D f(\theta)^{\frac{1}{\gamma}}(n(\theta)b_0(\theta))^{\frac{\gamma-1}{\gamma}}\mu(\mathrm{d}\theta)}{\rho - \lambda_0(1-\gamma)}\right)^\gamma \tag{4.2}$$

is a classical solution to (3.1) *on* $H^{b_0}_{++}$.

Proof. Notice first that $\mathcal{U}$ is Fréchet differentiable in $H^{b_0}_{++}$ and

$$[\nabla\mathcal{U}(z)](\theta) = f(\theta)z(\theta)^{-\gamma}, \qquad z \in H^{b_0}_{++}.$$

Hence, by straightforward computations

$$\mathcal{H}(q) = \frac{\gamma}{1-\gamma}\int_D f(\theta)^{\frac{1}{\gamma}}(n(\theta)q(\theta))^{\frac{\gamma-1}{\gamma}}\mu(\mathrm{d}\theta), \qquad q \in H_{++},$$

with optimizer

$$\hat{z}(q)(\theta) = \mathrm{argmax}_{z\in H_+}\big\{\mathcal{U}(z) - \langle Nz, q\rangle\big\} = \left(\frac{f(\theta)}{n(\theta)q(\theta)}\right)^{\frac{1}{\gamma}}. \tag{4.3}$$

Moreover

$$\nabla v(x) = \alpha\langle x, b_0\rangle^{-\gamma}b_0.$$

Plugging these expression into (3.1) and dividing all terms by $\frac{\langle x,b_0\rangle^{1-\gamma}}{1-\gamma}$, we get the following algebraic equation in $\alpha \in \mathbb{R}$

$$\rho\alpha = \lambda_0(1-\gamma)\alpha + \gamma\alpha^{\frac{\gamma-1}{\gamma}}\int_D f(\theta)^{\frac{1}{\gamma}}(n(\theta)b_0(\theta))^{\frac{\gamma-1}{\gamma}}\mu(\mathrm{d}\theta),$$

which has a unique positive solution provided by (4.2). ☐

5. The solution of the control problem

In order to produce an optimal control starting at $x_0 \in H^{b_0}_{++}$, we study the *closed loop equation* associated to the solution v and to the candidate optimal feedback map (4.3) under Assumption 4.1:

$$x'(t) = Lx(t) - N\Phi x(t), \quad x(0) = x_0, \tag{5.1}$$

where $\Phi : H \to H$ is the bounded linear positive operator

$$[\Phi z](\theta) = \left(\frac{f(\theta)}{\alpha n(\theta)b_0(\theta)}\right)^{\frac{1}{\gamma}}\langle z, b_0\rangle. \tag{5.2}$$

This linear equation admits a unique mild solution $\hat{x}$, which is also a weak solution. In particular, testing against $b_0 \in D(L^*)$ and taking into account the definition of α, it holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}(t), b_0\rangle &= \langle\hat{x}(t), L^*b_0\rangle - \langle N\Phi\hat{x}(t), b_0\rangle\\ &= \left(\lambda_0 - \alpha^{-\frac{1}{\gamma}}\int_D f(\theta)^{\frac{1}{\gamma}}\,(n(\theta)b_0(\theta))^{\frac{\gamma-1}{\gamma}}\,\mu(\mathrm{d}\theta)\right)\langle\hat{x}(t), b_0\rangle.\end{aligned}$$

Hence, taking int account the definition of α, providing

$$\alpha^{-\frac{1}{\gamma}}\int_D f(\theta)^{\frac{1}{\gamma}}\,(n(\theta)b_0(\theta))^{\frac{\gamma-1}{\gamma}}\,\mu(\mathrm{d}\theta) = \frac{\rho-\lambda_0(1-\gamma)}{\gamma}, \tag{5.3}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{x}(t), b_0\rangle = g\,\langle\hat{x}(t), b_0\rangle,$$

i.e.

$$\langle \hat{x}(t), b_0 \rangle = \langle x_0, b_0 \rangle \, e^{gt}, \quad t \geq 0,$$

where

$$g := \frac{\lambda_0 - \rho}{\gamma}. \tag{5.4}$$

By Assumption 4.1(i), we get

$$\hat{x}(t) \in H^{b_0}_{++}, \qquad \forall t \geq 0. \tag{5.5}$$

Theorem 5.1. *Let Assumption 4.1 hold, let $x_0 \in H^{b_0}_{++}$, and let v be the solution to* (3.1) *given as in Proposition 4.2. The control*

$$\hat{c}(t, \theta) := [\Phi \hat{x}(t)](\theta) = \left(\frac{f(\theta)}{\alpha n(\theta) b_0(\theta)} \right)^{\frac{1}{\gamma}} \langle x_0, b_0 \rangle e^{gt},$$

belongs to $\mathcal{A}^{b_0}_{++}(x_0)$ and is optimal for (P^{b_0}) starting at x_0. Moreover, $v(x_0) = V(x_0)$.

Proof. We want to apply Theorem 3.2. First, notice that, using the concept of weak solution (2.3) with $\psi = b_0$, we have

$$\frac{d}{\mathrm{d}t} \langle x^{x_0,c}(t), b_0 \rangle = \lambda_0 \langle x^{x_0,c}(t), b_0 \rangle - \langle Nc(s), b_0 \rangle \qquad \forall c \in \mathcal{A}^{b_0}_{++}(x_0),$$

i.e.

$$\langle x^{x_0,c}(t), b_0 \rangle = \langle x_0, b_0 \rangle e^{\lambda_0 t} - \int_0^t e^{\lambda_0 (t-s)} \langle Nc(s), b_0 \rangle \qquad \forall c \in \mathcal{A}^{b_0}_{++}(x_0).$$

Since both n and b_0 are nonnegative (Assumption 4.1(i) and (iii)), we have

$$0 \leq \langle x^{x_0,c}(t), b_0 \rangle \leq \langle x_0, b_0 \rangle e^{\lambda_0 t} \qquad \forall c \in \mathcal{A}^{b_0}_{++}(x_0).$$

Hence, for every $c \in \mathcal{A}^{b_0}_{++}(x_0)$,

$$\begin{aligned} 0 \ &\leq \ e^{-\rho t} v(x^{x_0,c}(t)) = \alpha e^{-\rho t} \frac{\langle x^{x_0,c}(t), b_0 \rangle^{1-\gamma}}{1-\gamma} \\ &\leq \ \frac{\alpha}{1-\gamma} \langle x_0, b_0 \rangle e^{-(\rho - \lambda_0(1-\gamma))t}. \end{aligned}$$

Therefore, using Assumption 4.1(ii), we see that (3.2) hold. This allows to apply part (i) of Theorem 3.2.

Let us now turn to part (ii) of Theorem 3.2, i.e. to prove the optimality of $\hat{c}$. By construction $x^{x_0,\hat{c}} = \hat{x}$. Hence, by (5.5) and Assumption 4.1(v), we obtain $\hat{c} \in \mathcal{A}^{b_0}_{++}(x_0)$. Moreover, again by construction, $\hat{c}$ verifies the optimality condition (3.6). So, Theorem 3.2(ii) applies and the proof is complete.

□

Assumption 5.2. *$L^* : H \to H$ admits an orthonormal basis of eigenvectors, i.e. there exists an orthomormal basis $\{b_n\}_{n \in \mathbb{N}}$ of H such that*

$$L^* b_n = \lambda_n b_n \qquad \forall n \in \mathbb{N}.$$

Moreover,

$$\lambda_k < g, \qquad \forall k \in \mathbb{N} \setminus \{0\}. \tag{5.6}$$

Theorem 5.3. *Let Assumption 4.1 and 5.2 hold, and let* $x_0 \in H^{b_0}_{++}$*. Then, setting* $\hat{x}_g := e^{-gt}\hat{x}(t)$*, we have the convergence in* H

$$\hat{x}_g(t) \to \langle x_0, b_0\rangle \sum_{k=0}^{\infty} q_k b_k, \qquad \textit{as } t \to \infty,$$

where

$$q_0 := 1, \qquad q_k := \frac{\zeta_k}{\lambda_k - g},$$

where

$$\zeta_k := \left\langle \left(\frac{f(\cdot)}{\alpha n(\cdot) b_0(\cdot)}\right)^{\frac{1}{\gamma}}, b_k \right\rangle$$

Moreover, the speed of convergence is exponential of order

$$s := \sup_{k\in\mathbb{N}\setminus\{0\}} \{\lambda_k - g\}.$$

Proof. The equation for $\hat{x}_g$ is

$$\hat{x}_g'(t) = (L - g)\hat{x}_g(t) - N\Phi\hat{x}_g(t). \tag{5.7}$$

Let $\hat{x}_g^{(k)}(t) = \langle \hat{x}_g(t), b_k\rangle$, so that, by Fourier series expansion

$$\hat{x}_g^{(k)}(t) = \sum_{k=0}^{\infty} \hat{x}_g^{(k)}(t) b_k.$$

From the fact that $\hat{x}_g$ is also a weak solution to (5.7), we get the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_g^{(k)}(t) = (\lambda_k - g)\hat{x}_g^{(k)}(t) - \langle x_0, b_0\rangle \zeta_k, \quad k \in \mathbb{N}.$$

We already know that, by definition of g,

$$\hat{x}_g^{(0)}(t) \equiv \langle x_0, b_0\rangle.$$

For the others term, when $k \in \mathbb{N} \setminus \{0\}$, we have

$$\hat{x}_g^{(k)}(t) = \langle x_0, b_k\rangle e^{(\lambda_k - g)t} + \langle x_0, b_0\rangle \zeta_k \frac{1 - e^{(\lambda_k - g)t}}{\lambda_k - g}.$$

The claim follows by (5.6). ☐

6. The economic problem

We apply the results of the previous section to an economic problem by taking the following specifications of the general framework above:

(a) $D = \mathbf{S}^1 := \{\xi \in \mathbb{R}^2 : \ |\xi| = 1\} \cong \mathbb{R}/\mathbb{Z}$; $\mathbf{S}^1$ is topologically identified with $[0, 2\pi] \subset \mathbb{R}$ when the extreme of the latter interval are identified; similarly, functions on $\mathbf{S}^1$ are identified with 2π-perodic functions on $\mathbb{R}$;
(b) $\mu =$ Hausdorff measure on the Borel σ-algebra of $\mathbf{S}^1$; i.e., through the identification $\mathbf{S}^1 \cong \mathbb{R}/\mathbb{Z}$, $\mu =$Lebesgue measure on $(0, 2\pi)$;
(c) $L = \sigma\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + A$, where $\sigma, A > 0$ are constant; accordingly, the integration with respect to this measure will be simply denoted by $\mathrm{d}\theta$;

(d) $f(\theta) = n(\theta)^\beta$, with $\beta \geq 0$.

In this case Assumption 2.1 is verified $L = L^*$ and this operator admits a spectral decomposition on the separable Hilbert space $\mathbf{L}^2(\mathbf{S}^1, \mathrm{d}\theta; \mathbb{R})$. Moreover, there is a doubly indexed sequence of eigenvectors and eigenvalues of $L = L^*$ which is explicit. Indeed,

$$b_0 = (2\pi)^{-1/2}\mathbf{1}_{\mathbf{S}^1}, \quad \lambda_0 = A,$$

and

$$\begin{cases} b_k^{(1)}(\theta) = \pi^{-1/2}\sin(k\theta), \\ b_k^{(2)}(\theta) = \pi^{-1/2}\cos(k\theta), \qquad \forall\theta \in \mathbf{S}^1,\ \forall k \in \mathbb{N}\setminus\{0\}. \\ \lambda_k = A - \sigma k^2, \end{cases}$$

This formal frame fits quite well optimal growth models with AK production function, one-dimensional geography and capital diffusion as in Boucekkine et al. [3, 5]. Indeed, calling $K(t, \cdot) = x(t)$ according to the usual notation for capital in AK models, formally the problem becomes:

$$V(K_0) := \sup_{\mathcal{A}(K_0)} \int_0^\infty e^{-\rho t}\left(\int_{\mathbf{S}^1} \frac{c(t,\theta)^{1-\gamma}}{1-\gamma} n(\theta)^\beta \mathrm{d}\theta\right)\mathrm{d}t,$$

where $\rho > 0$, $\gamma \in (0, 1)$, $\beta \geq 0$,

$$\mathcal{A}(K_0) := \big\{c \in \mathbf{L}^1_{loc}([0, +\infty); \mathbf{L}^2(\mathbf{S}^1, \mathrm{d}\theta; \mathbb{R}^+)) : \\ K(t,\theta) \geq 0 \textit{ and } K(t,\theta) \not\equiv 0 \text{ for a.e. } (t,\theta) \in \mathbb{R}_+ \times \mathbf{S}^1\big\},$$

under the state PDE-constraint

$$\text{(6.1)} \quad \begin{cases} \dfrac{\partial K}{\partial t}(t,\theta) = \sigma\dfrac{\partial^2 K}{\partial\theta^2}(t,\theta) + AK(t,\theta) - n(\theta)c(t,\theta), \qquad (t,\theta) \in \mathbb{R}_+ \times \mathbf{S}^1, \\ K(0,\theta) = K_0(\theta), \qquad \theta \in \mathbf{S}^1, \end{cases}$$

A few comments on the economic problem are needed at this stage. First of all, let us comment on the objective function,

$$\int_0^\infty e^{-\rho t}\left(\int_{\mathbf{S}^1} \frac{c(t,\theta)^{1-\gamma}}{1-\gamma} n(\theta)^\beta \mathrm{d}\theta\right)\mathrm{d}t,$$

which is indeed the social welfare function to be maximized by the central planner. This generalizes the social welfare function adopted by Boucekkine et al. [3], which is itself an extension of Boucekkine et al. [5]. In the latter, $n(\theta) \equiv 1$ and in the former $\beta = 1$. In this case, the planner sums all the utilities of all the individuals in location θ at time t. This corresponds to total utilitarianism or equivalently to the Benthamite case. When $\beta = 0$, one gets average utilitarianism or the Millian social welfare function: the planner do not consider all the individuals but only the average. In between, $\beta \in (0\ , 1)$, we get a continuum of intermediate configurations, we shall refer to this case as imperfect altruism: roughly speaking, as β increases, the planner takes more closely into account the welfare of his population.[2] Two other parameters of the social welfare functions are interesting to comment briefly on. Parameter ρ may be interpreted as the time discounting factor of the planner:

[2]A natural extension is to consider that β is a function of θ, which is a direct way to model regional or local particularism. We abstract away from this in this paper.

the bigger ρ, the more weight will be given to the present (or to the current generations) by the planner. More interestingly, parameter γ has a double role in our spatiotemporal frame: it captures on one hand the cost of intertemporal consumption substitution faced by individuals, as in the standard non-spatial settings, but it also indicates the degree of aversion to inequality of the planner on the other. Thus one has therefore to expect that as γ increases, the less unequal will be the consumption distribution over space. We shall check this in the last section devoted to numerical illustration.

Finally the state equation (6.1) gives the spatiotemporal dynamics of capital. $\sigma\frac{\partial^2 K}{\partial\theta^2}(t,\theta)$ is the diffusion term of the equation, it depicts the dynamics of capital through space with $\sigma > 0$ the diffusion speed. The rest of terms are obvious. In particular $n(\theta)c(t,\theta)$ is the so-called dilution effect of demographic size on capital accumulation. Notice that in this economy goods are produced and consumed locally, only capital moves cross locations (or more precisely, capital is moved by the planner according to the PDE above in line with Isard and Liossatos [12]).

Next, assuming that

(A1) $\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta+\gamma-1}{\gamma}}\mathrm{d}\theta < \infty,\ \int_{\mathbf{S}^1} n(\theta)^{\frac{2(\beta-1)}{\gamma}}\mathrm{d}\theta < \infty,$

(A2) $\rho > A(1-\gamma)$,

all the assumptions of the previous sections are verified. By expliciting the result of Theorem 5.1 with these specifications, we get the following.

Theorem 6.1. *Let (A1)–(A2) hold and assume that the solution $\hat{K}$ to the linear integro-PDE*

(6.2)
$$\begin{cases} \dfrac{\partial K}{\partial t}(t,\theta) = \sigma\dfrac{\partial^2 K}{\partial\theta^2}(t,\theta) + AK(t,\theta) - \alpha^{-1/\gamma}(2\pi)^{-1/2}\left(\int_{\mathbf{S}^1} K(t,\theta)\mathrm{d}\theta\right) n(\theta)^{\frac{\beta+\gamma-1}{\gamma}}, \\ K(0,\theta) = K_0(\theta) \end{cases}$$

where $t \in \mathbb{R}_+$, $\theta \in \mathbf{S}^1$, is such that $\hat{K} \geq 0$ and $\hat{K} \neq 0$. Then, the following claims hold true.

(i) The value function is explicitly given by

$$V(K_0) = \left(\frac{\gamma}{\rho - \lambda_0(1-\sigma)}\right)^{\gamma}\left(\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta+\gamma-1}{\gamma}}\mathrm{d}\theta\right)^{\gamma}\frac{(2\pi)^{-1/2}}{1-\gamma}\left(\int_{\mathbf{S}^1} K_0(\theta)\mathrm{d}\theta\right)^{1-\gamma}.$$

(ii) The control

$$\hat{c}(t,\theta) = \alpha^{-1/\gamma}(2\pi)^{-1/2}\left(\int_{\mathbf{S}^1} K_0(\theta)\mathrm{d}\theta\right) n(\theta)^{\frac{\beta-1}{\gamma}} e^{gt},$$

where $g := \frac{A-\rho}{\gamma}$ is optimal and the corresponding optimal capital is $\hat{K}$; g is therefore the optimal growth rate of the economy.

(iii) If, moreover, $A - \sigma < g$ and

$$\sum_{k=1}^{\infty}\left((\zeta_k^{(1)})^2 + (\zeta_k^{(2)})^2\right) < \infty,$$

where

$$\zeta_k^{(2)} := \frac{\rho - A(1-\gamma)}{\gamma}\left(\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta+\gamma-1}{\gamma}}\mathrm{d}\theta\right)^{-1}\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta-1}{\gamma}}\cos(k\theta)\mathrm{d}\theta$$

$$\zeta_k^{(1)} := \frac{\rho - A(1-\gamma)}{\gamma}\left(\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta+\gamma-1}{\gamma}}\mathrm{d}\theta\right)^{-1}\int_{\mathbf{S}^1} n(\theta)^{\frac{\beta-1}{\gamma}}\sin(k\theta)\mathrm{d}\theta,$$

then the detrended optimal capital

$$\hat{K}_g(t,\cdot) := e^{-gt}\hat{K}(t,\theta)$$

converges in $\mathbf{L}^2(\mathbf{S}^1, \mathrm{d}\theta; \mathbb{R}^+)$ *to the function*

$$\hat{K}_g^\infty(\theta) := (2\pi)^{-1/2}\left(\int_{\mathbf{S}^1} K_0(\theta)\mathrm{d}\theta\right)\left(1 + \sum_{k=1}^{\infty}\left(\zeta_k^{(1)}\sin(k\theta) + \zeta_k^{(2)}\cos(k\theta)\right)\right)$$

as $e^{-(g-A+\sigma)t}$.

A few comments are in order here. Of course, since we are dealing with an AK model, growth is endogenous and all variables grow at exponential rates. The growth rates are not location-dependent. This is not surprising since technology (though parameter A) is the same everywhere, and so are individual preferences. Second, as in Boucekkine et al. [3], we are able to identify the optimal stationary spatial distributions in closed-form and the corresponding convergence speeds, which certainly speaks about the flexibility of the analytical method developed. Third, and more importantly, β plays a central role in the shape of these distributions, in particular for consumption per capita. Indeed, this distribution is uniform if and only if the social welfare function is Benthamite ($\beta = 1$). This is an important property which was not of course reachable in the previous contributions to this area as they all rely on total utilitarianism. The next section goes deeper into this question via numerical exploration.

7. Some numerical illustrations

We shall now explore numerically some of the salient properties of the optimal long-term distributions theoretically uncovered above. We shall in particular, highlight the implications of two key parameters: the aversion to inequality parameter γ, and the altruism parameter β. To make the exposition more focused on the novelties brought by our generalization of social welfare, we shall concentrate on consumption, which is the control variable and the sole measure of welfare in our model. It can be shown that just as in Boucekkine et al. [3], long-term total and per-capita capital, production and investments are lower in the more populated areas. This has a simple explanation: to guarantee a reasonable level of consumption to everybody, the planner needs to maintain an higher level of aggregate consumption in more densely populated areas leading to lower investment at the same locations. It follows that in this model where the optimal spatiotemporal decisions are only driven by demographic heterogeneity across space, and not by technology discrepancy, the key variable is consumption and the determinants behind (notably, aversion to inequality and altruism).[3]

To start our numerical exercises, we need a benchmark (reasonable) calibration. Concretely, we choose: $\rho = 0.04$, γ between 0.6 and 0.9, β between 0 and 1 and $A = 0.08$. The value of ρ (the discount rate parameter) is chosen consistently with the data from Lopez [17]. Parameter γ, also measuring the inverse of the intertemporal substitution with the CRRA specification of preferences, can be calibrated using relative risk aversion consistently with the values found in individual choice experiments (see for example Cubitt et al. [8], and Tversky and Kahneman [22]). The value of A is set to generate long-term growth rates of 4% to 6%, which are consistent with those observed in developing countries (see e.g. World Bank Group [23]). As to the exogenous density distribution, we assume that we have a demographic center as depicted in Figure 1:

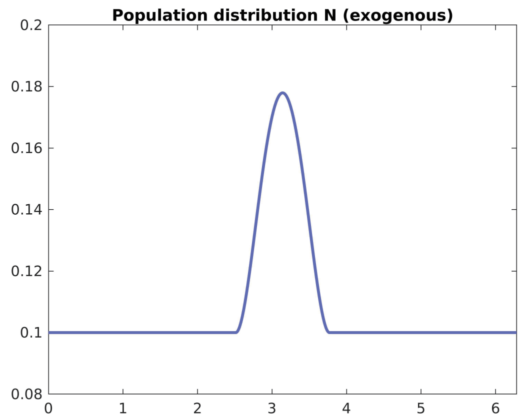

FIGURE 1

Because of the neoclassical dilution effect, the computed stationary distributions will always deliver that optimal consumption per capita goes down with population size over space except in the Benthamite case, which therefore features a case of egalitarianism. More interesting insights can be gained from the numerical exercises.

The implications of the aversion to inequality parameter, γ. Here we fix the altruism parameter β to 0, that is we stick to the Millian social welfare function. Needless to say, since we are concerned with consumption per capital, the Benthamite case, $\beta = 1$, as studied in Boucekkine et al. [3, 5] is irrelevant as by the theorem just above, this is a case where consumption is uniform across space, independently of γ. Figure 2 delivers the stationary distributions of consumption per capita detrended

[3]We could have followed Allen and Arkolakis [1] and assume that the size of population drives productivity at any location. In other words, we could have assumed population-based productivity heterogeneity. We don't to that to single out the pure population effect in our generalized context.

for $\gamma = 0.6, 0.75, 0.9$. First, as γ increases, consumption is in average bigger. This is due to the fact that, as outlined in the previous section, γ also measures the cost of intertemporal substitution in consumption: when it's high, savings and investment are lower and consumption is higher. But in our spatiotemporal frame, γ also measures the planner's aversion to inequality. One can check that as γ rises, consumption distributions becomes less unequal. A rough and quick verification is to compute the ratio between the periphery *plateau* consumption value and its value at the trough of the distribution: it decreases steadily.[4]

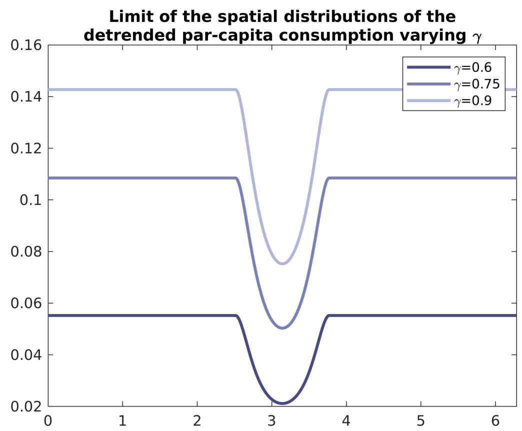

FIGURE 2

The implications of the altruism parameter, β. Figure 3 reports the stationary spatial distribution of detrended consumption capita. For this exercise, we fix $\gamma = 0.8$, and we play on the the altruism parameter. Precisely, we compute the distributions for the Benthamite ($\beta = 1$), Millian ($\beta = 0$) and an imperfect altruism ($\beta = 0.5$) cases. Not surprisingly, the optimal stationary distribution of consumption per capita is uniform. More interestingly, the Millian case ($\beta = 0$) delivers by far the largest spatial inequality in consumption while imperfect altruism displays intermediate results from this point of view. Indeed, the ratio between the periphery *plateau* consumption value and its value at the trough of the distribution is larger than 2, which exceeds the ratio between the corresponding population sizes. That is to say at low enough values of the altruism parameter β, the optimal spatiotemporal dynamics amplify the neoclassical dilution effect, even in the long-run! Including this parameter into the analysis seems therefore highly interesting from the normative point of view, and probably also from a more positive perspective in contexts of regional particularism.

[4]In our calibrated example, this ratio move from 2.75 at $\gamma = 0.6$, to 1.85 when $\gamma = 0.75$ and finally it is equal to 0.75 for $\gamma = 0.9$. Empirical standard deviations deliver the same ranking of course.

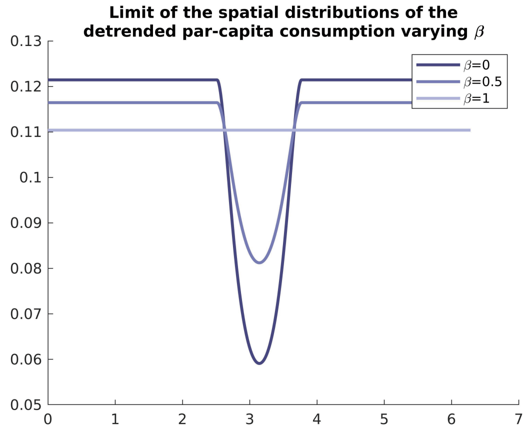

FIGURE 3

8. CONCLUSION

In this paper, we have generalized the social welfare function typically considered in the recent literature on spatiotemporal growth with capital diffusion. Adapting the dynamic programming method in infinite dimension used in Boucekkine et al. [3], we have been able to solve in closed-form for the optimal controls and for the corresponding spatial stationary distributions as well, either in the short or in the long-run. We have found several interesting results. In particular, we prove that the Benthamite case is the unique one for which the optimal stationary detrended consumption spatial distribution is uniform. Interestingly enough, we also find that as the social welfare function gets closer to the Millian case, the optimal spatiotemporal dynamics amplify the typical neoclassical dilution population size effect, even in the long-run.

Further extensions with space-dependent altruism featuring regional particularism are worth exploring. We could have also studied much more in detail the interplay between altruism and aversion to inequality (that is between β and γ). We leave it for future work.

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