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STRONGLY AGREEABLE PROGRAMS FOR THE ROBINSON-SOLOW-SRINIVASAN MODEL

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Abstract. In this work we consider different optimality criterions for the Robinson-Solow-Srinivasan model. In particular, it is shown that a program is strong agreeable if and only if it is weakly maximal and good.

1. Introduction

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 4–14, 21, 22, 28, 31, 33, 36–38, 41, 50, 52] and the references mentioned therein. These problems arise in engineering [1, 29, 47], in models of economic growth [10, 15, 23–27, 32, 35, 42, 44–47, 49, 54], in the game theory [16, 40, 51], in optimal control with PDE [17, 39, 43, 53] in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3] and in the theory of thermodynamical equilibrium for materials [30, 34]. In this paper we study the infinite horizon problem related to a discrete-time optimal control system describing the Robinson-Solow-Srinivasan model. We consider different optimality criterions for the Robinson-Solow-Srinivasan model. In particular, it is shown that a program is strong agreeable if and only if it is weakly maximal and good.

It should be mentioned that discrete-time optimal control problems arising in economic dynamics usually are studied under assumptions that all their good programs converge to a turnpike which is an interior point of the set of admissible pairs [49, 51]. In this paper we study a large class of control systems for which the turnpike is not necessarily an interior point of the set of admissible pairs. This makes the situation more difficult and less understood.

One of the main topics in the infinite horizon optimal control theory is to study the existence and properties of solutions of problems over an infinite horizon using different optimality criteria. In the present paper, studying infinite horizon problems, we deal with the notion of a good program introduced by D. Gale in [15] which is of great usage in optimal control and economic dynamics (see, for example, [10, 47, 49] and the references mentioned therein) and with the notion of an agreeable program [18–20].

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2. The Robinson-Solow-Srinivasan model

Let R^1 (R^1_+) be the set of real (non-negative) numbers and let R^n be the n-dimensional Euclidean space with non-negative orthant

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}.$$

For every pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, define their inner product by

$$xy = \sum_{i=1}^{n} x_i y_i$$

and let $x >> y$, $x > y$, $x \geq y$ have their usual meaning. Namely, for a given pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, we say that $x \geq y$, if $x_i \geq y_i$ for all $i = 1, \dots, n$, $x > y$ if $x \geq y$ and $x \neq y$, and $x >> y$ if $x_i > y_i$ for all $i = 1, \dots, n$.

Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For every $x \in R^n$, denote by $\|x\|$ its Euclidean norm in R^n.

Let $a = (a_1, \dots, a_n) >> 0$, $b = (b_1, \dots, b_n) >> 0$, $d \in (0, 1)$,

$$c_i = b_i/(1 + da_i), \ i = 1, \dots, n. \tag{2.1}$$

We assume the following:

$$\text{There exists } \sigma \in \{1, \dots, n\} \text{ such that for all}$$
$$i \in \{1, \dots, n\} \setminus \{\sigma\}, \ c_\sigma > c_i. \tag{2.2}$$

A sequence $\{x(t), y(t)\}_{t=0}^{\infty}$ is called a program if for each integer $t \geq 0$

$$(x(t), y(t)) \in R^n_+ \times R^n_+, \ x(t+1) \geq (1-d)x(t),$$
$$0 \leq y(t) \leq x(t), \ a(x(t+1) - (1-d)x(t)) + ey(t) \leq 1. \tag{2.3}$$

Let T_1, T_2 be integers such that $0 \leq T_1 < T_2$. A pair of sequences

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$$

is called a program if $x(T_2) \in R^n_+$ and for each integer t satisfying $T_1 \leq t < T_2$ relations (2.3) is valid.

For the economic interpretation of the model defined above and, in particular, for the economic meaning of our notation see the seminal paper by M. Ali Khan and T. Mitra [23].

Assume that $w : [0, \infty) \to R^1$ is a continuous strictly increasing concave and differentiable function which represents the preferences of the planner.

Define

$$\Omega = \{(x, x') \in R^n_+ \times R^n_+ : \ x' - (1-d)x \geq 0$$
$$\text{and } a(x' - (1-d)x) \leq 1\} \tag{2.4}$$

and a correspondence $\Lambda : \Omega \to R^n_+$ given by

$$\Lambda(x, x') = \{y \in R^n_+ : \ 0 \leq y \leq x \text{ and } ey \leq 1 - a(x' - (1-d)x)\}. \tag{2.5}$$

For every $(x, x') \in \Omega$ set

$$u(x, x') = \max\{w(by) : \ y \in \Lambda(x, x')\}. \tag{2.6}$$

A golden-rule stock is $\widehat{x} \in R^n_+$ such that $(\widehat{x}, \widehat{x})$ is a solution to the problem: maximize $u(x, x')$ subject to
(i) $x' \geq x$; (ii) $(x, x') \in \Omega$.

It was shown in [23] that there exists a unique golden-rule stock

$$\widehat{x} = (1/(1 + da_\sigma))e(\sigma). \tag{2.7}$$

Set

$$\widehat{y} = \widehat{x}. \tag{2.8}$$

For $i = 1, \dots, n$ set

$$\widehat{q}_i = a_i b_i/(1 + da_i),\ \widehat{p}_i = w'(b\widehat{x})\widehat{q}_i. \tag{2.9}$$

It was shown in [23] that

$$w(b\widehat{x}) \geq w(by) + \widehat{p}x' - \widehat{p}x \tag{2.10}$$

for every $(x, x') \in \Omega$ and for every $y \in \Lambda(x, x')$.

A program $\{x(t), y(t)\}_{t=0}^\infty$ is good if there is a real number M such that

$$\sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) \geq M \text{ for every nonnegative integer } T.$$

A program $\{x(t), y(t)\}_{t=0}^\infty$ bad if

$$\lim_{T\to\infty} \sum_{t=0}^{T}(w(by(t)) - w(b\widehat{y})) = -\infty.$$

The following result was proved in [23].

Proposition 2.1. *Every program which is not good is bad.*

The following two results were obtained in [45]. They show that an asymptotic turnpike property holds for our infinite horizon problem.

Theorem 2.2. *Assume that the function w is strictly concave. Then for every good program $\{x(t), y(t)\}_{t=0}^\infty$,*

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Set

$$\xi_\sigma = 1 - d - (1/a_\sigma). \tag{2.11}$$

Theorem 2.3. *Assume that $\xi_\sigma \neq -1$. Then*

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x})$$

for every good program $\{x(t), y(t)\}_{t=0}^\infty$.

We use a notion of an overtaking optimal program introduced by Gale [15] and von Weizsacker [44]. This optimality criterion is used in optimal control [10, 47, 49].

A program $\{x^*(t), y^*(t)\}_{t=0}^\infty$ is overtaking optimal if

$$\limsup_{T\to\infty} \sum_{t=0}^{T}[w(by(t)) - w(by^*(t))] \leq 0$$

for every program $\{x(t), y(t)\}_{t=0}^{\infty}$ which satisfies $x(0) = x^*(0)$.

The following existence result was also obtained in [45].

Theorem 2.4. *Assume that for every good program $\{x(t), y(t)\}_{t=0}^{\infty}$,*

$$\lim_{t\to\infty}(x(t), y(t)) = (\widehat{x}, \widehat{x}).$$

Then for every point $x_0 \in R^n_+$ there is an overtaking optimal program $\{x(t), y(t)\}_{t=0}^{\infty}$ such that $x(0) = x_0$.

3. Turnpike results

The study of our infinite horizon problem is based on turnpike results which are presented in this section.

Let $z \in R^n_+$ and $T \geq 1$ be a natural number. Set

$$U(z, T) = \sup\{\sum_{t=0}^{T-1} w(by(t)) : \ (\{x(t)\}_{t=0}^{T}, \ \{y(t)\}_{t=0}^{T-1}) \text{ is a program such that } x(0) = z\}.$$

Note that $U(z, T)$ is a finite number [26].

Let $x_0, x_1 \in R^n_+$, T_1, T_2 be integers, $0 \leq T_1 < T_2$. Define

$$U(x_0, x_1, T_1, T_2) = \sup\{\sum_{t=T_1}^{T_2-1} w(by(t)) : \ (\{x(t)\}_{t=T_1}^{T_2}, \ \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = x_0, \ x(T_2) \geq x_1\}.$$

(Here we suppose that a supremum over empty set is $-\infty$.) Note that

$$U(x_0, x_1, T_1, T_2) < \infty$$

[26].

The next turnpike result was obtained in [26].

Theorem 3.1. *Assume that each good program $\{u(t), v(t)\}_{t=0}^{\infty}$ converges to the golden-rule stock $(\widehat{x}, \widehat{x})$:*

$$\lim_{t\to\infty}(u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let M, ϵ be positive numbers and $\Gamma \in (0, 1)$. Then there exist a natural number L and a positive number γ such that for each integer $T > 2L$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \leq Me$ and $az_1 \leq \Gamma d^{-1}$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0, \ x(T) \geq z_1, \quad \sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, z_1, 0, T) - \gamma$$

there are integers τ_1, τ_2 such that

$$\tau_1 \in [0, L], \ \tau_2 \in [T - L, T],$$

$$\|x(t) - \widehat{x}\|, \ \|y(t) - \widehat{x}\| \leq \epsilon \textit{ for all } t = \tau_1, \ldots, \tau_2 - 1.$$

Moreover if $\|x(0) - \widehat{x}\| \leq \gamma$ then $\tau_1 = 0$ and if $\|x(T) - \widehat{x}\| \leq \gamma$ then $\tau_2 = T$.

For every positive number M and every function $\phi : R^n_+ \to R^1$ define

$$\|\phi\|_M = \sup\{|\phi(z)| : z \in R^n \text{ and } 0 \le z \le Me\}.$$

Let integers T_1, T_2 satisfy $0 \le T_1 < T_2$, $w_i : R^n_+ \to R^1$, $i = T_1, \dots, T_2 - 1$ be bounded on bounded subsets of R^n_+ functions. For every pair of points $z_0, z_1 \in R^n_+$ define

$$U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1) = \sup\{\sum_{t=T_1}^{T_2-1} w_t(y(t)) :$$

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = z_0,\ x(T_2) \ge z_1\},$$

$$U(\{w_t\}_{t=T_1}^{T_2-1}, z_0) = \sup\{\sum_{t=T_1}^{T_2-1} w_t(y(t)) :$$

$$(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1}) \text{ is a program such that } x(T_1) = z_0\}.$$

(Here we assume that supremum over empty set is $-\infty$.) It is not difficult to see that the following result holds.

Lemma 3.2. *Let integers T_1, T_2 satisfy $0 \le T_1 < T_2$ and $w_i : R^n_+ \to R^1$, $i = T_1, \dots, T_2 - 1$ be bounded on bounded subsets of R^n_+ upper semicontinuous functions. Then the following assertions hold.*

1. For every point $z_0 \in R^n_+$ there exists a program $(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{T_1}^{T_2-1})$ such that

$$x(T_1) = z_0,\ \sum_{t=T_1}^{T_2-1} w_t(y(t)) = U(\{w_t\}_{t=T_1}^{T_2-1}, z_0).$$

2. For every pair of points $z_0, z_1 \in R^n_+$ such that $U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1)$ is finite there exists a program $(\{x(t)\}_{t=T_1}^{T_2}, \{y(t)\}_{t=T_1}^{T_2-1})$ such that $x(T_1) = z_0$, $x(T_2) \ge z_1$ and

$$\sum_{t=T_1}^{T_2-1} w_t(y(t)) = U(\{w_t\}_{t=T_1}^{T_2-1}, z_0, z_1).$$

Lemma 3.2 is deduced in a straightforward manner from the upper semicontinuous of the objective functions and the compactness of the set of trajectories on bounded intervals.

The following stability results were obtained in [48]. They show that the turnpike phenomenon is stable under small perturbations of the utility functions.

Theorem 3.3. *Suppose that for each good program $\{u(t), v(t)\}_{t=0}^{\infty}$,*

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : i = 1, \dots, n\}$, $\epsilon > 0$ and $\Gamma \in (0, 1)$. Then there exist a natural number L and a positive number γ such that for each integer $T > 2L$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le Me$ and $az_1 \le \Gamma d^{-1}$, each finite sequence of functions $w_i : R^n_+ \to R^1$, $i = 0, \dots, T - 1$ which are bounded on bounded subsets of R^n_+ and such that

$$\|w_i - w(b(\cdot))\|_M \le \gamma$$

for every integer $i \in \{0, \dots, T-1\}$ *and every program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *such that*

$$x(0) = z_0,\ x(T) \ge z_1,$$

$$\sum_{t=0}^{T-1} w_t(y(t)) \ge U(\{w_t\}_{t=0}^{T-1}, z_0, z_1) - \gamma$$

there exist integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1.$$

Moreover if $|x(0) - \widehat{x}\| \le \gamma$*, then* $\tau_1 = 0$ *and if* $\|x(T) - \widehat{x}\| \le \gamma$*, then* $\tau_2 = T$*.*

Theorem 3.4. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$*,*

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let $M > \max\{(a_i d)^{-1} : i = 1, \dots, n\}$ *and* $\epsilon > 0$*. Then there exist a natural number* L *and a positive number* γ *such that for each integer* $T > 2L$*, each* $z_0 \in R^n_+$ *satisfying* $z_0 \le Me$*, each finite sequence of functions* $w_i : R^n_+ \to R^1$*,* $i = 0, \dots, T-1$ *which are bounded on bounded subsets of* R^n_+ *and such that*

$$\|w_i - w(b(\cdot))\|_M \le \gamma$$

for each $i \in \{0, \dots, T-1\}$ *and each program* $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ *which satisfies*

$$x(0) = z_0,$$

$$\sum_{t=0}^{T-1} w_t(y(t)) \ge U(\{w_t\}_{t=0}^{T-1}, z_0) - \gamma$$

there are integers τ_1, τ_2 *such that*

$$\tau_1 \in [0, L],\ \tau_2 \in [T-L, T],$$

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \le \epsilon \text{ for all } t = \tau_1, \dots, \tau_2 - 1.$$

Moreover if $\|x(0) - \widehat{x}\| \le \gamma$ *then* $\tau_1 = 0$ *and if* $\|x(T) - \widehat{x}\| \le \gamma$ *then* $\tau_2 = T$*.*

4. Optimality criterions

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is called weakly optimal [10] if for each program

$$\{x(t), y(t)\}_{t=0}^{\infty}$$

satisfying $x(0) = x^*(0)$ the following inequality holds:

$$\liminf_{T\to\infty} \sum_{t=0}^{T} [w(by(t)) - w(by^*(t))] \le 0.$$

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is called weakly maximal [46] if for each integer $T > 0$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) = x^*(0),\ x(T) \ge x^*(T)$ the following inequality holds:

$$\sum_{t=0}^{T-1} [w(by(t)) - w(by^*(t))] \le 0.$$

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is called agreeable [18–20] if for all integers $t \geq 0$,

$$u(x^*(t), x^*(t+1)) = w(by^*(t))$$

and if for any natural number T_0 and any $\epsilon > 0$ there exists an integer $T_\epsilon > T_0$ such that for any integer $T > T_\epsilon$ and any program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ satisfying $x(0) = x^*(0)$ there exists a program $(\{x'(t)\}_{t=0}^{T}, \{y'(t)\}_{t=0}^{T-1})$ such that

$$x'(0) = x(0),\ x'(t) = x^*(t),\ t = 0, \ldots, T_0,$$

$$\sum_{t=0}^{T-1} w(by'(t)) \geq \sum_{t=0}^{T-1} w(by(t)) - \epsilon.$$

In [27] it was shown that the following properties hold:

(a) if $\{x(t), y(t)\}_{t=0}^{\infty}$ is a weakly maximal program and

$$\limsup_{t\to\infty} \|y(t)\| > 0$$

then $\{x(t), y(t)\}_{t=0}^{\infty}$ is a good program;

(b) every weakly optimal program is weakly maximal;

(c) every agreeable program is weakly maximal.

Proposition 4.1. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Then every agreeable program is good.

Proof. Assume that a program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is agreeable. We claim that it is good. Properties (a) and (c) imply that it is sufficient to show that

$$\limsup_{t\to\infty} \|y(t)\| > 0.$$

Let us prove that

$$\lim_{t\to\infty} y(t) = \widehat{x}.$$

Fix

$$M > \|x^*(0)\| + \max\{(a_i d)^{-1} :\ i = 1, \ldots, n\}. \tag{4.1}$$

Let $\epsilon > 0$. Theorem 3.4 implies that there exist a natural number L_1 and $\gamma > 0$ such that the following property holds:

(d) for each integer $T > 2L$, each $z_0 \in R^n_+$ satisfying $z_0 \leq Me$ and each program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0,$$

$$\sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, T) - \gamma$$

we have

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = L, \ldots, T - L - 1.$$

Let

$$S > 2L$$

be an integer. By the definition of an agreeable program, there exists an integer $T > S + L$ and a program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ such that

$$x(t) = x^*(t),\ t = 0, \dots, S + L,$$

$$\sum_{t=0}^{T-1} w(by(t)) \geq U(z_0, T) - \gamma.$$

It follows from property (d), (4.1) and the relations above that

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = L, \dots, T - L - 1$$

and

$$\|x(t) - \widehat{x}\|,\ \|y(t) - \widehat{x}\| \leq \epsilon \text{ for all } t = L, \dots, S.$$

Since ϵ is an arbitrary positive number and S is an arbitrary integer satisfying $S > 2L$ we conclude that

$$\lim_{t\to\infty} x(t) = \widehat{x},\ \lim_{t\to\infty} y(t) = \widehat{x}.$$

This completes the proof of Proposition 4.1. □

Let

$$M > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\}. \tag{4.2}$$

A program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ satisfying

$$x^*(0) \leq Me$$

is called strongly agreeable if for all integers $t \geq 0$,

$$u(x^*(t), x^*(t+1)) = w(by^*(t))$$

and if for every natural number T_0 and every positive number ϵ there exist $\delta > 0$ and an integer $T_\epsilon > T_0$ such that for each integer $T > T_\epsilon$ and each finite sequence of functions $w_i : R^n_+ \to R^1$, $i = 0, \dots, T-1$ which are bounded on bounded subsets of R^n_+ and such that

$$\|w_i - w(b(\cdot))\|_M \leq \delta$$

for each $i \in \{0, \dots, T-1\}$ there exists a program $(\{x(t)\}_{t=0}^{T}, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(t) = x^*(t),\ t = 0, \dots, T_0$$

and

$$\sum_{t=0}^{T-1} w_t(y(t)) \geq U(\{w_t\}_{t=0}^{T-1}, x(0)) - \epsilon.$$

The following theorem is our main result.

Theorem 4.2. *Suppose that for each good program* $\{u(t), v(t)\}_{t=0}^{\infty}$,

$$\lim_{t\to\infty} (u(t), v(t)) = (\widehat{x}, \widehat{x}).$$

Let

$$M > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\} \tag{4.2}$$

and $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *be a program satisfying*

$$x^*(0) \leq Me. \tag{4.3}$$

Then the following properties are equivalent:

(i) $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is strongly agreeable;*

(ii) $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is agreeable;*

(iii) $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is weakly maximal and good;*

(iv) $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is weakly maximal and satisfies*

$$\lim_{t\to\infty}(x^*(t), y^*(t)) = (\widehat{x}, \widehat{x}).$$

(v) $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ *is weakly maximal and satisfies*

$$\limsup_{t\to\infty} \|y^*(t))\| > 0.$$

5. Proof of Theorem 4.2

In the proof of Theorem 4.2 we use the following two auxiliary results.

Proposition 5.1 ([24]). *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $x, x' \in R^n_+$ satisfying*

$$\|x - \widehat{x}\|, \|x' - \widehat{x}\| \leq \delta$$

there exist $\bar{x} \geq x'$, $y \in R^n_+$ such that

$$(x, \bar{x}) \in \Omega,\ y \in \Lambda(x, \bar{x}),$$

$$\|y - \widehat{x}\| \leq \epsilon,\ \|\bar{x} - \widehat{x}\| \leq \epsilon.$$

Lemma 5.2 ([46]). *Assume that*

$$M_0 > \max\{(a_i d)^{-1} :\ i = 1, \dots, n\},$$

$(x, x') \in \Omega$ and that $x \leq M_0 e$. Then $x' \leq M_0 e$.

Proof of Theorem 4.2. Clearly, (i) implies (ii), (ii) implies (iii), (iii) implies (iv), (iv) implies (v) and (v) implies (iii). In order to complete the proof of the theorem it is sufficient to show that (iii) implies (i).

Assume that $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is weakly maximal and good. Then

$$\lim_{t\to\infty} x^*(t) = \lim_{t\to\infty} y^*(t) = \widehat{x}. \tag{5.1}$$

Let $T_0 \geq 1$ be an integer and $\epsilon \in (0, 1)$. Since $\widehat{x}$ is the golden-rule stock there exists

$$\delta_0 \in (0, \epsilon/8)$$

such that the following property holds:

(a) for each $(x, x') \in \Omega$ satisfying

$$\|x - \widehat{x}\|,\ \|x' - \widehat{x}\| \leq 2\delta_0$$

and each $y \in \Lambda(x, x')$ we have

$$w(by) \leq w(b\widehat{x}) + \epsilon/8.$$

Proposition 5.1 implies that there exists

$$\delta_1 \in (0, \delta_0)$$

such that the following property holds:

(b) for each $x, x' \in R^n_+$ satisfying

$$\|x - \widehat{x}\|, \|x' - \widehat{x}\| \leq \delta_1$$

there exist $\bar{x} \geq x'$, $y \in R^n_+$ such that

$$(x, \bar{x}) \in \Omega, \; y \in \Lambda(x, \bar{x}),$$
$$\|y - \widehat{x}\| \leq \delta_0, \; \|\bar{x} - \widehat{x}\| \leq \delta_0,$$
$$|w(by) - w(b\widehat{x})| \leq \delta_0/8.$$

By Theorem 3.4, there exist a natural number L_0 and a positive number

$$\delta_2 \in (0, \delta_1)$$

such that the following property holds:

(c) for each integer $T > 2L_0$, each $z_0 \in R^n_+$ satisfying $z_0 \leq Me$, each finite sequence of functions $w_i : R^n_+ \to R^1$, $i = 0, \dots, T-1$ which are bounded on bounded subsets of R^n_+ and such that

$$\|w_i - w(b(\cdot))\|_M \leq \delta_2$$

for each $i \in \{0, \dots, T-1\}$ and each program $(\{x(t)\}_{t=0}^T, \{y(t)\}_{t=0}^{T-1})$ which satisfies

$$x(0) = z_0,$$

$$\sum_{t=0}^{T-1} w_t(y(t)) \geq U(\{w_t\}_{t=0}^{T-1}, z_0) - 2\delta_2$$

we have

$$\|x(t) - \widehat{x}\| \leq \delta_1 \text{ for all } t = L_0, \dots, T - L_0 - 1.$$

In view of (5.1), there exists an integer $L_1 \geq 1$ such that

(5.2) $$\|x^*(t) - \widehat{x}\| \leq \delta_1 \text{ for all integers } t \geq L_1.$$

Set

(5.3) $$T_\epsilon = 2(L_0 + L_1 + T_0 + 4).$$

Choose a positive number $\delta < \delta_2$ such that

(5.4) $$\delta(L_0 + L_1 + T_0 + 4) < \epsilon/64.$$

Assume that an integer $T > T_\epsilon$, functions $w_i : R^n_+ \to R^1$, $i = 0, \dots, T-1$ are bounded on bounded subsets of R^n_+,

(5.5) $$\|w_i - w(b(\cdot))\|_M \leq \delta$$

for each $i \in \{0, \dots, T-1\}$ and that a program $(\{x(t)\}_{t=0}^T, \{y(t)\}_{t=0}^{T-1})$ satisfies

(5.6) $$x(0) = x^*(0),$$

and

(5.7). $$\sum_{t=0}^{T-1} w_t(y(t)) \geq U(\{w_t\}_{t=0}^{T-1}, x^*(0)) - \delta/2.$$

Lemma 5.2, (4.2), (4.3) and (5.6) imply that

(5.8) $$x(t) \leq Me, \; t = 0, \dots, T, \; y(t) \leq Me, \; t = 0, \dots, T-1,$$

(5.9) $$y^*(t) \leq x^*(t) \leq Me, \; t = 0, 1, \dots .$$

Property (c), (4.3), (5.3), (5.4) and (5.6) imply that

$$\|x(t) - \widehat{x}\| \le \delta_1 \text{ for all } t = L_0, \dots, T - L_0 - 1. \tag{5.10}$$

In view of (5.3) and (5.10),

$$\|x(L_0 + L_1 + T_0 + i) - \widehat{x}\| \le \delta_1,\ i = 0, 1, 2, 3, 4. \tag{5.11}$$

By (5.2),

$$\|x^*(L_0 + L_1 + T_0 + i) - \widehat{x}\| \le \delta_1,\ i = 0, 1, 2, 3, 4. \tag{5.12}$$

Property (a), (5.11) and (5.12) imply that

$$w(by(L_0 + L_1 + T_0 + i)) \le w(b\widehat{x}) + \epsilon/8,\ i = 0, 1, 2, 3, \tag{5.13}$$

$$w(by^*(L_0 + L_1 + T_0 + i)) \le w(b\widehat{x}) + \epsilon/8,\ i = 0, 1, 2, 3. \tag{5.14}$$

It follows from (5.11), (5.12) and property (b) that there exist

$$\bar{x}(L_0 + L_1 + T_0 + 1) \in R^n_+,\ \bar{y}(L_0 + L_1 + T_0) \in R^n_+$$

such that

$$\bar{x}(L_0 + L_1 + T_0 + 1) \ge x^*(L_0 + L_1 + T_0 + 1), \tag{5.15}$$

$$(x(L_0 + L_1 + T_0), \bar{x}(L_0 + L_1 + T_0 + 1)) \in \Omega, \tag{5.16}$$

$$\bar{y}(L_0 + L_1 + T_0) \in \Lambda(x(L_0 + L_1 + T_0), \bar{x}(L_0 + L_1 + T_0 + 1)), \tag{5.17}$$

$$\|\bar{x}(L_0 + L_1 + T_0 + 1) - \widehat{x}\| \le \delta_0, \tag{5.18}$$

$$\|\bar{y}(L_0 + L_1 + T_0) - \widehat{x}\| \le \delta_0, \tag{5.19}$$

$$|w(b\bar{y}(L_0 + L_1 + T_0)) - w(b\widehat{x})| \le \delta_0/8. \tag{5.20}$$

Set

$$\bar{x}(t) = x(t),\ t = 0, \dots, L_0 + L_1 + T_0,\ \bar{y}(t) = y(t),\ t = 0, \dots, L_0 + L_1 + T_0 - 1. \tag{5.21}$$

By (5.16), (5.17) and (5.21), $(\{\bar{x}(t)\}_{t=0}^{L_0+L_1+T_0+1}, \{\bar{y}(t)\}_{t=0}^{L_0+L_1+T_0})$ is a program. In view of (5.6), (5.15) and (5.21),

$$\bar{x}(0) = x^*(0),\ \bar{x}(L_0 + L_1 + T_0 + 1) \ge x^*(L_0 + L_1 + T_0 + 1). \tag{5.22}$$

Since the program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is weakly maximal it follows from (5.22) that

$$\sum_{t=0}^{L_0+L_1+T_0} w(by^*(t)) \ge \sum_{t=0}^{L_0+L_1+T_0} w(b\bar{y}(t)). \tag{5.23}$$

It follows from (5.14), (5.20), (5.21) and (5.23) that

$$\begin{aligned}\sum_{t=0}^{L_0+L_1+T_0-1} w(by^*(t)) &\geq \sum_{t=0}^{L_0+L_1+T_0} w(b\bar{y}(t)) - w(by^*(L_0+L_1+T_0))\\ &\geq \sum_{t=0}^{L_0+L_1+T_0-1} w(by(t)) + w(b\widehat{x}) - \delta_0/8 - w(b\widehat{x}) - \epsilon/8\\ &= \sum_{t=0}^{L_0+L_1+T_0-1} w(by(t)) - \delta_0/8 - \epsilon/8.\end{aligned} \tag{5.24}$$

It follows from (5.11), (5.12) and property (b) that there exist

$$\tilde{x}(L_0+L_1+T_0+1) \in R^n_+,\ \tilde{y}(L_0+L_1+T_0) \in R^n_+$$

such that

$$\tilde{x}(L_0+L_1+T_0+1) \geq x(L_0+L_1+T_0+1), \tag{5.25}$$

$$(x^*(L_0+L_1+T_0), \tilde{x}(L_0+L_1+T_0+1)) \in \Omega, \tag{5.26}$$

$$\tilde{y}(L_0+L_1+T_0) \in \Lambda(x^*(L_0+L_1+T_0), \tilde{x}(L_0+L_1+T_0+1)), \tag{5.27}$$

$$\|\tilde{x}(L_0+L_1+T_0+1) - \widehat{x}\| \leq \delta_0,\ \|\tilde{y}(L_0+L_1+T_0) - \widehat{x}\| \leq \delta_0, \tag{5.28}$$

$$|w(b\tilde{y}(L_0+L_1+T_0)) - w(b\widehat{x})| \leq \delta_0/8. \tag{5.29}$$

Set

$$\tilde{x}(t) = x^*(t),\ t = 0,\dots,L_0+L_1+T_0,\ \tilde{y}(t) = y^*(t),\ t = 0,\dots,L_0+L_1+T_0-1. \tag{5.30}$$

By (5.26), (5.27) and (5.30), $(\{\tilde{x}(t)\}_{t=0}^{L_0+L_1+T_0+1}, \{\tilde{y}(t)\}_{t=0}^{L_0+L_1+T_0})$ is a program.

For all integers $t = L_0+L_1+T_0+1,\dots,T-1$ set

$$\tilde{y}(t) = y(t), \tag{5.31}$$

$$\tilde{x}(t+1) = (1-d)\tilde{x}(t) + x(t+1) - (1-d)x(t). \tag{5.32}$$

By (5.25) and (5.32),

$$\tilde{x}(t) \geq x(t),\ t = L_0+L_1+T_0+1,\dots,T. \tag{5.33}$$

In view of (5.31)-(5.33), $(\{\tilde{x}(t)\}_{t=0}^{T}, \{\tilde{y}(t)\}_{t=0}^{T-1})$ is a program. It follows from (5.4), (5.5) and (5.31) that

$$\begin{aligned}\sum_{t=0}^{T-1} w_t(\tilde{y}(t)) - \sum_{t=0}^{T-1} w_t(y(t)) &\geq \sum_{t=0}^{L_0+L_1+T_0} w_t(\tilde{y}(t)) - \sum_{t=0}^{L_0+L_1+T_0} w_t(y(t))\\ &\geq \sum_{t=0}^{L_0+L_1+T_0} w(b\tilde{y}(t)) - \sum_{t=0}^{L_0+L_1+T_0} w(by(t))\\ &\quad - 2\delta(L_0+L_1+T_0+2)\\ &\geq \sum_{t=0}^{L_0+L_1+T_0} w(b\tilde{y}(t)) - \sum_{t=0}^{L_0+L_1+T_0} w(by(t)) - \epsilon/32.\end{aligned} \tag{5.34}$$

By (5.6), (5.7), (5.13), (5.24), (5.29), (5.30) and (5.34),

$$
\begin{aligned}
\sum_{t=0}^{T-1} w_t(\tilde{y}(t)) - \sum_{t=0}^{T-1} w_t(y(t)) &\geq \sum_{t=0}^{L_0+L_1+T_0} w(b\tilde{y}(t)) - \sum_{t=0}^{L_0+L_1+T_0} w(by(t)) - \epsilon/32 \\
&\geq \sum_{t=0}^{L_0+L_1+T_0-1} w(by^*(t)) + w(b\widehat{x}) - \delta_0/8 \\
&\quad - \sum_{t=0}^{L_0+L_1+T_0-1} w(by(t)) - w(b\widehat{x}) - \epsilon/8 - \epsilon/32 \\
&\geq -\delta_0/8 - \epsilon/8 - \epsilon/8 - \delta_0/8 - \epsilon/32 > -\epsilon/2.
\end{aligned} \tag{5.35}
$$

In view of (5.7) and (5.35),

$$
\sum_{t=0}^{T-1} w_t(\tilde{y}(t)) \geq -\epsilon/2 + \sum_{t=0}^{T-1} w_t(y(t)) \geq U(\{w_t\}_{t=0}^{T-1}, x^*(0)) - \epsilon.
$$

Thus the program $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ is strongly agreeable and property (iii) holds. Therefore (iii) implies (i) and this completes the proof of Theorem 4.2. □

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