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TRANSFERABLE-UTILITY β-CORE OF DISCONTINUOUS GAMES

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Abstract. In this paper we study the nonemptiness of the transferable-utility β-core of strategic-form games with possibly discontinuous payoff functions. Inspired by the pioneering work of Reny (1999), we handle the discontinuities by introducing the notion of *reaction security*. We show that every concave, bounded, compact, strongly separable and reaction-secure game has a nonempty transferable-utility β-core. In the special two-player setting, we show that certain assumptions are redundant and also provide a result on the nonemptiness of the β-core without the transferable utility assumption. We present applications of our results to oligopoly markets.

1. Introduction

The *core* is the standard cooperative solution concept for economic models involving the cooperative behavior of the agents, and is dated back to Edgeworth's work in the 19^{th}-century. An allocation, or a profile of actions, is in the core if it is not blocked by any coalition, that is, no group of players can make each of its members better off by acting for themselves. The core is extensively used in both exchange and production economies, and it is the main solution concept in the field of cooperative game theory.

It is common to model the economic problems involving cooperative behavior by a game in characteristic function from where each coalition's set of attainable payoffs is given or can be easily defined. However, when externalities are present, i.e., an individuals or group's action affects the well being of the others, then the characteristic function form is not directly applicable. For such problems, it is suitable to model the problem as a strategic-form game in which the action set of each individual is given and the payoffs are determined by the actions of the individuals, and then define a core concept for this game. Even though the core concept of this

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game can be defined as usual: a profile of actions is in the core of this game if it is not blocked by any coalition, the very definition of blocking has to take the actions of the complementary coalition into account due to the presence of externalities. Aumann [6, 7] introduces two core concepts for strategic form games: α-core and β-core. A pair of an action profile and a payoff profile is in the α-core of a game if the grand coalition's aggregate payoff from the action profile is equivalent to the aggregate payoff profile, and no coalition has an alternative action which makes all of its members better off, independently of the actions of the other players. The α-core is a "pessimistic" solution concept in terms of the perception of the members of the blocking coalition's capability to react the outsiders' actions. The β-core, on the other hand, is a stronger solution concept such that the blocking coalition is permitted to counteract to each action of the complementary coalition so as to achieve a higher aggregate payoff. It is as if a blocking coalition announces its intention to block, forces the complementary coalition to move first, and then responds, rather than the reverse order of moves. Hence, the members of the blocking coalition has an "optimistic" perspective in terms of their reaction capability to outsiders' actions.

Scarf [34], by using the methods he developed in his 1967 paper on the core of a characteristic function-form game, provides a result on the nonemptiness of the α-core of a strategic-form game. Until the work of Zhao [45], the β-core existence result has been absent.[1] Zhao [45] provides, under the transferable utility assumption, a result on the nonemptiness of the β-core of a game with continuous payoff functions. A transferable-utility β-core existence result has direct relevance to many economic problems in which agents have incentive to cooperate and transfers are allowed. For example, in oligopoly markets, acting cooperatively yields firms higher payoffs which can then be redistributed among them. In these markets, cooperation with side payments can be interpreted as overt collusion. However, this profitable merger will not take place unless firms could split the monopoly profits without any objection. An element of a β-core describes an allocation of the profits in a monopoly merger. Therefore, the nonemptiness of β-core provides a necessary condition for monopoly merger. In other words, the monopoly merger can only take place if the original market has a nonempty core.

Many economic problems are suitably modeled by games with discontinuous payoff functions. The seminal works of Dasgupta and Maskin [15], Simon-Zame [36]

[1]In his paper, Scarf provides a counterexmaple to the nonemptiness of the β-core. There are existence results for stronger solution concepts such as *strong Nash equilibrium* in the literature, see Ichiishi [19, Chapter 2] for a detailed discussion and [32] for a recent work. Zhao's work is the first paper which provides a direct existence result when the utilities are transferable. See also [44, 46] and [27] on a discussion of different cooperative solution concepts for strategic-form games.

and Reny [28] initiate a rich and evolving literature on the existence of an equilibrium in games with possibly discontinuous payoffs.[2] The purpose of this paper is to generalize Zhao's [1999b] result to games with possibly discontinuous payoff functions. In line with the recent literature on discontinuous games, we define a class of games with possibly discontinuous payoffs, which we call *reaction-secure* games, and provide sufficcient conditions such that these games have a nonempty transferable-utility β-core.[3]

The paper is organized as follows. Section 2 defines the basic concepts and notations, Section 3 presents two theorems on the nonemptiness of the β-core of discontinuous games of general n-player games with transferable utilities, and also it connects these results to hybrid solutions, and Section 4 concerns a special case of two-player games and it presents four propositions complementing the two theorems by showing that this special setup allows us to weaken (even drop some of) the convexity and separability assumptions, and a proposition on the nonemptiness of the nontransferable utility β-core of discontinuous games. Section 5 illustrates the applications of our results to Bertrand duopoly and Cournot oligopoly games, and Section 6 concludes.

2. Notational and conceptual preliminaries

A (*strategic form*) *game* is a list $G = (X_i, u_i)_{i\in N}$ where $N = \{1, \dots, n\}$ is the finite collection of players, X_i is a nonempty set of actions for player i and $u_i : X \to \mathbb{R}$ represents the payoff function of player i defined on the set of action profiles $X = \prod_{i\in N} X_i$. Let $G = (X_i, u_i)_{i\in N}$ be a game such that X_i is a nonempty subset of a finite dimensional Euclidean space.[4] for each player i. Then G is said to be

(i) *compact* if X_i is compact for each $i \in N$,

(ii) *concave* if X_i is convex and u_i is concave for each $i \in N$,

(iii) *bounded* if u_i is bounded for each $i \in N$.

Let $G = (X_i, u_i)_{i\in N}$ be a game. A *coalition* is an element S in $\mathcal{N} = 2^N \backslash \emptyset$. Let $\mathring{\mathcal{N}}$ denote the set of all coalitions excluding the grand coalition. The set of actions available to a coalition S is denoted as $X_S = \prod_{i\in S} X_i$, and the vector of utility functions of coalition S as $u_S = (u_i)_{i\in S}$.[5] For each coalition S, let $-S = N\backslash S$ denote the

[2]See [12] and [29] for two symposia, and [31] for a survey on the recent developments in the discontinuous games literature.

[3]Recently, an existence result for the α-core of a game with possibly discontinuous payoff functions has been provided by [38], with or without transferable utilities. Since the TU β-core is contained in the TU α-core, this paper also provides an existence result for the TU α-core.

[4]The results presented in this paper can easily be generalized to arbitrary topological vector spaces.

[5]We drop the subscript N for the grand coalition, and when it is clear from the context, we use i instead of $\{i\}$.

complementary coalition. An *imputation* is a payoff vector $v \in \mathbb{R}^n$. An imputation is *x-feasible* if $\sum_{i\in N} u_i(x) = \sum_{i\in N} v_i$. A coalition $S \in \mathring{\mathcal{N}}$ *blocks* an imputation $v \in \mathbb{R}^n$ if there exists $f : X_{-S} \to X_S$ such that $\sum_{i\in S} u_i\left(f_S(z_{-S}), z_{-S}\right) > \sum_{i\in S} v_i$ and the the grand coalition *blocks* v if there exists $x \in X$ such that $\sum_{i\in N} u_i\left(x\right) > \sum_{i\in S} v_i$.

Definition 2.1. A pair of an action profile and imputation $(x^*, v^*) \in X \times \mathbb{R}^n$ is in the *β-core*[6] of a game $G = (X_i, u_i)_{i\in N}$ if v^* is x^*-feasible and is not blocked by any coalition.

Note that a pair of an action profile and an imputation (x^*, v^*) is in the β-core of the game if x^* maximizes the payoff of the grand coalition, v^* is the maximum payoff of the grand coalition and no coalition has a reaction function which yields itself a strictly higher payoff depending on the complementary coalition's action. An action profile x is a *Nash equilibrium* of G if no player has a strategy which gives her strictly higher payoff than x assuming that the others do not change their actions. The (transferable utility) β-core differs from the Nash equilibrium in three aspects. First, the coalitions are allowed to act together. Second, coalition members can transfer their payoff among each other. Third, the complementary coalitions are allowed to punish the deviants while the deviants are allowed to react to the actions of the complementary coalitions. As we note in the introduction, another closely related solution concept is the *α-core*; the main difference is that it does not allow the blocking coalition to counteract the complementary coalition's action. By definition, for a game, its β-core is contained in its α-core, for both transferable and nontransferable utilities; see [7], [34] and [45] for a detailed comparison of the two core concepts.

Next, we introduce two weak continuity concepts, inspired by the pioneering work of Reny [28], which imposes topological assumptions on the game itself. For a bounded game G, let

$$\mathcal{X} = \left\{ (x, v) \in X \times \mathbb{R}^n \middle| \sum_{i\in N} v_i = \sum_{i\in N} u_i(x),\ \inf_{z\in X} u_i(z) \le v_i\ \forall i \in N \right\} \tag{2.1}$$

denote the set of all pairs of action profiles x and x-feasible imputations v such that the imputations are bounded below by players' lowest possible payoffs. It is clear that $\mathcal{X}$ contains the β-core of G. Let $\bar{\mathcal{X}}$ denote the (topological) closure of $\mathcal{X}$. Note that if the game is compact and bounded, then $\bar{\mathcal{X}}$ is compact.

[6]In this definition, and in the definitions of an imputation, feasibility and blocking concepts above, we assume the utilities are "transferable." Since all of our results but one (Proposition 4.8) assume transferable utilities, we abbreviated the notation and do not explicitly refer to transferable utility in the definitions for the convenience of the reader. Moreover, when we refer to the core of nontransferable utility games, we explicitly mention that the utilities are nontransferable; see Definitions 4.6 and 4.7 and Proposition 4.8.

Definition 2.2. A bounded, compact game G is *reaction-secure* (RS) if for each $(x,v)\in\bar{\mathcal{X}}$ that is not in the β-core of G, there exist an open neighborhood $U^{x,v}$ of (x,v), $y_N^{x,v}\in X$, $w_N^{x,v}\in\mathbb{R}$, and for each $S\in\mathring{\mathcal{N}}$ there exist $w_S^{x,v}\in\mathbb{R}$ and a function $\delta_S^{x,v}:X_{-S}\to X_S$ such that

(i) $\sum_{i\in N}u_i(y_N^{x,v})\geq w_N^{x,v}$

(ii) for all $S\in\mathring{\mathcal{N}}$ and for all $z_{-S}\in X_{-S}$, $\sum_{i\in S}u_i(\delta_S^{x,v}(z_{-S}),z_{-S})\geq w_S^{x,v}$,

(iii) for each $(x',v')\in U^{x,v}$ there exists $S\in\mathcal{N}$ such that $\sum_{i\in S}v'_i<w_S^{x,v}$.

Remark 2.3. If a pair of action profile and imputation (x,v) is not in the β-core of G, then by definition at least one coalition blocks v. The reaction-security (RS) imposes the following structure on the blockings: (a) an open neighborhood of (x,v) does not contain a point in the β-core of G, i.e. the payoff at each point in the neighborhood is blocked by at least one coalition, (b) the identity of the blocking coalition is allowed to vary, but each coalition's reaction function must remain unchanged. This type of security notion is inspired by the pioneering work of Reny [28]. His notion of better-reply-security assumes that every action profile which is not a Nash equilibrium of G (a) has a neighborhood which does not contain a Nash equilibrium, i.e. at each point in the neighborhood at least one player deviates at it, (b) the identity of the deviant is allowed to vary, but his deviation strategy must remain same in response to the remaining players' tremble on the neighborhood. A weakening of Reny's better-reply-security notion is provided by Barelli and Meneghel [8] and Reny [30] that allows each player has multiple deviation strategies which change upper semicontinuously in response to the remaining players' tremble on the neighborhood.[7] In particular, they define the following weak continuity concept.

> A bounded, compact game G is *correspondence-secure* (CS) if for each $x\in X$ that is not a Nash equilibrium of G, there exist an open neighborhood U^x of x, and for each player $i\in N$, $v_i^x\in\mathbb{R}$ and an upper semicontinuous, u.s.c. hereafter,[8] correspondence $\psi_i^x:U^x\twoheadrightarrow X_i$ with compact and nonempty values[9] such that
> (i) $u_i(z'_i,z_{-i})\geq v_i^x$ for each $i\in N$, $z\in U^x$ and $z'_i\in\psi_i^x(z)$,
> (ii) for each $x'\in U^x$ there exists $i\in N$ such that $u_i(x')<v_i^x$.

[7] See also [26], [39], [18], [13], [25], [2, 3], [21], [40] and [24] on the continuity postulate in economic theory, and generalizations and applications of Reny's better-reply-security notion.

[8] A correspondence with compact values whose range is a compact Hausdorff space is u.s.c. if and only if its graph is closed in the product topology; see Aliprantis-Border [1, Lemma 17.11, p. 561].

[9] Reny [30] assumes ψ_i^x also has convex values, but for finite dimensional spaces we can drop this assumption.

There are some similarities between the concepts of reaction-security and correspondence-security. Both assume that if a point is not an equilibrium, then a neighborhood of it does contain an equilibrium. Moreover, both put some structure on the deviation/blocking, the players/coalitions should deviate "nicely". Beyond these conceptual similarities, the two concepts are quite different. First, in the RS, the reaction functions are structure free, i.e. they do not have any continuity or convexity property. Second, the trembling of the others is replaced by all actions of the complementary coalition. Although this seems a strong assumption, it is consistent with the definition of blocking and Reny's insight. The CS assumes a structure on *deviations,* and the RS on *blockings*, and the definition of blocking, in contrast to deviation, already incorporates *all* actions of the complementary coalition. These arguments imply that there is no inclusion relation between the RS and CS concepts – Example 1 below illustrates this claim.

Note that if a pair of action profile and imputation (x^*, v^*) is in the β-core, then the following three conditions are satisfied: (i) $x^* \in \underset{x\in X}{\operatorname{argmax}} \sum_{i\in N} u_i(x)$, (ii) $\sum_{i\in N} v_i^* = \sum_{i\in N} u_i(x^*)$, and (iii) v^* is not blocked by any coalition. In order to guarantee a solution to the maximization problem defined in (i), we define the RS on $\bar{\mathcal{X}}$, not on $\mathcal{X}$.[10] Observe the maximization problem in part (i) is independent of the no blocking condition in part (iii), hence it is possible to separately analyze these two parts and obtain a different and simpler continuity concept as follows.

Let G be a compact and bounded game. If the aggregate payoff function $\bar{u}(x) = \sum_{i\in N} u_i(x)$ is upper semicontinuous, then $\bar{u}$ has a maximizer.[11] Fix a maximizer $\bar{x}$ of $\bar{u}$. Then, focusing only on the redistributions of the grand coalition's maximum aggregate payoff, which are called *imputations*, is enough to show the nonemptiness of the β-core. In particular, let

$$\mathcal{V} = \left\{ v \in \mathbb{R}^n | \sum_{i\in N} v_i = \sum_{i\in N} u_i(\bar{x}),\ \inf_{x\in X} u_i(x) \le v_i \ \forall i \in N \right\} \tag{2.2}$$

denote the set of bounded below and $\bar{x}$-feasible imputations. It is clear that the β-core of G is nonempty if and only if $\mathcal{V}$ contains an imputation which is not blocked by any coalition. Note that since G is bounded, $\mathcal{V}$ is compact.

Definition 2.4. A bounded, compact game $G = (X_i, u_i)_{i\in N}$ is RS_N if the aggregate payoff function $\bar{u}$ is upper semicontinuous and for each $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not

[10] This property of RS is similar to the original better-reply-security notion of [28] which works on the closure of the graph of the game.

[11] One can also impose weaker assumptions such transfer continuity of [37], or continuous neighborhood selection of [39], or continuous inlcusion property of [18], to guarantee that $\bar{u}$ has a maximizer.

in the β-core of G, there exist an open neighborhood U^v of v, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^v \in \mathbb{R}$ and a function $\delta_S^v : X_{-S} \to X_S$ such that

(i) $\sum_{i\in S} u_i(\delta_S^v(z_{-S}), z_{-S}) \geq w_S^v$ for each $S \in \mathring{\mathcal{N}}$ and $z_{-S} \in X_{-S}$,

(ii) for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i\in S} v_i' < w_S^v$.

Remark 2.5. Note that while the RS_N property focuses only on $\{\bar{x}\} \times \mathcal{V} \subset \bar{\mathcal{X}}$ and does not take the points in the closure of $\mathcal{X}$ into account, it explicitly imposes assumptions on the grand coalition's blocking behavior which is not imposed by the RS. It is easy to see that if $\bar{u}$ is upper semicontinuous, then every RS game is RS_N. However, the RS_N property does not imply RS – Example 1 below illustrates this claim.[12]

Let G be a bounded RS game. For each $S \in \mathring{\mathcal{N}}$ and $(x, v) \in \bar{\mathcal{X}}$ that is not in the β-core of G define the set of all securing reaction functions as

$$\Delta_S(x, v) = \{\delta_S^{x',v'} : X_{-S} \to X_S \mid (x', v') \in \bar{\mathcal{X}} \text{ is not in the } \beta\text{-core of } G \text{ and } w_S^{x',v'} \geq w_S^{x,v}\},$$

where the functions $\delta_S^{x',v'}$ and the values $w_S^{x,v}, w_S^{x',v'}$ are as defined in Definition 2.2. Now let G be a bounded RS_N game. For each $S \in \mathring{\mathcal{N}}$ and $v \in \mathcal{V}$ such that $(\bar{x}, v)$ is not in the β-core of G define the set of all securing reaction functions as

$$\Delta_S^N(v) = \{\delta_S^{v'} : X_{-S} \to X_S \mid v' \in \mathcal{V}, (\bar{x}, v') \text{ is not in the } \beta\text{-core of } G \text{ and } w_S^{v'} \geq w_S^v\}$$

where the functions $\delta_S^{v'}$ and the values $w_S^v, w_S^{v'}$ are as defined in Definition 2.4.

Definition 2.6. A bounded, compact RS game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ there exists $(x, v) \in \bar{\mathcal{X}}$ such that for each $\delta_S \in \Delta_S(x, v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i\in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S})$$

Definition 2.7. A bounded, compact RS_N game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ there exists $v \in \mathcal{V}$ such that for each $\delta_S \in \Delta_S^N(v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i\in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

[12] The following simple one player game may also be helpful in illustrating this claim. $X = [0, 1]$ and $u(0) = 1$, $u(1) = 0$, and $u(x) = x$ for all $x \in (0, 1)$. This 1-player game is not RS since 1 is not a maximizer of u and $(x, v) = (1, 1) \in \bar{\mathcal{X}}$. However, it is clear that the game is RS_N.

Remark 2.8. Note that the strong separability definition we provide here closely related to the following definition provided in Zhao [45, Definition 3, p.157]: "A game G is *strongly separable* if for each $S \in \mathring{\mathcal{N}}$ and each $i \in S$,

$$u_i(x^*(\hat{z}_{-S}), \hat{z}_{-S}) = \min_{z_{-S} \in X_{-S}} u_i(x^*(\hat{z}_{-S}), z_{-S}),$$

where for each $z_{-S} \in X_{-S}$, $x^*_S(z_{-S})$ solves $\max_{x_S \in X_S} \sum_{i \in S} u_i(x_S, z_{-S})$, and for given $x^*_S(\cdot)$, $\hat{z}_{-S}$ solves $\min_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(x^*(z_{-S}), z_{-S})$." While Zhao's definition requires the equality to hold for a specific reaction function, our definitions requires the equality to hold for a set of reaction functions, hence his definition imposes a weaker condition. On the other hand, while his definition requires the equality to hold for all members of a given coalition, our definitions impose the equality restriction on the aggregate utility of a given coalition, and also we do not require $\tilde{z}_{-S} = \hat{z}_{-S}$, hence our definition imposes weaker conditions. Having said this, if utilities are continuous, we can revise the sets of the securing reaction functions such that our definitions are weaker than Zhao's definition as follows. Assume the payoff functions are continuous. It is easy to see that we can set $\Delta_S(x, v) = \Delta^N_S(v') = \{x^*_S(\cdot)\}$ for all $S \in \mathring{\mathcal{N}}$ and $(x, v) \in \bar{\mathcal{X}}$ and $v' \in \mathcal{V}$ since RS and RS_N require only the existence of a securing reaction function which can be selected as the "best" reaction function without loss of generality.

3. Results: general n-player games

Our first result generalizes the main result of [45] by weakening the continuity assumption on the payoff functions.

Theorem 3.1. Every concave, bounded, compact, strongly separable and RS game has a nonempty β-core.

Before presenting the proof, we present three remarks: the first is on the structure of the discontinuity of a concave function, the second compares Theorem 3.1 to the relevant results in the antecedent literature and the third is on the method-of-proof.

Remark 3.2. It is well known that every real-valued concave function on a Euclidean space is continuous at each point of its domain's relative interior. Hence, discontinuities can occur only at the relative boundary of the domain. And, it is easy to define a concave function that is discontinuous at every point of the relative boundary of its domain. Ernst [16], in his generalizatino of [17], provides a nice characterization of the continuity properties of a concave function on the relative boundary of its domain. Before stating this result, we shall introduce some concepts. A subset X of $\mathbb{R}^m$ is called a *polytope* provided that it is the convex hull of a finite set of points And X is said to be *boundedly polyhedral* provided that its intersection with any polytope is a polytope. It is clear that any compact boundedly polyhedron is a polytope. Ernst [16, Theorem 2.4, p.3672] states that *given a convex and compact subset X of $\mathbb{R}^m$, every concave function on X is lower semicontinuous if and only if X is a polytope.* Since any convex and compact subset X of $\mathbb{R}$ is a polytope, Ernst's theorem implies that any concave function on X is lower semicontinuous. However, for $\mathbb{R}^m$, $m \geq 2$, this result is not true. Given a convex and compact subset X of $\mathbb{R}^m$ which is not polytope (such as unit ball), it is always possible to find a concave function on X which is not lower semicontinuous. Hence, our setup which eliminates *max* and replaces *min* with *inf* is crucial for games with possibly discontinuous payoff functions. And as Carter [14, p.334] states "This is not a mere curiosity. Economic life often takes place at the boundaries of convex sets, where the possibility of discontinuities must be taken into account."

Remark 3.3. Scarf [34, Theorem] provides a result on the nonemptiness of the α-core of continuous games with nontransferable utilities. Zhao [44, Theorem 1] and Zhao [45, Theorem 1] prove the nonemptiness of the α-core and the β-core of continuous games with transferable utilities, respectively. Uyanik [38, Theorems 1, 2, 3] generalizes Scarf's and Zhao's results on the nonemptiness of the α-core to games with possibly discontinuous preferences, with or without transferable utilities. The relationship between our results on the nonemptiness of the β-core with transferable utilities, Theorem 3.1 above and Theorem 3.5 below, and Theorems 2 and 3 in Uyanik [38] on the nonemptiness of the α-core with transferable utilities are similar to that of Theorem 1 in Zhao [44] and Theorem 1 in Zhao [45], with the exception that we allow discontinuous preferences and the nature of the discontinuities can be quite different in our results and in the results presented in the author's earlier work.

Remark 3.4. We now describe our method-of-proof and compare it to those used in Zhao [45] and Uyanik [38]. Zhao assumes the payoff functions are continuous and the action sets are compact, therefore he can work with "best-responses." In our setup, his approach does not work since the best-responses may not exist due to the discontinuities in the payoffs. Instead, we use the approach introduced by Reny [28] and assume the game has an empty β-core. Then, by using the reaction-security, which requires the blockings to be "nice", and the compactness of the action sets we obtain a finite collection points. By using these points we carefully construct an auxiliary TU game in characteristic function form. Then we use the separability of the game and the concavity of the payoffs in order to show that the TU game is balanced. Our line of arguments at this step uses the construction in Zhao's proof. Then, Bondareva-Shaply theorem imply that the TU game has a nonempty core.

This furnishes us a contradiction with the reaction-security of the original game. Our proof approach is similar to the proof approach of Uyanik [38] to extent that both construct auxiliary games. However, the construction of the auxiliary games and the line of the arguments in the proofs are quite distinct.

We now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. Let G be a concave, bounded, compact, strongly separable, RS_N game and has an empty β-core. Then, since G is RS, for each $(x, v) \in \bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x, v), $y_N^{x,v} \in X$, $w_N^{x,v} \in \mathbb{R}$, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^{x,v} \in \mathbb{R}$ and a function $\delta_S^{x,v} : X_{-S} \to X_S$ such that $\sum_{i\in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$, $\sum_{i\in S} u_i(\delta_S^{x,v}(z_{-S}), z_{-S}) \geq w_S^{x,v}$ for each $S \in \mathring{\mathcal{N}}$ and $z_{-S} \in X_{-S}$, and for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v'_i < w_S^{x,v}$. The family $\{U^{x,v} | \ (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | \ k = 1, \ldots, m\}$. Moreover, for each $S \in \mathring{\mathcal{N}}$, the pair $(x_S, v_S) \in \bar{\mathcal{X}}$ is identified[13] by strong separability. Hence, there are $(|\mathcal{N}| - 1)$-many additional open sets, that is, for each $S \in \mathring{\mathcal{N}}$, we have U^{x_S,v_S}. Define $U^{x_{m+j},v_{m+j}} = U^{x_S,v_S}$ when S is the jth member of the set $\mathring{\mathcal{N}}$. Let $K = \{1, \ldots, m, m + 1, \ldots, m + |\mathcal{N}| - 1\}$. Define for all $k \in K$, $U^k = U^{x_k,v_k}$, and for all $S \in \mathring{\mathcal{N}}$, $w_S^k = w_S^{x_k,v_k}$ and $\delta_S^k = \delta_S^{x_k,v_k}$, and $y_N^k = y_N^{x_k,v_k}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathring{\mathcal{N}}$,

$$W(S) = \max_{k\in K} w_S^k,$$

$$W(N) = \max\left\{\max_{k\in K} \sum_{i\in N} u_i(y_N^k), \ \max_{l\in L} \sum_{i\in N} u_i(x^l)\right\},$$

where L and $x^l \in X$ are defined as follows. First, for each coalition $S \in \mathring{\mathcal{N}}$, pick

$$k_S \in \operatorname*{argmax}_{k\in K} \ w_S^k.$$

Define for each coalition $S \in \mathring{\mathcal{N}}$,

$$\delta_S = \delta_S^{k_S}.$$

Let L denote the number of minimal balanced collection of coalitions that does not include N. Since number of all collection of coalitions is $2^{|\mathcal{N}|}$ and $\mathcal{N}$ is a finite set, L is finite. Let $\mathcal{T} = \{\mathcal{B}^l\}_{l\in L}$ denote the set of all minimal balanced collection of coalitions which does not include N. For each $\mathcal{B}^l \in \mathcal{T}$, let $\lambda^l = \{\lambda_S^l\}_{S\in\mathcal{B}^l}$ be

[13]We abuse the notation and use subscripts for both indices and coordinates when the context is clear.

the corresponding unique balancing weights.[14] By construction, for each $S \in \mathcal{B}^l$, $W(S) = w_S^{k_S} \le \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S})$ for each $z_{-S} \in X_{-S}$. Define $x^l \in X$ as

$$x_i^l = \sum_{S\in\mathcal{B}^l : i\in S} \lambda_S^l \delta_{S,i}(\tilde{z}_{-S}) \quad \text{for all } i \in N,$$

where $\tilde{z}_{-S}$ is determined by strong separability.[15]

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \le W(N)$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Since the only minimal balanced collection of coalition which includes N is $\mathcal{B} = \{N\}$, we shall prove the above inequality for all $\mathcal{B} \in \mathcal{T}$. Pick $\mathcal{B} \in \mathcal{T}$ with the (unique) balancing weights $\lambda = \{\lambda_S\}_{S\in\mathcal{B}}$. (note that the balancing weights of a minimal balanced collection of coalitions are always unique). We shall show that there exists $x' \in X$ such that

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \le \sum_{i\in N} u_i(x') \le W(N).$$

Recall that in defining $W(N)$, we define an action profile for each balanced collection. Now, define x^* as

$$x_i^* = \sum_{S\in\mathcal{B} : i\in S} \lambda_S \delta_{S,i}(\tilde{z}_{-S}) \quad \text{for all } i \in N.$$

By construction of W, $\sum_{i\in N} u_i(x^*) \le W(N)$. Therefore, showing

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \le \sum_{i\in N} u_i(x^*)$$

will be sufficient. From the construction of W,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \le \sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}),$$

and since W is strongly separable, Lemma 7.1 in the Appendix implies there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}) \le \sum_{S\in\mathcal{B}} \lambda_S \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Therefore,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \le \sum_{i\in N} \sum_{S\in\mathcal{B} : i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

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[14]Note that the balancing weights of a minimal balanced collection of coalitions are unique; see Kannai [20, p. 361] and Appendix below.

[15]Note that, by construction, $\delta_S \in \Delta_S(x_k, v_k)$ for all $k \in K$.

hence showing the following inequality implies the desired result.

$$\sum_{S\in\mathcal{B}:i\in S}\lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_i(x^*) \text{ for each } i\in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S \in X$ for each $S\in\mathcal{B}$ containing player 1 as follows. If $i\in S$, then

$$y_i^S = \delta_{S,i}(\tilde{z}_{-S}).$$

If $i\notin S$, then

$$y_i^S = \frac{\sum \lambda_E \delta_{E,i}(\tilde{z}_{-E})}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E\in\mathcal{B}$ which contain player i but not player 1. From Scarf [34, p.179],

$$x^* = \sum_{S\in\mathcal{B}:1\in S}\lambda_S y^S.$$

Pick a coalition $S\in\mathcal{B}$ containing player 1. Then by construction of y^S and the strong separability,

$$\inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S\in\mathcal{B}:1\in S}\lambda_S \inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq \sum_{S\in\mathcal{B}:1\in S}\lambda_S u_1(y^S) \leq u_1(x^*).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B}\in\mathcal{T}$, $\sum_{S\in\mathcal{B}}\lambda_S W(S)\leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^*\in\mathbb{R}^n$ such that $\sum_{i\in N} v_i^* = W(N)$ and $\sum_{i\in S} v_i^* \geq W(S)$ for all $S\in\mathcal{N}$. By construction of $W(N)$, there exists $\bar{x}\in X$ such that $\sum_{i\in N} v_i^* = \sum_{i\in N} u_i(\bar{x})$. In particular, $\bar{x} = y_N^k$, or x^l for some $k\in K, l\in L$. Hence, $(\bar{x}, v^*)\in\bar{\mathcal{X}}$. Since G is RS and $(\bar{x}, v^*)\in U^k$ for some $k\in K$, there exists $S\in\mathcal{N}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k\leq W(S)$. This furnishes us a contradiction. □

We next present our second main result which replaces the assumption that the game is RS in Theorem 3.1 with the assumption that the game is RS_N.

Theorem 3.5. Every concave, bounded, compact, strongly separable and RS_N game has a nonempty β-core.

Proof of Theorem 3.5. Let G be a concave, bounded, compact, strongly separable and RS_N game. Since the aggregate payoff function $\bar{u}$ is u.s.c. and X is compact, therefore there exists $\bar{x}\in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Moreover, the game is bounded, hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is RS_N, for each $v \in \mathcal{V}$, there exist an open neighborhood U^v of v, and for each $S \in \mathring{\mathcal{N}}$ there exist $w^v_S \in \mathbb{R}$ and a function $\delta^v_S : X_{-S} \to X_S$ such that $\sum_{i \in S} u_i(\delta^v_S(z_{-S}), z_{-S}) \geq w^v_S$ for each $S \in \mathring{\mathcal{N}}$, $z_{-S} \in X_{-S}$, and for each $v' \in U^v$ there exists $S \in \mathring{\mathcal{N}}$ such that $\sum_{i \in S} v'_i < w^v_S$. The family $\{U^v |\ v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} |\ k = 1, \ldots, m\}$.

Moreover, for each $S \in \mathring{\mathcal{N}}$, there exists $x_S \in \mathcal{V}$ identified by strong separability. Hence, there are $(|\mathcal{N}| - 1)$-many additional open sets, that is, for each $S \in \mathring{\mathcal{N}}$, we have U^{x_S}. Define $U^{x_{m+j}} = U^{x_S}$ when S is the jth member of the set $\mathring{\mathcal{N}}$. Let $K = \{1, \ldots, m, m+1, \ldots, m + |\mathcal{N}| - 1\}$. Define for all $k \in K$, $U^k = U^{v_k}$, and for all $S \in \mathcal{N}$, $w^k_S = w^{v_k}_S$ and $\delta^k_S = \delta^{v_k}_S$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $S \in \mathcal{N} \backslash N$,

$$W(S) = \max_{k \in K} w^k_S, \quad \text{and} \quad W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

Next, for each coalition $S \in \mathring{\mathcal{N}}$, pick

$$k_S \in \operatorname*{argmax}_{k \in K} w^k_S,$$

and define for each coalition $S \in \mathring{\mathcal{N}}$,

$$\delta_S = \delta^{k_S}_S.$$

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Pick a minimal balanced collection of coalitions $\mathcal{B}$ with the (unique) balancing weights $\lambda = \{\lambda_S\}_{S \in \mathcal{B}}$ (recall that the balancing weights of a minimal balanced collection of coalitions are always unique). For $\mathcal{B} = \{N\}$, there is nothing to prove. Otherwise, since $\mathcal{B}$ is a minimal balanced collection of coalition, it does not contain N. We shall show that there exists $x' \in X$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq \sum_{i \in N} u_i(x') \leq \bar{w} = W(N).$$

where the last inequality follows from the definition of $\bar{w}$. By construction, for each $S \in \mathcal{B}$, $W(S) = w^{k_S}_S \leq \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S})$ for each $z_{-S} \in X_{-S}$. Define $x^* \in X$ as

$$x^*_i = \sum_{S \in \mathcal{B} : i \in S} \lambda_S \delta_{S,i}(\tilde{z}_{-S}),$$

where $\tilde{z}_{-S}$ is determined by strong separability. By the construction above, $\sum_{i\in N} u_i(x^*) \leq W(N) = \bar{w}$. Therefore, showing

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} u_i(x^*)$$

will be sufficient. From the construction of W,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}),$$

and since W is strongly separable, Lemma 7.1 in the Appendix implies there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{S\in\mathcal{B}} \lambda_S \inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{S\in\mathcal{B}} \lambda_S \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Therefore,

$$\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq \sum_{i\in N} \sum_{S\in\mathcal{B}:i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

hence showing the following inequality implies the desired result.

$$\sum_{S\in\mathcal{B}:i\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_i(x^*) \text{ for each } i \in N.$$

At this level of generality, it is sufficient to demonstrate that the above inequality holds for player 1, since by a suitable renaming of players any particular player can be made the first. We now define $y^S \in X$ for each $S \in \mathcal{B}$ containing player 1 as follows. If $i \in S$, then

$$y_i^S = \delta_{S,i}(\tilde{z}_{-S}).$$

If $i \notin S$, then

$$y_i^S = \frac{\sum \lambda_E \delta_{E,i}(\tilde{z}_{-E})}{\sum \lambda_E}.$$

where in both the numerator and the denominator the summation is taken over all $E \in \mathcal{B}$ which contain player i but not player 1. From Scarf [34, p.179],

$$x^* = \sum_{S\in\mathcal{B}:1\in S} \lambda_S y^S.$$

Pick a coalition $S \in \mathcal{B}$ containing player 1. Then by construction of y^S and the strong separability,

$$\inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq u_1(y^S).$$

Therefore, from the concavity of u_1,

$$\sum_{S\in\mathcal{B}:1\in S} \lambda_S \inf_{z_{-S}\in X_{-S}} u_1(\delta_S(\tilde{z}_{-S}), z_{-S}) \leq \sum_{S\in\mathcal{B}:1\in S} \lambda_S u_1(y^S) \leq u_1(x^*).$$

Therefore, since $\mathcal{B}$ is arbitrarily chosen, for each $\mathcal{B} \in \mathcal{T}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i\in N} v_i^* = W(N)$ and $\sum_{i\in S} v_i^* \geq W(S)$ for all $S \in \mathcal{N}$. Since G is RS_N and $v^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

Remark 3.6. We end this section by presenting a result on hybrid solution due to [43, 45]. For some game theoretic situations only some subsets of the players can behave cooperatively, due to factors such as transaction costs, social and legal restrictions. A solution concept for such models is called hybrid solution which assumes that the players are partitioned into coalitions and that they will cooperate within each coalition but compete (in the Nash sense) among coalitions. Now, let $G = (X_i, u_i)_{i\in N}$ be a game. A given coalition structure (a partition of N) $\Delta = \{S_1, \ldots, S_m\}$ induces a game $G_\Delta = \{X_S, \sum_{i\in S} u_i\}_{S\in\Delta}$ among partition members and m parametric games

$$G_S(x_{-S}) = \{X_i, u_i(\cdot, x_{-S})\}_{i\in S}.$$

Definition 3.7. Let $G = (X_i, u_i)_{i\in N}$ be a game and $\Delta = \{S_1, \ldots, S_m\}$ a coalition structure. A *hybrid solution* for G is a pair of an action profile and imputation $(x^*, v^*) \in X \times \mathbb{R}^n$ such that for each $S \in \Delta$,

(i) x^* is a Nash equilibrium of G_Δ,

(ii) (x_S^*, v_S^*) is in the (transferable utility) β-core of $G_S(x_{-S}^*)$.

The following result follows from Theorems 3.1, 3.5, and Barelli and Meneghel [8, Theorem 2.2, p.816].

Corollary 3.8. A bounded, concave game G has a hybrid solution if G_Δ is correspondence-secure (CS) and for each $S \in \Delta$ and $x_{-S} \in X_{-S}$, $G_S(x_{-S})$ is compact, strongly separable and either RS, or RS_N.

Note that [45] allows each coalition in Δ to use other distribution rules such as α-core. It is of interest to investigate further hybrid solutions in discontinuous games.

Proof of Corollary 3.8. Since $G_S(x_{-S})$ is a concave game for each $S \in \Delta$ and $x_{-S} \in X_{-S}$, u_i is concave for each player $i \in S$, hence $\sum_{i\in S} u_i(\cdot, x_{-S})$ is concave. Then, since G_Δ is CS, Barelli and Meneghel [8, Theorem 2.2, p.816] implies there exists $x^* \in X$ such that $x_S^* \in \operatorname{argmax}_{x_S\in X_S} \sum_{i\in S} u_i(x_S, x_{-S}^*)$ for each $S \in \Delta$. And since all conditions of Theorem 3.1 are satisfied (if the game is RS, and Theorem 3.5 if the game is RS_N), for each $S \in \Delta$ there exists $v_S^* \in \mathbb{R}^{|S|}$ such that (x_S^*, v_S^*) is in the β-core of $G_S(x_{-S}^*)$. □

4. Results: two-player games

This section studies the nonemptiness of the β-core of two-player games. As we noted above, the two-player setup allows us to weaken (even drop some of) the convexity and separability assumptions: Propositions 4.1 and 4.2 show that in the two-player setup, the convexity structure can simply be dropped (since the clever argument originally introduced by Scarf is not needed in this special setup), and Propositions 4.4 and 4.5 show that the strong separability in the first two propositions can be replaced by a stronger continuity assumption and quasiconcavity of the payoff functions.

Our first result in this section drops the entire convexity structure, both convexity of the actions sets and the concavity of the payoff functions.

Proposition 4.1. A bounded, compact, strongly separable and RS 2-player game has a nonempty β-core.

Proof of Proposition 4.1. Assume G has an empty β-core. Then, since G is RS, for each $(x,v)\in\bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x,v), $y_N^{x,v}\in X$, $w_N^{x,v}\in\mathbb{R}$, and for each $i\in N$ there exist $w_i^{x,v}\in\mathbb{R}$ and a function $\delta_i^{x,v}:X_j\to X_i$ for $i\neq j$ such that $\sum_{i\in N}u_i(y_N^{x,v})\geq w_N^{x,v}$, $u_i(\delta_i^{x,v}(z_j),z_j)\geq w_i^{x,v}$ for each $i\neq j$ and $z_j\in X_j$, and for each $(x',v')\in U^{x,v}$ there exists $S\in\mathcal{N}$ such that $\sum_{i\in S}v'_i<w_S^{x,v}$. The family $\{U^{x,v}|\ (x,v)\in\bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k}|\ k=1,\dots,m\}$. And for each $i\in N$, $(x_i,v_i)\in\bar{\mathcal{X}}$ is identified by strong separability. Let $U^k=U^{x_k,v_k}$, and for all $S\in\mathcal{N}$, $w_S^k=w_S^{x_k,v_k}$, $y_N^k=y_N^{x_k,v_k}$, and for $i=1,2$, $\delta_i^k=\delta_i^{x_k,v_k}$ for all $k\in K=\{1,\dots,m,\dots,m+2\}$.

Now define a TU game $W:\mathcal{N}\to\mathbb{R}$ as follows. For all $i\in N$,

$$W(\{i\})=\max_{k\in K}w_i^k,$$

$$W(N)=\max\left\{\max_{k\in K}\sum_{i\in N}u_i(y_N^k),\ \sum_{i\in N}u_i(x^*)\right\},$$

where $x^*\in X$ is defined as follows. First, define for each coalition $S\in\mathcal{N}$,

$$k_S\in\operatorname*{argmax}_{k\in K}w_S^k,\quad\text{and}\quad\delta_S=\delta_S^{k_S}.$$

By construction, for each $i\neq j$, $W(\{i\})=w_i^{k_i}\leq u_i(\delta_i(z_j),z_j)$ for each $z_j\in X_j$. Define $x^*\in X$ as

$$x_i^*=\delta_i(\tilde{z}_j)\quad\text{for } i\neq j,$$

where $\tilde{z}_j$ is determined by strong separation.

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S\in\mathcal{B}}\lambda_S W(S)\le W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B}=\{N\}$, or $\{\{1\},\{2\}\}$. For $\mathcal{B}=\{N\}$, there is nothing to prove. Let $\mathcal{B}=\{\{1\},\{2\}\}$. The definition of infimum and strong separation imply for each $i\neq j$,

$$\inf_{z_j\in X_j} u_i(\delta_i(z_j),z_j)\le u_i(\delta_i(\tilde{z}_j),\tilde{z}_j)=\inf_{z_j\in X_j} u_i(\delta_i(\tilde{z}_j),z_j).$$

Hence,

$$w_i^{k_i}\le u_i(\delta_i(\tilde{z}_j),\delta_j\tilde{z}_i)=u_i(x^*).$$

Therefore, $\sum_{i\in N}u_i(x^*)\le W(N)$ implies $W(1)+W(2)\le W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^*\in\mathbb{R}^2$ such that $\sum_{i\in N}v_i^*=W(N)=\sum_{i\in N}$ and $v_i^*\ge W(\{i\})$ for all $i\in N$. By construction of $W(N)$, there exists $\bar{x}\in X$ such that $\sum_{i\in N}v_i^*=\sum_{i\in N}u_i(\bar{x})$. In particular, $\bar{x}=y_N^k$ for some $k\in K$, or $\bar{x}=x^*$. Hence, $(\bar{x},v^*)\in\bar{\mathcal{X}}$. Since G is RS and $(\bar{x},v^*)\in U^k$ for some $k\in K$, there exists $S\in\mathcal{N}$ such that $\sum_{i\in S}v_i^*<w_S^k$. By construction, $w_S^k\le W(S)$. This furnishes us a contradiction. □

We next show that RS property in Proposition 4.1 can be replaced by RS_N property.

Proposition 4.2. A bounded, compact, strongly separable and RS_N 2-player game has a nonempty β-core.

Proof of Proposition 4.2. Since the aggregate payoff function $\bar{u}$ is u.s.c. and X is compact, therefore there exists $\bar{x}\in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is RS_N, for each $v\in\mathcal{V}$, there exist an open neighborhood U^v of v, and for each $i\in N$ there exist $w_i^v\in\mathbb{R}$ and a function $\delta_i^v:X_j\to X_i$ such that $u_i(\delta_i^v(z_j),z_j)\ge w_i^v$ for each $i\neq j$, and $z_j\in X_j$, and for each $v'\in U^v$ there exists $i\in N$ such that $v_i'<w_i^v$. The family $\{U^v|\ v\in\mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k}|\ k=1,\dots,m\}$. And for each $i\in N$, $v_i\in\mathcal{V}$ is identified by strong separability. Let $U^v=U^{v_k}$, and for all $i\in N$, $w_i^k=w_i^{v_k}$ and $\delta_i^k=\delta_i^{v_k}$ for all $k\in K=\{1,\dots,m,\dots,m+2\}$.

Now define a TU game $W:\mathcal{N}\to\mathbb{R}$ as follows. For all $i\in N$,

$$W(\{i\})=\max_{k\in K}w_i^k,\quad\text{and } W(N)=\bar{w}=\max_{x\in X}\sum_{i\in N}u_i(x).$$

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S\in\mathcal{B}}\lambda_S W(S)\le W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B}=\{N\}$, or $\{\{1\},\{2\}\}$. For $\mathcal{B}=\{N\}$, there is nothing to prove.

Let $\mathcal{B} = \{\{1\},\{2\}\}$. By construction, $W(\{1\}) = w_1^{k_1} \leq u_1(\delta_1(z_2), z_2)$ for all $z_2 \in X_2$ and $W(\{2\}) = w_2^{k_2} \leq u_2(z_1, \delta_2(z_1))$ for all $z_1 \in X_1$. And strong separability implies for each $i \in N$, there exists $\tilde{z}_{-i} \in X_{-i}$ such that

$$w_i^{k_i} \leq u_i(\delta_i(\tilde{z}_{-i}), \tilde{z}_{-i}) = \inf_{z_{-i} \in X_{-i}} u_i(\delta_i(\tilde{z}_{-i}), z_{-i}) \ \text{ for } i = 1, 2.$$

Hence,

$$w_1^{k_1} \leq u_1(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1)) \ \text{ and } w_2^{k_2} \leq u_2(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1)).$$

Therefore, $\sum_{i\in N} u_i(\delta_1(\tilde{z}_2), \delta_2(\tilde{z}_1))) \leq \bar{w}$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i\in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. Since G is RS_N and $v^* \in U^k$ for some $k \in K$, there exists $i \in N$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. $\square$

We next introduce some new concepts for our next two propositions.

Definition 4.3. A bounded, compact game G is *strongly reaction-secure* (strongly RS) if for each $(x,v) \in \bar{\mathcal{X}}$ that is not in the β-core of G, there exist an open neighborhood $U^{x,v}$ of (x,v), $y_N^{x,v} \in X$, $w_N^{x,v} \in \mathbb{R}$, and for each $S \in \mathring{\mathcal{N}}$ there exist $w_S^{x,v} \in \mathbb{R}$ and an upper semicontinuous correspondence $\delta_S^{x,v} : X_{-S} \twoheadrightarrow X_S$ with nonempty and compact values such that

(i) $\sum_{i\in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$

(ii) for all $S \in \mathring{\mathcal{N}}$, all $z_{-S} \in X_{-S}$ and all $z_S \in \delta_S^{x,v}(z_{-S})$, $\sum_{i\in S} u_i(z_S, z_{-S}) \geq w_S^{x,v}$,

(iii) for all $(x',v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i' < w_S^{x,v}$.

Note that strong-RS property replaces the functions $\delta_S^{x,v}$ in the RS-property with nonempty- and compact-valued u.s.c. correspondences. If the payoff functions are continuous, then the "best response correspondences" satisfy this assumption. Hence, the strong RS is substantially weaker than the continuity assumption. Indeed, it is direct analogue of the CS property defined above. Define *strongly* RS_N analogously.

We now present a result showing that, in the two-player setup, strengthening the continuity assumption allows us to drop the separability assumption when the payoff functions are quasiconcave.

Proposition 4.4. A bounded, compact and strongly RS 2-player game has a nonempty β-core if for each player i, X_i is convex and u_i is quasiconcave on X_i.

Proof of Proposition 4.4. Assume G has an empty β-core. Then, since G is strongly RS, for each $(x,v) \in \bar{\mathcal{X}}$, there exist an open neighborhood $U^{x,v}$ of (x,v), $y_N^{x,v} \in X$,

$w_N^{x,v} \in \mathbb{R}$, and for each $i \in N$ there exist $w_i^{x,v} \in \mathbb{R}$ and a nonempty- and compact-valued u.s.c. correspondence $\delta_i^{x,v} : X_j \twoheadrightarrow X_i$ for $i \neq j$ such that $\sum_{i\in N} u_i(y_N^{x,v}) \geq w_N^{x,v}$, and $u_i(z_i, z_j) \geq w_i^{x,v}$ for each $i \neq j$ and $z_j \in X_j$, and $z_i \in \delta_i^{x,v}(z_j)$, and for each $(x', v') \in U^{x,v}$ there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i' < w_S^{x,v}$. The family $\{U^{x,v} | \ (x, v) \in \bar{\mathcal{X}}\}$ is an open covering of $\bar{\mathcal{X}}$ which, by compactness of $\bar{\mathcal{X}}$, contains a finite subcovering $\{U^{x_k,v_k} | \ k = 1, \ldots, m\}$. Let $U^k = U^{x_k,v_k}$, and for all $S \in \mathcal{N}$, $w_S^k = w_S^{x_k,v_k}$, $y_N^k = y_N^{x_k,v_k}$, and for $i = 1, 2$, $\delta_i^k = \delta_i^{x_k,v_k}$ for all $k \in K = \{1, \ldots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k\in K} w_i^k,$$

$$W(N) = \max \left\{ \max_{k\in K} \sum_{i\in N} u_i(y_N^k), \ \sum_{i\in N} u_i(x^*) \right\},$$

where $x^* \in X$ is defined as follows. First, define for each player $i \in N$,

$$k_i \in \operatorname*{argmax}_{k\in K} w_i^k, \quad \text{and} \quad \delta_i = \mathrm{co}(\delta_i^{k_i}),$$

where $\mathrm{co}(\delta_i^{k_i})$ is the convex hull of $\delta_i^{k_i}$. Since $\delta_i^{k_i}$ is u.s.c., has nonempty and compact values, and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, δ_i is u.s.c. and has nonempty and compact values. Moreover, since u_i is quasiconcave for each $i \in N$, $u_i(z_i, z_j) \geq w_i^{k_i}$ for each $z_j \in X_j$ and $z_i \in \delta_i(z_j)$ for $i \neq j$. Now, define a correspondence $\delta : X \twoheadrightarrow X$ as $\delta(z) = (\delta_1(z_2), \delta_2(z_1))$. Since δ is u.s.c. and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_i^* \in \delta_i(x_j^*), \ i \neq j.$$

Hence, $w_i^{k_i} \leq u_i(z_i, z_j)$ for all $z_j \in X_j$ and $z_i \in \delta_2(z_j)$ for $i \neq j$ implies $w_i^{k_i} \leq u_i(x^*)$ for $i = 1, 2$.

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. Recall that, $w_1^{k_1} \leq u_1(x^*)$ and $w_2^{k_2} \leq u_2(x^*)$. Therefore, $\sum_{i\in N} u_i(x^*) \leq W(N)$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathbb{R}^2$ such that $\sum_{i\in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all $i \in N$. By construction of $W(N)$, there exists $\bar{x} \in X$ such that $\sum_{i\in N} v_i^* = \sum_{i\in N} u_i(\bar{x})$. In particular, $\bar{x} = y_N^k$ for some $k \in K$, or $\bar{x} = x^*$. Hence, $(\bar{x}, v^*) \in \bar{\mathcal{X}}$. Since G is strongly RS and $(\bar{x}, v^*) \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $\sum_{i\in S} v_i^* < w_S^k$. By construction, $w_S^k \leq W(S)$. This furnishes us a contradiction. □

The next result replaces the strong RS property in Proposition 4.4 by strong RS_N property.

Proposition 4.5. A bounded, compact and strongly RS_N 2-player game has a nonempty β-core if for each player i, X_i is convex and u_i is quasiconcave on X_i.

Proof of Proposition 4.5. Since the aggregate payoff function $\bar{u}$ is u.s.c. and X is compact, therefore there exists $\bar{x} \in X$ which maximizes the aggregate payoff function $\bar{u}$ of the grand coalition. Hence, $\mathcal{V}$ is a well-defined compact set. Now, assume G has an empty β-core. Then, since G is strongly RS_N, for each $v \in \mathcal{V}$, there exist an open neighborhood U^v of v, and for each $i \in N$ there exist $w_i^v \in \mathbb{R}$ and u.s.c. correspondence $\delta_i^v : X_j \twoheadrightarrow X_i$ with nonempty and compact values such that $u_i(z_i, z_j) \geq w_i^v$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta_i^v(z_j)$, and for each $v' \in U^v$ there exists $i \in N$ such that $v'_i < w_i^v$. The family $\{U^v | \; v \in \mathcal{V}\}$ is an open covering of $\mathcal{V}$ which, by compactness of $\mathcal{V}$, contains a finite subcovering $\{U^{v_k} | \; k = 1, \dots, m\}$. Let $U^k = U^{v_k}$, and for all $i \in N$, $w_i^k = w_i^{v_k}$ and $\delta_i^k = \delta_i^{v_k}$ for all $k \in K = \{1, \dots, m\}$.

Now define a TU game $W : \mathcal{N} \to \mathbb{R}$ as follows. For all $i \in N$,

$$W(\{i\}) = \max_{k \in K} w_i^k, \quad \text{and } W(N) = \bar{w} = \max_{x \in X} \sum_{i \in N} u_i(x).$$

We shall show that for each minimally balanced collection of coalitions $\mathcal{B}$, $\sum_{S \in \mathcal{B}} \lambda_S W(S) \leq W(N)$. Note that there are only two minimal balanced collection of coalitions: $\mathcal{B} = \{N\}$, or $\{\{1\}, \{2\}\}$. For $\mathcal{B} = \{N\}$, there is nothing to prove. Let $\mathcal{B} = \{\{1\}, \{2\}\}$. Define for each player $i \in N$,

$$k_i \in \operatorname*{argmax}_{k \in K} w_i^k, \quad \text{and} \quad \delta_i = \operatorname{co}(\delta_i^{k_i}),$$

where $\operatorname{co}(\delta_i^{k_i})$ is the convex hull of $\delta_i^{k_i}$. Since $\delta_i^{k_i}$ is u.s.c., has nonempty and compact values, and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, δ_i is u.s.c. and has nonempty and compact values. And since u_i is quasiconcave for each $i \in N$, $u_i(z_i, z_j) \geq w_i^{k_i}$ for each $z_j \in X_j$ and $z_i \in \delta_i(z_j)$ for $i \neq j$. Now, define a correspondence $\delta : X \twoheadrightarrow X$ as $\delta(z) = (\delta_1(z_2), \delta_2(z_1))$. Since δ is *usc*, and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_i^* \in \delta_i(x_j^*), \; i \neq j.$$

Hence, $w_i^{k_i} \leq u_i(z_i, z_j)$ for all $z_j \in X_j$ and $z_i \in \delta_2(z_j)$ for $i \neq j$ implies $w_i^{k_i} \leq u_i(x^*)$ for $i = 1, 2$. Therefore, $\sum_{i \in N} u_i(x^*) \leq \bar{w}$ implies $W(1) + W(2) \leq W(N)$.

Bondareva-Shapley Theorem provided in the Appendix implies W has a nonempty core, i.e. there exists $v^* \in \mathcal{V}$ such that $\sum_{i \in N} v_i^* = W(N)$ and $v_i^* \geq W(\{i\})$ for all

$i \in N$. Since G is strongly RS_N and $v^* \in U^k$ for some $k \in K$, there exists $i \in \mathcal{N}$ such that $v_i^* < w_i^k$. By construction, $w_i^k \leq W(\{i\})$. This furnishes us a contradiction. □

We end this section by presenting a result on the nonemptiness of the β-core of a two-player game with possibly discontinuous payoff functions and nontransferable utilities (NTU).[16]

Definition 4.6. Let G be a game. A coalition S β^{NTU}*-blocks* an action profile $x \in X$ if $\forall z_{-S} \in X_{-S}$, $\exists x'_S \in X_S$ such that[17] $u_S(x'_S, z_{-S}) \gg u_S(x)$. An action profile $x^* \in X$ is in the β^{NTU}*-core* of G if x^* is not β^{NTU}-blocked by any coalition.

Definition 4.7. A two-player game G is *strongly* RS^{NTU} if for each $x \in X$ that is not in the β^{NTU}-core of G, there exist an open neighborhood U^x of x, $y_N^x \in X$, $v_N^x \in \mathbb{R}^2$, and for each $i \in N$ there exist $v_i^x \in \mathbb{R}$ and a nonempty- and compact-valued, u.s.c. correspondence $\delta_i^x : X_j \twoheadrightarrow X_i$ for $i \neq j$ such that

(i) $u_i(z_i, z_j) \geq v_i^x$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta_i^x(z_j)$, and $u_N(y_N^x) \geq v_N^x$,

(ii) for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $u_S(x') \ll v_S^x$.

Proposition 4.8. A compact and strongly RS^{NTU} 2-player game has a nonempty β^{NTU}-core if for each player i, X_i is convex and u_i is quasiconcave on X_i.

Proof of Proposition 4.8. Assume G has an empty β^{NTU}-core. Then, since G is strongly RS^{NTU}, for each $x \in X$, there exist an open neighborhood U^x of x, $y_N^x \in X$, $v_N^x \in \mathbb{R}^2$, and for each $i \in N$ there exist $v_i^x \in \mathbb{R}$ and a nonempty valued, u.s.c. correspondence $\delta_i^x : X_j \twoheadrightarrow X_i$ for $i \neq j$ such that $u_i(z_i, z_j) \geq v_i^x$ for each $i \neq j$, $z_j \in X_j$ and $z_i \in \delta_i^x(z_j)$, and $u_N(y_N^x) \geq v_N^x$, and for each $x' \in U^x$ there exists $S \in \mathcal{N}$ such that $x' \ll v_S^x$. The family $\{U^x |\ x \in X\}$ is an open covering of X which, by compactness of X, contains a finite subcovering $\{U^{x_k} \,|\, k = 1, \ldots, m\}$. Let $U^k = U^{x_k}$, and for all $S \in \mathcal{N}$, $v_S^k = v_S^{x_k}$, $y_N^k = y_N^{x_k}$ and for each $i \in N$, $\delta_i^k = \delta_i^{x_k}$, for all $k \in K = \{1, \ldots, m\}$.

Now define an NTU game $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^2$ as follows. For all $i \in N$,

$$V(\{i\}) = \bigcup_{k \in K} \{v \in \mathbb{R}^2 |\ v_i \leq v_i^k\},$$

$$V(N) = \bigcup_{k,k' \in K} \left(\{v \in \mathbb{R}^2 |\ v \leq v_N^k\} \cup \{v \in \mathbb{R}^2 |\ v \leq u(x_1^k, x_2^{k'})\} \right),$$

[16] An analogous result for continuous payoff functions is provided in Ichiishi [19, Remark 2.3.2, p.37].

[17] We use the following convention for comparing two vectors: for $a, b \in \mathbb{R}^n$, $a \gg b$ denotes $a_i > b_i$ for all i, $a > b$ denotes $a_i \geq b_i$ for all i and $a \neq b$, and $a \geq b$ denotes $a_i \geq b_i$ for all i.

where $(x_1^k, x_2^{k'}) \in X$ is defined as follows. First, define for each player $i \in N$ and $k \in K$,

$$\delta_i^k = \text{co}(\delta_i^k),$$

where $\text{co}(\delta_i^k)$ is the convex hull of δ_i^k. Since δ_i^k is u.s.c. and X_i is subset of a finite dimensional Euclidean space for each $i \in N$, and $k \in K$, δ_i^k is *usc*. And since u_i is quasiconcave on X_i for each $i \in N$, $u_i(z_i, z_j) \geq v_i^k$ for each $z_j \in X_j$ and $z_i \in \delta_i^k(z_j)$ for $i \neq j$, $k \in K$. Now, for each $k, k' \in K$, define a correspondence $\delta^{k,k'} : X \twoheadrightarrow X$ as $\delta^{k,k'}(z) = (\delta_1^k(z_2), \delta_2^{k'}(z_1))$. Since $\delta^{k,k'}$ is *usc*, and has nonempty and convex values, Kakutani's fixed point theorem implies there exists $x^* \in X$ such that

$$x_1^k \in \delta_1^k(x_2^*) \text{ and } x_2^{k'} \in \delta_2^{k'}(x_1^*).$$

Hence, $v_1^k \leq u_1(x_1^k, x_2^{k'})$ and $v_2^{k'} \leq u_2(x_1^k, x_2^{k'})$.

By construction, the NTU game V is balanced. To see this, note that there are three balanced collection of coalitions for this game of which two contain N. For these collections, there is nothing to prove. The only balanced collection of coalitions which does not include N is $\mathcal{B} = \{\{1\}, \{2\}\}$. Pick $v \in V(S)$ for all $S \in \mathcal{B}$. Then, there exists $k_1, k_2 \in K$ such that $v_1 \leq v_1^{k_1}$ and $v_2 \leq v_2^{k_2}$. We showed above that $(v_1^{k_1}, v_2^{k_2}) \leq u(x_1^{k_1}, x_2^{k_2})$, and hence $v \in V(N)$. Therefore V is balanced.

It is clear that conditions (i)-(ii) of Scarf's Theorem provided in the Appendix are satisfied. And since for each coalition S, the set $V(S)$ is constructed by using finitely many points, condition (iii) of Scarf's Theorem is satisfied. Hence, V has a nonempty core, i.e. there exists $v^* \in V(N)$ such that $v^* \notin \text{int}V(S)$ for all $S \in \mathcal{N}$. Since $v^* \in V(N)$, by construction there exists $x^* \in X$ such that $v^* \leq u(x^*)$. Also, since G is strongly RS^{NTU} and $x^* \in U^k$ for some $k \in K$, there exists $S \in \mathcal{N}$ such that $u_S(x^*) \ll v_S^k$. Since $\{v_S^k\} \times \mathbb{R}^{|-S|} \subset V(S)$, $v^* \in \text{int}V(S)$. This furnishes us a contradiction. □

5. Applications

Our first example illustrates a duopoly game with nonempty β-core which is strongly RS_N but neither RS, nor CS.[18]

Example 1. (Bertrand Duopoly Game) Assume there are two firms which have a common action set $[0, T]$ where T is a large positive number. Assume firm 1 has zero marginal cost, $c_1 = 0$, whereas firm 2 has a strictly positive marginal cost $c_2 = c \in (0, T/2)$. The demand function $D : [0, T] \to \mathbb{R}_+$ is defined as

$$D(p) = \max\{\alpha - p, 0\}$$

[18] We intentionally choose a simple framework in the first example to: (1) illustrate different continuity concepts we define above, and (2) show that the these continuity concepts are non-nested.

where $\alpha \in (2c, T)$. Since the good is identical, the firm with the lowest cost captures the market. When the firms set the same price, then they share the demand equally. Therefore, the profit function of firm $i \neq j$ is defined as

$$\pi_i(p_1, p_2) = \begin{cases} D(p_i)(p_i - c_i) & \text{if } p_i < p_j, \\ \frac{1}{2} D(p_i)(p_i - c_i) & \text{if } p_i = p_j, \\ 0 & \text{if } p_i > p_j. \end{cases}$$

It is easy to check that this game does not have a Nash equilibrium.[19]

We next prove that this game has a nonempty β-core by verifying that it satisfies the assumptions of Proposition 4.5. It is clear that this game is bounded and compact and each firm's profit function is quasiconcave in its own action. It remains to show that it is strongly RS_N. The aggregate payoff function is $\bar{\pi}(p_1, p_2) = D(p_i)(p_i - c_i)$ if $p_i < p_j$ and $0.5D(p_i)(2p_i - c)$ if $p_i = p_j$. Therefore $\{(\alpha/2, p_2) \mid p_2 > \alpha/2\}$ is the set of maximizers of $\bar{\pi}$ and the maximum aggregate profit is $\alpha^2/4$. Hence, in equilibrium (if exists), the low cost firm produces the entire output and then the firms share the profit. Then, the set of imputations is defined as

$$\mathcal{V} = \{v \mid v_1 + v_2 = \alpha^2/4, v_1 \geq 0, v_2 \geq -\alpha c\}.$$

Note that firm 1 cannot block any imputation in $\mathcal{V}$ since firm 2 can always set its price to 0 which yields firm 1's a maximum profit of zero. Firm 2 blocks all imputations which gives it a negative payoff by setting its price equal to c. Since grand coalition cannot block any imputation by construction, we only need to check the imputations which gives firm 2 negative profit. For each $v \in \mathcal{V}$ such that $v_2 < 0$, define $\varepsilon^v = -v_2/3$, $U^v = \{v' \in V| \ \|v' - v\| < \varepsilon^v\}$, $\delta_1^v(z_2) = \{0\}, \delta_2^v(z_1) = \{0\}$ for every $z \in X$ and $w_1^v = w_2^v = 0$. It is clear that δ_i^v is an u.s.c. correspondence with nonempty and compact values, and conditions (i) and (ii) of the strong-RS_N property are satisfied. Therefore, this game is strongly RS_N. Hence, by Proposition 4.5, this game has a non-empty β-core. In fact, a careful algebra implies that the following pair of an action profile and imputation

$$(p^*, v^*) = \left(\left(\frac{\alpha}{2}, \alpha \right), \left(\frac{\alpha^2}{4}, 0 \right) \right)$$

is in the β-core of this game.

We next show that this game is neither CS, nor RS. Since this game satisfies all the assumptions of [8, Theorem 2.2] except CS and it does not have a Nash equilibrium, therefore it is not CS. In order to see that it is not RS, note that

[19] More precisely, this game has no Nash equilibrium in pure-strategies. It is known that this game has a Nash equilibrium in mixed-strategies, see [9].

$(\hat{p}, \hat{v}) = ((\alpha/2, \alpha/2), (\alpha^2/4, 0)) \in \bar{\mathcal{X}}$ is not in the β-core of G since $\sum_{i=1}^{2} \pi_i(\hat{p}) < \alpha^2/4$. But, no coalition blocks $\hat{v}$.

The strong separability assumption is a strong assumption and it may not be easy to verify for some games. The next example illustrates that oligopoly games with monotonically decreasing demand functions satisfy this property.

Example 2. (Cournot Oligopoly Game) Let $N = \{1, \ldots, n\}$ be the set of firms, $X_i = [0, \hat{y}_i]$ be the production set of firm i where $\hat{y}_i > 0$ denotes firm i ' s capacity constraint, and $\pi_i : X \to \mathbb{R}$ be the profit function of firm i which is defined as

$$\pi_i(x) = f_i(x_i, \tilde{x}_{-i}) = p(x_i + \tilde{x}_{-i})x_i - c_i(x_i),$$

where $\tilde{x}_{-i} = \sum_{j \in N \setminus \{i\}} x_j$, $p : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse demand function and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is firm i's cost function. We prove that if this game is bounded and RS (or RS_N), p is a monotonically decreasing function, and f_i is concave in the two argument for each firm i, then the game has a nonempty β-core.

We first show that the game is strongly separable. Assume the game is RS. Pick $S \in \mathring{\mathcal{N}}$, $\delta_S \in \{\delta_S^{x,v} | (x, v) \in \bar{\mathcal{X}}$ is not in the β-core of $G\}$, and $z_{-S} \in X_{-S}$. Note that $\hat{y}_{-S} \geq z_{-S}$ implies $p(\sum_{i \in S} \delta_{S,i}(\hat{y}_{-S}) + \sum_{i \in -S} \hat{y}_{-S,i}) \leq p(\sum_{i \in S} \delta_{S,i}(\hat{y}_{-S}) + \sum_{i \in -S} z_{-S,i})$. It follows from

$$\pi_i(\delta_S(\hat{y}_{-S}), x_{-S}) =$$
$$p\left(\sum_{j \in S} \delta_{S,j}(\hat{y}_{-S}) + \sum_{j \in -S} x_{-S,j}\right) \delta_{S,i}(\hat{y}_{-S}) - c_i(\delta_{S,i}(\hat{y}_{-S})) \text{ for all } x_{-S} \in X_{-S}$$

that

$$\pi_i(\delta_S(\hat{y}_{-S}), \hat{y}_{-S}) \leq \pi_i(\delta_S(\hat{y}_{-S}), z_{-S}).$$

Since $z_{-S} \in X_{-S}$ is arbitrarily chosen,

$$\sum_{i \in S} \pi_i(\delta_S(\hat{y}_{-S}), \hat{y}_{-S}) = \sum_{i \in S} \inf_{z'_{-S} \in X_{-S}} \pi_i(\delta_S(\hat{y}_{-S}), z'_{-S}).$$

Hence G is strongly separable. An analogous implies that RS can be replaced by RS_N.

Next, we show that π_i is concave on X if and only if f_i is concave in the two argument for each $i \in N$. Pick $i \in N, x, y \in X$ and $\delta \in (0, 1)$. Then, $\pi_i(\delta x + (1 - \delta)y) \geq \delta\pi_i(x) + (1 - \delta)\pi_i(y)$ if and only if $p(\delta x_i + (1 - \delta)y_i + \delta \sum_{j \in N \setminus \{i\}} x_j + (1 - \delta) \sum_{j \in N \setminus \{i\}} y_j)(\delta x_i + (1 - \delta)y_i) - c_i(\delta x_i + (1 - \delta)y_i) \geq \delta(p(x_i + \sum_{j \in N \setminus \{i\}} x_j)x_i - c_i(x_i)) + (1 - \delta)(p(y_i + \sum_{j \in N \setminus \{i\}} y_j)y_i - c_i(y_i))\}$ if and only if $f_i(\delta x + (1 - \delta)y) \geq \delta f_i(x) + (1 - \delta)f_i(y)$.

Therefore, all assumptions of Theorems 3.1 and 3.5 above are satisfied, hence this oligopoly game has a nonempty β-core.

6. Concluding remarks

This paper provides sufficient conditions for the nonemptiness of the β-core of games with transferable utilities and possibly discontinuous payoff functions. In our results, we assume concave payoff functions, which can have discontinuities only at their relative boundaries. We noted in Section 2 that equilibria often exist at the boundaries of the domain, therefore the possibility of discontinuities of a concave function must be taken into account. Yet, there are interesting economic problems that are modeled by payoff functions which are not concave. Our results on the special two-player setting drop, or weaken, the concavity assumption, and also prove that transferability of the utilities is not necessary for a nonemptiness result in this special setting. A possible direction of research is to extend our results by weakening the concavity assumption in general n-player setting.Moreover, generalizing our nonemptiness result for two-player games with nontransferable utilities to general n-player setting is an open problem, even under continuous payoff functions. Finally, there is a rich and evolving literature on games with continuum of players, studying the existence of both cooperative and non-cooperative solutions; see for example the work of [23] and [22] on noncooperative solutions, and the work of [4, 5], [41] and [42] on cooperative solutions of strategic-form games. Generalization of our results to games with continuum of agents is of interest.

7. Appendix

We first present a lemma that is useful in proving the results above.

Lemma 7.1. If a bounded RS (RS_N) game G is strongly separable, then for each $S \in \mathring{\mathcal{N}}$ there exists $(x, v) \in \bar{\mathcal{X}}$ ($v \in \mathcal{V}$) such that for each $\delta_S \in \Delta_S(x, v)$ ($\delta_S \in \Delta_S^N(v)$) there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\inf_{z_{-S} \in X_{-S}} \sum_{i \in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

Proof of Lemma 7.1. Let G be a bounded, RS and strongly separable game. Then for each $S \in \mathring{\mathcal{N}}$, there there exists $(x, v) \in \bar{\mathcal{X}}$ such that for each $\delta_S \in \Delta_S(x, v)$ there exists $\tilde{z}_{-S} \in X_{-S}$ such that

$$\sum_{i \in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}) = \sum_{i \in S} \inf_{z_{-S} \in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}),$$

and, from the definition of infimum,

$$\inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i\in S} u_i(\delta_S(\tilde{z}_{-S}), \tilde{z}_{-S}).$$

Therefore,

$$\inf_{z_{-S}\in X_{-S}} \sum_{i\in S} u_i(\delta_S(z_{-S}), z_{-S}) \leq \sum_{i\in S} \inf_{z_{-S}\in X_{-S}} u_i(\delta_S(\tilde{z}_{-S}), z_{-S}).$$

The proof is analogous for RS_N games. □

We next present two classical results on the core of characteristic function form games with the transferable and nontransferable utilities.

A *transferable-utility (TU) game* is a function $W : \mathcal{N} \to \mathbb{R}$, where $N = \{1, \ldots, n\}$ is the set of players and $\mathcal{N} = 2^{N\backslash\emptyset}$ the set of coalitions. The *core* of a TU game W is defined as

$$\text{Core}(W) = \left\{ v \in \mathbb{R}^n | \sum_{i\in N} v_i \leq W(N) \text{ and } \sum_{i\in S} v_i \geq W(S)\ \forall S \in \mathcal{N} \right\}.$$

A collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *balanced* if for each coalition S, there exists a nonnegative scalar λ_S with $\lambda_S = 0$ if $S \notin \mathcal{B}$ such that for each $i \in N$, $\sum_{S:i\in S} \lambda_S = 1$. A balanced collection of coalitions $\mathcal{B} \subset 2^{\mathcal{N}}$ is *minimal* if it does not have a balanced proper subcollection. Note that the balancing weights of every minimally balanced collection of coalitions are unique; see Kannai [20, p. 361] for references and further details. Next, we state the influential theorem of Bondareva [10] and Shapley [35] which is used to prove our results.

Theorem (Bondareva-Shapley). *A TU game W has a nonempty core if and only if* $\sum_{S\in\mathcal{B}} \lambda_S W(S) \leq W(N)$ *for every minimally balanced collection of coalition* $\mathcal{B}$.

A *nontransferable-utility (NTU) game* is a nonempty-valued correspondence $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^n$. The *core* of an NTU game V is defined as

$$\text{Core}(V) = V(N)\backslash \left(\bigcup_{S\in\mathcal{N}} \text{int}V(S) \right).$$

where $\text{int}V(S)$ is the (topological) interior of the set $V(S)$. An NTU game V is *balanced* if for all balanced collection of coalitions $\mathcal{B}$, $\bigcap_{S\in\mathcal{B}} V(S) \subset V(N)$. We end this paper by stating the beautiful theorem of [33] which is used to prove our results.

Theorem (Scarf). *A balanced NTU game V has a nonempty core if for each coalition S*,

(i) $V(S)$ *is closed,*

(ii) $v' \in \mathbb{R}^n$, $v \in V(S)$ *and* $v'_S \leq v_S$ *imply* $v' \in V(S)$,

(iii) *there exists* $M_S \in \mathbb{R}^{|S|}$ *such that* $v \in V(S)$ *implies* $v_S \leq M_S$.

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