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ON THE CONSUMPTION-SAVING TRADEOFF UNDER NON-EXPONENTIAL DISCOUNTING AND PARTIAL COMMITMENT

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Abstract. We consider a simple saving problem in continuous time, in which the decision maker has a CES utility function, a non-exponential discounting function (similar to discrete-time β-δ preferences), and partial commitment ability. The interest rate is constant and there is a non-negativity constraint on wealth. We demonstrate (i) that there may exist multiple equilibria consisting of linear consumption-saving policies and (ii) that the interest rate, the time-preference, and the commitment ability alone do not pin down whether the decision maker saves or dissaves. These observations complement related results for models in which the decision maker lacks any commitment power.

1. Introduction

The optimal choice between consumption and saving forms the heart of many intertemporal decision problems in economics. In its most elementary specification, namely when a single decision maker with a constant rate of pure time-preference has access to an asset with a constant interest rate, it follows readily (e.g., from the Euler equation) that consumption and wealth must be increasing (decreasing) if the time-preference rate falls short of (exceeds) the interest rate. In many other situations it is also true that the optimal choice between saving and dissaving depends on a simple condition involving only the interest rate and the time-preference of the agent. One exception occurs when the decision maker has a non-exponential discounting function and lacks commitment power. In this case, the agent's behavior is usually described as the outcome of a Markov-perfect Nash equilibrium in a game between multiple selves, of which the agent is assumed to consists. As shown by Phelps and Pollak [11, p. 196] this game can have two equilibria, one in which the agent saves in every period and another one in which she dissaves. Hence, whether the agent saves or dissaves is not fully determined by the interest rate and the agent's time-preference. Phelps and Pollak [11] restrict their analysis to equilibria with linear strategies and they demonstrate that multiplicity of such equilibria can only arise if an optimal solution under full commitment fails to exist.

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Krusell and Smith [7], too, consider a single decision maker with a non-exponential discounting function and no commitment power, but in contrast to Phelps and Pollak [11] they go beyond linear strategies. Using a constructive proof they show that, in general, there is a high degree of equilibrium indeterminacy. Their results seem to support the claim that this model does not allow for robust qualitative predictions about the optimal saving behavior of the agent. Cao and Werning [3], however, point out that the construction used by Krusell and Smith [7] is valid only locally and, consequently, that their results apply to a model in which the agent is restricted to choose asset holdings only from an endogenously determined interval. For the original model without this restriction, Cao and Werning [3] demonstrate that there exists a simple function of the time-preference parameters such that, whenever the value of this function exceeds the interest rate, all equilibria feature dissaving whereas in the opposite case all equilibria display saving. In view of the afore-mentioned findings from Phelps and Pollak [11] it is clear that Cao and Werning [3] must restrict the parameters in such a way that the problem under full commitment has an optimal solution. As a consequence, there cannot exist multiple equilibria consisting of *linear* strategies.

The contributions mentioned above provide the starting point for the present investigation. We stick to the simple formulation of Phelps and Pollak [11] (linear technology, constant elasticity of intertemporal substitution), but we relax the assumption that the agent has no commitment power at all. Instead, we assume *partial commitment* as in Roberds [13], Schaumburg and Tambalotti [14], or Debortoli and Nunes [4]. For clarity of exposition we use the continuous-time framework from Sorger [15], which treats time-preference and commitment power as two distinct characteristics of the agent. In this respect our model differs crucially from those proposed by Harris and Laibson [5, 6] and Cao and Werning [2], in which the commitment horizon of a self coincides with the point in time at which the time-preference changes.

Our findings are all derived from a single example and can be summarized as follows. First, whether it is optimal to save or to dissave depends not only on the interest rate and on the time-preference of the agent but also on her commitment ability. To understand this intuitively, suppose that the agent applies a lower time-preference rate in the distant future than in the near future, as suggested by experimental evidence (see, e.g., Ainslie [1], Loewenstein and Prelec [9], Loewenstein and Thaler [10], or Thaler [18]). If she has access to a strong commitment technology, her behavior will obviously be more strongly influenced by her patience applied in the distant future than by her impatience regarding the near future. In other words, even if time-preference and commitment power are conceptually two different characteristics of the agent (as highlighted by the approach used by Sorger [15] and in the present paper), the effects of these characteristics on the agent's behavior cannot be disentangled. Second and more surprisingly, even for a fixed specification of interest rate, time-preference, and commitment ability, there may exist a linear equilibrium in which the agent saves *and* another one in which she dissaves. More specifically, in contrast to the models studied by Phelps and Pollak [11], Krusell and Smith [7], and Cao and Werning [2,3], ours can have multiple and qualitatively different equilibria with *linear* strategies even under parameter constellations for which an optimal solution under full commitment exists.

The rest of the paper is organized as follows. Section 2 describes the model and defines the solution concept under partial commitment. Section 3 presents and discusses our results. In the appendix we demonstrate that essentially the same results can arise for other specifications of non-exponential discounting, too.

2. Model formulation

2.1. **The basic problem.** We consider a single agent whose lifetime is the infinite interval $\mathbf{T} = [0, +\infty)$ and who faces the tradeoff between consumption and saving. The saving vehicle is an asset with a constant interest rate $r > 0$. The agent's wealth and consumption rate at time $t \in \mathbf{T}$ are denoted by $x(t)$ and $c(t)$, respectively. Hence, it holds for all $t \in \mathbf{T}$ that

$$\dot{x}(t) = rx(t) - c(t). \tag{2.1}$$

We assume that both wealth and consumption must remain non-negative at all times, that is, the constraints

$$x(t) \geq 0\,,\ c(t) \geq 0 \tag{2.2}$$

are imposed for all $t \in \mathbf{T}$.

The agent has an instantaneous utility function of the form

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \tag{2.3}$$

where $\sigma \in (0, 1) \cup (1, +\infty)$ denotes the inverse of the elasticity of intertemporal substitution.[1] Furthermore, we assume that the agent lacks full commitment power and that she has a non-exponential discounting function. Following the bulk of the literature on decision making under non-exponential discounting and lack of commitment, we describe the behavior of the agent by an equilibrium of a game between her multiple selves.[2] In the remainder of this section we compactly describe the details of the approach from Sorger [15], which shares some common features with Harris and Laibson [5, 6] and Cao and Werning [2].

2.2. **Commitment ability.** To formalize limited commitment ability, we assume that the agent consists of a dynasty of countably many autonomous *selves* indexed by $\ell \in \{0, 1, 2, \ldots\}$. Let λ be a strictly positive real number and let $(T_\ell)_{\ell=0}^{+\infty}$ be a sequence of independent random variables, each of which is exponentially distributed with expected value $\mathbb{E}(T_\ell) = 1/\lambda$. Furthermore, we define $t_0 = 0$ and $t_{\ell+1} = t_\ell + T_\ell$ for all $\ell \geq 0$. Self ℓ lives during the interval $[t_\ell, +\infty)$ but controls consumption decisions only during the interval $[t_\ell, t_{\ell+1})$. Correspondingly, we refer to $[t_\ell, +\infty)$ as the lifetime of self ℓ and to $[t_\ell, t_{\ell+1})$ as its control interval. The arrival of a new self at time t_ℓ forms a decision point of the agent, at which she can re-optimize her

[1]The case $\sigma = 1$ would correspond to logarithmic utility and was considered in the general equilibrium setting discussed in [15]. The logarithmic case is actually simpler to handle but is less interesting for the present study because the optimal saving rate is independent of the interest rate.

[2]See, for example, Strotz [17], Phelps and Pollak [11], Pollak [12], Roberds [13], Laibson [8], Schaumburg and Tambalotti [14], Debortoli and Nunes [4], and Sorger [16, chapter 6].

consumption plan for the future. More specifically, we assume that the ℓ-th self of the agent chooses at time t_ℓ a consumption-saving policy to which the agent adheres during the control interval $[t_\ell, t_{\ell+1})$. The end of this interval, $t_{\ell+1}$, will be called self ℓ's commitment horizon and forms the next decision point for the agent.

It follows from our assumptions that the expected length of every control interval is $1/\lambda$ and, hence, that the parameter λ is a measure of the commitment ability of the agent. Small values of λ correspond to an effective commitment technology whereas large values of λ correspond to weak commitment ability. The limiting cases $\lambda \to 0$ and $\lambda \to +\infty$ describe full commitment and no commitment, respectively.

2.3. **Time-preference.** Even if self ℓ cannot make consumption-saving decisions beyond its commitment horizon $t_{\ell+1}$, it cares about consumption throughout its entire lifetime $[t_\ell, +\infty)$. This lifetime is partitioned into self ℓ's youth $[t_\ell, t_\ell + S_\ell)$ and self ℓ's old age $[t_\ell + S_\ell, +\infty)$. The duration of young age, S_ℓ, is an exponentially distributed random variable with expected value $\mathbb{E}(S_\ell) = 1/\mu$, where μ is a strictly positive real number. We assume that $\{T_\ell \,|\, \ell = 0, 1, 2, \ldots\} \cup \{S_\ell \,|\, \ell = 0, 1, 2, \ldots\}$ is a set of independent random variables.

The time-preference of the agent is described by the discounting functions used by her selves. The discounting function of self ℓ is denoted by d_ℓ. Utility derived at time s is discounted back to time $t \leq s$ by the factor $d_\ell(s; t)$. Mimicking the popular discrete-time β-δ model, which has also been used by Phelps and Pollak [11], Krusell and Smith [7], and Cao and Werning [3], we assume that the pure rate of time-preference of the agent, denoted by $\rho > 0$, is constant but that utility derived during the old age of a self is discounted by an additional factor $\beta > 0$. Formally, we assume that the discounting function is given by

$$d_\ell(s;t) = \begin{cases} e^{-\rho(s-t)} & \text{if } t_\ell \leq t \leq s < t_\ell + S_\ell \text{ or } t_\ell + S_\ell \leq t \leq s, \\ \beta e^{-\rho(s-t)} & \text{if } t_\ell \leq t < t_\ell + S_\ell \leq s, \end{cases} \tag{2.4}$$

If $\beta = 1$, this is the standard exponential discounting function. If β is different from 1, however, the discounting function is non-exponential and optimal solutions which are based on it are not dynamically consistent. The discounting function defined in (2.4) has been used by Cao and Werning [2], Harris and Laibson [5, 6], and Sorger [15] and is illustrated in figure 1 for the case where $\beta < 1$ and $t = 0$.

The transition from self ℓ's youth to its old age can be considered as a shock to the agent's time-preference. The parameter μ determines how early in each self's life the shock occurs. For small values of μ, the probability of an early shock is small, and young age is expected to last very long. If μ is large, on the other hand, preference shocks are likely to occur early in the self's life. Both limiting cases $\mu \to 0$ and $\mu \to +\infty$ correspond to exponential discounting at rate ρ.

The assumption that commitment ability and time-preference are described by two independent processes $(T_\ell)_{\ell=0}^{+\infty}$ and $(S_\ell)_{\ell=0}^{+\infty}$, respectively, was introduced by Sorger [15]. Harris and Laibson [5, 6] and Cao and Werning [2], on the other hand, essentially assume that $T_\ell = S_\ell$ holds for all $\ell \in \{0, 1, 2, \ldots\}$. In this situation, the limiting case $\mu \to +\infty$ is referred to instantaneous gratification.

2.4. **Consumption-saving policies.** We distinguish between two possible *modes* for the agent according to whether the self in charge of consumption-saving decisions

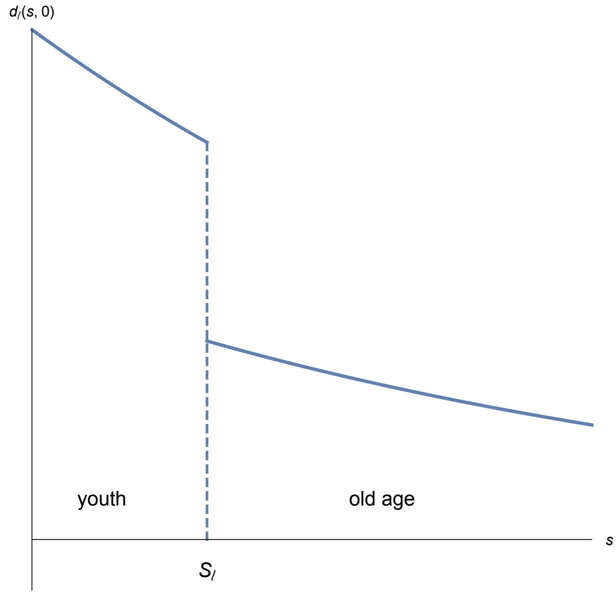

FIGURE 1. The discounting function specified in (2.4) for $\beta < 1$.

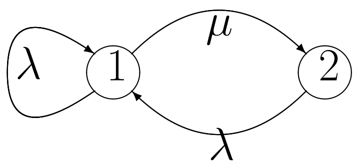

FIGURE 2. Mode transitions for the agent

is young or old. We call these two modes ‘mode 1’ and ‘mode 2’, respectively. Figure 2 visualizes the transitions between the two modes. A transition from mode 1 to mode 2 happens if a preference shock occurs, and transitions to mode 1 (either from mode 1 or from mode 2) occur at decision points, i.e., whenever a new self arrives. The difference between a transition from mode 2 to 1 and a transition from mode 1 to itself is that the former corresponds to the arrival of a new self during the old age of the previous self, whereas the latter corresponds to the arrival of a new self during the youth of the previous self. The rates at which the mode transitions occur are indicated in the figure.

Let us denote the mode of the agent at time $t \in \mathbf{T}$ by $m(t) \in \{1, 2\}$. The *state* of the agent at time t is the pair $(x(t), m(t))$ consisting of her wealth and her mode. A *stationary consumption-saving policy* is a function $g : \mathbf{Z} \mapsto [0, +\infty)$, where $\mathbf{Z} = [0, +\infty) \times \{1, 2\}$ is the state space of the agent. If a self uses such a consumption function, it chooses consumption $c(t)$ according to

$$c(t) = g(x(t), m(t)). \tag{2.5}$$

In other words, the rate of consumption at time $t \in \mathbf{T}$ depends on the agent's asset holdings at time t, $x(t)$, and on the age of the decision-making self at time t, $m(t)$. If an agent uses the consumption-saving policy g, she applies the strategy $g(\cdot, 1)$ when she is controlled by a young self, and she applies the strategy $g(\cdot, 2)$ when she is controlled by an old self.

2.5. **Equilibrium.** To model the behavior of the agent, we adopt the so-called 'sophisticated' approach from the literature on dynamic inconsistency. This means that we consider Markov-perfect equilibria of the intra-personal game played by the separate selves of the agent.

Self ℓ chooses $c(t)$ for all $t \in [t_\ell, t_\ell + T_\ell)$. In doing this, self ℓ correctly anticipates that later selves will act according to *their* preferences, which are inconsistent with self ℓ's preferences because the discounting function is non-exponential. If all selves after ℓ use the consumption-saving policy g, then self ℓ's preference ordering over consumption streams can be represented by the utility functional

$$\mathbb{E}\left[\int_{t_\ell}^{t_\ell+T_\ell} d_\ell(t; t_\ell) u(c(t))\, \mathrm{d}t + \int_{t_\ell+T_\ell}^{+\infty} d_\ell(t; t_\ell) u(g(x(t), m(t)))\, \mathrm{d}t\right], \tag{2.6}$$

where the expectation is taken with respect to the distributions of S_ℓ, T_ℓ, and the mode process $m(\cdot)$. Note that the second term inside the brackets in (2.6) depends on self ℓ's decisions only via $x(t_{\ell+1})$, which is the agent's wealth at time $t_{\ell+1} = t_\ell + T_\ell$ and which forms the initial wealth for self $\ell + 1$.

Self ℓ seeks to maximize the objective functional in (2.6) subject to the constraints (2.1) and (2.2) and a historically given initial capital stock $x(t_\ell) = x$. Let us denote this optimization problem by $P_\ell(x; g)$. A stationary consumption-saving policy g qualifies as a *stationary Markov-perfect equilibrium* if the decision rule (2.5) generates an optimal solution of problem $P_\ell(x; g)$ for all $\ell \in \{0, 1, 2, \ldots\}$ and all $x \geq 0$.

3. Results and implications

In this section we derive a condition on two positive real numbers γ_1 and γ_2, which is sufficient for the stationary consumption-saving policy g defined by

$$g(x, m) = \gamma_m x \tag{3.1}$$

for all $x \in [0, +\infty)$ and all $m \in \{1, 2\}$ to qualify as a stationary Markov-perfect equilibrium. We will then use this condition to obtain results on equilibrium multiplicity and on the optimal consumption-saving tradeoff.

From now on we restrict the parameters in the following way:

$$\sigma < 1\,,\ (1 - \sigma) r < \rho. \tag{3.2}$$

The first inequality in (3.2) says that the elasticity of intertemporal substitution is larger than 1 and implies that the utility function in (2.3) takes only non-negative values. As a consequence, the integrals in (2.6) are well defined (possibly equal to $+\infty$). The second inequality in (3.2) requires the rate of time-preference to be sufficiently large relative to the interest rate r. Since wealth cannot grow at a rate larger than r (due to (2.1) and (2.2)), utility cannot grow at a rate larger than $(1-\sigma)r$ (due to (2.1)-(2.3)). It follows therefore from the inequality $(1-\sigma)r < \rho$ that the objective functional in (2.6) is indeed finite for all feasible solutions. This condition (more precisely, its discrete-time equivalent) must necessarily be *violated* for multiple linear equilibria to exist in Phelps and Pollak [11].[3] By imposing the second assumption in (3.2) we therefore ensure that equilibrium multiplicity in the present setting must have a different source than in Phelps and Pollak [11].

It is also worth mentioning that the parameter restrictions in (3.2) imply that the transversality condition

$$\lim_{s\to+\infty} \mathbb{E}_t \left[d_\ell(s;t)u'(c(s))x(s)\right] = 0 \tag{3.3}$$

holds whenever linear consumption-saving policies such as (3.1) are employed. Indeed, we have $u'(c) = c^{-\sigma} \geq 0$ and, hence,

$$0 \leq d_\ell(s;t)u'(c(s))x(s) \leq \max\{\gamma_1^{-\sigma}, \gamma_2^{-\sigma}\} d_\ell(s;t)x(s)^{1-\sigma}.$$

Since wealth cannot grow at a rate faster than r and the discounting function declines exponentially at rate ρ, it follows from $(1-\sigma)r < \rho$ that (3.3) is satisfied.

Let us determine the continuation value function for an arbitrary self, say self ℓ, under the assumption that all later selves use a consumption-saving policy of the conjectured form. Given an arbitrary time $t \geq t_{\ell+1}$, there are four cases to be distinguished depending on whether self ℓ is young or old at time t and on whether the self in control at time t is young or old. The latter criterion is described by the mode $m(t)$. To capture the age of self ℓ at time $t \in \mathbf{T}$ we write $a_\ell(t) = 1$ if self ℓ is young at time t and $a_\ell(t) = 2$ when self ℓ is old at time t. With this notation, we can write the continuation value function as

$$\begin{aligned} &V_{am,\ell}(x,t) \\ = \;& \mathbb{E}\left[\int_t^{+\infty} d_\ell(s;t)u(\gamma_{m(s)}x(s))\,\mathrm{d}s \;\middle|\; a_\ell(t) = a,\, m(t) = m,\, x(t) = x\right], \end{aligned}$$

where it is understood that $t \geq t_{\ell+1}$ and that wealth evolves according to equation (2.1), that is,

$$\dot{x}(s) = [r - \gamma_{m(s)}]x(s).$$

Since the problem is stationary and all selves are ex ante identical (except for their time of birth), the term $V_{am,\ell}(x,t)$ is independent of ℓ and t. In what follows, we will therefore drop the subscript ℓ and the argument t. The following equations

[3] Cao and Werning [3] also make an assumption that is equivalent to $(1-\sigma)r < \rho$ and emphasize that it rules out multiplicity of equilibria with linear consumption-saving policies. Note that both Phelps and Pollak [11] and Cao and Werning [3] assume that the agent has no (i.e., not even partial) commitment power.

hold:[4]

$$\begin{aligned}
(3.4)\qquad \rho V_{11}(x) &= u(\gamma_1 x) + V_{11}'(x)(r-\gamma_1)x + \mu[V_{12}(x) - V_{11}(x)] \\
&\quad + \mu[\beta V_{21}(x) - V_{11}(x)], \\
(3.5)\qquad \rho V_{12}(x) &= u(\gamma_2 x) + V_{12}'(x)(r-\gamma_2)x + \lambda[V_{11}(x) - V_{12}(x)] \\
&\quad + \mu[\beta V_{22}(x) - V_{12}(x)], \\
(3.6)\qquad \rho V_{21}(x) &= u(\gamma_1 x) + V_{21}'(x)(r-\gamma_1)x + \mu[V_{22}(x) - V_{21}(x)], \\
(3.7)\qquad \rho V_{22}(x) &= u(\gamma_2 x) + V_{22}'(x)(r-\gamma_2)x + \lambda[V_{21}(x) - V_{22}(x)].
\end{aligned}$$

To interpret these equations it will be convenient to refer to figure 3. The four nodes in the graph correspond to the cases

$$(a, m) \in \{(1,1), (1,2), (2,1), (2,2)\}.$$

Consider for example the node in figure 3 which is labelled 11. It describes a situation in which self ℓ is still young ($a = 1$) and the agent is in mode $m = 1$. Now consider the time interval $[t, t + \mathrm{d}t)$ of infinitesimal length $\mathrm{d}t$. The change of total expected utility for self ℓ during this interval consists of four different components that correspond to the four terms on the right-hand side of equation (3.4):

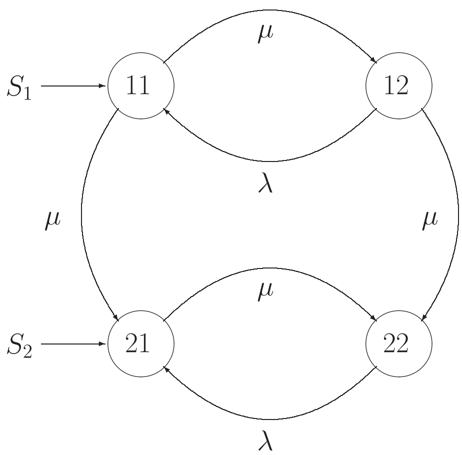

FIGURE 3. Computation of the continuation value function

(i) *Consumption:* Since the agent consumes at rate $c(s) = \gamma_1 x(s)$ when she is in mode 1 at time $s \in [t, \mathrm{d}t)$, the contribution of consumption to total expected utility during the interval $[t, t + \mathrm{d}t)$ is approximately equal to $\mathrm{d}V_{11}(x) = u(\gamma_1 x)\mathrm{d}t$.
(ii) *Saving/dissaving:* Since the agent is in mode 1 for all $s \in [t, t + \mathrm{d}t)$, wealth changes during this interval by approximately $\mathrm{d}x = (r - \gamma_1)x\mathrm{d}t$. Multiplication by the appropriate shadow price of wealth, $V_{11}'(x)$, yields the corresponding contribution to total expected utility, $\mathrm{d}V_{11}(x) = V_{11}'(x)(r - \gamma_1)x\mathrm{d}t$.
(iii) *Preference shock to the self in control:* If the self which is in control of consumption at time t experiences a preference shock, the system moves from node 11

[4]The derivatives of the functions $V_{am}(x)$ with respect to x are denoted by $V_{am}'(x)$.

in figure 3 to node 12.[5] The probability of a preference shock during the interval $[t, t+\mathrm{d}t)$ is approximately equal to $\mu \mathrm{d}t$ and the corresponding change in total expected utility is $V_{12}(x)-V_{11}(x)$. The contribution to total expected utility of self ℓ of a preference shock to the self in control is therefore $\mathrm{d}V_{11}(x) = \mu[V_{12}(x) - V_{11}(x)]\mathrm{d}t$. (iv) *Preference shock to self ℓ:* If self ℓ experiences a preference shock, the system moves from node 11 in figure 3 to node 21. The probability of such a shock during $[t, t+\mathrm{d}t)$ is approximately equal to $\mu\mathrm{d}t$. The corresponding change in total expected utility is $\beta V_{21}(x) - V_{11}(x)$. Hence, the contribution to total expected utility of self ℓ of a preference shock to self ℓ is $\mathrm{d}V_{11}(x) = \mu[\beta V_{21}(x) - V_{11}(x)]\mathrm{d}t$.
Note that the arrival of a new self does not have any effect in node 11, because it does neither change the mode of the agent nor the age of self ℓ. To summarize, the expected rate at which self ℓ derives total utility at time t is given by the right-hand side of equation (3.4). Because self ℓ applies the pure rate of time-preference ρ, this value has to coincide with $\rho V_{11}(x)$. Equations (3.5)-(3.7) have analogous interpretations.

We conjecture that

$$V_{am}(x) = \frac{A_{am}x^{1-\sigma}}{1-\sigma} \tag{3.8}$$

holds for all $(a, m) \in \{1, 2\}^2$ and all $x \geq 0$. It is straightforward to verify that this conjecture is correct if and only if the coefficients A_{am}, $(a, m) \in \{1, 2\}^2$, satisfy the equations

$$\begin{aligned}
\frac{\rho A_{11}}{1-\sigma} &= \frac{\gamma_1^{1-\sigma}}{1-\sigma} + A_{11}(r-\gamma_1) + \frac{\mu}{1-\sigma}(A_{12} - A_{11}) \\
&\quad + \frac{\mu}{1-\sigma}(\beta A_{21} - A_{11}), && (3.9)\\
\frac{\rho A_{12}}{1-\sigma} &= \frac{\gamma_2^{1-\sigma}}{1-\sigma} + A_{12}(r-\gamma_2) + \frac{\lambda}{1-\sigma}(A_{11} - A_{12}) \\
&\quad + \frac{\mu}{1-\sigma}(\beta A_{22} - A_{12}), && (3.10)\\
\frac{\rho A_{21}}{1-\sigma} &= \frac{\gamma_1^{1-\sigma}}{1-\sigma} + A_{21}(r-\gamma_1) + \frac{\mu}{1-\sigma}(A_{22} - A_{21}), && (3.11)\\
\frac{\rho A_{22}}{1-\sigma} &= \frac{\gamma_2^{1-\sigma}}{1-\sigma} + A_{22}(r-\gamma_2) + \frac{\lambda}{1-\sigma}(A_{21} - A_{22}). && (3.12)
\end{aligned}$$

Our next step is to solve the optimization problem of self ℓ. The Hamilton-Jacobi-Bellman equations for this problem are

$$\begin{aligned}
\rho W_1(x) &= \max_{c>0} \big\{u(c) + W_1'(x)(rx-c) + \lambda[V_{11}(x) - W_1(x)] \\
&\quad + \mu[\beta W_2(x) - W_1(x)]\big\}, && (3.13)\\
\rho W_2(x) &= \max_{c>0} \big\{u(c) + W_2'(x)(rx-c) + \lambda[V_{21}(x) - W_2(x)]\big\}, && (3.14)
\end{aligned}$$

[5]Note that preferences of self ℓ are not affected by such a shock.

where $W_m(x)$ denotes the maximal expected lifetime utility of self ℓ when the agent is in mode m and has wealth x. The interpretation of these two equations is similar to that of equations (3.4)-(3.7). Consider for example equation (3.13), which describes the optimal value function if the agent is in mode 1, that is, if self ℓ has not yet experienced its preference shock. The first term in the curly brackets on the right-hand side of this equation is the flow of utility that is derived from consumption. In contrast to equations (3.4)-(3.7), however, consumption is chosen optimally by self ℓ rather than given by the consumption-saving policy g. The second term describes the effect of a change in wealth. The third term captures the arrival of a new self. Such an arrival occurs at rate λ and, when it happens, self ℓ loses control and total expected utility changes from $W_1(x)$ to $V_{11}(x)$. This event corresponds to the arrow labelled S_1 in figure 3. Finally, the fourth term captures a preference shock to self ℓ. Such a shock occurs at rate μ and, when it happens, self ℓ remains in control over consumption but switches from being young to being old. Thus, the change in total expected utility is $\beta W_2(x) - W_1(x)$. Self ℓ tries to maximize the sum of these four terms through its choice of the consumption rate c. The maximal value that can be attained must coincide with the left-hand side of equation (3.13) for the same reason that we have explained in our discussion of equation (3.4). This completes the interpretation of equation (3.13). Equation (3.14) has an analogous interpretation. In this case, however, no preference shock can occur to self ℓ anymore and the arrival of a new self leads to node 21 in figure 3 (arrow S_2) rather than to node 11.

The first-order optimality condition for the optimization problems in equations (3.13)-(3.14) is

$$c^{-\sigma} = W_m'(x). \tag{3.15}$$

If we conjecture the functional form

$$W_m(x) = \frac{B_m x^{1-\sigma}}{1-\sigma},$$

where B_1 and B_2 are positive coefficients, then it follows from (3.15) that $c = (B_m)^{-1/\sigma}x$ and, hence, $\gamma_m = (B_m)^{-1/\sigma}$ hold. Substituting this result as well as (3.8) into the Hamilton-Jacobi-Bellman equations and dividing by $x^{1-\sigma}$ we obtain after simplification

$$[\rho - (1-\sigma)r + \lambda + \mu]\gamma_1^{-\sigma} - \beta\mu\gamma_2^{-\sigma} = \sigma\gamma_1^{1-\sigma} + \lambda A_{11}, \tag{3.16}$$

$$[\rho - (1-\sigma)r + \lambda]\gamma_2^{-\sigma} = \sigma\gamma_2^{1-\sigma} + \lambda A_{21}. \tag{3.17}$$

Let us summarize the arguments made so far in a formal theorem.

Theorem 3.1. *Let condition (3.2) be satisfied and suppose that there exist positive numbers γ_1, γ_2, A_{11}, A_{12}, A_{21}, and A_{22} satisfying equations (3.9)-(3.12) and (3.16)-(3.17). Then it follows that the stationary consumption-saving policy defined in (3.1) qualifies as a stationary Markov-perfect equilibrium.*

Proof. The functions W_m and V_{am} for $(a, m) \in \{1, 2\}^2$ satisfy the equilibrium conditions (3.4)-(3.7) and (3.13)-(3.14) by construction. The first-order condition (3.15) holds for $c = \gamma_m x$. Positivity of the coefficients γ_1, γ_2, A_{11}, A_{12}, A_{21}, and A_{22} ensures that the consumption-saving policy g and the (continuation-)value functions

W_m and V_{am} are feasible. Finally, it has been mentioned before that the parameter restriction (3.2) ensures that the transversality condition (3.3) holds. □

Equations (3.9)-(3.12) are linear with respect to the coefficients A_{11}, A_{12}, A_{21}, and A_{22} and can therefore easily be solved analytically provided that the system matrix is non-singular. However, by substituting the resulting solutions A_{11} and A_{21} into equations (3.16)-(3.17) one obtains a system of two non-linear equations for γ_1 and γ_2, which is in general not analytically solvable. Nevertheless, it is easy to illustrate various possibilities by numerical examples. Throughout the rest of this section we assume that

$$\sigma = 1/2$$

holds. Together with the parameter restriction (3.2) this requires that the inequality $2\rho > r$ is satisfied.

To begin with, let us consider the special case of full commitment $\lambda \to 0$, which can be solved analytically. It is readily seen from (3.17) that for $\lambda = 0$ it must hold that

$$\gamma_2 = 2\rho - r > 0.$$

Substituting this result into (3.16) we obtain the following quadratic equation for $y = \gamma_1^{1/2}$:

$$y^2 + \frac{2\beta\mu y}{\sqrt{2\rho - r}} - 2(\rho + \mu) + r = 0.$$

Because of $2\rho > r$ it follows that this equation has exactly one positive solution, and since $y = \gamma_1^{1/2} > 0$ must hold, we therefore obtain

$$\gamma_1 = \left[-\frac{\beta\mu}{\sqrt{2\rho - r}} + \sqrt{\frac{\beta^2\mu^2}{2\rho - r} + 2(\rho + \mu) - r} \right]^2 > 0.$$

With these values for γ_1 and γ_2 one can easily solve the system (3.9)-(3.12) for the corresponding coefficients A_{11}, A_{12}, A_{21}, and A_{22}. According to theorem 3.1 this solution defines a stationary Markov-perfect equilibrium provided that $A_{am} > 0$ holds for all $(a, m) \in \{1, 2\}^2$. Finally, let us mention that the case $\lambda = 0$ corresponds to full commitment, such that the problem of the agent is not a game but a simple optimization problem. This maximization problem has a strictly concave objective functional and linear constraints so that it can have at most one optimal solution.

To illustrate these arguments, suppose that the remaining parameters are given by $\rho = 1$, $\beta = 1/2$, $\mu = 1$, and $r = 3/2$. Then we obtain

γ_1	γ_2	A_{11}	A_{12}	A_{21}	A_{22}
1.05051	0.50000	0.95658	0.94281	1.37398	1.41421

Obviously, this solution satisfies $\gamma_2 < \gamma_1 < r$ so that the agent saves throughout her lifetime. To summarize, under the present parameter specifications there exists a unique equilibrium, this equilibrium consists of a linear consumption-saving policy, and the agent saves for all $t \in \mathbf{T}$.

Now let us change the commitment parameter λ from 0 to 1 while maintaining all other parameter values as before. Thus, we consider the situation described by $\sigma = \beta = 1/2$, $\rho = \lambda = \mu = 1$, and $r = 3/2$. In this case we obtain two feasible solutions of (3.9)-(3.12) and (3.16)-(3.17), which are stated in the following table.

γ_1	γ_2	A_{11}	A_{12}	A_{21}	A_{22}
1.08959	0.53493	0.94997	0.93943	1.34338	1.36726
1.67076	4.02933	0.84533	0.77949	1.07216	0.94327

In the equilibrium corresponding to the first line it holds that $\gamma_2 < \gamma_1 < r$, whereas in the second line we have $\gamma_2 > \gamma_1 > r$. As a consequence, there is saving in the first equilibrium and dissaving in the second one.

What are the implications of the above example? First of all note that changing λ from 0 to 1 affects the commitment ability of the agent but not her time-preference. We can therefore conclude that the optimal tradeoff between consumption and saving depends not only on the interest rate and the time-preference of the agent but also on her commitment ability. Intuitively, this is not hard to understand because an improvement of the commitment technology means that the self in charge will put more emphasis on the distant future, which is discounted more heavily than the near future. This effect, however, is not detectable if the commitment horizon coincides with the time of the preference shock which, to the best of our knowledge, is true for all models in the literature except for Sorger [15] and the present paper.

A second implication of the above example is that, even with a fixed commitment technology, the size of the interest rate together with the specification of time-preference do not pin down whether the agent saves or dissaves. This is reflected by the existence of two equilibria for the situation in which $\lambda = 1$ holds, one featuring saving and the other one dissaving. Although this observation has already been made by Phelps and Pollak [11] and Cao and Werning [3], there are important differences between their examples and ours. In contrast to Phelps and Pollak [11], we obtain this result even in the case where the problem under commitment has a solution, i.e., under the restriction $(1-\sigma)r < \rho$. And in contrast to Cao and Werning [3], our example is one of multiplicity of *linear* equilibria whereas they show that, in addition to a unique linear equilibrium, there may exist others involving discontinuous strategies.

It is worth pointing out that our example involves a value of σ, which is smaller than 1, namely, $\sigma = 1/2$. We have also tried to detect multiple linear equilibria for parameter constellations with $\sigma > 1$, but were unsuccessful. Hence, there seems to be a similarity to the results derived by Cao and Werning [3], who could show that the ambiguity of saving versus dissaving only arises for σ smaller than some threshold that is less than 1.

Finally, to check for the robustness of our results, we have carried out an analogous analysis for an alternative non-exponential discounting function, namely

$$d_\ell(s;t) = \begin{cases} e^{-\rho(s-t)} & \text{if } t_\ell \le t \le s < t_\ell + S_\ell, \\ e^{-\rho(S_\ell - t)} e^{-\delta(s-S_\ell)} & \text{if } t_\ell \le t < t_\ell + S_\ell \le s, \\ e^{-\delta(s-t)} & \text{if } t_\ell + S_\ell \le t \le s. \end{cases} \tag{3.18}$$

This discounting function is illustrated in figure 4 for the case where $\rho > \delta$ and $t = 0$ hold. Note that, in contrast to the specification from (2.4), this discounting function is continuous with respect to s but it involves different pure rates of time-preference. In the example depicted in figure 4 it holds that $\rho > \delta$ so that self ℓ applies a higher time-preference rate during its youth than during old age. The results for this specification, which are briefly described in the appendix, are qualitatively equivalent to those that we have described above.

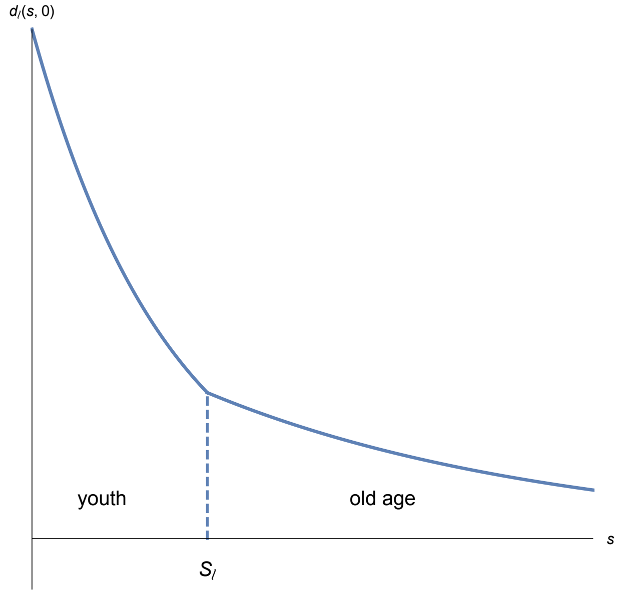

FIGURE 4. The discounting function from (3.18) for $\rho > \delta$.

Appendix

The analysis of the model in which the discounting function is given by (3.18) proceeds analogously to the analysis of the case where (2.4) holds so that we restrict ourselves to the main steps.

The key equations (3.9)-(3.12) and (3.16)-(3.17) are now given by

$$\begin{aligned}
\frac{\rho A_{11}}{1-\sigma} &= \frac{\gamma_1^{1-\sigma}}{1-\sigma} + A_{11}(r-\gamma_1) + \frac{\mu}{1-\sigma}(A_{12}-A_{11}) + \frac{\mu}{1-\sigma}(A_{21}-A_{11}),\\
\frac{\rho A_{12}}{1-\sigma} &= \frac{\gamma_2^{1-\sigma}}{1-\sigma} + A_{12}(r-\gamma_2) + \frac{\lambda}{1-\sigma}(A_{11}-A_{12}) + \frac{\mu}{1-\sigma}(A_{22}-A_{12}),\\
\frac{\delta A_{21}}{1-\sigma} &= \frac{\gamma_1^{1-\sigma}}{1-\sigma} + A_{21}(r-\gamma_1) + \frac{\mu}{1-\sigma}(A_{22}-A_{21}),\\
\frac{\delta A_{22}}{1-\sigma} &= \frac{\gamma_2^{1-\sigma}}{1-\sigma} + A_{22}(r-\gamma_2) + \frac{\lambda}{1-\sigma}(A_{21}-A_{22}),
\end{aligned}$$

and

$$\begin{aligned}
[\rho-(1-\sigma)r+\lambda+\mu]\gamma_1^{-\sigma} - \mu\gamma_2^{-\sigma} &= \sigma\gamma_1^{1-\sigma} + \lambda A_{11},\\
[\delta-(1-\sigma)r+\lambda]\gamma_2^{-\sigma} &= \sigma\gamma_2^{1-\sigma} + \lambda A_{21}.
\end{aligned}$$

The solution procedure is completely analogous to the case presented in the main text. For $\lambda = 0$ it is again possible to solve the equations analytically. Evaluating this solution for the parameters $\rho = 2$, $\delta = \mu = 1$, and $r = 3/2$ yields

γ_1	γ_2	A_{11}	A_{12}	A_{21}	A_{22}
1.28890	0.50000	0.85496	0.84853	1.34578	1.41421

If instead of $\lambda = 0$ we use $\lambda = 1$, then there exist two feasible solutions given by

γ_1	γ_2	A_{11}	A_{12}	A_{21}	A_{22}
1.38803	0.56230	0.83593	0.82675	1.29202	1.33357
2.61940	4.35878	0.71988	0.68199	0.98199	0.89513

It is obvious that this example gives rise to the same conclusions as those presented in the main text for the specification (2.4).

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