Pure and Applied Functional Analysis
Volume 6, Number 4, 2021, 851–855

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

INERTIA IMPLIES DISORDER: KINKS, INCOMPLETENESS, AND INTRANSITIVITY

EDWARD E. SCHLEE

Abstract. If every point in the consumption set $X = \mathbb{R}^\ell_+$ is supportable as a demand by more than one normalized positive price vector, and the asymmetric part of the reflexive binary relation that generates the demand is open, then the binary relation is incomplete or intransitive, that is, *non-ordered*.

1. Introduction

Classical demand theory in economics posits that consumers choose a consumption point that is undominated according to a given binary relation, subject to a budget constraint asserting that the consumption point be affordable at given prices and wealth. Economists interpret the binary relation as describing a consumer's preferences over consumption points. I define *inertia* in the theory of the consumer to mean that the consumer's choice does not change as the consumer's budget set changes (in particular ways). Such inertia has been empirically documented;[1] it is the limiting case of something more general, that consumers' choices sometimes change little when prices change, an observation that for example lies at the heart of the equity premium puzzle as formulated in [12]—asset prices vary a lot, aggregate consumption only a little. Geometrically, inertia at a point in my sense implies "kinks" on the boundary of a consumer's better-than-set at that point.

Rigotti and Shannon [14] demonstrate the possibility of "robust indeterminacies" in competitive equilibria. Specializing their Theorem 6 to the case of a single-consumer economy, every consumption point is supported as a demand for more than one normalized price vector; that is, inertia exists globally. They assume a particular form of *incomplete preferences* attributed to Bewley [1].[2] Incompleteness means that the binary relation that rationalizes demand is not complete: two or more consumption points might be unranked by the binary relation. Bewley's [1] model applies to a setting in which there is a single physical good and a finite number $S \geq 2$ of states of the world, a special case of Debreu's consumer model in [4, Chapter 7], absent completeness. A consumption choice then is a point $x \in \mathbb{R}^S_+$. In Bewley's model, a consumption x is strictly better than y if and only if, for some real-valued function u on $\mathbb{R}_+$, $\sum_s u(x_s)\pi_s > \sum_s u(y_s)\pi_s$ for every $(\pi_1, ..., \pi_S) \in \Pi \subset$

2020 *Mathematics Subject Classification.* 46N10.
Key words and phrases. Inertia, kinks, un-ordered preferences.

[1]Some evidence is mentioned in [14].

[2]Bewley [1] circulated as a working paper in 1986. I note that I use "inertia" differently than Bewley and Rigotti-Shannon.

$\{r \in \mathbb{R}^S_+ | \sum_{s=1}^S r_s = 1\}$, $\Pi \neq \emptyset$. The set Π is interpreted as the set of a consumer's beliefs about the unknown state in S. When Π contains more than one element, the boundary of a consumer's better-than-set for any point is kinked; see Figure 1 in Rigotti and Shannon [14].

Clearly, other models of incomplete preferences can generate kinks and so inertia (almost) everywhere.[3] I ask a converse: suppose that every consumption is supported as a demand with more than one normalized price, that is, the boundary of a consumer's better-than-set at any point is kinked; does it follow that the relation describing the consumer's preferences is incomplete? The answer is no. What does follow is that it is incomplete *or* intransitive.

2. Inertia and non-ordered preferences

For $X = \mathbb{R}^\ell_+$, let $\succsim \subset X \times X$ be a *reflexive* binary relation ($x \succsim x$ for every $x \in X$) with asymmetric part $\succ$ and symmetric part $\sim$. The interpretation is that X is a *consumption set* for a consumer–the set of consumption points that a consumer could choose, ignoring the budget constraint. Let $\Delta = \{p \in \mathbb{R}^\ell_{++} \mid \sum_i p_i = 1\}$, the set of normalized positive prices. For $p \in \Delta$ and $w \geq 0$ the consumer's budget set is $B(p,w) = \{x \in X \,|\, p \cdot x \leq w\}$. The consumer's *demand correspondence* is $d(p,w) = \{y \in B(p,w) \,|\, z \succ y \Rightarrow z \notin B(p,w)\}$ and the *inverse demand correspondence* is $g(x) = \{p \in \Delta \,|\, x \in d(p, p \cdot x)\}$. A relation $\succsim \subset X \times X$ is *complete* if, for every x and y in X, either $x \succsim y$ or $y \succsim x$; it is *transitive* if, for every x, y, and z in X with $x \succsim y$ and $y \succsim z$, it follows that $x \succsim z$; it is *locally nonsatiated* if, for every $x \in X$, and open neighborhood N of x, there is a $y \in X \cap N$ with $y \succ x$. Local nonsatiation ensures that every demand point exhausts the consumer's budget: $x \in d(p,w)$ implies $p \cdot x = w$.

Definition. A reflexive binary relation $\succsim \subset X \times X$ displays **inertia at $\mathbf{x} \in \mathbf{X}$** if $g(x)$ contains at least two elements. It displays **inertia** if it displays inertia at every $x \in X$.

Theorem. Suppose that $\succsim \subset X \times X$ is reflexive and locally nonsatiated, and $\succ$ is open. The relation $\succsim$ displays inertia only if it is not complete or not transitive.[4]

The Theorem is a consequence of this Proposition.

Proposition. If $\succsim$ is represented by a locally nonsatiated continuous function, then $\succsim$ cannot display inertia.[5]

Proof. Suppose that (i) $\succsim$ is representable by a locally nonsatiated continuous function U and (ii) for every $x \in \mathbb{R}^\ell_+$, $g(x)$ is not empty. I will show that there is some $\hat{x} \in X$ with $g(\hat{x})$ a singleton, so $\succsim$ does not display inertia.

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[3] For example, the extension in [13] of the multi-expected-utility model in [5] for preferences over *distributions* of consequences to preferences over Savage *acts*.

[4] I do not in addition assume that $\succsim$ is closed, since I want to allow the possibility that $\succsim$ is both incomplete and transitive. Schmeidler [15] proves a remarkable theorem: if X is a connected topological space, $\succsim \subset X \times X$ is closed, $\succ$ is open, $\succsim$ is transitive, and $x \succ y$ for some x and y in X, then $\succsim$ is complete.

[5] A real-valued function U on X *represents* $\succsim$ if $x \succsim y$ if and only if $U(x) \geq U(y)$. A representation U of $\succsim$ is locally nonsatiated if $\succsim$ is locally nonsatiated.

To begin, it follows from (i) and (ii) that the function U in (i) is strictly increasing ($x > y$ implies $U(x) > U(y)$) and quasiconcave (for any x and y in X and $\lambda \in [0,1]$, $U(\lambda x + (1-\lambda)y) \geq \min\{U(x), U(y)\}$). If U is not strictly increasing, then there are points x and y in $\mathbb{R}^\ell_+$ with $x > y$ and $U(x) \leq U(y)$. By (ii) there is a price $p \in \Delta$ with $x \in d(p, p\cdot x)$. But since $U(y) \geq U(x)$ and $p\cdot y \leq p\cdot x$, $y \in d(p, p\cdot x)$; since $p\cdot y < p\cdot x$, local nonsatiation fails. If U is not quasiconcave, then there are points x and y in $\mathbb{R}^n_+$ and a real number $\lambda \in (0,1)$ such that $U(\lambda x + (1-\lambda)y) < \min\{U(x), U(y)\}$. For any $p \in \Delta$, $\min\{p\cdot x, p\cdot y\} \leq p\cdot(\lambda x + (1-\lambda)y)$. So $\lambda x + (1-\lambda)y \notin d(p, p\cdot(\lambda x + (1-\lambda)y))$ for any $p \in \Delta$, and $\succsim$ does not satisfy (ii). It follows that U is quasiconcave and strictly increasing if (i) and (ii) hold.

Let $x_{-\ell} = (x_1, ..., x_{\ell-1})$. For each u in the range of U, define a real-valued function $h_u \subset \mathbb{R}^{\ell-1}_+ \times \mathbb{R}_+$ by $(x_{-\ell}, x_\ell) \in h_u$ if and only if $U(x_{-\ell}, x_\ell) = u$, and let D_u be the domain of h_u. Since U is strictly increasing, h_u is indeed a function; it characterizes the indifference surface passing through any point x with $U(x) = u$. Fix $x^* \in \mathbb{R}^\ell_{++}$ and set $u^* = U(x^*)$. Since $x^* >> 0$, and since U is continuous and strictly increasing, $x^*_{\ell-1} \in int(D_{u^*})$, the interior of D_{u^*}.[6] Let B be an open ball centered on $x^*_{-\ell}$ and contained in D_{u^*}. Since U is quasiconcave and strictly increasing, h_{u^*} is a convex function on the convex set B, and so is differentiable (Lebesgue) almost-everywhere on B. Let $\hat{x}_{-\ell} \in B$ be a point of differentiability of h_{u^*}. Let $\hat{x}_\ell = h_{u^*}(\hat{x}_{-\ell})$, $\hat{x} = (\hat{x}_{-\ell}, \hat{x}_\ell)$, and $\hat{p} \in g(\hat{x})$.

For any $x_{-\ell} \in D_{u^*}$, $\hat{x}_{-\ell}$ minimizes $\hat{p}_{-\ell} \cdot x_{-\ell} + \hat{p}_\ell h_{u^*}(x_{-\ell})$ on D_{u^*}.[7] Since h_{u^*} is differentiable at $\hat{x}_{-\ell}$, and $\hat{x}_{-\ell} \in int(D_{u^*})$, the first order necessary condition for minimization must hold for $i = 1, ..., \ell-1$: $\hat{p}_i + \hat{p}_\ell \frac{\partial h_{u^*}}{\partial x_i}(\hat{x}_{-\ell}) = 0$. But these $\ell - 1$ equalities can be satisfied by at most $\hat{p} \in \Delta$, so $g(\hat{x}) = \{\hat{p}\}$, and $\succsim$ does not display inertia. □

Proof of the Theorem. Suppose that $\succsim$ is complete, transitive, and locally nonsatiated, $\succ$ is open, and $g(x)$ is not empty for every $x \in X$. By a standard result, since $X = \mathbb{R}^\ell_+$ and $\succsim \subset X \times X$ is complete, transitive, and continuous, there is a continuous function U which represents it (e.g. [4, Chapter 4]). Since $\succsim$ is locally nonsatiated, so is U. By the Proposition, $g(x)$ is a singleton for some $x \in X$, so $\succsim$ does not display inertia. □

An alternative proof of the Proposition could potentially be constructed from a result in [2] that every quasiconcave function is Lebesgue-almost-everywhere differentiable. A difficulty is that a point of differentiability can be a critical point; and

[6]Suppose that $x^*_{-\ell}$ is not in the interior of D_{u^*}. I will show that if U is continuous, then it cannot be strictly increasing. Since $x^*_{\ell-1}$ is in D_{u^*}, it must be in the boundary of D_{u^*}. So there is a sequence z^n in $\mathbb{R}^{\ell-1}_{++}$ not in D_{u^*} that converges to $x^*_{-\ell}$. For each n we cannot have $U(z^n, 0) \leq u^* \leq U(z^n, x^*_\ell + 1)$; otherwise, since U is continuous, there would be a point x_ℓ between 0 and $x^*_\ell + 1$ with $U(z^n, x_\ell) = u^*$ and so $z^n \in D_{u^*}$. It follows that there is a converging subsequence $z^{n(k)}$ with either $U(z^{n(k)}, 0) > u^*$ for every k, or $U(z^{n(k)}, x^*_\ell + 1) < u^*$ for every k. Since U is continuous, in the first case $U(z^{n(k)}, 0) \to U(x^*_{-\ell}, 0) \geq u^* = U(x^*)$, and in the second case $U(z^{n(k)}, x^*_\ell + 1) \to U(x^*_{-\ell}, x^*_\ell + 1) \leq u^* = U(x^*)$; since $x^*_\ell > 0$, in either case U cannot be strictly increasing.

[7]Specifically, $\hat{x} \in d(\hat{p}, \hat{p} \cdot \hat{x})$ only if $\hat{x}$ minimizes $\hat{p} \cdot x$ on $\{x \in X | x \sim \hat{x}\}$, a standard result that follows if $X = \mathbb{R}^\ell$ and $\succsim$ is complete, transitive, and locally nonsatiated.

it is clear that a critical points can sometimes be supported by more than one price in Δ.

The Theorem does not assert that $\succsim$ is incomplete. To illustrate why incompleteness does not follow, suppose that $\succsim$ exhibits inertia, is not complete, and the other assumptions of the Theorem hold. Define $\succsim^*$ by $x \succsim^* y$ if $\neg(y \succ x)$. Then $\succsim$ and $\succsim^*$ generate the same demand, but $\succsim^*$ is complete (and so by the Theorem is intransitive).[8] In this sense, from the viewpoint of demands, completeness or its lack cannot be inferred from the demand correspondence unless transitivity is maintained.[9]

It is clear from the proof of the Proposition that there is more to say about the extent of inertia when preferences are complete and transitive. To formulate one such result, let U be a continuous, strictly increasing, and quasiconcave representation of $\succsim \subset X \times X$ (so that $g(x)$ is nonempty for every $x \in X$). As in the proof, let h_u be a function describing the indifference surface given implicitly by $U(x_{-\ell}, x_\ell) = u$, let D_u be its domain, and μ be $(\ell - 1)$-dimensional Lebesgue measure. Then for any open convex subset S of D_u, h_u is convex on S, and so differentiable μ-almost everywhere; applying the argument in the proof to each point of differentiability, the set of points that exhibit inertia in S is of μ-measure zero. The conclusion extends to countable unions of such convex sets, illustrating a sense in which inertia is rare in the standard consumer choice model with complete and transitive preferences.[10]

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[8] Kim and Richter [8, p. 384] and McKenzie [11, p. 830] use $\succsim^*$ to point out that incompleteness cannot be inferred from demands.

[9] Khan and Schlee [7, Section 3.8] give some background on non-transitive consumer theory. Some argue that a consumer with intransitive (and continuous) preferences can fall prey to a "money pump," induced to trade from a consumption point x back to x *minus* some quantity of goods in a sequence of trades (when $x \succ y \succsim z \succsim x$), and see this conclusion as a foundation for maintaining transitivity (for example, [3, p. 145] and [16]). These arguments appear to depend on implicit assumptions about how a consumer chooses in sequential decision problems that are not directly implied by intransitivity; Fishburn [6, Section 2.7] and Machina [9] point to such implicit assumptions.

[10] Machina [10] categorizes different sorts of kinks in complete and transitive decision models under risk.

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Manuscript received January 2 2019
revised February 8 2020

E. E. Schlee
Department of Economics, Arizona State University[11]
E-mail address: ed.schlee@asu.edu

[11] I thank M. Ali Khan and Edi Karni for many hours of discussions about incomplete preferences, and a referee for expositional comments.