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COOPERATIVE GREY GAMES: AN APPLICATION TO A FLOW SITUATION

EMAD FATHI HUSSIEN QASIM, SIRMA ZEYNEP ALPARSLAN GOK, OSMAN PALANCI, AND GERHARD WILHELM WEBER

Abstract. Natural and man-made disasters swept through various parts of the world and received a lot of attention over the past decades. Meanwhile, many parts of the world suffer from the lack of basic necessities, including shelter, water, food, education, access to basic health care and safety. Effective logistics operations are a critical component of addressing these needs to continue improving their own organization's efforts that are consistently part of a strategic model for successful humanitarian relief operations. In the sequel, uncertainty may affect a given location in humanitarian logistics such as lack an information about availability of resources, infrastructure to address needs, transport planning, reception and distribution of emergency supplies, type and quantity of the resources, and way of procurement and storage of the supplies.

In this paper, we give an application to a flow situation related with a logistic network after an earthquake. To do this, we use cooperative grey game theory with grey solutions to improve humanitarian logistics and to find a fair cost allocation between private organizations for supporting the relief operations under uncertainty.

1. Introduction

Istanbul is the largest city in Turkey and the heart of the country. It is also one of the largest agglomerations in Europe and the fifth largest city in the world in terms of population within city limits. Interestingly, Istanbul is a transcontinental city as it is located on the Bosphorus waterway in Northwest Turkey between the Black Sea and the Sea of Marmara. This means that the commercial centre is in Europe while the rest of the city is in Asia. Istanbul has grown very rapidly over the past one hundred years, although it has always had a large population. Istanbul has remained one of the largest cities in the world for most of its long history.

Geologically, Turkey is located at the boundary area where the Arabian Plate and African Plate are moving north towards the Eurasian Plate. A large scale fault line called North Anatolian Fault (NAF) is formed more than 1000 km long from East to West in the Northern territory of Turkey, and historically, many strong earthquakes have occurred along this fault line. In recent years (1939 and 1992), there was heavy damage to property, including the collapse of a number of buildings

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and infrastructures. The main industrial and commercial capital of Turkey Istanbul with a population of close to 15 million is predicted to be affected by a worst-case scenario earthquake of 7.5 in the coming decades.

In order to manage the potential earthquake disaster in Istanbul, it is necessary to prepare a seismic disaster prevention/mitigation plan, emergency rescue plan and restoration plan of the earthquake stricken area from middle to long-term points of view.

On the other hand, there are a lot of uncertainties about a disaster, when it happens, or how much it affects us, in addition to the fact that rescue demand is uncertain as well. Once rescue demand is requested, rescue equipment and relief personnel must be transported efficiently to minimize the damage in affected areas. At the preparation strategy planning stage of earthquake emergency supplying, the authorities must contemplate all doable circumstances prior to and create prudent and diligent emergency supplying plans to distribute the equipment. Disasters result in massive demands that often outstrip resources. In order to manage the potential earthquake disaster in Istanbul, it is necessary to prepare a seismic disaster prevention/mitigation plan, emergency rescue plan and restoration plan of the earthquake stricken area from middle to long-term points of view. Controlling the flow of those resources, to provide relief to the affected people is called emergency logistics [4, 7, 10]. Quick response to the urgent relief needs right after natural disasters through efficient emergency logistics distribution is vital to the mitigation of the disaster impact in the affected areas [10]. Logistics support is one of the major activities in disaster response. Commodities such as food, shelter and medicine must be sent from the supply centers to the affected area as quick as possible to support rescue operation and help wounded people. Furthermore, important or hazardous materials must be transferred from the affected areas to safety areas [12]. In recent years, Emergency Logistics is a new field of logistics which plays a key role in relief after disasters, has got much attention and become a new efficient methodology in dealing and analysis of many different situations that can be modeled as Operations Research (OR) problems. The problems arising in logistics are related to transportation, inventory, supply chains, distribution, location, routing, etc. [2].

Most of the facilities and distribution options that perform the functions of supply of materials, transformation of these materials into intermediate and finished products, and the distribution of these finished products to customers altogether constitute a logistic network, which is the subject of Supply Chain Management (SCM) [9]. Supply chain management (SCM) is the broad range of activities required to plan, control and execute a product's flow, from acquiring raw materials and production through distribution to the final customer, in the most streamlined and cost-effective way possible. For uncertainty and disturbed environments and the unrecognized characteristics of the events, it is hard to make urgent decision for relief demands in the context of emergency management.

A flow is a way of sending objects from one vertex (place) to another in a network. The objects that travel or flow through the network are called flow units or just units. For example, flow units can be a commodity, finished goods,or information. The nodes from which units enter through a network are called source nodes, and nodes to which the flow units are routed to are called sink nodes. Source nodes offer a supply, which is represented by the number of units available at the node. Sink nodes usually have demand, which is represented by the number of units that must be routed to them [9].

A game-theoretical approach can be used to tackle the SCM situations and problems from a cooperative that we focus on it in this study.

The most important issue for a logistic network is how to distribute and transport inventory on time under uncertainty. The cooperation in an SCM can reduce and increase both the logistic network ' s total cost and gain, respectively [9].

Games, which are derived from a flow situation under uncertainty, are called grey flow games. Cooperative Grey Game Theory conduces to completing the analysis of OR problems when there is more than one player in the corresponding situation. Therefore, after optimizing a particular system by means of OR techniques, in the system there are two or more players, who have to collaborate to be able to achieve an optimal result. That result then will tell how to distribute the extra benefits, or how costs are saved by cooperation among those agents seems reasonable and necessary. Thereupon, cooperative grey games can play a role in the complete analysis of the situation [8].

In literature the classical theory of two-person cooperative games is extended to two-person cooperative games with interval uncertainty in [1]. Further, the core, balancedness, superadditivity and related topics are studied. Moreover, solutions called ψ^{α}-values are introduced and characterizations are given. In [5], the stability and stabilization of a grey system whose state matrix is triangular is studied. The displacement operator and established transfer developed by the author are an indispensable tool for the grey system. In the sequel, a new class of cooperative games where the set of players is finite and the coalition values are grey numbers, is developed in [8]. In [8], an interesting solution concept, the grey Shapley value, is introduced and characterized with the properties of additivity, efficiency, symmetry and dummy player.

In this paper, we prepare a cooperative game theoretical model for emergency logistics to coordinate logistic supports for relief operations. The situation of the model relies on the logistic network consisting of one provider that gives the help materials when natural disasters, three countries as distributors and one retail merchant of the country stricken by the disaster (presumably Istanbul). The model essentially integrates two issues, namely, a provision network downside and transportation downside. By modifying the cooperative scientific theory approach, we have a tendency to provide some resolution ideas and compare the allocations. Herewith, the planned model seems to be economically associated, demonstrating the importance is economical for the importance of cooperation between countries, once a disaster happened. Furthermore, we model the grey flow problem by using Cooperative Grey Game Theory as an approach to solve the logistic network problem.

The rest of paper is organized as follows: We give some basic notions and solution concepts from cooperative grey games and flow situations in Section 2. The definition of the grey solutions necessary for this study is offered in Section 3. In Section 4, we present our cooperative grey flow game model, constructed after an earthquake as a natural disaster in reality it is a societal problem, and present several solutions for maximizing the transported commodities. Moreover, we give an example and compare our solutions. Finally, Section 5 completes this paper with a conclusion and an outlook to future studies.

2. Preliminaries

In this section, we formally give some basic concepts and notions from cooperative games and flow situations in order to provide the reader with all the necessary background to follow this paper.

Grey system theory [5], originally developed by Professor Deng in 1982, has become a very effective method of solving uncertainty problems under discrete data and incomplete information. Grey system theory has now been applied to various areas such as forecasting, system control, decision-making and computer graphics. Here, we give some basic definitions regarding relevant mathematical background of grey numbers in grey system theory:

A grey number takes an unknown distribution between fixed lower and upper bounds, denoted as $\otimes \in [\underline{a}, \overline{a}]$, , where $\underline{a}$ and $\overline{a}$, are respectively, the lower and upper bounds for $\otimes$.

Let $\otimes_1 \in [\underline{a}, \overline{a}]$,$\otimes_2 \in [\underline{b}, \overline{b}]$ and α is a positive real number, then

$$\otimes_1 \in [\underline{a}, \overline{a}] + \otimes_2 \in [\underline{b}, \overline{b}] \Leftrightarrow \otimes_1 + \otimes_2 \in [\underline{a} + \underline{b}, \overline{a} + \overline{b}].$$

The scalar multiplication of α and $\otimes$ is defined as follows:

$$\alpha\otimes \in [\alpha\underline{a}, \alpha\overline{a}].$$

We denote by $\mathcal{G}(\mathbb{R})$ the set of interval grey numbers in $\mathbb{R}$. Let $\otimes_1, \otimes_2 \in \mathcal{G}(\mathbb{R})$ with $\otimes_1 \in [\underline{a}, \overline{a}]$,$\otimes_2 \in [\underline{b}, \overline{b}]$, $\mid \otimes_1 \mid = \underline{a} - \overline{a}$ and $\alpha \in \mathbb{R}_+$. In general, the difference of $\otimes_1$ and $\otimes_2$ is defined as follows:

$$\otimes_1 \ominus \otimes_2 = \otimes_1 + (-\otimes_2) \in [\underline{a} - \underline{b}, \overline{a} - \overline{b}].$$

Different from the above subtraction we use a partial subtraction operator. We define $\otimes_1 \ominus \otimes_2$, only if $|\overline{a} - \underline{a}| \geq |\overline{b} - \underline{b}|$, by $\otimes_1 - \otimes_2 = [\underline{a} - \underline{b}, \overline{a} - \overline{b}]$ [1].

A cooperative grey game is an ordered pair $\langle N, w' \rangle$ with the player set $N = \{1, ..., n\}$ in which $w' = \otimes : 2^N \to \mathcal{G}(\mathbb{R})$ is the grey payoff characteristic function such that $w'(\emptyset) = \otimes_\emptyset \in [0, 0]$, grey payoff function $w'(S) = \otimes_S \in [\underline{A_S}, \overline{A_S}]$ refers to the valuing area of the grey expectation benefit which belongs to a coalition $S \in 2^N$, where $\underline{A_S}$ and $\overline{A_S}$ represent the maximum and minimum possible profits of the coalition S. So, a cooperative grey game can be considered as a classical cooperative game with grey profits $\otimes$.

Grey solutions are useful to solve reward/cost sharing problems with grey data using cooperative grey games as a tool. Building blocks for grey solutions are grey payoff vectors, i.e., vectors whose components belong to $\mathcal{G}(\mathbb{R})$. Finally, we denote by $\mathcal{G}(\mathbb{R})^N$ the set of all such grey payoff vectors. We denote by $\mathcal{G}G^N$ the family of all cooperative grey games [8].

A cooperative grey flow network can be described by a graph with node set W and arc set E a cooperative grey flow network derives games are called cooperative grey flow games. There are two distinguished nodes: the source (S_o) and the sink

(S_i). It is allowed that several arcs have the same end points. Consider a directed network $D = (W, E; t)$ and $t : E \to \mathcal{G}(\mathbb{R})$ is the data transmission time. Then the cooperative grey flow game $\Gamma_f = (E, w')$, associated with the network D, is defined as follows:

- The player set is $N = 1, 2, ..., n$;
- The arc set is $E = \{(i, j) \mid i, j \in N, i \neq j\}$;
- For each coalition $S \subseteq E$, $w'(S)$ is the value of the grey flow from (S_o) to (S_i) in the network of D consisting only of arcs belonging to S.

3. Grey solutions

Let us recall the definition of the grey solutions that is critical during this study [8, 11].

3.1. **The Grey Shapley value.** Now, we introduce some theoretical notions from the theory of cooperative grey games. For $w, w_1, w_2 \in IG^N$ and $w^{'}, w_1^{'}, w_2^{'} \in \mathcal{G}G^N$ we say that $w_1^{'} \in w_1 \leq w_2^{'} \in w_2$ if $w_1^{'}(S) \leq w_2(S)$, where $w_1^{'}(S) \in w_1(S)$ and $w_2^{'}(S) \in w_2(S)$, for each $S \in 2^N$. For $w_1^{'}, w_2^{'} \in \mathcal{G}G^N$ and $\lambda \in \mathbb{R}_+$ we define $\langle N, w_1^{'} + w_2^{'} \rangle$ and $\langle N, \lambda w^{'} \rangle$ by $(w_1^{'} + w_2^{'})(S) = w_1^{'}(S) + w_2^{'}(S)$ and $(\lambda w^{'})(S) = \lambda w^{'}(S)$ for each $S \in 2^N$. So, we conclude that $\mathcal{G}G^N$ endowed with "$\leq$" has a cone structure with respect to addition and multiplication with non-negative scalars above. For $w_1^{'}, w_2^{'} \in \mathcal{G}G^N$ where $w_1^{'} \in w_1, w_2^{'} \in w_2$ with $|w_1(S)| \geq |w_2(S)|$ for each $S \in 2^N$, $\langle N, w_1^{'} - w_2^{'} \rangle$ is defined by $(w_1^{'} - w_2^{'})(S) = w_1^{'}(S) - w_2^{'}(S) \in w_1(S) - w_2(S)$. We call a game $\langle N, w^{'} \rangle$ grey size monotonic if $\langle N, |w| \rangle$ is monotonic, i.e., $|w|(S) \leq |w|(T)$ for all $S, T \in 2^N$ with $S \subset T$. For further use we denote by $SM\mathcal{G}G^N$ the class of grey size monotonic games with player set N. The grey marginal operators and the grey Shapley value are defined on $SM\mathcal{G}G^N$. We denote by $\prod(N)$ the set of permutations $\sigma : N \to N$ of N. The grey marginal operator $m^\sigma : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ corresponding to σ, associates with each $w^{'} \in SM\mathcal{G}G^N$ the grey marginal vector $m^\alpha(w^{'})$ of $w^{'}$ with respect to σ, defined by

$$m_i^\sigma(w^{'}) := w^{'}(P^\sigma(i) \cup \{i\}) - w^{'}(P^\sigma(i))$$

$$\in [\underline{A_{P^\sigma(i)\cup\{i\}}} - \underline{A_{P^\sigma(i)}}, \overline{A_{P^\sigma(i)\cup\{i\}}} - \overline{A_{P^\sigma}(i)}] \text{ for each } i \in N,$$

where $P^\sigma(i) = \{r \in N | \sigma^{-1}(r) < \sigma^{-1}(i)\}$, and $\sigma^{-1}(i)$ denotes the entrance number of player i. For grey size monotonic games $\langle N, w^{'} \rangle$, $w^{'}(T) - w^{'}(S) \in w(T) - w(S)$ is defined for all $S, T \in 2^N$ with $S \subset T$, since $|w(T)| = |w|(T) \geq |w|(S) = |w(S)|$. We notice that for each $w^{'} \in SM\mathcal{G}G^N$ the grey marginal vectors $m^\sigma(w^{'})$ are defined for each $\sigma \in \prod(N)$, because the monotonicity of $|w|$ implies $\overline{A_{S\cup\{i\}}} - \underline{A_{S\cup\{i\}}} \geqslant \overline{A_S} - \underline{A_S}$, which can be rewritten as $\overline{A_{S\cup\{i\}}} - \overline{A_S} \geq \underline{A_{S\cup\{i\}}} - \underline{A_S}$. So, $w^{'}(S \cup \{i\}) - w^{'}(S) \in w(S \cup \{i\}) - w(S)$ is defined for each $S \subset N$ and $i \notin S$. Next, we notice that all the grey marginal vectors of a grey size monotonic game are efficient grey payoff vectors.

The grey Shapley value $\Phi^{'} : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is defined by

$$\Phi^{'}(w^{'}) := \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^{\sigma}(w^{'})$$

$$\in \left[\frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^{\sigma}(\underline{A}), \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^{\sigma}(\overline{A}) \right], \tag{3.1}$$

for each $w^{'} \in SM\mathcal{G}G^N$. We can write the previous equation as follows:

$$\Phi^{'}_i(w^{'}) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} \left[w^{'}(P^{\sigma}(i) \cup \{i\}) - w^{'}(P^{\sigma}(i)) \right]$$

$$\in \left[\frac{1}{n!} \sum_{\sigma\in\Pi(N)} \underline{A_{P^{\sigma}(i)\cup\{i\}}} - \underline{A_{P^{\sigma}(i)}}, \frac{1}{n!} \sum_{\sigma\in\Pi(N)} \overline{A_{P^{\sigma}(i)\cup\{i\}}} - \overline{A_{P^{\sigma}(i)}} \right]. \tag{3.2}$$

3.2. **The Grey Banzhaf value.** The Banzhaf value arises from the subjective belief that every player is equally possible to join any coalition. On the opposite hand, the Shapley value arises from the idea that for each player, the coalition he joins is equally likely to be of any size which all coalitions of a given size are equally likely [11].

The Grey Banzhaf value $\beta : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$, $\forall w^{'} \in SM\mathcal{G}G^N$ is defined as

$$\beta(w^{'}) = \frac{1}{2^{|N|-1}} \sum_{i\in S} \left[w^{'}(S) - w^{'}(S \setminus \{i\}) \right]. \tag{3.3}$$

3.3. **The $\mathcal{G}CIS$-value.** The Centre-of-gravity of the Imputation-Set value, shortly denoted by CIS-value [6], assigns to every player its individual worth, and distributes the remainder of the worth of the grand coalition N equally among all players [3].

The grey CIS-value assings every player to its individual grey worth, and distributes the remainder of the grey worth of the grand coalition N equally among all players [11]. The $\mathcal{G}CIS$-value $\mathcal{G}CIS : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is defined by

$$\mathcal{G}CIS_i(w^{'}) = w^{'}(\{i\}) + \frac{1}{|N|} \left[w^{'}(N) - \sum_{j\in N} w^{'}(\{j\}) \right], \text{ for all } i \in N. \tag{3.4}$$

3.4. **The $\mathcal{G}ENSC$-value.** The Grey Egalitarian Non-Separable Contribution value, shortly denoted by $\mathcal{G}ENSC$-value [11], assigns to every game $w^{'}$ the $\mathcal{G}CIS$-value of its dual game, i.e.,

$$\mathcal{G}ENSC_i(w^{'}) = \mathcal{G}CIS_i(w^{'*})$$

$$= \frac{1}{|N|} \left[w^{'}(N) + \sum_{j\in N} w^{'}(N \setminus \{j\}) \right] - w^{'}(N \setminus \{i\})), \text{ for all } i \in N. \tag{3.5}$$

The $\mathcal{G}ENSC$-value assigns to every player in a game its grey marginal contribution to "the grand coalition" and distributes the remainder equally among the players [3, 11].

3.5. **The $\mathcal{G}ED$-solution.** The Grey Equal Division solution, shortly denoted by $\mathcal{G}ED$-solution [3, 11], is given by

$$\mathcal{G}ED : \mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N,$$

$$\mathcal{G}ED_i(w^{'}) = \frac{w^{'}(N)}{|N|}, \text{ for all } i \in N. \tag{3.6}$$

4. Properties of grey solutions

In this section, we state several theoretical results of the grey solutions. First, we give some properties of the grey Shapley value on the class of grey size monotonic games. Let $w_1' = \otimes_1 \in \left[\underline{A_1}, \overline{A_1}\right], w_2' = \otimes_2 \in \left[\underline{A_2}, \overline{A_2}\right]$.

Proposition 4.1 ([8])**.** The grey Shapley value $\Phi : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is additive.

Let $w' \in SM\mathcal{G}G^N$ and $i, j \in N$. Then, i and j are called *symmetric players*, if $w'(S \cup \{j\}) - w'(S) = w'(S \cup \{i\}) - w'(S)$, for each S with $i, j \notin S$. The proof of the following proposition is straightforward.

Proposition 4.2 ([8])**.** Let $i, j \in N$ be symmetric players in $w' \in SM\mathcal{G}G^N$. Then, $\Phi_i'(w') = \Phi_j'(w')$.

Let $w' \in SM\mathcal{G}G^N$ and $i \in N$. Then, i is called a *dummy player* if $w'(S \cup \{i\}) = w'(S) + w'(\{i\})$, for each $S \in 2^{N \setminus \{i\}}$.

Proposition 4.3 ([8])**.** The grey Shapley value $\Phi' : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ has the dummy player property, i.e., $\Phi_i'(w) = w'(\{i\})$ for all $w' \in SM\mathcal{G}G^N$ and for all dummy players i in w'.

Proposition 4.4 ([8])**.** The grey Shapley value $\Phi' : SM\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ is efficient, i.e., $\sum_{i \in N} \Phi_i'(w') = w'(N)$.

Let $S \in 2^N \setminus \{\emptyset\}$, $\otimes \in \mathcal{G}(\mathbb{R})$ and let u_S be the classical unanimity game based on S. The cooperative grey game $< N, \otimes u_S >$ is defined by $(\otimes u_S)(T) = \otimes u_S(T)$ for each $T \in 2^N \setminus \{\emptyset\}$, and its Shapley value is given by

$$\Phi_i'(\otimes u_S) = \begin{cases} \otimes / |S|, & i \in S, \\ [0, 0], & i \notin S. \end{cases}$$

We denote by $K\mathcal{G}G^N$ the additive cone generated by the set

$$K = \left\{\otimes_S u_S | S \in 2^N \setminus \{\emptyset\}, \otimes_S \in \mathcal{G}(\mathbb{R})\right\}.$$

So, each element of the cone is a finite sum of elements of K along nonnegative factors. We notice that $K\mathcal{G}G^N \subset SM\mathcal{G}G^N$, and axiomatically characterize the restriction of the grey Shapley value to the cone $K\mathcal{G}G^N$.

The following Theorem is introduced by [8].

Theorem 4.1 ([8])**.** There is a unique solution $\Phi' : K\mathcal{G}G^N \to \mathcal{G}(\mathbb{R})^N$ satisfying the properties of additivity, efficiency, dummy-player and symmetry. This solution is the grey Shapley value.

Proof. From Propositions 4.1, 4.2, 4.3 and 4.4, and $K\mathcal{G}G^N \subset SM\mathcal{G}G^N$, we obtain that Φ' satisfies the four properties on $K\mathcal{G}G^N$.

Conversely, let Ψ' be an grey value satisfiying the four properties on $K\mathcal{G}G^N$. We have to show that $\Psi' = \Phi'$. Take $w' \in K\mathcal{G}G^N$. Notice that w' can be written as $w' = \sum_{S\in 2^N\setminus\{\emptyset\}} \otimes_S u_S$. Then, for each $S \in 2^N \setminus \{\emptyset\}$ and $\otimes_S \in \mathcal{G}(\mathbb{R})$ we have $\Psi'(\sum_{S\in 2^N\setminus\{\emptyset\}} \otimes_S u_S) = \sum_{S\in 2^N\setminus\{\emptyset\}} \Psi'(\otimes_S u_S)$ and $\Phi'(\sum_{S\in 2^N\setminus\{\emptyset\}} \otimes_S u_S) = \sum_{S\in 2^N\setminus\{\emptyset\}} \Phi'(\otimes_S u_S)$ by additivity. Therefore, we need to show that for each $S \in 2^N \setminus \{\emptyset\}$ and $\otimes_S \in \mathcal{G}(\mathbb{R})$, $\Psi'(\otimes_S u_S) = \Phi'(\otimes_S u_S)$. Take $S \in 2^N \setminus \{\emptyset\}$ and $\otimes_S \in \mathcal{G}(\mathbb{R})$. Note that for any $i \in N \setminus S$,

$$(\otimes_S u_S)(T \cup \{i\}) - (\otimes_S u_S)(T) = (\otimes_S u_S)(\{i\}) \in [0, 0].$$

By the dummy player property, we have

$$\Psi'_i(\otimes_S u_S) = \Phi'_i(\otimes_S u_S) \in [0, 0], \text{ for all } i \in N \setminus S. \tag{4.1}$$

Now, suppose that $i, j \in S$, $i \neq j$. Then the symmetry property implies that

$$\Phi'_i(\otimes_S u_S) = \Phi'_j(\otimes_S u_S) \text{ for all } i, j \in S, \tag{4.2}$$

and, similarly,

$$\Psi'_i(\otimes_S u_S) = \Psi'_j(\otimes_S u_S) \text{ for all } i, j \in S.$$

By efficieny, from Eqns. (4.1) and (4.2) we obtain for any $S \in 2^N \setminus \{\emptyset\}$ and for any $\otimes_S \in \mathcal{G}(\mathbb{R})$ we have:

$$\Psi'_i(\otimes_S u_S) = \Phi'_i(\otimes_S u_S) = \frac{\otimes_S}{|S|}. \tag{4.3}$$

Hence, $\Psi'(w) = \Phi'(w)$ for all $w' \in K\mathcal{G}G^N$ by Eqns. (4.1) and (4.3). □

Next, we give some results from equal surplus sharing solutions, which are introduced in [11]. We discuss a class of grey solutions that consists of all convex combinations of the $\mathcal{G}ED$-solution, the $\mathcal{G}CIS$-value and the $\mathcal{G}ENSC$-value, i.e., for $\alpha, \beta \in [0, 1]$, we consider grey solutions $\mathcal{G}\phi^{\alpha,\beta}$ given by

$$\mathcal{G}\phi^{\alpha,\beta}(w') = \alpha\mathcal{G}ENCIS^{\beta}(w') + (1-\alpha)\mathcal{G}ED(w'),$$

where $\mathcal{G}ENCIS^{\beta}(w')$ is given by (1). We denote the class of all grey solutions that are obtained in this way by $\mathcal{G}\Phi := \left\{\mathcal{G}\phi^{\alpha,\beta} : \alpha, \beta \in [0, 1]\right\}$. Clearly, the interesting solutions in this class are the $\mathcal{G}CIS$-value, which is obtained by taking $\alpha = \beta = 1$ (i.e., $\mathcal{G}CIS(w') = \phi^{1,1}(w')$), the $\mathcal{G}ENSC$-value, which is obtained by taking $\alpha = 1$, $\beta = 0$ (i.e., $\mathcal{G}ENSC(w') = \mathcal{G}\phi^{1,0}(w')$) and the $\mathcal{G}ED$-solution, which is obtained by taking $\alpha = 0$ (i.e., $\mathcal{G}ED(w') = \mathcal{G}\phi^{0,\beta}$, $\beta \in [0, 1]$). We thus can write $\mathcal{G}\phi^{\alpha,\beta}$ as

$$\begin{aligned}
\mathcal{G}\phi^{\alpha,\beta}(w') &= \alpha\mathcal{G}\phi^{1,\beta}(w') + (1-w')\mathcal{G}\phi^{0,1}(w') \\
&= \alpha\beta\mathcal{G}\phi^{1,1}(w') + \alpha(1-\beta)\mathcal{G}\phi^{1,0}(w') + (1-\alpha)\mathcal{G}\phi^{0,1}(w')
\end{aligned}$$

for $\alpha, \beta \in [0, 1]$. The following Propositions are introduced by [11]. Proposition 4.5 gives an expression of the solutions $\mathcal{G}\phi^{\alpha,\beta}$ in the sense that they give each player

i in a grey game w' some value $\lambda_i^{\alpha,\beta}(w')$, and the remainder of $w'(N)$ is equally divided among all players.

Proposition 4.5 ([11])**.** For every $w' \in SM\mathcal{G}G^N$ and $\alpha, \beta \in [0,1]$ it holds that

$$\mathcal{G}\phi_i^{\alpha,\beta}\left(w'\right) = \lambda_i^{\alpha,\beta}\left(w'\right) + \tfrac{1}{|N|}(w'(N) -_{j\in N} \lambda_j^{\alpha,\beta}\left(w'\right)),$$

where $\lambda_i^{\alpha,\beta}(w') = \alpha\left(\beta w'(\{i\}) - (1-\beta)\, w'(N\backslash\{i\})\right)$ for $i \in N$ such that $|w'|(i) = |w'|(N\backslash\{i\}) = |w'|(j)$ for all $i, j \in N$ with $i \neq j$.

Proof. For $w' \in SM\mathcal{G}G^N$ and $\alpha, \beta \in [0,1]$ we have

$$\begin{aligned}
\mathcal{G}\phi_i^{\alpha,\beta}\left(w'\right) &\in \alpha\mathcal{G}ENCIS^{\beta}\left(w'\right) + (1-\alpha)\,\mathcal{G}ED\left(w'\right) \\
&= \alpha\left(\beta w'(\{i\}) - (1-\beta)\, w'(N\backslash\{i\})\right) \\
&\quad + \frac{\alpha}{|N|}\left(w'(N) - \sum_{j\in N}(\beta w'(j) - (1-\beta)\, w'(N\backslash\{j\}))\right) \\
+\frac{1-\alpha}{|N|} w'(N) &= \alpha\left(\beta w'(\{i\}) - (-\beta)\, w'(N\backslash\{i\})\right) \\
&\quad + \frac{1}{|N|}\left(w'(N) - \sum_{j\in N}\alpha(\beta w'(j) - (1-\beta)\, w'(N\backslash\{j\}))\right) \\
&= \lambda_i^{\alpha,\beta}\left(w'\right) + \tfrac{1}{|N|}(w'(N) -_{j\in N} \lambda_j^{\alpha,\beta}\left(w'\right)).
\end{aligned}$$

□

Proposition 4.6 ([11])**.** For every $\alpha, \beta \in [0,1]$ and $w' \in SM\mathcal{G}G^N$ it holds that $\mathcal{G}\phi_i^{\alpha,\beta}(w'^*) = \mathcal{G}\phi^{\alpha,1-\beta}(w')$.

Proof. For $w' \in SM\mathcal{G}G^N$ and $\alpha, \beta \in [0,1]$ we have

$$\begin{aligned}
\mathcal{G}\phi_i^{\alpha,\beta}\left(w'^*\right) &= \lambda_i^{\alpha,\beta}\left(w'^*\right) + \tfrac{1}{|N|}(w'^*(N) -_{j\in N} \lambda_j^{\alpha,\beta}\left(w'^*\right)) \\
&= \alpha\left(\beta w'^*(\{i\}) - (1-\beta)\, w'^*(N\backslash\{i\})\right) \\
&+ \frac{1}{|N|}\left(w'^*(N) - \sum_{j\in N}\alpha(\beta w'^*(j) - (1-\beta)\, w'^*(N\backslash\{j\}))\right) \\
&= \alpha(\beta w'(N)) - \beta w'(N\backslash\{i\}) - (1-\beta)\, w'(N) + (1-\beta)\, w'(\{i\})) \\
&+ \frac{1}{|N|}\left(w'(N) - \sum_{j\in N}\alpha(\beta w'(N) - \beta w'(N\backslash\{j\}) - (1-\beta)\, w'(N) + (1-\beta)\, w'(j)\right) \\
&= w'(N)\left(\alpha\beta - \alpha(1-\beta) + \frac{1}{|N|} - \frac{|N|\,\alpha\beta}{|N|} + \frac{|N|\,\alpha(1-\beta)}{|N|}\right) \\
&+ \alpha\left((1-\beta)\, w'(i) - \beta\left(w'(N\backslash\{i\})\right) + \frac{1}{|N|}\sum_{j\in N}\left(\beta w'(N\backslash\{j\}) - (1-\beta)\, w'(j)\right)\right) \\
&= \frac{1}{|N|} w'(N) + \alpha\left((1-\beta)\, w'(\{i\}) - \beta w'(N\backslash\{i\})\right)
\end{aligned}$$

$$-\frac{1}{|N|}\left(\sum_{j\in N}\alpha\left(\left(1-\beta\right)w'\left(\{j\}\right)-\beta w'\left(N\backslash\{j\}\right)\right)\right)$$
$$=\mathcal{G}\phi^{\alpha,1-\beta}\left(w'\right).$$

□

5. An application

In this paper, we construct a model based on a possible logistic network after an earthquake in Istanbul, Turkey. Suppose that there is an earthquake in Istanbul and the needed commodities are to be sent to Istanbul. While the commodities are being transported to Istanbul on ground and by airways, some countries get benefits from the transportation because they own the routes. In our model, we choose these intermediate countries as distributors and construct the cooperative grey flow game in the application to the logistic network. In this way, the aim of the model is to obtain the transported maximum commodities to the affected city (Istanbul), and to provide the allocation of gains between the distributors fairly. For this model, firstly we set the logistic network scenario by using some parameters, and construct the cooperative grey flow game. To allocate the gains between distributor countries fairly, some solution concepts from Cooperative Grey Game Theory are used. These are the grey Shapley value and the grey solutions. Then, we compare the results and determine the solution that the players want to use it.

Consider the logistic network consisting of one supplier that applies the aid materials after natural disasters (D: Deutschland; Germany), three countries as distributors (U: Ukraine, G: Greece, A: Azerbaijan), and one retailer of the country affected by the earthquake (I: Istanbul). An illustration of our scenario is presented in Figure 1. We have two different paths, namely, as on ground and by airway. Each country has different types of vehicles; these are planes and trucks. Cargo planes supply aid materials and rescue equipment as commodities. Trucks supply commodities of medical, personal materials, food, etc.

Ukraine has one Airbus 310-300 cargo plane, $[30, 36]$ tons of capacity; and two trucks; each one has $[23, 25]$ tons of capacity. Greece has two Antonov An-12 cargo planes which each one has $[15, 17]$ tons of capacity; it also has one truck, $[37, 40]$ tons of capacity. Azerbaijan has two cargo planes, Ilyushin model, each one has $[43, 45]$ tons of capacity; furthermore, it has two trucks, each has a capacity of $[23, 25]$ tons.

Ground and airway distances between countries (supplier, distributors and retailer) are given in Figure 1. The numbers shown on the arcs are the capacities of the vehicles when they use these paths and the times of the transportation. Three countries own the arcs (C_1, C_2, C_3). In closer detail, C_1 (the upper arcs) owns the arcs (D, U) and (U, I) with the capacities shown. C_2 (the middle arcs) owns the arcs $(D, G), (G, I), (A, G)$ and (G, I) with the capacities. C_3 (the bottom arcs) owns the arcs (D, A) and (A, I) with the capacities shown.

There are two nodes that are distinguished from the others and are called the source s and the sink t, which have already been previously defined. There is also a finite and non-empty set understood as the player set. The arcs are considered and owned by the players. Moreover, a coalition owns the arcs of its members. The set

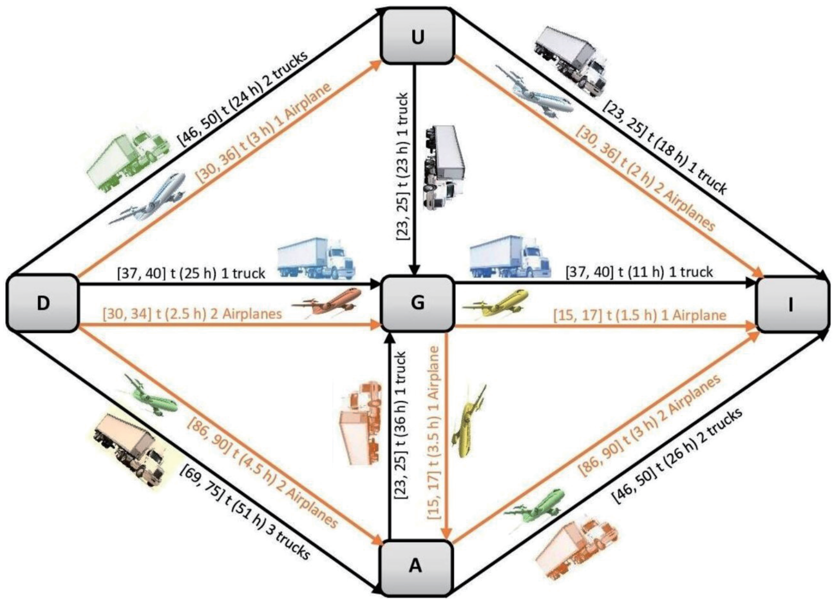

FIGURE 1. The illustration of our model.

of coalitions is denoted by C. An n-person cooperation grey flow game is a function w' from the set of coalitions to the set of real numbers. For a coalition $S \in C$, $w'(S)$ is defined as the maximum grey flow value for coalition S and through the network of its members if it operates on its own. This means that $w'(S)$ stands for the maximum grey flow that S can sustain by using its own portion of the network. The function w' just defined is called the characteristic function of the grey game. The flowchart of our model is given in Figure 2.

TABLE 1. The constructed grey flow game.

$S =$	ϕ	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{1,2,3\}$
$w'(S) \in$	[0, 0]	[50, 61]	[15, 17]	[86, 90]	[91, 103]	[116, 124]	[136, 151]	[229, 250]

TABLE 2. Grey marginal vectors of the model.

σ	$m_1^\sigma(w')$	$m_2^\sigma(w')$	$m_3^\sigma(w')$
$\sigma_1 = (1,2,3)$	$m_1^{\sigma_1}(w') \in [50, 61]$	$m_2^{\sigma_1}(w') \in [41, 42]$	$m_3^{\sigma_1}(w') \in [138, 147]$
$\sigma_2 = (1,3,2)$	$m_1^{\sigma_2}(w') \in [50, 61]$	$m_2^{\sigma_2}(w') \in [93, 99]$	$m_3^{\sigma_2}(w') \in [86, 90]$
$\sigma_3 = (2,1,3)$	$m_1^{\sigma_3}(w') \in [76, 86]$	$m_2^{\sigma_3}(w') \in [15, 17]$	$m_3^{\sigma_3}(w') \in [138, 147]$
$\sigma_4 = (2,3,1)$	$m_1^{\sigma_4}(w') \in [113, 126]$	$m_2^{\sigma_4}(w') \in [15, 17]$	$m_3^{\sigma_4}(w') \in [101, 107]$
$\sigma_5 = (3,1,2)$	$m_1^{\sigma_5}(w') \in [50, 61]$	$m_2^{\sigma_5}(w') \in [93, 99]$	$m_3^{\sigma_5}(w') \in [86, 90]$
$\sigma_6 = (3,2,1)$	$m_1^{\sigma_6}(w') \in [113, 126]$	$m_2^{\sigma_6}(w') \in [30, 34]$	$m_3^{\sigma_6}(w') \in [86, 90]$

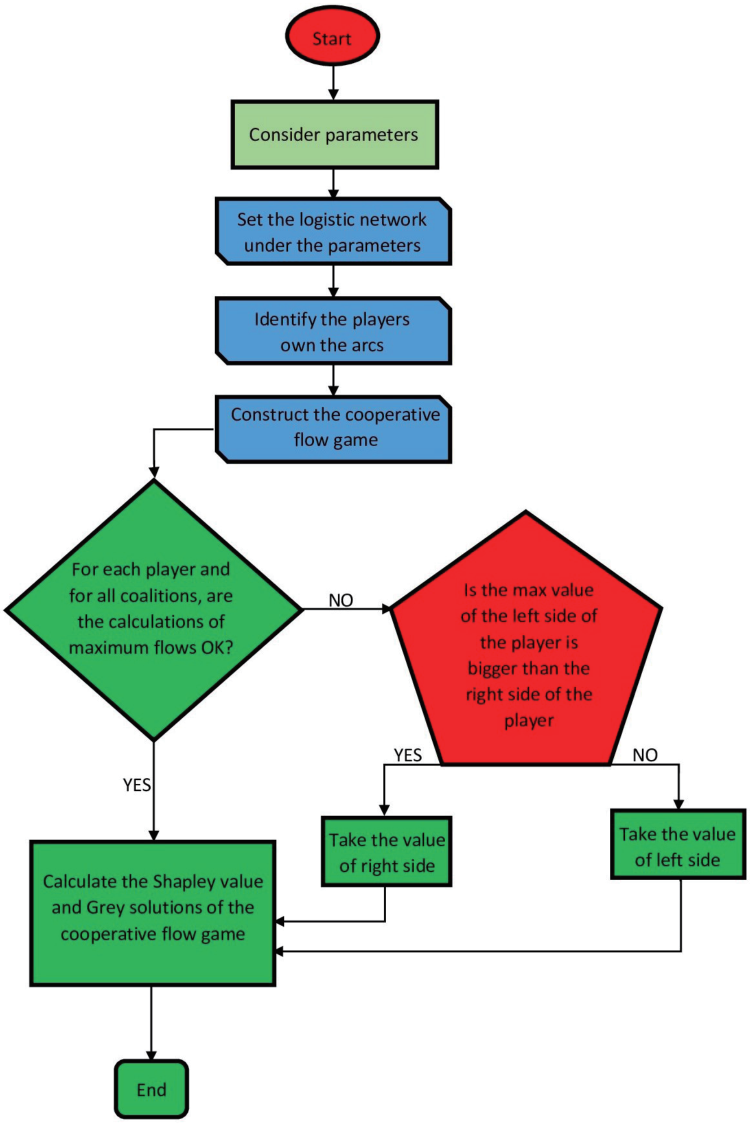

FIGURE 2. The flowchart of the model.

The average of the six grey marginal vectors is the Grey Shapley value of this game which can be shown as:

$$\Phi^{'}(w^{'}) \in ([96, 107.67], [73, 80.67], [129, 136.67])\,.$$

Now, Let us look at how the Grey Banzhaf value for this game. For player 1, we have:

$$\beta_1(w^{'}) \in \frac{1}{2^2}\sum_{1\in S}\left[w^{'}(S) - w^{'}(S\backslash\{1\})\right]$$

$$\beta_1(w^{'}) \in \frac{1}{4}\Big[w^{'}(\{1\}) + w^{'}(\{1,2\}) + w^{'}(\{1,2,3\}) + w^{'}(\{1,3\}) - w^{'}(\{2\}) - w^{'}(\{3\}) - w^{'}(\{2,3\})\Big]$$

$$\beta_1(w^{'}) \in [85.25, 96] .$$

The Grey Banzhaf values of the other players can be examined similarly as follows:

$$\beta_2(w^{'}) \in [62.25, 69] , \beta_3(w^{'}) \in [118.25, 125] .$$

At that rate, the Grey Banzhaf value is

$$\beta(w^{'}) \in ([85.25, 96] , [62.25, 69] , [118.25, 125]).$$

Now, we want to calculate $\mathcal{G}CIS$-value, $\mathcal{G}ENSC$-value and $\mathcal{G}ED$-solution. We find the $\mathcal{G}CIS$-value of our game as follows:

$$\mathcal{G}CIS_1(w^{'}) \in w^{'}(\{1\}) + \frac{1}{3}\left[w^{'}(\{1,2,3\}) - \sum_{j\in N} w^{'}(\{j\})\right] ,$$

$$\mathcal{G}CIS_1(w^{'}) \in w^{'}(\{1\}) + \frac{1}{3}\Big[w^{'}(\{1,2,3\}) - (w^{'}(\{1\}) + w^{'}(\{2\}) + w^{'}(\{3\}))\Big] = [96, 107.67] .$$

$$\mathcal{G}CIS_2(w^{'}) \in w^{'}(\{2\}) + \frac{1}{3}\Big[w^{'}(\{1,2,3\}) - (w^{'}(\{1\}) + w^{'}(\{2\}) + w^{'}(\{3\}))\Big] = [73, 80.67] .$$

$$\mathcal{G}CIS_3(w^{'}) \in w^{'}(\{3\}) + \frac{1}{3}\Big[w^{'}(\{1,2,3\}) - (w^{'}(\{1\}) + w^{'}(\{2\}) + w^{'}(\{3\}))\Big] = [129, 136.67] .$$

Then, the $\mathcal{G}CIS$-value is obtained by

$$\mathcal{G}CIS(w^{'}) \in ([96, 107.67] , [73, 80.67] , [129, 136.67]).$$

We calculate the $\mathcal{G}ENSC$-value of our game as follows:

$$\mathcal{G}ENSC_1(w^{'}) \in \frac{1}{3}\left[w^{'}(\{1,2,3\}) + \sum_{j\in N} w^{'}(N \setminus \{j\})\right] - w^{'}(N \setminus \{1\}),$$

$$\mathcal{G}ENSC_1(w^{'}) \in \frac{1}{3}\Big[w^{'}(\{1,2,3\}) + (w^{'}(\{1,2\}) + w^{'}(\{1,3\}) + w^{'}(\{2,3\}))\Big] - w^{'}(\{2,3\}) = [96, 107.67] .$$

$$\mathcal{G}ENSC_2(w^{'}) \in \frac{1}{3}\Big[w^{'}(\{1,2,3\}) + (w^{'}(\{1,2\}) + w^{'}(\{1,3\}) + w^{'}(\{2,3\}))\Big] - w^{'}(\{1,3\}) = [73, 80.67] .$$

$$\mathcal{G}ENSC_3(w^{'}) \in \frac{1}{3}\Big[w^{'}(\{1,2,3\}) + (w^{'}(\{1,2\}) + w^{'}(\{1,3\}) + w^{'}(\{2,3\}))\Big] - w^{'}(\{1,2\}) = [129, 136.67] .$$

Then, the $\mathcal{G}ENSC$-value is obtained by

$$\mathcal{G}ENSC(w') \in ([96, 107.67], [73, 80.67], [129, 136.67]).$$

Finally, we calculate the $\mathcal{G}ED$-solution of our game as follows:

$$\mathcal{G}ED_1(w') = \mathcal{G}ED_2(w') = \mathcal{G}ED_3(w') \in \frac{w'(\{1,2,3\})}{3} = [99.33, 108.33].$$

So, we have

$$\mathcal{G}ED(w') \in ([99.33, 108.33], [99.33, 108.33], [99.33, 108.33]).$$

Table 3 Displays the grey solutions of our model.

Table 3. The Grey Solutions.

Grey Solutions	Player 1	Player 2	Player 3
Grey Shapley value	$\in [96, 107.67]$	$\in [73, 80.67]$	$\in [129, 136.67]$
Grey Banzhaf value	$\in [85.25, 96]$	$\in [62.25, 69]$	$\in [118.25, 125]$
$\mathcal{G}CIS$-value	$\in [96, 107.67]$	$\in [73, 80.67]$	$\in [129, 136.67]$
$\mathcal{G}ENSC$-value	$\in [96, 107.67]$	$\in [73, 80.67]]$	$\in [129, 136.67]$
$\mathcal{G}ED$-value	$\in [99.33, 108.33]$	$\in [99.33, 108.33]$	$\in [99.33, 108.33]$

6. Conclusion and outlook

Following a natural disaster, the most immediate concern is to fulfill the basic and urgent needs of survivors. This includes food, water, shelter and basic healthcare. Large natural disasters can leave hundreds of thousands of people without adequate nourishment, protection or medical care for those injured. In uncertainties in an SCM occur because of several limitations. Moreover, data may not be available or may not be easy to communicate in large-scale if emergencies are in mutual casualty. Transportation can be limited because of transportation time, injury seriousness, on-field treatment, and medical center service load, which calls for the design of new cooperative games under uncertainty that can be modeled and used in this area.

In this research, we have considered the earthquake emergency logistic problem for one affected city and one supplier country under uncertainty. We have modeled the emergency problem for the situation given after an earthquake occurred in Istanbul by using cooperative grey flow games. Several grey solutions concepts are proposed for this model.

As a future work, our model can be extended to look at the sharing issues in OR among multiple cities, countries, etc. What is more, based on the current paper, the related issue of vehicle routing can became an extra research subject, too.

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E. Qasim
Department of Mathematics, Faculty of Education, Omar Al-Mukhtar University, Al-Beida, Libya
E-mail address: emad.qasim@omu.edu.ly

S. Z. Alparslan Gok
Department of Mathematics, Faculty of Arts and Sciences, Suleyman Demirel University, 32260 Isparta, Turkey
E-mail address: zeynepalparslan@yahoo.com

O. Palanci
Department of Mathematics, Faculty of Arts and Sciences, Suleyman Demirel University, 32260 Isparta, Turkey
E-mail address: osmanpalanci@sdu.edu.tr

G. W. Weber
Faculty of Engineering Management, Poznan University of Technology, Poznan, Poland; Institute of Applied Mathematics, METU, 06800 Ankara, Turkey
E-mail address: gerhard.weber@put.poznan.pl