*Pure and Applied Functional Analysis*
Volume 6, Number 4, 2021, 817–834

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL


# DISCONTINUOUS QUASI-VARIATIONAL RELATIONS WITH APPLICATIONS

RICHARD P. MCLEAN

Abstract. The Ky Fan equilibrium problem is a fundamental result of nonlinear analysis with myriad applications in optimization theory, fixed point theory, mathematical economics, and game theory. It is the goal of this paper to provide generalizations of the Ky Fan problem in terms of relations by applying recent weakened notions of continuity that have proved to be very useful in establishing the existence of Nash equilibrium in games with discontinuous payoffs. Our results apply to quasi-variational and generalized quasi-variational relations in the framework of topological vector spaces.

## 1. Introduction

A number of important applications of game theory involve discontinuous payoff functions. Building on previous work of Dasgupta and Maskin [10], Simon [3], Baye et al [5] and others, Reny [27] derived a number of existence results for games with discontinuous payoffs by relaxing upper semicontinuity of payoffs (such as Simon's [31] reciprocal upper semicontinuity or Dasgupta and Maskin's [10] upper semicontinuity of the sum of payoffs) and lower semicontinuity of payoffs (such as the notion of payoff security). If strategy sets are convex and payoffs are quasiconcave in own actions, then these relaxations of upper and lower semicontinuity can be applied to derive pure-strategy existence results.[1] Several recent papers have investigated the extent to which these results for games with discontinuous payoff functions can be extended to the case in which a player's preference order need not be representable by a utility function. Reny [28] introduces the notions of point security and correspondence security for games in which players' preference relations are complete, reflexive, and transitive, and generalizes existence results for strategic-form games with payoff functions found in Reny [27], Barelli and Meneghel [4], and McLennan et al. [22]. Carmona and Podczeck [8] introduce the notions of point target security and correspondence target security and provide several further generalizations of these results. For related results in games and models of abstract economies in which agents' preferences need not be representable by utility functions, see Reny [29], Nessah and Tian [24], and He and Yannelis [15].

2020 *Mathematics Subject Classification.* Primary 47J20, 49J52; Secondary 91A10.

*Key words and phrases.* quasi-variational relation, generalized quasi-variational relation, generalized game.

[1]For an excellent survey of this literature including extensive references, see Carmona [7]. For more recent results, see Reny [28].

Contemporaneous with this research program in game theory, a number of authors have studied various related equilibrium problems derived from the seminal work of Ky Fan. The Ky Fan equilibrium problem is a fundamental result of nonlinear analysis with myriad applications in optimization theory, fixed point theory, mathematical economics, and game theory to name just a few. The problem has the following statement:

**Ky Fan Equilibrium Problem**: Given a set $X$ and a function $f : X \times X \to \mathbb{R}$, find $\overline{x} \in X$ such that $f(\overline{x}, y) \leq 0$ for each $y \in X$.

The seminal result concerning the existence of a solution to the Ky Fan equilibrium problem is the following.

**Theorem 1.1** (Fan, [13]). *Suppose that $X$ is a compact, convex, non-empty subset of a Hausdorff topological vector space and that $f : X \times X \to \mathbb{R}$ satisfies*

1. *$x \mapsto f(x, y)$ is lower semi-continuous for each $y \in X$.*
2. *$y \mapsto f(x, y)$ is quasi-concave for each $x \in X$.*
3. *$f(x, x) \leq 0$ for each $x \in X$.*

*Then there exists $\overline{x} \in X$ such that $f(\overline{x}, y) \leq 0$ for each $y \in X$.*

As an application, Theorem 1.1 provides a classic existence proof for Nash equilibrium using the well known Nikaido-Isoda mapping. Given a game with players $N = \{1, \ldots, n\}$, strategy sets $X_i$, and payoff functions $u_i : X \to \mathbb{R}$ where $X = X_1 \times \cdots \times X_n$, define $f : X \times X \to \mathbb{R}$ as

$$f(x, y) = \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x).$$

A strategy profile $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in X$ is a Nash equilibrium if and only if $\overline{x}$ solves the Ky Fan equilibrium problem for $f$ and $X$. For example, if each $X_i$ is non-empty, convex, and compact, each $x_{-i} \mapsto u_i(x_{-i}, y_i)$ is lower semi-continuous for each $y_i$, $x \mapsto \sum_{i=1}^{n} u_i(x)$ is upper semi-continuous and $y \mapsto \sum_{i=1}^{n} u_i(x_{-i}, y_i)$ is quasi-concave for each $x_{-i}$, then the game has an equilibrium as an application of Theorem 1.1.

Fan's theorem has been generalized in many directions (for a comprehensive survey, see Tarafdar and Chowdhury [33]) and the focus of much recent work has been on the ways in which the lower semi-continuity and quasi-concavity assumptions of Theorem 1.1 can be relaxed. Such generalizations have proved fruitful in establishing the existence of Nash equilibrium in strategic form games and qualitative games via the Nikaido-Isoda mapping. For recent contributions, see Theorems 7.1 and 7.2 in Nessah and Tian [23], Theorem 2 in Prokopovych and Yannelis [26] and Proposition 1 in Scalzo [30].

Extensions of the Ky Fan equilibrium problem to the case of general relations may be found in Luc [20], Lin and Wang [19], Balaj and Luc [2], Balaj and Lin [3], Luc, Sarabi, and Soubeyran [21], Balaj and Lin [2], Hung and Kieu [16], and Yang [39]. It is the goal of this paper to provide further generalizations in terms of relations by applying recent weakened notions of continuity that have proved to be very useful in establishing the existence of Nash equilibrium in games with discontinuous payoffs.

## 2. Preliminary notation and definitions

Let $X$ and $Y$ be topological spaces and let $\varphi : X \to Y$ be a correspondence. Let dom$\varphi$ denote the set $\{x \in X | \varphi(x) \neq \varnothing\}$. If $Y$ is a vector space, let con$\varphi$ denote the correspondence with values $con\varphi(x)$ where $con\varphi(x)$ denotes the convex hull of $\varphi(x)$. If $Y$ is a topological vector space, we will follow Reny [28] and call a correspondence $\varphi : X \to Y$ *co-closed* if the corresondence $x \in X \mapsto con\varphi(x)$ has closed graph.

We say that $\varphi : X \to Y$

1. is compact (non-empty) valued if $\varphi(x)$ is a compact (non-empty) subset of $Y$ for each $x \in X$.
2. is upper hemi-continuous at $x \in X$ if for every open set $V \subseteq Y$ with $\varphi(x) \subseteq V$, there exists an open set $U \subseteq X$ with $x \in U$ such that $\varphi(x') \subseteq V$ for all $x' \in U$.
3. is lower hemi-continuous at $x \in X$ if for for every open set $V \subseteq Y$ with $\varphi(x) \cap V \neq \varnothing$, there exists an open set $U \subseteq X$ with $x \in U$ such that $\varphi(x') \cap V \neq \varnothing$ for all $x' \in U$.
4. is upper hemi-continuous (lower hemi-continuous) if $\varphi$ is upper hemi-continuous (lower hemi-continuous) at each $x \in X$.
5. has open lower sections if the set $\varphi^{-1}(y) := \{x \in X | y \in \varphi(x)\}$ is open in $X$ for each $y \in Y$.
6. has the local intersection property if for each $x \in X$ with $\varphi(x) \neq \varnothing$, there exists an open set $U(x)$ containing $x$ such that $\cap_{x' \in U(x)} \varphi(x') \neq \varnothing$.[2]
7. has the continuous inclusion property if for each $x \in X$, there exists an open set $U(x)$ containing $x$ and a co-closed correspondence $d : U(x) \to X$ such that $d(x') \subseteq \varphi(x')$.[3]

**Remark.** If $\varphi : X \to Y$ has open lower sections, then $\varphi$ has the local intersection property. If $\varphi$ has the local intersection property, then $\varphi$ has the continuous inclusion property.

We also record two fixed point theorems that are crucial for our results.

**Proposition 2.1** (Balaj and Muresan [1])**.** *Suppose that $X$ is a non-empty, convex subset of a Hausdorff topological vector space. Suppose that $T : X \to X$ is a correspondence with nonempty convex values having the local intersection property. Suppose that there exists a non-empty, convex compact subset $M \subseteq X$ and a compact subset $C \subseteq X$ such that, for each $x \in X \backslash C$, there exists an open set $U(x)$ containing $x$ such that*

$$\left[ \bigcap_{x' \in U(x)} T(x') \right] \cap M \neq \varnothing.$$

*Then $T$ has a fixed point.*

**Proposition 2.2** (He and Yannelis [15])**.** *Suppose that $X$ is a non-empty, convex, compact subset of a Hausdorff locally convex topological vector space. Suppose that*

[2]Here we follow the terminology in Wu and Shen [38] and Prokopovych [25]. The same concept appears as Definition 6 in Tian and Zhou [36] under the name *transfer open lower sections.*

[3]A similar idea is found in Uyanik [37] under the name continuous neighborhood selection property.

*$T : X \to X$ is a non-empty valued, convex valued corresondence with the continuous inclusion property. Then $T$ has a fixed point.*

## 3. Quasi-variational relation problems without continuity

The *quasi-variational equilibrium problem* is more general than the Ky Fan equilibrium problem and has been the subject of much research in the last two decades.

**Quasi-Variational Equilibrium Problem (QVEP)**: Given a set $X$, a function $f : X \times X \to \mathbb{R}$ and a correspondence $K : X \to X$, find $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and $f(\overline{x}, y) \leq 0$ for each $y \in K(\overline{x})$.

For a basic existence theorem, we have the following.

**Theorem 3.1.** *Suppose that $X$ is a non-empty, compact, convex subset of a Hausdorff topological vector space. Suppose that $K : X \to X$ is a correspondence and $f : X \times X \to \mathbb{R}$ is a function. Suppose that*

(i) *$x \mapsto f(x, y)$ is lower semi-continuous for each $y \in X$.*
(ii) *$y \mapsto f(x, y)$ is quasi-concave for each $x \in X$.*
(iii) *$f(x, x) \leq 0$ for each $x \in X$.*
(iv) *The correspondence $K$ is non-empty valued and convex valued with open lower sections.*
(v) *The set $F$ of fixed points of $K$ is closed in $X$.*

*Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and*

$$f(\overline{x}, y) \leq 0 \text{ for all } y \in K(\overline{x}).$$

**Remark.** While Browder's Theorem and assumption (iv) ensure that the set $F$ is non-empty, assumption (v) in Theorem 3.1 cannot be dropped. For example, suppose that $X = [0, 1]$,

$$\begin{aligned} K(x) &= [0, \tfrac{1}{2}] \text{ if } x \neq \tfrac{1}{2} \\ &= [0, \tfrac{1}{2}[ \text{ if } x = \tfrac{1}{2} \end{aligned}$$

and

$$\begin{aligned} f(x, y) &= 0 \text{ if } y \leq x \\ &= 1 \text{ if } y > x \end{aligned}$$

Then for each $x \in X$ with $x \in K(x)$, there exists a $y \in K(x)$ such that $f(x, y) > 0$. In this example, conditions (i), (ii), (iii), and (iv) of Theorem 3.1 are satisfied but the set of fixed points of $K$ is not closed.

The equilibrium problem for generalized games introduced in Debreu [12] is a special case of the QVEP. Again consider a game with players $N = \{1, .., n\}$, strategy sets $X_i$, and payoff functions $u_i : X_1 \times \cdots \times X_n \to \mathbb{R}$. To each player. we associate a feasible action corrrespondence $K_i : X \to X_i$ where $K_i(x) \subseteq X_i$ is the set of actions available to $i$ given the strategy profile $x$. A profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X$ is an equilibrium of the generalized game if for each player $i$,

$$\overline{x}_i \in \arg \max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i).$$

Defining $f(x,y) = \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x)$ and $K(x) = \times_{i \in N} K_i(x)$, it is clear that $\overline{x} \in X$ is an equilibrium in the generalized game if and only if $\overline{x}$ solves the QVEP problem for $f$ and $K$.

In this paper, we are concerned with a generalization of the QVEP in which the function $f$ is replaced with a relation. This extension is stated as follows:

**Quasi-Variational Relation Problem (QVRP)** : Given a set $X$ and a relation $R \subseteq X \times X$ and a correspondence $K : X \to X$, find $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y) \in R \text{ for all } y \in K(\overline{x}).$$

In addition to the QVEP and the Nikaido-Isoda approach to generalized games, the QVRP includes as special cases many important problems in optimization theory and non-linear analysis including:

- The quasi-variational inclusion probem where $R = \{(x,y) \in X \times X | 0 \in \psi(x,y)\}$ where $\psi : X \times X \to Z$ is a correspondence and $Z$ is a vector space.
- The quasi-variational inequality problem where $X = \mathbb{R}^n$ and for a given function $g : \mathbb{R}^n \to \mathbb{R}^n$, one defines $R = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | g(x) \cdot (y - x) \geq 0\}$.
- The nonlinear implicit complementarity problem of stochastic impulse control where $X = \mathbb{R}^n$ and for given functions $g : \mathbb{R}^n \to \mathbb{R}^n$ and $m : \mathbb{R}^n \to \mathbb{R}^n$, one defines $R = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | g(x) \cdot (y - x) \geq 0\}$ and $K(x) = \{y \in \mathbb{R}^n | y \geq m(x)\}$.
- The Walrasian equilibrium problem where $X = \{x \in \mathbb{R}^n_+ | \sum_i x_i = 1\}$ and for a given function $g : X \to \mathbb{R}^n$ satisfying $x \cdot g(x) \leq 0$ for all $x \in X$, one defines $R = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n | g(x) \cdot (y - x) \leq 0\}$ and $K(x) = X$ for all $x \in X$.

Our main goal is to investigate the extent to which the existence of a solution to the quasi-variational relation problem can be established when both the correspondence $K$ and the relation $R$ may exhibit discontinuities. In particular, we are interested in relaxing continuity of $R$ by employing recent ideas from the theory of discontinuous games.

**Definition 3.2.** Suppose that $X$ and $Y$ are topological spaces and $R \subseteq X \times Y$ is a relation.

(i) $R$ has closed lower sections if $\{x \in X : (x,y) \in R\}$ is closed in $X$ for each $y \in Y$.

(ii) $R$ is transfer semi-continuous if $x \in X$ and $(x,y) \notin R$ imply that there exists an open set $U(x)$ containing $x$ and $y^* \in Y$ such that $(x', y^*) \notin R$ for all $x' \in U(x)$.

**Remark.** If $R$ is a closed set in $X \times Y$, then $R$ has closed lower sections. If $R$ has closed lower sections, then the relation $R$ is transfer semi-continuous. The nomenclature of Definition 3.2 is motivated by properties of lower semi-continuous functions. If $f : X \to \mathbb{R}$ is lower semi-continuous and if $R = \{(x,y) \in X \times \mathbb{R} | y \geq f(x)\}$, then $R$ is closed and therefore satisfies (i) and (ii) of the definition.

Transfer semi-continuity of a relation $R$ generalizes several well known concepts in the literature. For example, a function $f : X \times Y \to \mathbb{R}$ is $\lambda$-transfer lower semi-continuous (Tian [34]) if and only if the relation $R = \{(x, y) \in X \times Y | f(x, y) \leq \lambda\}$ is transfer semi-continuous.

**Definition 3.3.** Suppose that $X$ is a topological space, $Q : X \to X$ is a correspondence and $R \subseteq X \times X$ a relation. Then $R$ is transfer semi-continuous with respect to $Q$ if $x \in X, y \in Q(x)$ and $(x, y) \notin R$ imply that there exists an open set $U(x)$ containing $x$ and $y^* \in Q(x)$ such that $(x', y^*) \notin R$ for all $x' \in U(x)$.[4]

**Theorem 3.4.** *Suppose that $X$ is a non-empty, convex subset of a Hausdorff topological vector space. Suppose that $K : X \to X$ and $Q : X \to X$ are correspondences and $R \subseteq X \times X$ is a relation and define $P(x) = \{y \in X : (x, y) \notin R\}$. Suppose that*

(i) *$R$ transfer semi-continuous with respect to $Q$.*
(ii) *$Q$ is non-empty valued, $Q$ has open lower sections, and $conQ(x) \subseteq K(x)$ for each $x \in X$.*
(iii) *The set $F$ of fixed points of $K$ is closed.*
(iv) *For each $x \in X$, $x \notin con(P(x) \cap Q(x))$.*
(v) *There exists a non-empty, convex compact subset $M \subseteq X$ and a compact subset $C \subseteq X$ such that, for each $x \in X \backslash C$, there exists an open set $U(x)$ containing $x$ such that*

$$\left[\bigcap_{x' \in U(x)} \left[Q(x') \cap P(x')\right]\right] \cap M \neq \varnothing.$$

*Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and*

$$(\overline{x}, y) \in R \text{ for all } y \in Q(\overline{x}).$$

*Proof.* We argue by contradiction. Suppose that the conclusion does not hold. Then for every $x \in X$ with $x \in K(x)$, we have $Q(x) \cap P(x) \neq \varnothing$. Next define the non-empty valued correspondence $T : X \to X$ where

$$\begin{aligned} T(x) &= Q(x) \cap P(x) \text{ if } x \in F \\ &= Q(x) \text{ if } x \notin F. \end{aligned}$$

We will show that $T$ has the local intersection property**.** Suppose that $y \in T(x)$. We must show that there exists an open set $U(x)$ in $X$ containing $x$ such that $\cap_{x' \in U(x)} T(x') \neq \varnothing$. Suppose that $x \notin F$. Since $y \in Q(x)$ and $Q$ has open lower sections, there exists an open set $U_1(x)$ containing $x$ such that $y \in Q(x')$ for all $x' \in U_1(x)$. Choosing an open set $U_2(x) \subseteq X \backslash F$ containing $x$, it follows that $y \in Q(x') = T(x')$ for all $x' \in U_1(x) \cap U_2(x)$. Now suppose that $x \in F$. Since $y \in Q(x) \cap P(x)$ and $R \subseteq X \times X$ is transfer semi-continuous with respect to $Q$, there exists an open set $U_1(x)$ containing $x$ and $y^* \in Q(x)$ such that $y^* \in P(x')$ for all $x' \in U_1(x)$. Since $y^* \in Q(x)$ and $Q$ has open lower sections, there exists an open set $U_2(x)$ containing $x$ such that $y^* \in Q(x')$ for all $x' \in U_2(x)$. Therefore, $U_1(x) \cap U_2(x)$ is an open set containing $x$ and $y^* \in Q(x') \cap P(x') \subseteq T(x')$ for all $x' \in U_1(x) \cap U_2(x)$. Defining $\eta(x) = conT(x)$ for each $x \in X$, it follows that

[4]For related notions of transfer continuity for functions, see Tian and Zhou [35], [36].

the non-empty valued, convex valued correspondence $\eta : X \to X$ has the local intersection property. Note that $Q(x) \cap P(x) \subseteq T(x) \subseteq \eta(x)$ for each $x$ so condition (v) implies that for each $x \in X \backslash C$, there exists an open set $U(x)$ containing $x$ such that we have $\bigcap_{x' \in U(x)} \eta(x') \cap M \neq \varnothing$. Applying Proposition 2.1, it follows that there exists $\overline{x} \in X$ such that $\overline{x} \in \eta(\overline{x})$. Note that $\overline{x} \in F$ for otherwise, $\overline{x} \notin K(\overline{x})$ and $\overline{x} \in \eta(\overline{x}) \subseteq conQ(x) \subseteq K(x)$. But $\overline{x} \in F$ implies that $\overline{x} \in \eta(\overline{x}) \subseteq con(Q(\overline{x}) \cap P(\overline{x}))$ violating condition (iv).

As a special case of Theorem 3.4, we obtain the single relation case of Theorem 3.1 of Lin and Ansari [18]. □

**Corollary 3.5.** *Suppose that $X$ is a non-empty, convex subset of a Hausdorff topological vector space. Suppose that $K : X \to X$ and $Q : X \to X$ are correspondences and $R \subseteq X \times X$ is a relation and define $P(x) = \{y \in X : (x, y) \notin R\}$. Suppose that*

(i) *For each $y \in X$, the set $\{x \in X : (x, y) \in R\}$ is closed in $X$.*
(ii) *$Q$ is non-empty valued, $Q$ has open lower sections and $conQ(x) \subseteq K(x)$ for each $x \in X$.*
(iii) *The set $F$ of fixed points of $K$ is closed.*
(iv) *For any finite set $\{x_1, .., x_n\} \subseteq X$ and for each $x^*$ in the convex hull of $\{x_1, .., x_n\}$, there exists an $i$ such that $(x^*, x_i) \in R$.*
(v) *There exists a non-empty, convex compact subset $M \subseteq X$ and a compact subset $C \subseteq X$ such that, for each $x \in X \backslash C$,*

$$Q(x) \cap P(x) \cap M \neq \varnothing.$$

*Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and*

$$(\overline{x}, y) \in R \text{ for all } y \in Q(\overline{x}).$$

As pointed out in Remark 3.1 of Lin and Ansari [18], one can replace condition (iv) with the assumptions that $P(x)$ is convex for each $x$ and $(x, x) \in R$ for each $x$ thus obtaining Theorems 1.1 and 3.1 as further corollaries.

## 4. A generalization with local convexity

In Theorem 3.4, we assume that the correspondence $Q$ has open lower sections. Note, however, that the Hausdorff topological vector space in Theorem 3.4 need not be locally convex. If we strengthen this assumption and assume in addition local convexity, then we can weaken both the assumptions that the relation $R$ is transfer continuous and that $Q$ has open lower sections. This alternative result is possible as an application of Proposition 2.2.

We begin with a generalization of transfer continuity inspired by another recent result from the theory of discontinuous games due to Reny[29].

**Definition 4.1.** Suppose that $X$ is a topological space, $Q : X \to X$ is a correspondence and $R \subseteq X \times X$ is a relation. Let $P(x) = \{y \in X | (x, y) \notin R\}$. The relation $R$ is correspondence secure with respect to $Q$ if whenever $x \in X$, $y \in Q(x)$ and $(x, y) \notin R$, there exists an open set $U(x)$ containing $x$ and a co-closed correspondence $d : U(x) \to X$ such that $d(x') \subseteq Q(x') \cap P(x')$ for all $x' \in U(x)$.

**Remark.** Mimicking the argument in the proof of Theorem 3.4, it follows that, if $Q : X \to X$ has open lower sections and if $R \subseteq X \times X$ is transfer semi-continuous with respect to $Q$, then $R$ is correspondence secure with respect to $Q$. To see this, suppose that $y \in Q(x)$ and $(x, y) \notin R$. Then there exists an open set $U_1(x)$ containing $x$ and $y^* \in Q(x)$ such that $(x', y^*) \notin R$ for all $x' \in U_1(x)$. Since $y^* \in Q(x)$ and $Q$ has open lower sections, there exists an open set $U_2(x)$ containing $x$ such that $y^* \in Q(x')$ for all $x' \in U_2(x)$. Therefore, $U_1(x) \cap U_2(x) = U(x)$ is an open set containing $x$ and $y^* \in Q(x') \cap P(x')$ for all $x' \in U(x)$. Now define $d(x') = \{y^*\}$ for all $x' \in U(x)$.

However, neither open lower sections nor transfer semi-continuity alone implies correspondence security of $R$ with respect to $Q$ as shown by the the next two examples.

**Example.** Let $X = [0, 1]$. Let

$$\begin{aligned} Q(x) &= \{1\} \text{ if } 0 \le x < \frac{1}{2} \\ &= [0,1] \text{ if } x = \frac{1}{2} \\ &= \{0\} \text{ if } \frac{1}{2} < x \le 1. \end{aligned}$$

Suppose that $f$ is defined as follows:

$$\begin{aligned} f(x, y) &= 1 \text{ if } 0 \le x < 1 \text{ and } y = 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Defining $R = \{(x, y|f(x, y) \le 0\}$, it follows that $R$ is transfer semi-continuous with respect to $Q$ but $Q$ does not have open lower sections. In addition, $R$ is not correspondence secure with respect to $Q$. Since $Q$ is not lower hemicontinuous, $Q$ does not have open lower sections. To see that $R$ is transfer semi-continuous with respect to $Q$, note that $x \in Q(x)$ implies that $x = \frac{1}{2}$. Suppose that $y \in Q(\frac{1}{2})$ but $f(\frac{1}{2}, y) > 0$. Then $y = 0$. Let $U(\frac{1}{2})$ be an open set in $X$ satisfying $U(\frac{1}{2}) \subseteq [0, 1[$ and $\frac{1}{2} \in U(\frac{1}{2})$. Then $y \in Q(\frac{1}{2})$ and $(x', y) \notin R$ for all $x' \in U(\frac{1}{2})$. In addition, $R$ is not correspondence secure with respect to $Q$ since $0 \in Q(\frac{1}{2})$ and $(\frac{1}{2}, 0) \notin R$ but $Q(x) \cap \{y \in X|(x, y) \notin R\} = \varnothing$ if $x < \frac{1}{2}$ so there does not exist an open set $U(\frac{1}{2})$ containing $\frac{1}{2}$ and a co-closed correspondence $d : U(\frac{1}{2}) \to X$ such that $d(x') \subseteq Q(x') \cap P(x')$ for all $x' \in U(\frac{1}{2})$.

**Example.** Let $X = [0, 1]$. Let

$$Q(x) = [0, 1] \text{ if } 0 \le x \le 1.$$

Suppose that $f$ is defined as follows:

$$\begin{aligned} f(x, y) &= 1 \text{ if } 0 \le x < \frac{1}{2} \text{ and } y = 1 \\ &= 1 \text{ if } \frac{1}{2} \le x \le 1 \text{ and } y = 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$

Defining $R = \{(x, y|f(x, y) \leq 0\}$, it follows that $Q$ has open lower sections but $R$ is not transfer semi-continuous with respect to $Q$. In addition, $R$ is not correspondence secure with respect to $Q$. To see that $R$ is not transfer semi-continuous with respect to $Q$, note that $0 \in Q(\frac{1}{2})$ and $(\frac{1}{2}, 0) \notin R$. Let $U(\frac{1}{2})$ be an open set in $X$ with $\frac{1}{2} \in U(\frac{1}{2})$ suppose that $y \in [0, 1]$. If $0 < y < 1$, then $(\frac{1}{2}, y) \in R$. If $y = 1$, then $(x', 1) \in R$ for all $x' \in U(\frac{1}{2})$ with $x' > \frac{1}{2}$. If $y = 0$, then $(x', 0) \in R$ for all $x' \in U(\frac{1}{2})$ with $x' < \frac{1}{2}$. In addition, $R$ is not correspondence secure with respect to $Q$ since $0 \in Q(\frac{1}{2})$ and $(\frac{1}{2}, 0) \notin R$ but

$$\begin{aligned} Q(x) \cap \{y \in X|(x, y) \notin R\} &= \{1\} \text{ if } 0 \leq x < \frac{1}{2} \\ &= \{0\} \text{ if } \frac{1}{2} \leq x \leq 1 \end{aligned}$$

so there does not exist an open set $U(\frac{1}{2})$ containing $\frac{1}{2}$ and a co-closed correspondence $d : U(\frac{1}{2}) \to X$ such that $d(x') \subseteq Q(x') \cap P(x')$ for all $x' \in U(\frac{1}{2})$.

**Theorem 4.2.** *Suppose that $X$ is a non-empty, compact, convex subset of a Hausdorff locally convex topological vector space. Suppose that $K : X \to X$ and $Q : X \to X$ are correspondences and $R \subseteq X \times X$ is a relation and define $P(x) = \{y \in X : (x, y) \notin R\}$. Suppose that*

(i) *$R$ is correspondence secure with respect to $Q$.*
(ii) *$Q$ is non-empty valued with the continuous inclusion property and $conQ(x) \subseteq K(x)$ for each $x \in X$.*
(iii) *The set $F$ of fixed points of $K$ is closed.*
(iv) *For each $x \in X$, $x \notin con(P(x) \cap Q(x))$.*

*Then there exists $\overline{x} \in X$ such that $\overline{x} \in K(\overline{x})$ and*

$$(\overline{x}, y) \in R \text{ for all } y \in Q(\overline{x}).$$

*Proof.* We argue by contradiction. Suppose that the conclusion does not hold. Then $Q(x) \cap P(x) \neq \varnothing$ for every $x \in X$ with $x \in K(x)$. Next define the non-empty valued correspondence $T : X \to X$ where

$$\begin{aligned} T(x) &= Q(x) \cap P(x) \text{ if } x \in F \\ &= Q(x) \text{ if } x \notin F. \end{aligned}$$

We will show that $T$ has the continuous inclusion property. Suppose that $y \in T(x)$. If $x \notin F$, then $F$ closed and the continuous inclusion property for $Q$ imply that there exists an open set $U(x) \subseteq X \backslash F$ containing $x$ and a co-closed correspondence $d_x : U(x) \to X$ such that $d_x(x') \subseteq Q(x') = T(x')$ for each $x' \in U(x)$. If $x \in F$, then correspondence security of $R$ with respect to $Q$ ensures the existence of an open set $U(x)$ containing $x$ and a co-closed correspondence $d_x : U(x) \to X$ such that $d_x(x') \subseteq Q(x') \cap P(x') \subseteq T(x')$ for each $x' \in U(x)$. Let $\eta(x) = conT(x)$. Then $\eta : X \to X$ is non-empty valued and has the continuous inclusion property. Applying Proposition 2.2, it follows that there exists $\overline{x} \in X$ such that $\overline{x} \in \eta(\overline{x})$. Note that $\overline{x} \in F$ for otherwise, $\overline{x} \notin K(\overline{x})$ and $\overline{x} \in \eta(\overline{x}) \subseteq conQ(x) \subseteq K(x)$. But $\overline{x} \in F$ implies that

$$\overline{x} \in \eta(\overline{x}) \subseteq con(Q(\overline{x}) \cap P(\overline{x}))$$

violating condition (iv).

The next example satisfies the assumptions of Theorem 4.2, but $R$ is neither transfer semi-continuous with respect to $Q$ nor does $Q$ have open lower sections so the assumptions of Theorem 3.4 are not satisfied. □

**Example.** Let $X = [0,1]$. Let $Q = K$ where

$$\begin{aligned} K(x) &= [0,1] \text{ if } x = 0 \\ &= [x, \frac{1}{2}] \text{ if } 0 < x \leq \frac{1}{2} \\ &= [\frac{1}{2}, x] \text{ if } \frac{1}{2} \leq x < 1 \\ &= [0,1] \text{ if } x = 1 \end{aligned}$$

Suppose that $f$ is defined as follows:

$$\begin{aligned} f(x,y) &= 1 \text{ if } y = \frac{1}{4} + \frac{x}{2} \text{ and } x \in [0, \frac{1}{2}[\cup]\frac{1}{2}, 1] \\ &= 0 \text{ otherwise.} \end{aligned}$$

Defining $R = \{(x, y|f(x,y) \leq 0\}$, it follows that $K$ has does not have open lower sections since $K$ is not lower hemi-continuous. However, $K$ does have the continuous inclusion property since $K$ is compact valued, convex valued and upper hemi-continuous. Furthermore, $R$ is not transfer semi-continuous with respect to $K$. To see this, note that $\frac{3}{8} \in K(\frac{1}{4})$ and $(\frac{1}{4}, \frac{3}{8}) \notin R$ and that $K(\frac{1}{4}) = [\frac{1}{4}, \frac{1}{2}]$. Let $U(\frac{1}{4})$ be an open set in $X$ with $\frac{1}{4} \in U(\frac{1}{4})$ suppose that $y \in [\frac{1}{4}, \frac{1}{2}]$. Then $(x', \frac{3}{8}) \in R$ for each $x' \in U(\frac{1}{4})$ with $x' \neq \frac{1}{4}$. If $y > \frac{3}{8}$, then $(x', y) \in R$ for all $x' \in U(\frac{1}{4})$ with $x' < \frac{1}{4}$. If $y < \frac{3}{8}$, then $(x', y) \in R$ for all $x' \in U(\frac{1}{4})$ with $x' > \frac{1}{4}$. However, $R$ is correspondence secure with respect to $K$. Suppose that $y \in K(x)$ and $(x, y) \notin R$. Then $x \in [0, \frac{1}{2}[\cup]\frac{1}{2}, 1]$. If $x \in [0, \frac{1}{2}[$, then $K(x) \cap \{y \in X|(x,y) \notin R\} = \{\frac{1}{4} + \frac{x}{2}\}$. Let U(x) be an open set in $X$ satisfying U(x)$\subseteq [0, \frac{1}{2}[$ and $x \in U(x)$. For each $x' \in U(x)$, define $d(x') = \{\frac{1}{4} + \frac{x'}{2}\}$. Then $d : U(x) \to R$ is co-closed and $d(x') \subseteq K(x') \cap \{y \in X|(x', y) \notin R\}$ for each $x' \in U(x)$. A similar argument applies if $x \in ]\frac{1}{2}, 1]$. Note that $x = \frac{1}{2}$ is the unique solution to the quasi-variatonal relation problem.

## 5. Applications in game theory

5.1. **Generalized Games.** Consider a generalized game with players $N = \{1, .., n\}$, strategy sets $X_i$ and payoff functions $u_i : X_1 \times \cdots \times X_n \to \mathbb{R}$. Suppose that each $X_i$ is a compact, convex, non-empty subset of a Hausdorff TVS. To each player, we associate a feasible action corrrespondence $K_i : X \to X_i$ where $K_i(x) \subseteq X_i$ is the set of actions available to $i$. Defining

$$R = \{(x, y) \in X \times X| \sum_{i=1}^{n} u_i(x_{-i}, y_i) - \sum_{i=1}^{n} u_i(x) \leq 0$$

and $K(x) = \times_{i \in N} K_i(x)$, it is clear that $\overline{x} \in X$ is an equilibrium in the generalized game if and only if $\overline{x}$ solves the QVRP problem for $R$ and $K$.

Suppose that

(a) For every real $\lambda$, each $u_i$ is $\lambda$-transfer lower semi-continuous in $x_{-i}$ with respect to $K_i$: if $y_i \in K_i(x)$ and $u_i(x_{-i}, y_i) > \lambda$ then there exists an open set $U_i(x)$ containing $x$ and $y_i^* \in K_i(x)$ such that $u_i(x'_{-i}, y_i) > \lambda$ for all $x' \in U_i(x)$.

(b) The aggregate payoff $x \mapsto \sum_{i=1}^n u_i(x)$ is upper semi-continuous.

(c) Each $K_i$ is non-empty valued and convex valued with open lower sections and the set of fixed points of $x \mapsto K(x) = K_1(x) \times \cdots \times K_n(x)$ is closed.

(d) For each $x$, $\{y \in X : (x, y) \notin R\}$ is convex.

Then there exists a profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X_1 \times \cdots \times X_n$ such that

$$\overline{x}_i \in \arg \max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i) \text{ for each } i.$$

To apply Theorem 3.4, we need to use assumptions (a) and (b) to establish that $R$ is transfer semi-continuous with respect to $K$. The argument adapts the proof of Lemma 1 in Prokopovych and Yannelis [26]. Suppose that $(x, y) \in X \times X$ and that $\sum_{i=1}^n u_i(x_{-i}, y_i) - \sum_{i=1}^n u_i(x) > 0$. Then

$$\sum_{i=1}^{n} u_i(x_{-i}, y_i) > \lambda > \sum_{i=1}^{n} u_i(x)$$

for some $\lambda$. So there exist numbers $\lambda_i$ such that $\sum_i \lambda_i = \lambda$ and $u_i(x_{-i}, y_i) > \lambda_i$ for each $i$. Applying Assumption (a), there exists for each $i$ an open set $U_i(x)$ containing $x$ and $y_i^* \in K_i(x)$ such that $u_i(x'_{-i}, y_i^*) > \lambda_i$ for all $x' \in U_i(x)$. Applying assumption (b), there exists an open set $V(x)$ containing $x$ such that $\lambda > \sum_{i=1}^n u_i(x')$ for all $x' \in V(x)$. Therefore, $\sum_{i=1}^n u_i(x'_{-i}, y_i^*) > \sum_i \lambda_i = \lambda > \sum_{i=1}^n u_i(x')$ for all $x' \in [\cap_i U_i(x)] \cap V(x)$ implying that $R$ is transfer semi-continuous with respect to $K$.

5.2. **Generalized Skew Symmetric Games.** To introduce this idea, let $S$ be a nonempty set. A function $\varphi : S \times S \to \mathbb{R}$ is *skew-symmetric* if $\varphi(x, y) = -\varphi(y, x)$ for all $(x, y) \in S \times S$. Obviously, skew symmetry implies that $\varphi(x, x) = 0$ for all $x \in S$. A relation $\succsim$ in $S \times S$ has a skew-symmetric representation if there exists a skew-symmetric function $\varphi : S \times S \to \mathbb{R}$ satisfying

$$y \succsim x \Leftrightarrow \varphi(x, y) \leq 0.$$

From the definition, it follows that every relation $\succsim$ admitting a skew-symmetric representation is reflexive and complete, and if $\succsim$ admits a utility representation $u : S \to \mathbb{R}$, then $\succsim$ admits the skew-symmetric representation $\varphi(x, y) = u(x) - u(y)$.

A qualitative game is a collection $G = (X_i, \succsim_i)_{i=1}^n$, where $n$ is a finite number of players, $X_i$ is a nonempty set of actions for player $i$, and $\succsim_i$ is a preference relation for player $i$ defined on the set $X := \times_{i=1}^n X_i$ of action profiles, i.e., $\succsim_i$ is a binary relation in $X \times X$.

We say that a qualitative game $G = (X_i, \succsim_i)_{i=1}^n$ is a skew symmetric game (SSYM) if for each $i$ there exists a skew symmetric map $\varphi_i : X \times X \to \mathbb{R}$ satisfying

$$y \succsim_i x \Leftrightarrow \varphi_i(x, y) \leq 0, \quad (x, y) \in X \times X.$$

A Nash equilibrium of an SSYM game $(X_i, \varphi_i)_{i=1}^n$ is a strategy profile $(x_1, ..., x_n) \in \times_{i=1}^n X_i$ such that for each $i$,

$$\varphi_i((y_i, x_{-i}), x) \leq 0, \quad \text{for all } y_i \in X_i.$$

For example, suppose that each $X_i$ is a compact, nonempty, convex subset of $\mathbb{R}^{m_i}$ for some $m_i \geq 1$. In addition, suppose that each $\varphi_i$ is continuous on $X \times X$ and $y_i \mapsto \varphi_i((y_i, x_{-i}), x)$ is quasiconcave for each $x \in X$. Now define

$$\mu_i(z) := \arg\max_{x_i \in X_i} \varphi_i((x_i, z_{-i}), z), \qquad \text{for each } z \in X.$$

Then, combining Berge's Maximum Theorem and the Kakutani Fixed Point Theorem, it follows that there exists $\overline{x} \in X$ such that

$$\overline{x} \in \mu_1(\overline{x}) \times \cdots \times \mu_n(\overline{x}),$$

i.e.,

$$\varphi_i((x_i, \overline{x}_{-i}), \overline{x}) \leq \varphi_i(\overline{x}, \overline{x}) = 0.$$

Discontinuous SSYM games are studied in Carbonell-Nicolau and McLean [6].

There is also an obvious approach to existence using some version of the Ky-Fan inequality. Let

$$f(x, y) = \sum_{i=1}^n \varphi_i((y_i, x_{-i}), x).$$

Then $\overline{x} \in X$ is an equilibrium if and only if

$$f(\overline{x}, y) \leq 0, \text{ for each } y \in X.$$

Suppose that the SSYM game $G = (X_i, \varphi_i)_{i=1}^N$ where each $X_i$ is a non-empty, compact, convex subset of a Hausdorff locally convex TVS. To each player, we associate a feasible action corrrespondence $K_i : X \to X_i$. Define a relation $R \subseteq X \times X$ as

$$R = \{(x, y) \in X \times X | \sum_{i=1}^n \varphi_i((y_i, x_{-i}), x) \leq 0\}$$

and a correspondence $K : X \to X$ with $K(x) = K_1(x) \times \cdots \times K_n(x)$ for each $x \in X$. Suppose that

(i) $R$ is correspondence secure with respect to $K$.
(ii) $K$ is non-empty valued and convex valued with the continuous inclusion property.
(iii) For each $x$, $\{y \in X : (x, y) \notin R\}$ is convex.

Then there exists a profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n) \in X_1 \times \cdots \times X_n$ such that

$$\overline{x}_i \in \arg\max_{y_i \in K_i(\overline{x})} u_i(\overline{x}_{-i}, y_i) \text{ for each i.}$$

## 6. Generalized quasi-variational relation problems

Two other significant problems in non-linear analysis are the generalized equilibrium problem and the generalized quasi-variational equilibrium problem.

**Generalized Equilibrium Problem (GEP)**: Given sets $X$ and $Z$, a function $g : X \times X \times Z \to \mathbb{R}$ and a correspondence $\varphi : X \to Z$, find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in X.$$

A problem that includes the generalized equilibrium problem and the quasi-equilibrium problem as special cases is the generalized quasi-variational equilibrium problem.

**Generalized Quasi-Variational Equilibrium Problem (GQVE)**: Given sets $X$ and $Z$, a function $g : X \times X \times Z \to \mathbb{R}$ and two correspondences $K : X \to X$ and $\varphi : X \to Z$, find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$g(\overline{x}, y, \overline{z}) \leq 0 \text{ for all } y \in K(\overline{x}).$$

If $g(x, y, z) = f(x, y)$ for some function $f$, then the problem specializes to the quasi-equilibrium problem and if in addition $K(x) = X$ for all $x$, we recover the Ky Fan equilibrium problem as a special case. If $g(x, y, z) = z \cdot (x - y)$, then the GVEP specializes to the generalized quasi-variational inequality problem. The GQVE problem was introduced in Chan and Pang [9] (see also Wu and Shen [38]) as part of their study of the generalized quasi-variational inequality problem in finite dimensions

The GQVE problem can be further generalized to relations. (see e.g., Yang [39] or Hung and Kieu [16]).

**Generalized Quasi-Variational Relation Problem (GQVR)**: Given sets $X$ and $Z$, a relation $R \subseteq X \times X \times Z$ and two correspondednces $K : X \to X$ and $\varphi : X \to Z$, find $\overline{x} \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that $\overline{x} \in K(\overline{x})$ and

$$(\overline{x}, y, \overline{z}) \in R \text{ for all } y \in K(\overline{x}).$$

The GQVR problem includes the quasi-variational inclusion problem studied in, e.g., Hai et al. [14], Lin [17] and the references therein. Our goal is to prove an existence result for the GQVR problem with minimal continuity assumptions. The assumptions that we do impose are again motivated by weakened contnuity ideas that have been developed for Nash equilibrium existence results for discontinuouus games.

We begin with a result that does not assume local convexity and generalizes Theorem 3.4 above by extending the notion of transfer semi-continuity.

**Definition 6.1.** Suppose that $X$ and $Z$ topological spaces. Suppose that $Q : X \to X$ is a correspondence, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Then $R$ is transfer semi-continuous with respect to $Q$ and $\varphi$ if for all $(x, y, z) \in X \times X \times Z$ such that $y \in Q(x), z \in \varphi(x)$ and $(x, y, z) \notin R$, there exists an open set $V(x, z) \subseteq X \times Z$ containing $(x, z)$ and $y^* \in Q(x)$ such that $(x', y^*, z') \notin R$ for all $(x', z') \in V(x, z)$.

**Theorem 6.2.** *Suppose that $X$ is a non-empty, convex subset of a Hausdorff TVS and that $Z$ is a Hausdorff TVS. Suppose that $K : X \to X$ and $Q : X \to X$ are correspondences, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Define $P(x,z) = \{y \in X | (x,y,z) \notin R\}$ and suppose that*

(i) *$R$ is transfer semi-continuous with respect to $Q$ and $\varphi$*
(ii) *$Q$ is non-empty valued with open lower sections and $conQ(x) \subseteq K(x)$ for each $x \in X$.*
(iii) *$\varphi$ is non-empty valued and convex valued with the local intersection property.*
(iv) *The set $F = \{(x,z) \in X \times Z | (x,z) \in K(x) \times \varphi(x)\}$ is closed.*
(v) *For each $(x,z) \in X \times Z$, $x \notin con(P(x,z) \cap Q(x))$.*
(vi) *There exists a non-empty, convex compact subset $M \subseteq X \times Z$ and a compact subset $C \subseteq X \times Z$ such that, for each $(x,z) \in (X \times Z) \backslash C$, there exists an open set $U(x,z)$ containing $(x,z)$ such that*

$$\left[ \bigcap_{(x',z') \in U(x,z)} [\, [P(x',z') \cap Q(x')] \times \varphi(x') ] \right] \cap M \neq \varnothing.$$

*Then there exists $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\overline{x} \in K(\overline{x}), \overline{z} \in \varphi(\overline{x})$ and*

$$(\overline{x}, y, \overline{z}) \in R \text{ for all } y \in Q(\overline{x}).$$

*Proof.* We again argue by contradiction.

Suppose that the conclusion of the theorem is false. Then for every $(x,z) \in X \times Z$ with $x \in K(x)$ and $z \in \varphi(x)$, there exists a $y \in K(x)$ such that $(x,y,z) \notin R$. Therefore, $P(x,z) \cap K(x) \neq \varnothing$ for every $(x,z) \in F$. Define a non-empty valued correspondence $T : X \times Z \to X$ as

$$\begin{aligned} T(x,z) &= P(x,z) \cap Q(x) \text{ if } (x,z) \in F \\ &= Q(x) \text{ if } (x,z) \notin F. \end{aligned}$$

We show that $T$ has the local intersection property. Suppose that $y \in T(x,z)$. we must show that for each $(x,z)$, there exists an open set $U(x,z)$ in $X \times Z$ containing $(x,z)$ such that

$$\bigcap_{(x',z') \in U(x,z)} T(x') \neq \varnothing.$$

If $(x,z) \in F$, then $y \in P(x,z) \cap Q(x)$. Since $R \subseteq X \times X \times Z$ is transfer semi-continuous in $(x,z)$ with respect to $Q$ and $\varphi$, there exists an open set $V_1(x,z) \subseteq X \times Z$ containing $(x,z)$ and $y^* \in Q(x)$ such that $(x',y^*,z') \notin R$ for all $(x',z') \in V_1(x,z)$. Since $y^* \in Q(x)$ and $Q$ has open lower sections, there exists an open $V_2(x) \subseteq X$ containing $x$ such that $y^* \in Q(x')$ for all $x' \in V_2(x)$. Therefore, $U(x,z) = V_1(x,z) \cap [V_2(x) \times Z] \subseteq X \times Z$ is an open set containing $(x,z)$ and $y^* \in P(x',z') \cap Q(z') \subseteq T(x',z')$ for all $(x',z') \in U(x,z)$. If $(x,z) \notin F$, then $y \in Q(x)$ so there exists an open set $U_1(x) \subseteq X$ containing $x$ such that $y \in Q(x')$ for all $x' \in U_1(x)$. Choosing an open set $U_2(x,z) \subseteq X \backslash F$ containing $(x,z)$, it follows that $y \in Q(x') = T(x',z')$ for all $(x',z') \in [U_1(x) \times Z] \cap U_2(x,z)$.

Next, define a non-empty valued, convex valued correspondence $\eta : X \times Z \to X \times Z$ as $\eta(x,z) = conT(x,z) \times \varphi(x)$ and note that $\eta$ has the local intersection property. In addition, note that $[P(x,z) \cap Q(x)] \times \varphi(x) \subseteq \eta(x,z)$ for each $(x,z)$ so condition (vi) implies that for each $(x, z) \in (X \times Z) \backslash C$, there exists an open set $U(x, z)$ containing $(x, z)$ such that we have

$$\left[ \bigcap_{(x',z') \in U(x,z)} \eta(x', z') \right] \cap M \neq \varnothing.$$

Applying Proposition 2.1, it follows that there exists $(\overline{x}, \overline{z}) \in X \times Z$ such that $(\overline{x}, \overline{z}) \in \eta(\overline{x}, \overline{z})$. Note that $(\overline{x}, \overline{z}) \in F$ for otherwise, $\overline{x} \notin K(\overline{x})$ and $\overline{x} \in conQ(\overline{x}) \subseteq K(\overline{x})$. But $(\overline{x}, \overline{z}) \in F$ implies that $\overline{x} \in con[P(\overline{x}, \overline{z}) \cap Q(\overline{x})]$ violating condition (v).

To generalize Theorem 4.2, we need to extend the notion of correspondence security. □

**Definition 6.3.** Suppose that $X$ and $Z$ are topological spaces. Suppose that $Q : X \to X$ is a correspondences, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Let $P(x, z) = \{y \in X | (x, y, z) \notin R\}$. Then $R$ is correspondence secure with respect to $Q$ and $\varphi$ if whenever $x \in X, y \in Q(x), z \in \varphi(x)$ and $(x, y, z) \notin R$, there exists an open set $U(x, z)$ containing $(x, z)$ and a co-closed correspondence $d : U(x, z) \to X$ such that $d(x', z') \subseteq Q(x') \cap P(x', z')$ for all $(x', z') \in U(x, z)$.

**Remark.** If $Q$ has open lower sections and if $R$ is transfer semi-continuous with respect to $Q$ and $\varphi$, then $R$ is correspondence secure with respect to $Q$ and $\varphi$.

**Theorem 6.4.** *Suppose that $X$ is a non-empty, convex, compact subset of a locally convex Hausdorff TVS and that $Z$ is a locally convex Hausdorff TVS. Suppose that $K : X \to X$ and $Q : X \to X$ are correspondences, $\varphi : X \to Z$ is a correspondence and $R \subseteq X \times X \times Z$ is a relation. Define $P(x, z) = \{y \in X | (x, y, z) \notin R\}$ and suppose that*

rm (i) *$R$ is correspondence secure with respect to $Q$ and $\varphi$.*
(ii) *$Q$ is non-empty valued with the continuous inclusion property and $conQ(x) \subseteq K(x)$ for each $x \in X$.*
(iii) *$\varphi$ is non-empty valued and convex valued with the continuous inclusion property.*
(iv) *The set $F = \{(x, z) \in X \times Z | (x, z) \in K(x) \times \varphi(x)\}$ is closed.*
(v) *For each $(x, z) \in X \times Z$, $x \notin con(P(x, z) \cap Q(x))$.*

*Then there exists $\overline{x} \in X$ and $\overline{z} \in Z$ such that $\overline{x} \in K(\overline{x}), \overline{z} \in \varphi(\overline{x})$ and*

$$(\overline{x}, y, \overline{z}) \in R \text{ for all } y \in Q(\overline{x}).$$

*Proof.* We argue by contradiction. Suppose that the conclusion of the theorem is false. Define a non-empty valued correspondence

$$\begin{aligned} T(x, z) &= Q(x) \cap P(x, z) \text{ if } (x, z) \in F \\ &= Q(x) \text{ if } (x, z) \notin F. \end{aligned}$$

Using an argument analogous to that of Theorem 4.2, we can use (i) and (ii) to establish that $T$ has the continuous inclusion property. Defining $\eta(x, z) = conT(x, z) \times \varphi(x)$, it follows (using (iii)) that $\eta : X \times Z \to X \times Z$ is a non-empty valued, convex valued correspondence with the continuous inclusion property so applying Proposition 2.2, we conclude that there exists $(\overline{x}, \overline{z}) \in X \times Z$ with $(\overline{x}, \overline{z}) \in \eta(\overline{x}, \overline{z})$.

Note that $(\overline{x}, \overline{z}) \in F$ for otherwise, $\overline{x} \notin K(\overline{x})$ and $\overline{x} \in conQ(\overline{x}) \subseteq K(\overline{x})$. But $(\overline{x}, \overline{z}) \in F$ implies that $\overline{x} \in con(P(\overline{x}, \overline{z}) \cap Q(\overline{x}))$ violating condition (v).

As an application of Theorem 6.4, we consider an optimization-based approach to generalized games. Let $X_i$ be a non-empty, compact, convex subset of a Banach space $V_i$ with dual $V_i^*$ and let $u_i : X \to \mathbb{R}$ denote the payoff function of player $i$. Let $K_i : X \to X_i$ denote the feasible set correspondence for player $i$. For $x \in X$, let $N_{K_i(x)}(x_i) \subseteq V_i^*$ denote the normal cone to $K_i(x)$ at $x_i$, i.e.,

$$N_{K_i(x)}(x_i) = \{z \in V_i^* | \langle z, y - x_i \rangle \leq 0 \text{ for all } y \in K_i(x)\}.$$

For each $x_{-i} \in X_{-i}$ and $y_i \in X_i$, let $\partial_i^+ u_i(x_{-i}, y_i)$ denote the superdifferential of $z_i \mapsto u_i(x_{-i}, z_i)$ evaluated at $y_i$. Note that $\partial_i^+ u_i(x_{-i}, y_i)$ is a convex, weak*-closed set in $V_i^*$ for each $x_{-i}$ and $y_i$. Then a strategy profile is a Nash equilibrium if for each $i$ we have $\overline{x}_i \in K_i(\overline{x})$ and

$$\partial_i^+ u_i(\overline{x}_{-i}, \overline{x}_i) \cap N_{K_i(\overline{x})}(\overline{x}_i) \neq \varnothing.$$

To formulate the problem as a GQVE problem, let

$$\begin{aligned} g(x, y, z) &= \sum_i \langle z_i, y_i - x_i \rangle \\ \varphi(x) &= \prod_i \varphi_i(x_{-i}, x_i) = \prod_i \partial_i^+ u_i(x) \\ K(x) &= \prod_i K_i(x). \end{aligned}$$

Then a strategy profile $\overline{x} = (\overline{x}_1, .., \overline{x}_n)$ is a Nash equilibrium if the GQVE problem has a solution, i.e., if there exists an $x \in X$ and $\overline{z} \in \varphi(\overline{x})$ such that

$$\sum_i \langle \overline{z}_i, y_i - \overline{x}_i \rangle \leq 0 \text{ for all } y \in K(\overline{x}).$$

Suppose that for each $i$, $x \mapsto \partial_i^+ u_i(x)$ is non-empty valued with the continuous inclusion property and that $K$ is non-empty valued and convex valued with the continuous inclusion property. Furthermore, suppose (letting $Q = K$) that conditions (i) and (iv) of Theorem 6.4 are satisfied. Condition (v) is trivially satisfied so applying Theorem 6.4, we conclude that the QQVE problem has a solution. □

## Acknowledgments

The author thanks the referee for helpful comments.

## References

[1] M. Balaj and S. Muresan, *Generalizations of the Fan-Browder fixed point theorem and minimax inequalities*, Arch. Math. (Brno) **41** (2005), 399-407.

[2] M. Balaj and D. T. Luc, *On mixed variational relation problems*, Comput. Math. Appl. **60** (2010), 2712–2722.

[3] M. Balaj and L. J. Lin, *Generalized variational relation problems with applications*, J. Optim. Theor Appl. **148** (2011), 1–13.

[4] P. Barelli and I. Meneghel , *A note on the equilibrium existence problem in discontinuous games*, Econometrica **81** (2013), 813–824.

[5] M. Baye, G. Tian and J. Zhou, *Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs*, Rev. Econ. Stud. **60** (1993), 935–948.

[6] O. Carbonell-Nicolau and R. P. McLean , *Nash and Bayes-Nash equilibria in strategic-form games with intransitivities*, Econom. Theory **68** (2019), 935–965.

[7] G. Carmona, *Existence and Stability of Nash Equilibrium*, World Scientific Publishing, Singapore, 2013.

[8] G. Carmona and K. Podczeck, *Existence of Nash equilibrium in ordinal games with discontinuous preferences*, Econom. Theory **61** (2016), 457–478.

[9] D. Chan and J.-S. Pang, *The generalized quasi-variational inequality problem*, Math. Oper. Res. **7** (1982), 211–222.

[10] P. Dasgupta and E. Maskin , *The existence of equilibrium in discontinuous economic games, I: Theory*, Rev. Econ. Stud. **53** (1986), 1–26.

[11] P. Dasgupta and E. Maskin , *The existence of equilibrium in discontinuous economic games, II: Applications*, Rev. Econ. Stud. **53** (1986), 27–41.

[12] G. Debreu, *A social equilibrium existence theorem*, Proc. Natl. Acad. Sci. USA **38** (1952), 386-393.

[13] K. Fan, *A Minimax Inequality and Applications*, in: Inequalities, vol.3, O.Shisha (ed.), Academic Press, New York, 1972.

[14] N. Hai, P. Khanh and N. Quan, *On the existence of solutions to quasivariational inclusion problems*, J. Global Optim. **45** (2009), 565–581.

[15] W. He and N.C. Yannelis, *Equilibria with discontinuous preferences*, J. Math. Anal. Appl. **450** (2017), 1421–1433.

[16] N. Hung and P. Kieu, *On the existence and essential components of solution sets for systems of generalized quasi-variational relation problems*, J. Inequal. Appl. **2014** (2014): 250.

[17] L. J. Lin, *Some results on systems of quasi-variational inclusion problems and systems of generalized quasi-variational inclusion problems*, Nonlinear Anal. **72** (2010), 37–49.

[18] L. J. Lin and Q. Ansari, *Systems of quasi-variational relations with applications*, Nonlinear Analy. **72** (2010), 1210–1220.

[19] L. J. Lin and S. Y. Wang, *Simultaneous variational relation problems and related applications*, Comput. Math. Appl. **58** (2009),1711–1721.

[20] D. T. Luc, *An abstract problem in variational analysis*, J. Optim. Theory Appl., **138** (2008), 65–76.

[21] D. T. Luc, E. Sarabi and A. Soubeyran, *Existence of solutions in variational relation problems without convexity*, J. Math. Anal. Appl. **364** (2010), 544–555.

[22] A. McLennan, P. K. Monteiro and R. Tourky, *Games with discontinuous payoffs: A strengthening of Reny's existence theorem*, Econometrica **79** (2011), 1643–1664.

[23] R. Nessah, and G. Tian , *Existence of solution of minimax inequalities, equilibria in games and fixed points without convexity and compactness assumptions*, J. Optim. Theory Appl. **157** (2013), 75–95.

[24] R. Nessah, R., and G. Tian, *On the existence of Nash equilibrium in discontinuous games*, Econom. Theory **61** (2016), 515–540.

[25] P. Prokopovych, *On equilibrium existence in payoff secure games*, Econom. Theory **483** (2011), 5–16.

[26] P. Prokopovych and N. C. Yannelis, *On the existence of mixed strategy Nash equilibria*, J. Math. Econom. **52** (2014), 87–97.

[27] P. J. Reny, *On the existence of pure and mixed strategy Nash equilibria in discontinuous games*, Econometrica **67** (1999), 1029–1056.

[28] P. J. Reny, *Nash equilibrium in discontinuous games*, Econom. Theory **61** (2016a), 553–569.

[29] P. J. Reny, *Equilibrium in discontinuous games without complete or transitive preferences*, Econom. Theory Bull. **4** (2016b), 1–4.

[30] V. Scalzo, *Equilibrium existence in games: slight single deviation property and Ky Fan minimax inequality*, J. Math. Econom. **82** (2019), 8197–201.

[31] L. K. Simon, *Games with discontinuous payoffs*, Rev. Econ. Stud. **54** (1987), 569–597.

[32] M. H. Shih and K. K, Tan, *Generalized quasi-variational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl. **108** (1985), 333–343.

[33] E. Tarafdar and M. Chowdhury, *Topological Methods for Set-valued Non-linear Analysis*, World Scientific, Singapore, 2008.

[34] G. Tian, *Generalizations of the FKKM theorem and the Ky Fan minimax inequality, with applications to maximal elements, price equilibrium, and complementarity*, J. Math. Anal. Appl. **170** (1992), 457–471.

[35] G. Tian and J. Zhou, *The Maximum theorem and the existence of Nash equilibrium of (generalized) games without lower semicontinuities*, J. Math. Anal. Appl. **166** (1992), 351–364.

[36] G. Tian and J. Zhou, *Transfer Continuities, generalizations of the Weierstrass and maximum theorems: a full characterization*, J. Math. Econom. **24** (1995), 281–303.

[37] M. Uyanik, *Existence and computation of equilibria in games and economies, unpublished dosctoral dissretation*, Baltimore, Johns Hopkins University, 2016.

[38] X. Wu and S. Shen, *A Further Generalization of Yannelis-Prabhakar's Continuous Selection Theorem and its Applications*, J. Math. Anal. Appl.c**197** (1996), 61–74.

[39] Z. Yang, *Existence of solutions of generalized quasi-variational relation problems by the inductive relations and some applications*, J. Syst. Sci. Complex. **29** (2016) 219–227.

*Manuscript received December 29 2019*
*revised March 17 2020*

Richard P. McLean
Department of Economics, Rutgers University, New Brunswick, NJ 08901
*E-mail address*: rpmclean@economics.rutgers.edu