Pure and Applied Functional Analysis
Volume 6, Number 4, 2021, 807–815

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON THE FUNDUMENTAL THEOREM FOR YOUNG MEASURES

TORU MARUYAMA

Abstract. Let Ω be a measure space and X a topological space with the Borel σ-field. Measurable functions $u_n : \Omega \to X (n = 1, 2, \dots)$ and a real-valued continuous function $f : X \to \mathbb{R}$ are assumed to be given. We examine the convergence of the sequence $\{f \circ u_n\}$ of the compositions of f and $u_n (n = 1, 2, \dots)$ in view of the criteria (1) the weak*-convergence in $\mathfrak{L}^\infty(\Omega, \mathbb{R})$ and (2) the weak-convergence in $\mathfrak{L}^1(\Omega, \mathbb{R})$. The theory of the narrow convergence of the Young measures is effectively made use of.

1. Introduction

Consider a sequence $\{u_n : \Omega \to X\}$ of measurable mappings, where Ω is a measure space and X is a topological space endowed with the Borel σ-field.

Given a continuous function $f : X \to \mathbb{R}$, we would like to examine the convergence properties of the sequence $\{f \circ u_n\}$ of compositions. The criteria of convergence are (1) the weak*- convergence in $\mathfrak{L}^\infty(\Omega, \mathbb{R})$ and (2) the weak-convergence in $\mathfrak{L}^1(\Omega, \mathbb{R})$.

Under certain assumptions, there is a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that

$$f \circ u_{n'} \to \int_X f(x) d\nu_\omega \quad \text{as} \quad n' \to \infty,$$

where $\{\nu_\omega | \omega \in \Omega\}$ is a "measurable family" of Borel probability measures. The criteria of convergence are stated above.

This fact, called "the fundamental theorem for Young measures", was discovered and proved by several authors including Balder [1], Ball [2], Evans [4], and so on. Applications to nonlinear partial differential equations are neatly explained in Evans [4].

The object of the present paper is to give a proof of the fundamental theorem from the viewpoint of Maruyama [6], in which I tried to provide an overview of the basic structure of the theory of Young measures. Its framework is a little bit more general than the previous works by Ball, Evans, and others.

2. Basic concepts and facts

I start by explaining briefly some basic materials concerning the theory of Young measures, which are prerequisites for the analysis developed in the following sections. For the details, please consult Maruyama [6].

2020 *Mathematics Subject Classification.* 28A50, 28A33.
Key words and phrases. Young measure, measurable family, disintegration, uniform tightness .

Let $(\Omega, \mathcal{E}, \mu)$ be a measure space and X a Hausdorff topological space endowed with the Borel σ-field $\mathcal{B}(X)$. The projection of the product space $\Omega \times X$ into Ω(resp.X) is denoted by π_Ω(resp.π_X).

A measure γ on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ which satisfies

$$\gamma \circ \pi_\Omega^{-1} = \mu \tag{2.1}$$

is called a Young measure. The set of all the Young measures is denoted by $\mathfrak{Y}(\Omega, \mu; X)$.

A set $\{\nu_\omega | \omega \in \Omega\}$ of finite measures on $(X, \mathcal{B}(X))$ is called a measurable family if the mapping

$$\omega \mapsto \nu_\omega(B)$$

is measurable for all $B \in \mathcal{B}(X)$.

Throughout this paper, we always assume that :

- **(A)** $(\Omega, \mathcal{E}, \mu)$ **is a finite complete measure space, and** $\mathcal{E}$ **is countably generated.**
- **(B)** X **is a locally compact metrizable Souslin space.**

We denote by $\mathfrak{M}(X)$ the space of all the Radon signed measures on X. Since X is Souslin, any (positive) finite measure on $(X, \mathcal{B}(X))$ is a Radon measure.[1] $\mathfrak{M}(X)$ is a normed vector space with the norm $||\nu|| = |\nu|(X)$ (total variation), $\nu \in \mathfrak{M}(X)$. $\mathfrak{M}(X)$ is isomorphic to the dual space of $\mathfrak{C}_\infty(X, \mathbb{R})$; i.e. $\mathfrak{C}_\infty(X, \mathbb{R})' \cong \mathfrak{M}(X)$ where $\mathfrak{C}_\infty(X, \mathbb{R})$ is the space of all real-valued continuous functions on X which vanish at infinity (with the sup-norm $|| \cdot ||_\infty$). Hence we can define the weak*-topology on $\mathfrak{M}(X)$ via this dual relation.

$\mathfrak{L}^1(\Omega, \mathfrak{C}_\infty(X, \mathbb{R}))$ is the space of all the $\mathfrak{C}_\infty(X, \mathbb{R})$-valued integrable functions, and $\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ is the space of all the $\mathfrak{M}(X)$-valued essentially bounded measurable functions, where $\mathfrak{M}(X)$ is endowed with the Borel σ-field generated by the weak*-topology. Then it is established that[2]

$$\mathfrak{L}^1(\Omega, \mathfrak{C}_\infty(X, \mathbb{R}))' \cong \mathfrak{L}^\infty(\Omega, \mathfrak{M}(X)). \tag{2.2}$$

The set of Carathéodory functions $\varphi : \Omega \times X \to \mathbb{R}$ which satisfiy (i) $x \mapsto \varphi(\omega, x) \in \mathfrak{C}_\infty(X, \mathbb{R})$ and (ii) $\int_\Omega \sup_{x \in X} |\varphi(\omega, x)| d\mu < \infty$ is denoted by $\mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X)$. It is clear that $\mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X)$ can be identified with $\mathfrak{L}^1(\Omega, \mathfrak{C}_\infty(X, \mathbb{R}))$.

A measurable family $\nu = \{\nu_\omega | \omega \in \Omega\}$ with finite essential sup of $||\nu_\omega||$ may be regarded as an element of $\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$.

The set of positive(resp.probability) Radon measures is denoted by $\mathfrak{M}_+(X)$ (resp. $\mathfrak{M}_+^1(X)$).

It is an established fact that any Young measure γ can be represented in the form

$$\gamma(A) = \int_\Omega \{ \int_X \chi_A(\omega, x) d\nu_\omega \} d\mu, \quad A \in \mathcal{E} \otimes \mathcal{B}(X) \tag{2.3}$$

by means of some measurable family $\{\nu_\omega \in \mathfrak{M}_+^1(X) |\, \omega \in \Omega\}$, and such a $\{\nu_\omega\}$ is unique.[3] This representation is called the disintegration of γ.

[1] Maruyama [5] pp.392–395. Schwartz[7] 122–124.

[2] Bourbaki [3] chap.VI, Warga [9] chap.IV.

[3] Maruyama [6] Theorem 2.7, 2.8. .

Hence $\mathfrak{Y}(\Omega, \mu; X)$ corresponds with $\mathfrak{P}(\Omega, \mu; X)$, the set of measurable families consisting of probability measures, in one-to-one way via (2.3). If we define

$$\Phi : \{\nu_\omega | \omega \in \Omega\} \mapsto \gamma,$$

Φ is a mapping of $\mathfrak{P}(\Omega, \mu; X)$ onto $\mathfrak{Y}(\Omega, \mu; X)$.

We write the relation (2.3) symbolically as

$$\gamma = \int_\Omega \delta_\omega \otimes \nu_\omega d\nu. \tag{2.4}$$

We introduce the relative weak*- topology on $\mathfrak{P}(\Omega, \mu; X)$ from $\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$.A topology on $\mathfrak{Y}(\Omega, \mu; X)$ generated by the mappings

$$\gamma \mapsto \int_{\Omega \times X} \varphi(\omega, x) d\gamma = \int_\Omega \int_X \varphi(\omega, x) d\nu_\omega d\mu,\ \varphi \in \mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X) \tag{2.5}$$

is called the narrow topology. The mapping Φ is a homeomorphism between $\mathfrak{P}(\Omega, \mu; X)$ and $\mathfrak{Y}(\Omega, \mu; X)$.

A subset H of $\mathfrak{Y}(\Omega, \mu; X)$ is said to be uniformly tight if there exists a compact set K_ε in X for each $\varepsilon > 0$ such that

$$\sup_{\gamma \in H} \gamma(\Omega \times (X \backslash K_\varepsilon)) \leqq \varepsilon.$$

If H is uniformly tight, H is sequentially compact in the narrow topology.[4]

In the case X is not locally compact, the concept of "vanishing at infinity" does not make sense. Taking account of such a general case, the generalized narrow topology is defined on $\mathfrak{Y}(\Omega, \mu; X)$ by the family of mappings

$$\gamma \mapsto \int_{\Omega \times X} \varphi(\omega, x) d\gamma, \quad \varphi \in \mathfrak{G}_{\mathfrak{C}}(\Omega, \mu; X),$$

instead of $\mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X)$.$\mathfrak{G}_{\mathfrak{C}}$ is the set of Carathéodory functions which satisfy

$$\int_\Omega \sup_{x \in X} |\varphi(\omega, x)| d\mu < \infty.$$

However, under assumption [B], the two topologies coincide for $H \subset \mathfrak{Y}(\Omega, \mu; X)$ if H is relatively compact with respect to the generalized narrow topology.[5]

3. Statement of the fundamental theorem

We now state the so-called "fundamental theorem" in our setting.

Theorem 3.1. *We assume (A) and (B) in section 2. A sequence $\{u_n : \Omega \to X\}$ of $(\mathcal{E}, \mathcal{B}(X))$ - measurable functions is assumed to be given. F is a closed set in X.*

[4] Maruyama [6] Theorem 3.5.
[5] Maruyama [6] Theorem 3.5.

(I) *Suppose that* $\rho(u_n(\omega), F) \to 0$ *in measure as* $n \to \infty$,[6] *where* ρ *is a metric on* X *compatible with the given topology. Then the following propositions hold good.*

(i) *There exists a subsequence* $\{u_{n'}\}$ *of* $\{u_n\}$ *and a measurable family* $\nu = \{\nu_\omega | \omega \in \Omega\} \in \mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ *such that*

a. $w^*\text{-}\lim_{n' \to \infty} f(u_{n'}) = \int_X f(x) d\nu_\omega$ *for any* $f \in \mathfrak{C}_\infty(X, \mathbb{R})$,

b. *supp* $\nu_\omega \subset F$ *a.e.,*

c. $\nu_\omega(X) \leqq 1$ *a.e.*

(ii) *For any real-valued continuous function* $f \in \mathfrak{C}(X, \mathbb{R})$ *such that* $\{f(u_{n'})\}$ *is weakly sequentially compact,*

$$w\text{-}\lim_{n' \to \infty} f(u_{n'}) \to \int_X f d\nu_\omega \ \textit{in}\ \mathfrak{L}^1(\Omega, \mathbb{R}).$$

(II) *Assume, in addition, that for any* $\varepsilon > 0$, *there exists some compact set* $K_\varepsilon \subset X$ *such that*

$$\sup_n \mu\{\omega \in \Omega | u_n(\omega) \notin K_\varepsilon\} \leqq \varepsilon.$$

Then the following propositions also hold good.

(i) *The same results as in (I) hold good with* $\nu_\omega(X) = 1$ *a.e.*

(ii) *For any* $f \in \mathfrak{C}_\infty(X, \mathbb{R})$,

$$w\text{-}\lim_{n' \to \infty} f(u_{n'}) = \int_X f d\nu_\omega \ \textit{in}\ \mathfrak{L}^1(\Omega, \mathbb{R}).$$

The above result can be extended to the case where Ω is σ-finite. In this case, there exists an increasing sequence $\{\Omega_m\}$ of $\mathcal{E}$-measurable sets of finite measures such that $\Omega = \bigcup_{m=1}^\infty \Omega_m$. $f \in \mathfrak{C}_\infty(X, \mathbb{R})$ is given. Applying the Theorem 3.1 to Ω_1, we obtain a subsequence $\{u_{1,n}\}$ of $\{u_n\}$ and a measurable family $\nu^1 = \{\nu^1_\omega | \omega \in \Omega_1\} \in \mathfrak{L}^\infty(\Omega_1, \mathfrak{M}(X))$ such that

a'. $w^*\text{-}\lim_{n \to \infty} f(u_{1,n}) = \int_X f(x) d\nu^1_\omega$ in $\mathfrak{L}^\infty(\Omega, \mathbb{R})$,

b'. supp $\nu^1_\omega \subset F$ a.e. in Ω_1,

c'. $\nu^1_\omega(X) \leqq 1$ a.e. in Ω_1.

Next we apply the Theorem 3.1 to the sequence $\{u_{1,n}\}$ and Ω_2 to get a further subsequence $\{u_{2,n}\}$ of $\{u_{1,n}\}$ and $\nu^2 \in \mathfrak{L}^\infty(\Omega_2, \mathfrak{M}(X))$ which satisfies the conditions similar to a', b', and c' on Ω_2 rather than Ω_1.By the uniqueness, ν^2 is an extension of ν^1 a.e. Repeat this process infinitely and obtain a subsequence $\{u_{n'}\}$ of $\{u_n\}$ by

[6] $\rho(u_n(\omega), F) = \inf_{x \in F} \rho(u_n(\omega), x)$.(1) is equivalent to $\lim_{n \to \infty} \mu\{\omega \in \Omega \big| u_n(\omega) \notin V\} = 0$ for any open set V containing F.

Cantor's diagonal process. Finally define a measurable family ν_ω by

$$\nu_\omega(E) = \begin{cases} \nu^1_\omega(E) & \text{for} \quad \omega \in \Omega_1, \\ \nu^m_\omega(E) & \text{for} \quad \omega \in \Omega_m \backslash \Omega_{m-1} \quad (m \geqq 2) \end{cases}$$

for each $E \in \mathcal{B}(X)$.

Thus we obtain a small generalization of the Theorem 3.1 to a σ-finite measure space.

In Ball[2] and Evans[4], both Ω and X are assumed to be subsets of some Euclidean spaces, and μ is specified as the usual Lebesgue measure.

4. Proof of the theorem

We prove the fundamental theorem in several steps.

1. Define a sequence $\nu^n = \{\nu^n_\omega | \omega \in \Omega\} : \Omega \to \mathfrak{M}(X)$ of measurable families by

$$\nu^n_\omega = \delta_{u_{n(\omega)}} \quad ; n = 1, 2, \ldots, \tag{4.1}$$

where $\delta_{u_{n(\omega)}}$ is the Dirac measure concentrating at $u_n(\omega)$. It is obvious that $||\nu^n_\omega|| = |\nu^n_\omega|(X)$(total variation) $= 1$. Since the unit ball S of $\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ is metrizable [7], $\{\nu^n\}$ has a subsequence $\{\nu^{n'}\}$ which w^* - converges to some $\nu^* \in \mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ with $||\nu^*|| \leqq 1$. [8]

We denote by $\gamma_{n'}$ and γ_* the measures on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ which enjoy the representations

$$\gamma_{n'} = \int_\Omega \delta_\omega \otimes \nu^{n'}_\omega d\mu, \tag{4.2}$$

$$\gamma_* = \int_\Omega \delta_\omega \otimes \nu^*_\omega d\mu.$$

We know that $\{\gamma_{n'}\}$ converges to γ_* in the narrow topology since $\nu^{n'} \to \nu^*$ in the weak*- topology as $n' \to \infty$ (cf.Maruyama [6] Theorem 2.7). In other words,

$$\int_{\Omega \times X} \varphi(\omega, x) d\gamma_{n'} \to \int_{\Omega \times X} \varphi(\omega, x) d\gamma_* \qquad \text{as} \quad n' \to \infty \tag{4.3}$$

for any $\varphi \in \mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X)$.

Obviously $g(\omega)f(x) \in \mathfrak{G}_{\mathfrak{C}_\infty}(\Omega, \mu; X)$ for any $f \in \mathfrak{C}_\infty(X, \mathbb{R})$ and $g \in \mathfrak{L}^1(\Omega, \mathbb{R})$. Consequently it follows from (4.3) that

---

[7] We assume that σ - field $\mathcal{E}$ on Ω is countably generated and X is a metrizable locallly compact Souslin space. Hence $\mathfrak{L}^1(\Omega, \mathfrak{C}_\infty(X, \mathbb{R}))$ is separable. Consequently the unit ball S of the dual space $\mathfrak{L}^1(\Omega, \mathfrak{C}_\infty(X, \mathbb{R}))' \cong \mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ is w^*-metrizable.

[8] In the case X is compact, $||\nu^*|| = 1$. (cf.Maruyama [5] Theorem 2.6.)

$$\begin{aligned}(4.4)\qquad &\int_{\Omega\times X} g(\omega)f(x)d\gamma_{n'}\\ &=\int_\Omega g(\omega)\int_X f(x)d\nu_\omega^{n'}d\mu\\ &=\int_\Omega f(u_{n'}(\omega))g(\omega)d\mu \quad \text{(by (4.1))},\\ &\longrightarrow \int_{\Omega\times X} g(\omega)f(x)d\gamma_* \quad \text{(by (4.3))}\\ &=\int_\Omega g(\omega)\int_X f(x)d\nu_\omega^* d\mu \qquad \text{as} \quad n'\to\infty.\end{aligned}$$

Since this holds good for any $g \in \mathfrak{L}^1(\Omega, \mathbb{R})$, we obtain

$$(4.5)\qquad f(u_{n'}(\omega)) \to \int_X f(x)d\nu_\omega^* \qquad \text{as} \quad n'\to\infty$$

with respect to the weak* -topology in $\mathfrak{L}^\infty(\Omega, \mathbb{R})$.

2. We next show that supp $\nu_\omega \in F$ *a.e.*

Since $\rho(u_{n'}, F) \to 0$ in measure (as $n' \to \infty$), there exists a subsequence $\{u_{n''}\}$ of $\{u_{n'}\}$ such that $\rho(u_{n''}(\omega), F) \to 0$ *a.e.*(as $n'' \to \infty$). Define a correspondence(= multi-valued mapping) $\Gamma_p : \Omega \twoheadrightarrow X (p = 1, 2, \dots)$ by

$$(4.6)\qquad \Gamma_p(\omega) = cl.\{u_{n''}(\omega) \big| n'' \geqq p\}\,; p = 1, 2, \dots.$$

Then each Γ_p is a closed - valued and measurable correspondence. Needless to say, $u_{n''}(\omega) \in \Gamma_p(\omega)$. We also define a function $f_p : \Omega \times X \to \bar{\mathbb{R}}$ by

$$(4.7)\qquad f_p(\omega, x) = \begin{cases} 0 & \text{for} \quad x \in \Gamma_p(\omega), \\ \infty & \text{for} \quad x \notin \Gamma_p(\omega). \end{cases}$$

Then f_p is a positive normal integrand defined on $\Omega \times X$. Since $\{\gamma_{n''}\}$ converges to γ_* in the narrow topology, we obtain

$$\begin{aligned}\int_{\Omega\times X} f_p(\omega,x)d\gamma_* &\leqq \liminf_{n''}\int_{\Omega\times X} f_p(\omega,x)d\gamma_{n''}\\ &=\liminf_{n''}\int_\Omega f_p(\omega, u_{n''}(\omega))d\mu \quad \text{(by(4.1))}\\ &=0 \quad \text{(by(4.7))}\end{aligned}$$

(cf. Maruyama [6] Theorem 3.2).

Thus we must have

$$\int_X f_p(\omega,x)d\nu_\omega^* = 0 \quad a.e.,$$

which implies that

$$\nu_\omega^*(X\backslash\Gamma_p(\omega)) = 0 \quad a.e., \quad i.e. \quad \operatorname{supp}\nu_\omega^* \subset \Gamma_p(\omega) \quad a.e.$$

Taking account of the relation [9]

$$\bigcap_{p=1}^{\infty} \Gamma_p(\omega) = L_s(u_{n''}(\omega)) \subset F \quad a.e.,$$

we obtaion

$$\text{supp}\, \nu_\omega^* \subset F \quad a.e.$$

3. Suppose that $f{:}X \to \mathbb{R}$ is a continuous function such that $\{f(u_n(\omega))\}$ is weakly sequentially compact in $\mathfrak{L}^1(\Omega, \mathbb{R})$. Then there exisits a subsequence $\{f(u_{n'})\}$ which weakly converges to, say, $h(\omega) \in \mathfrak{L}^1(\Omega, \mathbb{R})$. For the sake of simplicity , we assume that $\{f(u_n)\}$ itself weakly converges to h. We may also assume, without loss of generality, that f is nonnegative.[10] Define a sequence $\{\theta^{(k)} : \mathbb{R} \to \mathbb{R}\}(k = 1, 2, \dots)$ in $\mathfrak{C}_\infty(\mathbb{R}, \mathbb{R})$ by

$$\theta^{(k)}(t) = \begin{cases} |t| & \text{for} \quad |t| \leqq k, \\ -k|t| + k(k+1) & \text{for} \quad k \leqq |t| \leqq k+1, \\ 0 & \text{for} \quad |t| \geqq k+1. \end{cases} \tag{4.8}$$

We now prove that

$$(\theta^{(k)} \circ f)(u_n) \to f(u_n) \quad \text{as} \quad k \to \infty \tag{4.9}$$

weakly and uniformly in n.

Let φ be any element of $\mathfrak{L}^\infty(\Omega, \mathbb{R})$. Then we obtain the evaluation

$$\begin{aligned} &\left| \int_\Omega \varphi\{(\theta^{(k)} \circ f)(u_n) - f(u_n)\} d\mu \right| \\ &\qquad \leqq ||\varphi||_\infty \int_\Omega |(\theta^{(k)} \circ f)(u_n) - f(u_n)| d\mu \qquad \text{(Hölder's inequality)} \\ &\qquad \leqq ||\varphi||_\infty \times 2 \int_{\{\omega \in \Omega |\ |f(u_n(\omega))| \geqq k\}} |f(u_n(\omega))| d\mu \\ &\qquad \leqq \text{constant} \int_{\{\omega \in \Omega |\ |f(u_n(\omega))| \geqq k\}} |f(u_n(\omega))| d\mu. \end{aligned}$$

Since $\{f(u_n)\}$ is weakly sequentially compact in $\mathfrak{L}^1(\Omega, \mathbb{R})$ by assumption, there exists some $K > 0$, for each $\varepsilon > 0$,which satisfies

$$\sup_n \int_{\{\omega \in \Omega | f(u_n(\omega)) \geqq K\}} f(u_n(\omega)) d\mu \leqq \varepsilon \tag{4.10}$$

(note that $f \geqq 0$).

This completes the proof of (4.9).[11]

---

[9] $L_s(u_{n''}(\omega))$ is the topological limit sup of the set $\{u_{n''}(\omega)\}$. See Maruyama [6] footnote 66.

[10] If not, we represent f as $f = f^+ - f^-$(where f^+ and f^- are the positive part and the negative part of f, respectively). Both of $\{f^+(u_n)\}$ and $\{f^-(u_n)\}$ are weakly sequentially compact. So we have only to show what we need for f^+ and f^-, and then combine the results.

[11] For the characterization of the weak compactness in $\mathfrak{L}^1(\Omega, \mathbb{R})$, see Maruyama [5] pp.275-278.

Let η be any element of $\mathfrak{L}^\infty(\Omega,\mathbb{R})$. Since $\mu\Omega<\infty$, it is obvious that $\eta\in\mathfrak{L}^1(\Omega,\mathbb{R})$. By the result already established in the step **1**, it holds good that

$$\lim_{n\to\infty}\int_\Omega \eta\cdot(\theta^{(k)}\circ f)(u_n)d\mu=\int_\Omega \eta(\omega)\cdot\int_X(\theta^{(k)}\circ f)(x)d\nu^*_\omega d\mu. \tag{4.11}$$

(We use $\theta^{(k)}\circ f$ instesd of f.) It follows that

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega \eta\cdot\int_X(\theta^{(k)}\circ f)(u_n)d\nu^*_\omega d\mu\\
&\underset{(4.11)}{=}\lim_{k\to\infty}\lim_{n\to\infty}\int_\Omega\eta\cdot(\theta^{(k)}\circ f)(u_n)d\mu\\
&=\lim_{n\to\infty}\lim_{k\to\infty}\int_\Omega\eta\cdot(\theta^{(k)}\circ f)(u_n)d\mu\\
&\underset{(4.9)}{=}\lim_{n\to\infty}\int_\Omega\eta\cdot f(u_n)d\mu\\
&\underset{(4.5)}{=}\int_\Omega\eta(\omega)\int_X f(x)d\nu^*_\omega d\mu\\
&=\int_\Omega\eta\cdot hd\mu\quad(w\text{-}\lim_{n\to\infty}f(u_n)=h).
\end{aligned}\tag{4.12}$$

Consequently we obtain

$$h(\omega)=\int_X f(x)d\nu^*_\omega.$$

4. If we assume the condition (II), then the set

$$\gamma_n=\int_\gamma\delta_\omega\otimes\delta_{u_n}(\omega)d\mu\ :\ n=1,2,\ldots\ldots$$

of Young measures is uniformly tight with respect to the narrow topology.[12] Hence $\{\gamma_n\}$ is sequentially compact in the narrow topology.[13] There exist some subsequence $\{\gamma_{n'}\}$ of $\{\gamma_n\}$ and a Young measure $\gamma^*\in\mathfrak{Y}(\Omega,\mu;X)$ such that

$$\gamma_{n'}\to\gamma^*\ \text{ in the narrow topology as } n'\to\infty.$$

In other words,

$$\int_{\Omega\times X}\psi(\omega,x)d\gamma_{n'}=\int_\Omega\psi(\omega,u_{n'}(\omega))d\mu\longrightarrow\int_{\Omega\times X}\psi(\omega,x)d\gamma^*\quad\text{as}\quad n'\to\infty \tag{4.13}$$

for any $\psi\in\mathfrak{G}_{\mathfrak{C}_\infty}(\Omega,\mu:X)$. γ^* can be represented in the form of disintegration by means of a measurable family $\nu^*=\{\nu^*_\omega|\omega\in\Omega\}\in\mathfrak{P}(\Omega,\mu;X)$; *i.e.*

$$\gamma^*=\int_\Omega\delta_\omega\otimes\nu^*_\omega d\mu\quad\text{where }\nu^*_\omega\in\mathfrak{M}^1_+(X)\ a.e.$$

If we specify $\psi(\omega,x)$ as

$$\psi(\omega,x)=\varphi(\omega)\cdot f(x),\quad\varphi\in\mathfrak{L}^1(\Omega,\mathbb{R}),\quad f\in\mathfrak{C}_\infty(X,\mathbb{R}),$$

[12]Maruyama [6] Theorem 3.6.
[13]Maruyama [6] Theorem 3.5.

then $\psi(\omega, x) \in \mathfrak{G}_{\mathfrak{C}_\infty}((\Omega, \mu; X)$. It follows from (4.13) that

$$\int_\Omega \varphi(\omega) f(u_{n'}(\omega)) d\mu \longrightarrow \int_\Omega \varphi(\omega) \int_X f(x) d\nu^*_\omega d\mu \quad \text{as } n' \to \infty. \tag{4.14}$$

Since $\mu\Omega < \infty, \mathfrak{L}^\infty(\Omega, \mathbb{R}) \subset \mathfrak{L}^1(\Omega, \mathbb{R})$. Hence (4.14) holds good for any $\varphi \in \mathfrak{L}^\infty(\Omega, \mathbb{R})$. Thus we have proved that

$$w\text{-}\lim_{n' \to \infty} f(u_{n'}) = \int_X f(x) d\nu^*_\omega \, .$$

□

References

[1] E. J. Balder, *A general approach to lower semicontinuity and lower closure in optimal control theory*, SIAM J. Control and Optim. **22** (1984), 570–598.
[2] J. M. Ball, *A version of the fundamental theorem for Young measures*, in: PDEs and Continuum Models of Phase Transitions, M. Rascle, D. Serre and M. Slemrod (eds.), Springer-Verlag, Berlin/Heidelberg, 1989, pp. 207–215.
[3] N. Bourbaki, *Elements of Mathmatics ; Integration I*, Springer, Berlin, 2004.
[4] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations,* Amer.Math.Soc., Providence, 1990.
[5] T. Maruyama, *Integration and Functional Analysis*, Springer-Verlag,Tokyo, 2006(in Japanese).
[6] T. Maruyama, *Disintegration of Young measures and nonlinear analysis*, Advances in Mathematical Economics **23** (2020), 223–312.
[7] L. Schwartz, *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures*, Oxford, London, 1973.
[8] M. Valadier, *Young measures*, in: Methods of Nonconvex Analysis, A. Cellina (ed.), Springer-Verlag, Berlin / Heidelberg, 1990, pp. 152–188.
[9] J. Warga, *Optimal Control of Differential and Functional Equations*, Academic press, New York, 1972.

Manuscript received December 27 2019
revised January 18 2020

T. Maruyama
Keio University 2-15-45 Mita, Minato-ku Tokyo 108-8345, Japan
E-mail address: maruyama@econ.keio.ac.jp