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A NOTE ON SUSTAINABILITY IN CLOSED MULTISECTOR MODELS

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Abstract. In this note we consider the role of exhaustible resources in limiting the future consumption possibilities of an economy. A variation of the closed multisector model of David Gale is introduced. We identify conditions on the set of net output vectors that imply that the economy cannot sustain a constant positive consumption of any good in all periods. The result is applicable to activity analysis models. The main result follows from an application of duality theory of convex cones.

1. Introduction

In the substantial literature on the role of exhaustible resources in limiting future consumption prospects, an interesting analytical question has been to characterize (in infinite horizon models) substitution possibilities in a technology that ensure the maintenance of a positive constant level of consumption at each point of time in the future (or, in a discrete-time framework, a constant quantity of consumption in every period). Some of the sharpest results have been obtained in Cobb-Douglas economies (Solow (1974), Stiglitz (1974), Beckmann (1974) in the early *Symposium of Review of Economic Studies* and Mitra (1983)). Extensions to models that do not involve Cobb-Douglas production functions required challenging exploration (see Mitra (1978), Cass and Mitra (1991), Dasgupta and Mitra (1983) and the references cited there).

In a discrete-time model, it is intuitive that, if the range of input substitution possibilities is "limited" or "narrow", an economy in which exhaustible natural resources play an essential role in production processes will find it impossible to sustain a constant quantity of consumption level "forever". The modest purpose of this note is to confirm such an intuition for a class of multi-sector models.

In Section 3, we develop a variation of the closed linear model of production of Gale (1956). In our framework, there are n_1 producible goods that *can also be consumed*, and (unlike Gale (1956)), n_2 natural resources that are *essential* inputs in the net production of producible goods but are *not* directly consumed. The stocks of natural resources *cannot* be augmented by the available production processes. We write $n_1 + n_2 = n$ (both n_1 and n_2 are allowed to be any finite positive integers).

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A production *process* (or, an *activity*) converts (transforms) a (non-negative) n-vector x of inputs into a (non-negative) n-vector y of outputs. Let $\mathcal{T}$ be the set of all production processes ("the technology"). To clarify the role of resources, it is convenient to write $x = (\tilde{x}, s)$ and $y = (\tilde{y}, s')$, where $\tilde{x}$ (respectively, $\tilde{y}$) denotes the quantities of producible goods used as inputs (respectively, quantities of produced as outputs) and s (respectively, s') denotes the quantities of natural resources used as inputs (generated as outputs). Exhaustible natural resources are distinguished by the property that for *all* $(x, y) \in \mathcal{T}$, $s \geqq s'$ (see T.1). The *essential* role of the resources as inputs is captured by the assumption that for any producible good i, if $y_i > x_i$ then there is some resource k such that $s_k > s'_k$ (see T.5). The consumption possibilities consistent with $\mathcal{T}$ can best be described by the set of *net* output vectors $Z = \{z : z = y - x, \text{ where } (x, y) \in \mathcal{T}\}$. Now, if $\mathcal{T}$ is a convex cone, so is Z. By T.5, Z does not contain any semipositive $u \geq 0$. For our result a more restrictive property of Z seems needed: we assume (see T.6) that the closure of Z (denoted by $cl(Z)$) is a convex cone that does not contain any semipositive $u \geq 0$. When $\mathcal{T}$ happens to be a *polyhedral convex cone*, it is necessarily closed, and Z too, is a polyhedral (hence, closed) convex cone that does not contain (by T.5) any semipositive $u \geq 0$. The activity analysis models of von Neumann and Leontief are examples of the polyhedral case.

A result on the duality of convex cones (see Theorem 2.2, in Section 2, recalled for completeness of exposition) guarantees that there is a positive price system $p > 0$ such that $p \cdot z \leqq 0$ for all $z \in cl(Z)$ (see Lemma 3.2). This positive price system at which no process generates a positive profit plays a crucial role in calculations that establish our main result which we now state informally. Given the initial stocks of producible goods ($\tilde{X} > 0$) and natural resources $S > 0$, a *program* is a complete specification of decisions on the choice of inputs, consumptions, and production processes that meet the appropriate conditions of consistency and non-negativity (see (3.3) in Section 3). Let $\tilde{c}_t = \tilde{y}_t - \tilde{x}_t$ be the (non-negative) n_1-vector of consumptions in period t. generated by any program, and we write $\parallel \tilde{c}_t \parallel = \sum_{k=1}^{n_1} \tilde{c}_k$, and $p = (p_1, p_2)$ where p_1 is the positive n_1-vector of prices of producible goods and p_2 is the n_2-vector of prices of exhaustible resources. One can then show that for any finite T,

$$\sum_{t=1}^{T} (p_1 \cdot \tilde{c}_t) \leqq [p_1 \cdot \tilde{X} + p_2 \cdot S].$$

It follows that

$$\sum_{t=1}^{\infty} (p_1 \cdot \tilde{c}_t) = \lim_{T \to \infty} [\sum_{t=1}^{T} (p_1 \cdot \tilde{c}_t)] \leqq [p_1 \cdot \tilde{X} + p_2 \cdot S].$$

From a standard property of a convergent infinite series, we get:

$$\lim_{t \to \infty} (p_1 \cdot \tilde{c}_t) = 0,$$

which, in turn, implies (as $p_1 > 0$),

$$\lim_{t \to \infty} \parallel \tilde{c}_t \parallel = 0.$$

In Section 4 we provide a few remarks on the development of the literature initiated by the von Neumann model and on some results following Solow (1974)—but do not attempt a survey in either direction. At this moment it suffices to say that the researchers on development planning found the analytical framework of a closed linear model particularly appealing to explore optimal growth in labor surplus economies (see Chakravarty (1969, pages 185-189). We follow this line of interpretation: a look at the long-run prospects of a labor surplus economy facing constraints on the supply of natural resources that are essential as inputs. Our main result stresses the importance of the development of new technologies that are less reliant on exhaustible resources for generating positive, constant consumptions in all future periods. Better still is to study the question, not for an economy in isolation, but one in which there is internal production, as well as external trade with the outside world.

2. Duality for closed convex cones

Convex cones and their duals have been explored not only in huge mathematical literature on convexity and linear inequalities, but also in many articles and monographs that have been influential in the development of mathematical economics (see Uzawa (1958), Gale (1960), Karlin (1959) and Nikaido (1968)). Since there are variations in definitions and notation, we shall follow Nikaido (1968) in collecting very briefly the relevant concepts and mathematical results that are needed in the statement and proof of our main result in Section 4.

A non-negative (respectively, positive) real number a is denoted by $a \geqq 0$ (respectively, $a > 0$). A vector $x = (x_i) \in \mathbb{R}^n$ is called *non-negative* (written, $x \geqq 0$), if $x_i \geqq 0$ for all $i = 1, 2...n$. It is called *semipositive* (written, $x \geq 0$), if $x \geqq 0$ and $x_i > 0$ for some i, and *positive* (written, $x > 0$) if $x_i > 0$ for all $i = 1, 2...n$.

A (nonempty) subset K of $\mathbb{R}^n$ is a convex cone if it satisfies:

(i) $x + y \in K$ for any $x, y \in K$;
(ii) $\lambda x \in K$ for any $x \in K$ and $\lambda \geqq 0$.

The dual convex cone K^* of a convex cone K is defined as:

$$K^* = \{y \in \mathbb{R}^n : \ x \cdot y \geqq 0 \text{ for any } x \in K\}.$$

Observe that even though K may not be closed, K^* is always closed. Now, K^* has its dual $(K^*)^*$, usually denoted simply by K^{**}. For a convex cone K, we have:

(i) $K^{**} \supset K$;
(ii) $K^{**} = K$ if and only if K is closed.

An application of a separation theorem leads to the following conclusion (Theorem 3.5 of Nikaido (1968, p.35)):

Theorem 2.1. *Let X be a convex set in $\mathbb{R}^n$ that contains no positive vector. Then there is a semipositive vector $p \geq 0$ such that $p \cdot x \leq 0$ for all $x \in X$.*

Observe that in general, we *cannot* claim that p is positive (Nikaido (1968, p.35)). However, we have the following (see Nikaido (1968), Theorem 3.6):

Theorem 2.2. *Let K be a closed convex cone. If K contains no semipositive vector $u \geq 0$, then $-K^*$ contains a positive $p > 0$, and vice versa.*

A convex cone K is *polyhedral* if it is generated by a finite set of vectors $\{a^1, a^2, \ldots, a^s\}$, i.e., $K = \{y = \sum_{i=1}^{s} \lambda^i a^i : \lambda_i \geqq 0\}$. A polyhedral convex cone K is closed. A convex cone K is polyhedral if and only if K is the set of all solutions to a finite system of linear inequalities (see Nikaido (1968), Moore (2007)).

3. A closed linear model of production

We turn to a variation of Gale's closed linear model of production (1956). A finite number of goods are "produced" by means of *processes* denoted by a pair (x, y) of nonnegative n-vectors. We interpret x as an input vector that the process "converts" into an output vector y with a one-period lag in this conversion. We shall also refer to (x, y) as an input-output pair. The set of all technologically feasible processes (briefly, *the technology*) is described by a nonempty set $\mathcal{T}$ in $\mathbb{R}_+^{2n}$. Formally,

$$\mathcal{T} = \{(x, y) : \ x \text{ can be converted into } y\}.$$

A good k is an *exhaustible resource* if for all $(x, y) \in \mathcal{T}$, $y_k \leqq x_k$. In other words, *no* known process can augment the stock of an exhaustible resource. Of the n goods, there are n_1 producible goods that can be either used as an input or consumed. There are n_2 exhaustible resources, these are important inputs that are not consumed directly ($n_1 + n_2 = n$). For an input or output vector x or y, we have the following ordering convention: the first n_1 coordinates of x (or y) denote the quantities of producible goods (to be written $\tilde{x} = (\tilde{x}_i)$ or $\tilde{y} = (\tilde{y}_i)$, where $i = 1, 2, \ldots, n_1$). The following n_2 coordinates will denote quantities of exhaustible resources, to be written $s = (s_k)$ as input stock, and $s' = (s'_k)$ as an output stock where $k = n_1+1, n_1+2, \ldots, n_1+n_2$. We write $x = (\tilde{x}, s)$ and $y = (\tilde{y}, s')$ to explicitly distinguish between producible goods and resources. Formally,

T.1. *For any process $(x, y) \in \mathcal{T}$, where $x = (\tilde{x}, s)$, and $y = (\tilde{y}, s')$, the inequality "$s \geqq s'$" holds.*

We assume that all producible capital goods, as well as the resource can be stored costlessly (no depreciation when stored):

T.2. *For any $x = (\tilde{x}, s) \geqq 0$, $(x, x) \in \mathcal{T}$. Also, "$(0, y) \in \mathcal{T}$" implies that "$y = 0$"* (impossibility of free production).

T.3. "$((x = (\tilde{x}, s), y = (\tilde{y}, s')) \in \mathcal{T})$" *implies* "$((\tilde{x}_\sim, s_\sim), (\tilde{y}_\sim, s'_\sim) \in \mathcal{T})$ *where* $(\tilde{x}_\sim, s_\sim) \geqq (\tilde{x}, s)$, $0 \leqq \tilde{y}_\sim \leqq \tilde{y}$, *and* $0 \leqq s'_\sim \leqq s'$" *(free disposal).*

To think of the future consumption possibilities and resource exhaustion, it is important to introduce the set Z of *net outputs* generated by $\mathcal{T}$ defined by:

$$\begin{aligned} Z &= \{z : z = (y - x), \text{ where } (x, y) \in \mathcal{T}\}. \\ &= \{z : z = (\tilde{y} - \tilde{x},\ s' - s) \text{ where } ((\tilde{x}, s), (\tilde{y}, s')) \in \mathcal{T}\}. \end{aligned} \tag{3.1}$$

Note that the storage process (x, x) is assumed to be technologically feasible. So $0 \in Z$. The structure of Z is determined by the properties of $\mathcal{T}$.

T.4. *$\mathcal{T}$ is a closed convex cone.*

The importance of the role of resources in converting x to y is captured by the following assumption:

T.5. *Let "*$(x = (\tilde{x}, s), y = (\tilde{y}, s')) \in \mathcal{T}$*". If for any* i, $\tilde{y}_i > \tilde{x}_i$, *there is some (one or more)* k *such that* $s_k > s'_k$.

In other words, for a positive net production of any producible good i, a positive quantity of some (one or more) resource is needed.

Lemma 3.1. *Under T.4. and T.5,* Z *is a convex cone that does not contain any* $u \geq 0$.

Proof. Z is clearly a convex cone since $\mathcal{T}$ *is*. Consider $z = (\tilde{y} - \tilde{x}, s' - s) \in Z$. Observe that $s' - s \leqq 0$. On the other hand, $\tilde{y} - \tilde{x} \geq 0$ implies, by T.5., that $s' - s \leq 0$. □

We assume a stronger property of Z than that ensured by this Lemma 3.1.

Let $cl(Z)$ be the closure of Z. $cl(Z)$ is a convex cone (Nikaido (1969, Theorem 2.6). We assume:

T.6 $cl(Z)$ *is a convex cone that does not contain any* $u \geq 0$.

We now turn to the basic "duality" result: At a price system $p \geqq 0$, define the profit of a process $(x, y) \in \mathcal{T}$ as $\pi[(x, y), p] = p \cdot y - p \cdot x = p \cdot z$. We prove the existence of a *positive* $p > 0$ such that no process in $\mathcal{T}$ generates a positive profit.

Lemma 3.2. *There is* $p > 0$ *such that* $p \cdot z \leqq 0$ *for all* $z \in cl(Z.)$.

Proof. Since $cl(Z)$ is a closed convex cone that does not contain any $u \geq 0$, it follows from Theorem 2.2 that $-(cl(Z))^*$ contains a positive $p > 0$. Hence, $-p \cdot z \geqq 0$ for all $z \in cl(Z)$, or $p \cdot z \leqq 0$ for all $z \in cl(Z)$. □

We write $p = (p_1, p_2) > 0$ to stress that the n_1-vector $p_1 > 0$ denotes the vector of prices of producible goods and $p_2 > 0$ is the n_2-vector denoting the prices of exhaustible resources. We can write more explicitly

$$p \cdot z = p_1 \cdot (\tilde{y} - \tilde{x}) + p_2 \cdot (s' - s) \leqq 0. \tag{3.2}$$

The initial stocks of the producible goods $\tilde{X} > 0$ and the resources $S > 0$ are given. A *program from* $(\tilde{X}, S)$ is a sequence $(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{c}}) = (x_t, y_{t+1}, \tilde{c}_{t+1})$ satisfying:

$$\begin{aligned} \tilde{x}_0 &= \tilde{X},\ s_0 = S, \\ (x_t, y_{t+1}) &= (x_t = (\tilde{x}_t, s_t), y_{t+1} = (\tilde{y}_{t+1}, s'_{t+1})) \in \mathcal{T} \text{ for } t \geqq 0, \\ s_{t+1} &= s'_{t+1} \text{ for } t \geqq 0, \\ \tilde{c}_{t+1} &= (\tilde{y}_{t+1} - \tilde{x}_{t+1}) \geqq 0 \text{ for } t \geqq 0. \end{aligned} \tag{3.3}$$

A program specifies a sequence of decisions described as follows. The initial stocks $\tilde{X}$ and S are used in period 0 as inputs in a chosen process to generate the first-period output vector $(\tilde{y}_1, s'_1)$, where $s'_1 \leqq S$. A part $\tilde{c}_1$ of $\tilde{y}_1$, the vector of producible goods available, is chosen for consumption. The rest, the vector $\tilde{x}_1 (= \tilde{y}_1 - \tilde{c}_1)$ and the available stocks of the resources $s_1 = s'_1$ are used as inputs in a chosen process to generate the output vector $(\tilde{y}_2, s'_2)$ where s'_2 is the stock of resource, $s'_2 \leqq s'_1$, and the story is repeated.

Observe that as resources are not consumed, by setting $s_{t+1} = s'_{t+1}$ (use of the entire stocks of resources as inputs in period $t + 1$), the formulation introduces a "conservation" or (in Hurwicz's words) "nonwastefulness" clause in the definition. A program defines a corresponding sequence (s'_t) of resource vectors that is nonincreasing.

We now come to the main result on the long-run behavior of consumptions in our model.

For any non-negative vector $c \in \mathbb{R}^{n_1}_{+}$ we define the norm

$$\| c \| = \sum_{k=1}^{n_1} c_k. \tag{3.4}$$

Theorem 3.3. *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{c}) = (x_t, y_{t+1}, \tilde{c}_{t+1})$ *be a program from* $(\tilde{X}, S) > 0$. *Then*

$$\lim_{t\to\infty} \| \tilde{c}_t \| = 0. \tag{3.5}$$

Proof. We use (3.2) repeatedly. First, by the definition of a program in (3.3) we have

$$\tilde{c}_1 = \tilde{y}_1 - \tilde{x}_1 \text{ and } \Big((\tilde{X}, S), (\tilde{y}_1, s'_1)\Big) \in \mathcal{T}. \tag{3.6}$$

Hence, using (3.2) and the price vector $p = (p_1, p_2) > 0$

$$\begin{aligned} p_1 \cdot (\tilde{y}_1 - \tilde{X}) + p_2 \cdot (s'_1 - S) &\leqq 0 \text{ or,} \\ p_1 \cdot \tilde{y}_1 &\leqq p_1 \cdot \tilde{X} + p_2 \cdot (S - s'_1). \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$p_1 \cdot \tilde{c}_1 = p_1 \cdot (\tilde{y}_1 - \tilde{x}_1) \leqq p_1 \cdot \tilde{X} + p_2 \cdot (S - s'_1) - p_1 \cdot \tilde{x}_1. \tag{3.8}$$

Similarly,

$$\tilde{c}_2 = \tilde{y}_2 - \tilde{x}_2, \text{ and } ((\tilde{x}_1, s_1), (\tilde{y}_2, s'_2)) \in \mathcal{T} \text{ with } s_1 = s'_1. \tag{3.9}$$

Again, using (3.2),

$$\begin{aligned} p_1 \cdot (\tilde{y}_2 - \tilde{x}_1) + p_2 \cdot (s'_2 - s'_1) &\leqq 0 \text{ or,} \\ p_1 \cdot \tilde{y}_2 &\leqq p_1 \cdot \tilde{x}_1 + p_2 \cdot (s'_1 - s'_2). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10),

$$p_1 \cdot \tilde{c}_2 = p_1 \cdot (\tilde{y}_2 - \tilde{x}_2) \leqq p_1 \cdot \tilde{x}_1 + p_2 \cdot (s'_1 - s'_2) - p_1 \cdot \tilde{x}_2. \tag{3.11}$$

From (3.8) and (3.11) we get, after simplifying,

$$p_1 \cdot \tilde{c}_1 + p_1 \cdot \tilde{c}_2 \leqq p_1 \cdot \tilde{X} + p_2 \cdot (S - s'_2) - p_1 \cdot \tilde{x}_2.$$

Repeating the steps we get, for any finite T,

$$\sum_{t=1}^{T} (p_1 \cdot \tilde{c}_t) \leqq p_1 \cdot \tilde{X} + p_2 \cdot (S - s'_T) - p_1 \cdot \tilde{x}_T \leqq p_1 \cdot \tilde{X} + p_2 \cdot S.$$

It follows that:

$$\sum_{t=1}^{\infty} (p_1 \cdot \tilde{c}_t) = \lim_{T\to\infty} [\sum_{t=1}^{T} (p_1 \cdot \tilde{c}_t)] \leqq p_1 \cdot \tilde{X} + p_2 \cdot S.$$

In particular,

$$\lim_{t \to \infty} (p_1 \cdot \tilde{c}_t) = 0. \tag{3.12}$$

Let $\alpha = \min_k(p_{1k}) > 0$. Since $(p_1 \cdot \tilde{c}_t) \geqq \alpha \|\tilde{c}_t\| \geqq 0$, (3.5) follows from (3.12). □

Remark 3.1. We now turn to the "polyhedral" case. Replace T.4 by

T.4' $\mathcal{T}$ *is a polyhedral convex cone.*

It follows that (under T.4' and T.5):

Z is a polyhedral (hence, closed) convex cone that does not contain any $u \geq 0$.

Now, from Theorem 2.2 we get that $-Z^*$ contains a positive $p > 0$. Hence, $-p \cdot z \geqq 0$ for all $z \in Z$, or $p \cdot z \leqq 0$ for all $z \in Z$. Using the same arguments as in the above proof, we see that for any program $(\mathbf{x}, \mathbf{y}, \mathbf{c}) = (x_t, y_{t+1}, \tilde{c}_{t+1})$ from $(\tilde{X}, S) > 0$, $\lim_{t\to\infty} \parallel \tilde{c}_t \parallel = 0$.

In a more restricted situation (a single resource that is essential in *all* processes), we analyzed the polyhedral case earlier (Majumdar and Bar (2013)). The point that we should stress is this: under T.5, *any property* of $\mathcal{T}$ that ensures that Z *is a closed* convex cone enables us to invoke Theorem 2.2 and get the positive p satisfying $p \cdot z \leqq 0$ for all $z \in Z$. The *polyhedral* case T.4' merely identifies a class of prominent technologies.

Remark 3.2. Note that, even without assuming T.6, we get a semipositive price system $p \geq 0$ such that $p \cdot z \leqq 0$ for all $z \in Z$ by using Lemma 3.1 and Theorem 2.1. The proof of Theorem 3.3, however, rests crucially on the fact that the price system used in the calculations is positive (see the line below (3.12)). We needed the more restrictive T.6 to establish Lemma 3.2.

4. A leaf from the past

In his article, von Neumann (1945-46) assumed explicitly (page 2) that "the natural factors of production, including labor, can be expanded in unlimited quantities". In his paper, Gale (1956) announced two objectives: first, to provide a generalization in which the "condition (H)" ("each process involves each good in the economy as input or output", "a serious weakness") is dispensed with. Secondly, he wanted to provide an elementary treatment of the two main results of von Neumann as "the original proofs... were extremely involved, depending on fixed point theorem". Karlin (1959) provided a succinct account of the two main aspects of the von Neumann equilibrium (the existence of a process generating a maximal rate of balanced expansion, and the existence of a supporting price system) by using a compactness and a separation argument respectively. The notoriously difficult lemma in the von Neumann paper was proved in a simpler manner by Kakutani (1941, Theorem 2), using his celebrated fixed point theorem. In the subsequent development of intertemporal economics, Gale's framework or the "von Neumann-Gale model" (see Makarov and Rubinov (1977)) appeared regularly: often in contexts far removed from "balanced" growth. It should be mentioned that in their well-known paper, Kemeny, Morgenstern and Thompson (1956) also rejected the condition (H), but thought that assuming an unlimited supply of natural factors "was quite proper" for an economist,"because there are or have been many instances of economic development where it is true". Neither Gale nor Kemeny, Morgenstern and Thompson questioned the appropriateness of "balanced" growth (all production activities growing over time at the *same* rate) as even a remotely realistic description of economic development. But Koopmans (1965) did ("arbitrary and contrary to all experience about economic growth"). "Worse than that, it seems quixotic to ignore completely the historically given capital stock at the beginning,... and to assume that out of some fourth dimension one can pull forth a capital sock" that can use a von Neumann process. Finally, "a more unusual defect is that consumption is not treated as an end in itself". Our variation - with an explicit treatment of consumption and initial stock, is more in the tradition of McFadden's study (1967) of optimal growth in "reachable" economies. The von Neumann-Gale economies are members of this class, but those facing constraints from exhaustible resources typically are not.

We now turn to the issue of sustainability that appeared in Solow (1974): Our remarks are based on Mitra's investigation inspired by Solow (see Mitra (1978) and Cass and Mitra (1991)). We use the short summary of Majumdar and Bar (2013) in reviewing Mitra's result. Consider as above an input-output pair with one producible good ($n_1 = 1$) and one exhaustible resource ($n_2 = 1$). Let $(\tilde{x}, r) \in \mathbb{R}^2_+$ be such that x denotes the quantity of the producible good that is used an input ("capital") and r is the quantity of an exhaustible resource. Let $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ be the net output function for the input $\tilde{x}$. Let $F(\tilde{x}, r) = G(\tilde{x}, r) + \tilde{x}$ be the gross output function when there is no depreciation. We can describe the technology as:

$$\mathcal{T} = \{\big((\tilde{x}, s), (\tilde{y}, s')\big) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ :\ 0 \le \tilde{y} \le F(\tilde{x}, r),\ \text{for some } 0 \le r \le s - s'\}.$$

Given the initial stocks of the producible good $X > 0$ and of the resource $S > 0$. A program is a sequence $(\mathbf{x}, \mathbf{y}, \mathbf{c}) = ((\tilde{x}_t, s_t), (\tilde{y}_{t+1}, s'_{t+1}), \tilde{c}_{t+1})$ as defined in (3.3).

Assume the net input function $G(\tilde{x}, r)$ satisfies the following assumptions:

G.1. G *is continuous on* $\mathbb{R}^2_+$; *it is homogeneous of degree one, concave, twice continuously differentiable for* $(\tilde{x}, r) > 0$, $G_x > 0$, $G_r > 0$.

A program from $(X, S) > 0$ is said to *sustain a positive consumption level* if $\inf_{t \ge 1} c_t > 0$.

For any positive scalar $d > 0$, we define a set $Q(d)$ that represents all the quantities of the input $\tilde{x}$ so that together with some quantity of the exhaustible resource r, it is possible to produce output d from inputs $\tilde{x}, r$: $Q(d) = \{\tilde{x} \ge 0 : G(\tilde{x}, r) = d$ for some $r \ge 0\}$. The isoquant function corresponding to the output level d is $i_d\,(\tilde{x}) : Q(d) \to R_+$, i.e., for $\tilde{x} \in Q(d)$, $G(\tilde{x}, i_d(\tilde{x})) = d$. Given the assumptions in G.1., $i'_d < 0$, $i''_d < 0$ hold. For any $\theta > 0$ such that $\theta \in Q(d)$, the area under the isoquant curve between θ and $\theta + L$ for some $L \ge 0$, $\int_\theta^{\theta+L} i_d(\tilde{x}) d\tilde{x}$ is non-decreasing in L. Hence, as $L \to \infty$, $\int_\theta^{\theta+L} i_d(x) dx$ either converges to some finite limit or diverges to ∞. The integral $\int_\theta^\infty i_d(x) dx$ is called θ-area under the d-isoquant.

Definition 4.1. The production function $G(\tilde{x}, r)$ is said to be "regular" if, for every $d > 0$ and every $\theta \in Q(d)$, the θ-area under the d-isoquant is finite.

One of the main results due to Mitra is that *if* G is "regular", there is a *program from* $(X, S) > 0$ *that sustains a positive consumption level.* Moreover, under some additional conditions (satisfied, for example, when G has the Cobb-Douglas functional form), the converse is also true. That is, *if G is not "regular"*, there is *no* program from $(X, S) > 0$ that sustains a positive consumption level.

Consider, for example, a Cobb-Douglas production function $G(\tilde{x}, r) = \tilde{x}^{\alpha} r^{\beta}$ where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Then, for any $d > 0$, and $\theta \in Q(d) = (0, \infty)$ the d-isoquant is given by $i_d(\tilde{x}) = (d^{1/\beta})/\tilde{x}^{\frac{\alpha}{\beta}}$. So, the θ-area under the d-isoquant is finite if and only if $\frac{\alpha}{\beta} > 1$. Thus, a positive consumption level is sustainable if and only if $\alpha > \beta$. This result was obtained by Solow (1974).

5. Bon voyage

In a long and outstanding career, Professor Ali Khan has displayed an intimidating range in research and an inspiring commitment to scholarship. He has followed many routes: "standard" or "nonstandard":

"For thousands of years I roamed the paths of earth,
From waters round Sri Lanka, in dead of night, to seas up the Malabar Coast.
Much have I wandered. I was there in the gray world of Ashoka,
And of Bimbisara, pressed on through the city of Vidarbha.
I am a weary heart surrounded by life's frothy ocean."
(Banalata Sen by Jibanananda Das, translated by Clinton Seeley).

However, unlike the traveler of the poet, Professor Khan is never weary: there is always an alluring Cinnamon Island on the horizon for him to set sail.

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