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DOES FINANCIAL DEVELOPMENT AMPLIFY SUNSPOT FLUCTUATIONS?

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Abstract. Does financial development amplify or contract sunspot fluctuations? To address this question, we explore a two-sector dynamic general equilibrium model with financial frictions and sector-specific production externalities. We first derive a condition for indeterminacy of equilibria to occur, and then, a sunspot variable is introduced in the economy with financial frictions. The outcome shows that if labor intensity in the consumption good sector from the social perspective is very large, financial development is more likely to magnify sunspot fluctuations, whereas if labor intensity in the intermediate good sector from the social perspective is very large, financial development is more likely to contract sunspot fluctuations.

1. Introduction

Whether financial development (which relaxes financial constraints) amplifies the effect of sunspot variables has not been fully investigated in economic theory. In this paper, by applying a dynamic general equilibrium model with two production sectors, we demonstrate that financial development can magnify self-fulfilling business fluctuations caused by sunspot variables.

Over the past twenty years, many researchers have paid attention to indeterminacy of equilibria in dynamic general equilibrium models (e.g., Benhabib and Farmer, 1994, 1996; Boldrin and Rustichini, 1994; Benhabib et al., 2000; Nishimura and Venditti; 2004, 2007; Dufourt et al., 2015). Sunspots are defined as extrinsic random variables that may impact agents' expectations without directly affecting economic fundamentals (Shell, 1977; Azariadis, 1981; Cass and Shell, 1983). It is widely known that when indeterminacy arises, extrinsic uncertainty randomizes multiple equilibria and endogenous business fluctuations can occur. In this literature, production externalities have been one of the important features of the model, as they are a source of inefficiency that causes indeterminacy. Benhabib and Nishimura (1998), for instance, prove that indeterminacy can occur in a dynamic general equilibrium model with two production sectors even when production externalities are very small.

In the current paper, to explore whether financial development amplifies the effect of sunspots, financial frictions are explicitly introduced in the discrete-time version

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of Benhabib and Nishimura's model, in which consumption goods are produced in the first sector and intermediate goods are produced in the second sector. In our model, agents receive an idiosyncratic productivity shock in each period when producing capital goods. Those who have higher productivity purchase intermediate goods and initiate an investment project to produce capital goods. They face financial constraints and can borrow only up to a certain proportion of their savings when initiating an investment project. Meanwhile, those who have lower productivity lend their all savings in the financial market. Therefore, borrowers and lenders endogenously appear in equilibrium.

In our model, the extent of financial constraints measures the extent of financial development, following the literature (e.g., Aghion and Banerjee, 2005). In equilibrium, whereas capital accumulation is promoted and the gross product in the steady state monotonically increases as the financial sector becomes better developed, indeterminacy of equilibria is more likely to occur if the consumption sector is more labor intensive from the social perspective but less labor intensive from the private perspective relative to the intermediate sector. Given these results, we introduce a sunspot variable in the model and examine whether financial development amplifies or contracts sunspot fluctuations. Our findings are as follows. Under high labor intensity in the consumption good sector from the social perspective, financial development is more likely to magnify sunspot fluctuations, whereas under high labor intensity in the intermediate good sector from the social perspective, financial development is more likely to contract sunspot fluctuations.

The remainder of this paper proceeds as follows. In section 2, we present the model in which there are two production sectors in the economy, with each sector subject to sector-specific production externalities, and in which infinitely lived agents face financial constraints. In section 3, equilibrium in the model is derived. We also characterize a steady state of the dynamical system in this section. In section 4, the dynamic property in the neighborhood of the steady state is investigated, and the condition for the economy to exhibit indeterminacy of equilibria is obtained. In section 5, we introduce a sunspot variable in the model and explore whether financial development amplifies sunspot fluctuations. Section 6 concludes the paper.

2. Model

Consider a closed economy that consists of an infinitely lived representative firm and a continuum of infinitely lived agents with the total population normalized to one. The economy goes in discrete time indexed by t from $t = 0$ to $t = +\infty$. The representative firm produces both consumption and intermediate goods with different technologies in each period. The intermediate goods are assumed to be numeraire throughout the analysis. The infinitely lived agents are potential capital producers but receive uninsured idiosyncratic productivity shocks in each period that affect productivity in capital production.

2.1. **Agents.**

2.1.1. *Timing of events.* At the beginning of period t, an agent earns incomes: a wage income, returns to her savings, and a lump-sum profit from the representative firm. The market for consumption goods in period t opens at the beginning of the period and is closed before an idiosyncratic productivity shock in period t is realized. Accordingly, the agent must make a decision about consumption and savings at the beginning of period t without knowing her productivity in capital production. At the end of period t, the agent receives an idiosyncratic productivity shock. The agent can adopt two saving methods: lending her savings in the financial market or initiating an investment project. The agent chooses one of the saving methods with knowing her productivity. Lending one unit of savings in period t yields a claim to r_{t+1} units of intermediate goods in period $t+1$, where r_{t+1} is the gross real interest rate, whereas purchasing one unit of intermediate goods in period t for the investment project creates Φ_t units of capital used in period $t+1$. One unit of capital is sold at price q_{t+1} to the production sector in period $t+1$. Capital is perishable in one period. Although the agent can borrow in the financial market, she faces a financial constraint, which implies that she can only borrow up to a certain proportion of her savings.

Φ_t is the productivity shock on the probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sample space, $\mathcal{F}$ is a $\sigma-$algebra on Ω, and P is the probability measure.[1] Φ_t is a function of $\omega_t \in \Omega$, the support for which is given by $[0, \eta]$ where $\eta \in (0, \infty)$. The cumulative distribution function of Φ_t is given by $G(\Phi) := P(\{\omega_t \in \Omega \mid \Phi_t(\omega_t) \leq \Phi\})$, where $\{\omega_t \in \Omega \mid \Phi_t(\omega_t) \leq \Phi\} \in \mathcal{F}$. $G(\Phi)$ is continuously differentiable on the support and time-invariant. $\Phi_0, \Phi_1, \cdots$, are independent and identically distributed across both agents and time (the i.i.d. assumption). Because there is no insurance market for productivity shocks, no one can insure against low productivity. Define the history of ω_t as $\omega^{t-1} = \{\omega_0, \omega_1, ..., \omega_{t-1}\}$. Then, we obtain the probability space $(\Omega^t, \mathcal{F}^t, P^t)$, which is a Cartesian product of t copies of $(\Omega, \mathcal{F}, P)$ in which ω^{t-1} is an element of Ω^t. An individual who experiences this history can be identified by ω^{t-1}.

2.1.2. *Utility maximization.* An agent's expected lifetime utility in period t is given by

$$E\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} c_\tau(\omega^{\tau-1}) \middle| \, \omega^{t-1}\right],$$

where $\delta \in (0,1)$ is the subjective discount factor and $c_\tau(\omega^{\tau-1})$ is consumption. $E[.|\omega^{t-1}]$ is an expectation operator given the history ω^{t-1}. The agent in period t maximizes her lifetime utility subject to

$$p_\tau c_\tau(\omega^{\tau-1}) + x_\tau(\omega^\tau) + b_\tau(\omega^\tau)$$
$$= q_\tau \Phi_{\tau-1}(\omega_{\tau-1}) x_{\tau-1}(\omega^{\tau-1}) + r_\tau b_{\tau-1}(\omega^{\tau-1}) + w_\tau + \pi_\tau \tag{2.1}$$
$$b_\tau(\omega^\tau) \geq -\lambda a_\tau(\omega^{\tau-1}) \tag{2.2}$$
$$x_\tau(\omega^\tau) \geq 0 \tag{2.3}$$

for $\tau \geq t$. In (2.1), w_τ is a wage income, π_τ is a profit obtained from the production sector, and p_τ is the price of consumption goods. Additionally, $x_t(\omega^t)$ is an intermediate good used for an investment project, and $b_\tau(\omega^t)$ is lending if $b_\tau(\omega^t) > 0$

[1] One can assume $\Omega = [0, 1]$.

and borrowing if $b_\tau(\omega^t) < 0$. For capital production, a linear technology is assumed such as $\Phi_{\tau-1}(\omega_{\tau-1})x_{\tau-1}(\omega^{\tau-1})$, which is capital produced in period τ by an agent who initiates a project in period $\tau-1$ drawing productivity $\Phi_{\tau-1}$. In any case, $a_t(\omega^{t-1}) := x_t(\omega^t)+b_\tau(\omega^t)$ is the agent's savings in period t. When the agent makes a decision about consumption, $c_t(\omega^{t-1})$, and saving, $a_t(\omega^{t-1})$, in period t, she does not know her productivity $\Phi_t(\omega_t)$ in capital production, as previously explained. However, as noted from the expression $a_t(\omega^{t-1}) = x_t(\omega^t) + b_t(\omega^t)$, the agent knows $\Phi_t(\omega_t)$ when she makes a portfolio decision about investing in a project, lending, and/or borrowing in period t. Eq. (2.1) is effective for $\tau \geq 1$. It is assumed that the flow budget constraint is given by $a_0 = q_0X_0 + w_0 + \pi_0 - p_0c_0$ in period 0, where X_0 is the initial capital endowment that is commonly distributed across agents.

The agent faces a financial constraint given by inequality (2.2).[2] Inequality (2.2) implies that an agent can borrow in the financial market only up to a certain proportion of her savings, which is also regarded as her net worth. $\lambda \in (0,\infty)$ measures the extent of financial constraints; i.e., the financial constraint is more relaxed as λ becomes greater. Inequality (2.2) is converted into $b_\tau(\omega^\tau) \geq -\mu x_\tau(\omega^\tau)$, where $\mu = \lambda/(1+\lambda) \in (0,1)$. This constraint is more convenient than inequality (2.2), and we use it henceforth. The financial market approaches perfection as μ goes to 1. As in the literature, μ is assumed to measure the degree of financial development.[3] Finally, we impose a nonnegativity constraint given by inequality (2.3) on the purchase of intermediate goods.

2.1.3. *Optimal decision within a period.* We define a new variable as $\phi_t := r_{t+1}/q_{t+1}$. Agents know the productivity shocks for capital production realized in period t when they make an optimal portfolio decision about investment, lending, and borrowing in the same period. Therefore, an agent who receives $\Phi_t > \phi_t$ borrows up to the limit of the financial constraint and purchases intermediate goods to start an investment project, whereas an agent who receives $\Phi_t \leq \phi_t$ lends all her savings in the financial market and obtains interest of r_{t+1}.[4] One notes that ϕ_t is the cutoff for the productivity shocks that divide agents into lenders and borrowers in period t, and the agent's optimal decision within a period is given by

$$x_t(\omega^t) = \begin{cases} 0 & \text{if } \Phi_t(\omega_t) \leq \phi_t \\ \frac{a_t(\omega^{t-1})}{1-\mu} & \text{if } \Phi_t(\omega_t) > \phi_t, \end{cases} \tag{2.4}$$

and

$$b_t(\omega^t) = \begin{cases} a_t(\omega^{t-1}) & \text{if } \Phi_t(\omega_t) \leq \phi_t \\ -\frac{\mu}{1-\mu}a_t(\omega^{t-1}) & \text{if } \Phi_t(\omega_t) > \phi_t. \end{cases} \tag{2.5}$$

[2]Many researchers assume this type of financial constraint in the literature. See, for instance, Aghion et al. (1999), Aghion and Banerjee (2005), Aghion et al. (2005), and Kunieda and Shibata (2016).

[3]See, for instance, Aghion and Banerjee (2005).

[4]Agents who receive $\Phi_t = \phi_t$ are indifferent between initiating a project and lending in the financial market. We assume they lend their savings in the financial market.

From the portfolio decision given by (2.4) and (2.5), the flow budget constraint (2.1) is rewritten as

$$a_\tau(\omega^{\tau-1}) + p_\tau c_\tau(\omega^{\tau-1}) = R_\tau(\omega_{\tau-1})a_{\tau-1}(\omega^{\tau-2}) + w_\tau + \pi_\tau, \tag{2.6}$$

where $R_\tau(\omega_{\tau-1}) := \max\{r_\tau, (q_\tau\Phi_{\tau-1}(\omega_{\tau-1}) - r_\tau\mu)/(1-\mu)\}$. The maximization of the agent's lifetime utility subject to (2.6) yields the Euler equation as follows:

$$p_{t+1} = \delta E\left[R_{t+1}(\omega_t)|\omega^{t-1}\right] p_t. \tag{2.7}$$

The necessary and sufficient optimality conditions for the lifetime utility maximization problem are given by the Euler equation (2.7) as well as the transversality condition $\lim_{\tau\to\infty}\delta^\tau E[a_{t+\tau}(\omega^{t+\tau-1})/p_{t+\tau}|\omega^{t-1}] = 0$.

2.2. **Production.** The representative firm uses the following two Cobb-Douglas production technologies:

$$\bar{F}^1(l_t^1, z_t^1, \bar{l}_t^1, \bar{z}_t^1) = A(l_t^1)^{a_1}(z_t^1)^{b_1}(\bar{l}_t^1)^{\alpha_1}(\bar{z}_t^1)^{\beta_1},\ 0 < a_1, b_1, \alpha_1, \beta_1 < 1$$

for intermediate goods, and

$$\bar{F}^2(l_t^2, z_t^2, \bar{l}_t^2, \bar{z}_t^2) = B(l_t^2)^{a_2}(z_t^2)^{b_2}(\bar{l}_t^2)^{\alpha_2}(\bar{z}_t^2)^{\beta_2},\ 0 < a_2, b_2, \alpha_2, \beta_2 < 1$$

for consumption goods, where $a_i+\alpha_i+b_i+\beta_i = 1$ and $a_i+b_i =: e \in (0,1)$ for $i = 1, 2$. In the production technologies, l_t^i and z_t^i are labor and capital inputs, respectively. The components of production externalities with respect to labor and capital are given by $\bar{l}_t^i$ and $\bar{k}_t^i$, respectively. Note that both labor and capital exhibit positive externalities to the production technologies because both α_i and β_i are positive. Although $\bar{l}_t^i$ and $\bar{k}_t^i$ are exogenous when the firm solves the profit maximization problem, it holds that $l_t^i = \bar{l}_t^i$ and $k_t^i = \bar{k}_t^i$ in equilibrium.

The profit maximization problem that the firm faces is given as follows:

$$\max_{l_t^1, l_t^2, z_t^1, z_t^2} \Pi_t := \bar{F}^1(l_t^1, z_t^1, \bar{l}_t^1, \bar{z}_t^1) + p_t\bar{F}^2(l_t^2, z_t^2, \bar{l}_t^2, \bar{z}_t^2) - q_t z_t - w_t l_t, \tag{2.8}$$

where $z_t = z_t^1 + z_t^2$ is the aggregate capital and $l_t = l_t^1 + l_t^2$ is the population of agents. It is assumed that l_t is constant and normalized to $l_t = 1$. The first-order conditions for the profit maximization are given by

$$Aa_1\left(\frac{z_t^1}{l_t^1}\right)^{1-\theta_1} = p_t B a_2\left(\frac{z_t^2}{l_t^2}\right)^{1-\theta_2} = w_t. \tag{2.9}$$

and

$$Ab_1\left(\frac{z_t^1}{l_t^1}\right)^{-\theta_1} = p_t B b_2\left(\frac{z_t^2}{l_t^2}\right)^{-\theta_2} = q_t, \tag{2.10}$$

where $a_i + \alpha_i =: \theta_i \in (0,1)$ for $i = 1, 2$. From Eqs. (2.9) and (2.10), we have

$$z_t^1 = \frac{b_1 w_t}{a_1 q_t} l_t^1 \tag{2.11}$$

and

$$z_t^2 = \frac{b_2 w_t}{a_2 q_t} l_t^2. \tag{2.12}$$

Again from Eqs. (2.9) and (2.10), we also have

$$w_t = \Psi p_t^{\frac{1-\theta_1}{\theta_2-\theta_1}} =: w(p_t) \tag{2.13}$$

and

$$q_t = \Lambda p_t^{\frac{-\theta_1}{\theta_2-\theta_1}} =: q(p_t), \tag{2.14}$$

where $\Psi = [(Aa_1^{\theta_1}b_1^{1-\theta_1})^{\theta_2-1}(Ba_2^{\theta_2}b_2^{1-\theta_2})^{1-\theta_1}]^{1/(\theta_2-\theta_1)}$ and $\Lambda = [(Aa_1^{\theta_1}b_1^{1-\theta_1})^{\theta_2} (Ba_2^{\theta_2}b_2^{1-\theta_2})^{-\theta_1}]^{1/(\theta_2-\theta_1)}$.

One may say that if $a_1/b_1 < a_2/b_2 \iff a_1/(e-a_1) < a_2(e-a_2) \iff a_1 < a_2$, the consumption good sector is labor intensive relative to the intermediate good sector from the private perspective and if $\theta_1/(1-\theta_1) < \theta_2/(1-\theta_2) \iff \theta_1 < \theta_2$, the consumption good sector is labor intensive relative to the intermediate good sector from the social perspective.

3. Equilibrium

A competitive equilibrium is given by sequences of prices $\{w_t, q_t, p_t, r_{t+1}\}$ for all $t \geq 0$ and allocation $\{c_t(\omega^{t-1}), a_t(\omega^{t-1}), k_t(\omega^t), b_t(\omega^t)\}$ for all $t \geq 0$ and ω^t, and $\{z_t, l_t\}$ for all $t \geq 0$, so that (i) for each ω^t, the consumer maximizes her lifetime utility from time t onward, (ii) the representative firm maximizes its profits in each period, and (iii) the consumption and intermediate good markets, the financial market, the capital market, and the labor market clear.[5]

3.1. Market clearing conditions.

In each period, aggregate consumption is equal to the production of consumption goods, and thus, the consumption good market clearing condition is given by

$$C_t := \int_{\Omega^t} c_t(\omega^{t-1})dP^t(\omega^{t-1}) = F^2(z_t^2, l_t^2), \tag{3.1}$$

where $F^2(z_t^2, l_t^2) = \bar{F}^2(l_t^2, z_t^2, l_t^2, z_t^2)$. Eq. (2.4) implies that the intermediate goods are purchased by agents who draw higher productivity such that $\Phi_t(\omega_t) > \phi_t$. Accordingly, the intermediate good market clearing condition is given by

$$\int_{\Omega^t\times(\Omega\backslash\Xi_t)} x_t(\omega^t)dP^{t+1}(\omega^t) = F^1(z_t^1, l_t^1), \tag{3.2}$$

where $\Xi_t = \{\omega_t \in \Omega | \Phi_t(\omega_t) \leq \phi_t\}$ and $F^1(z_t^1, l_t^1) = \bar{F}^1(l_t^1, z_t^1, l_t^1, z_t^1)$. In the financial market, the aggregation of all debts and lending becomes zero, and thus, it holds that

$$\int_{\Omega^{t+1}} b_t(\omega^t)dP^{t+1}(\omega^t) = 0. \tag{3.3}$$

Capital is produced by agents who draw higher productivity such that $\Phi_t(\omega_t) > \phi_t$ as seen in (2.4) and used by the representative firm. Then, we have the capital market clearing condition as follows:

$$z_{t+1}^1 + z_{t+1}^2 = \int_{\Omega^t\times(\Omega\backslash\Xi_t)} \Phi_t(\omega_t)x_t(\omega^t)dP^{t+1}(\omega^t). \tag{3.4}$$

[5] To be precise, ω^{-1} is empty because c_0 is not subject to any history of the stochastic events.

Finally, the labor market clearing condition is given by

$$l_t^1 + l_t^2 = 1. \tag{3.5}$$

3.2. **Gross product in equilibrium.** From (2.11), (2.12), (3.5), and $z_t^1 + z_t^2 = z_t$, the production functions are expressed by

$$F^1(l_t^1, z_t^1) = \frac{a_2 q(p_t) z_t - b_2 w(p_t)}{e(a_2 - a_1)} \tag{3.6}$$

and

$$p_t F^2(l_t^2, z_t^2) = \frac{a_1 q(p_t) z_t - b_1 w(p_t)}{e(a_1 - a_2)}. \tag{3.7}$$

Eqs. (3.6) and (3.7) rewrite the gross product, $Y_t = F^1(l_t^1, z_t^1) + p_t F^2(l_t^2, z_t^2)$, as follows:

$$Y_t = \frac{q(p_t) z_t + w(p_t)}{e}. \tag{3.8}$$

3.3. **Cutoff.** The financial market clearing condition (3.3) determines ϕ_t in equilibrium as shown in Proposition 3.1.

Proposition 3.1. *The cutoff, ϕ_t, is given by*

$$G(\phi_t) = \mu. \tag{3.9}$$

Proof. See the Appendix.

Since the cumulative distribution function is strictly increasing over the support, ϕ_t is uniquely determined as $\phi_t = G^{-1}(\mu) =: \phi^*$ and increases as μ increases. This relationship between ϕ^* and μ implies that as the financial constraint is relaxed, the number of lenders increases and the number of capital producers (borrowers) decreases. As the number of capital producers decreases, inefficiency regarding the allocation of intermediate goods is corrected because the intermediate goods are intensively used by higher-productivity agents when ϕ^* increases.

3.4. **Dynamical system.** We obtain two lemmata that are useful for aggregating the flow budget constraint (2.6) across all agents.

Lemma 3.2.

$$\int_{\Omega^t} R_t(\omega_{t-1}) a_{t-1}(\omega^{t-2}) dP^t(\omega^{t-1}) = q(p_t) z_t. \tag{3.10}$$

Proof. See the Appendix.

One notes from Lemma 3.2 that the total income from all agents' savings is eventually equal to the value of total capital in the economy. By using Lemma 3.2, we aggregate the flow budget constraint (2.6) across all agents and obtain the relationship between the total saving and the intermediate goods produced in period t in Lemma 3.3 below.

Lemma 3.3.

$$\int_{\Omega^t} a_t(\omega^{t-1}) dP^t(\omega^{t-1}) = F^1(l_t^1, z_t^1) \tag{3.11}$$

Proof. See the Appendix.

The intermediate goods are used to produce capital by the higher-productivity agents, i.e., those who draw $\Phi_t(\omega_t)$ greater than ϕ_t, and thus, the i.i.d. assumption, Lemma 3.3, and Eq. (2.4) yield capital z_{t+1} as in Proposition 3.4.

Proposition 3.4.

$$z_{t+1} = \frac{H(\phi^*)}{1-\mu} F^1(l_t^1, z_t^1), \tag{3.12}$$

where $H(\phi^*) = \int_{\phi^*}^{\eta} \Phi_t(\omega_t) dG(\Phi)$.

Proof. See the Appendix.

Substituting Eq. (3.6) into Eq. (3.12) yields

$$z_{t+1} = \frac{H(\phi^*)}{1-\mu}\left(\frac{a_2 q(p_t) z_t - b_2 w(p_t)}{e(a_2 - a_1)}\right), \tag{3.13}$$

From Eqs. (3.8) and (3.13), we have

$$Y_{t+1} = -\frac{H(\phi^*) q(p_{t+1})}{e(1-\mu)(a_1 - a_2)}(a_2 Y_t - w(p_t)) + \frac{w(p_{t+1})}{e}. \tag{3.14}$$

The expected return, $E[R_{t+1}(\omega_t)|\omega^{t-1}]$, can be computed by using $G(\phi^*) = \mu$ and $\phi^* = r_{t+1}/q(p_{t+1})$ in Proposition 3.5 below.

Proposition 3.5.

$$E[R_{t+1}(\omega_t)|\omega^{t-1}] = q(p_{t+1})\frac{H(\phi^*)}{1-\mu}. \tag{3.15}$$

Proof. See the Appendix.

Eq. (2.7) is rewritten by Eqs. (2.14) and (3.15) as follows:

$$p_{t+1} = \left(\frac{\Lambda\delta H(\phi^*)}{1-\mu}\right)^{\frac{\theta_2-\theta_1}{\theta_2}} (p_t)^{\frac{\theta_2-\theta_1}{\theta_2}} =: s(p_t). \tag{3.16}$$

3.5. **Steady state.** From Eq. (3.16), we obtain the price of consumption goods in the steady state, $\bar{p}$, as

$$\bar{p} = \left(\frac{\Lambda\delta H(\phi^*)}{1-\mu}\right)^{\frac{\theta_2}{\theta_1}-1}. \tag{3.17}$$

It follows from Eqs. (2.13), (2.14), (3.13), and (3.17) that the capital stock, $\bar{z}$, in the steady state can be computed as

$$\bar{z} = \frac{b_2 \Psi \Lambda^{\frac{1-\theta_1}{\theta_1}} \left(\frac{\delta H(\phi^*)}{1-\mu}\right)^{\frac{1}{\theta_1}}}{a_2 + \delta e(a_1 - a_2)}. \tag{3.18}$$

Eqs. (2.13), (2.14), (3.14), and (3.17) yield the gross product, $\bar{Y}$, in the steady state as follows:

$$\bar{Y} = \Psi\left(\frac{1+\delta(a_1 - a_2)}{a_2 + \delta e(a_1 - a_2)}\right)\left(\frac{\delta\Lambda H(\phi^*)}{1-\mu}\right)^{\frac{1-\theta_1}{\theta_1}}. \tag{3.19}$$

The extent of financial market imperfections measured by μ affects the price of consumption goods, capital accumulation, and the gross product in the steady state as demonstrated in Proposition 3.6 below.[6]

Proposition 3.6. *As the financial constraint is relaxed, the following hold:*

- *If the production of the intermediate goods is more labor (capital) intensive than that of the consumption goods from the social perspective, i.e., $\theta_1 > \theta_2$ ($\theta_1 < \theta_2$) from the social perspective, the price of consumption goods in the steady state decreases (increases), i.e., $\partial\bar{p}/\partial\mu < 0$ ($\partial\bar{p}/\partial\mu > 0$).*
- *The capital stock in the steady state increases, i.e., $\partial\bar{z}/\partial\mu > 0$.*
- *The gross product in the steady state increases, i.e., $\partial\bar{Y}/\partial\mu > 0$.*

Proof. See the Appendix.

We can intuitively understand the results of Proposition 3.6. As the financial constraint is relaxed, allocative efficiency with regard to the use of intermediate goods is corrected. As the financial constraint is relaxed, less productive agents are excluded from capital production activities, and capital producers who draw higher productivity use more intermediate goods. Then, aggregate productivity regarding capital production in the economy becomes higher. Accordingly, total production of capital in the steady state, $\bar{z}$, increases. Capital and labor are complementary in production in both production sectors. Therefore, it is straightforward to show that if $\bar{z}$ increases, the capital price decreases because the marginal product of capital decreases, whereas the wage rate increases because the marginal product of labor increases. If the capital price decreases and the wage rate increases, the relative price of final goods in the labor-intensive sector increases, and the relative price of final goods in the capital-intensive sector decreases. This means that the price of consumption goods, $\bar{p}$, decreases (increases) as μ increases if $\theta_1 > \theta_2$ ($\theta_1 < \theta_2$). This reverse Stolper-Samuelson property helps us understand the effect that the relaxation of the financial constraint has on the price of consumption goods, $\bar{p}$, in the steady state.[7]

4. Local dynamics

By inserting Eq. (3.16) into (3.14), we obtain a dynamical system in this economy with respect to Y_t and p_t as follows:

$$\text{(4.1)} \qquad \begin{cases} Y_{t+1} = J(Y_t, p_t) \\ p_{t+1} = \left(\frac{\Lambda\delta H(\phi^*)}{1-\mu}\right)^{\frac{\theta_2-\theta_1}{\theta_2}} (p_t)^{\frac{\theta_2-\theta_1}{\theta_2}}, \end{cases}$$

[6]Throughout the analysis, we exclusively focus on the case of imperfect specialization in which the economy consistently produces both intermediate and consumption goods. One can show that the steady state that we have derived exists in the area of imperfect specialization. By continuity, both intermediate and consumption goods are produced in the neighborhood of the steady state.

[7]Note that the causality from the prices of input goods to the prices of final goods is opposite to that of the standard Stolper-Samuelson theorem, in which the prices of final goods affect the prices of input goods. In a closed economy without any aggregate shocks, the prices of input goods are determined by the extant capital and labor together with technologies, and thus, the prices of input goods affect the prices of final goods.

where

$$J(Y_t, p_t) = -\frac{H(\phi^*)q(s(p_t))}{e(1-\mu)(a_1-a_2)}(a_2 Y_t - w(p_t)) + \frac{w(s(p_t))}{e}.$$

The linearization of the dynamical system (4.1) around the steady state yields

$$\begin{pmatrix} Y_{t+1}-\bar{Y} \\ p_{t+1}-\bar{p} \end{pmatrix} = \begin{pmatrix} -\frac{a_2}{\delta e(a_1-a_2)} & J_p(\bar{Y},\bar{p}) \\ 0 & 1-\frac{\theta_1}{\theta_2} \end{pmatrix} \begin{pmatrix} Y_t-\bar{Y} \\ p_t-\bar{p} \end{pmatrix}, \tag{4.2}$$

where $J_p(Y,p) := \partial J(Y,p)/\partial p$. From Eq. (4.2), we obtain the eigenvalues, κ_1 and κ_2, of this dynamical system as follows:

$$\kappa_1 = -\frac{a_2}{\delta e(a_1-a_2)} \tag{4.3}$$

and

$$\kappa_2 = 1 - \frac{\theta_1}{\theta_2}. \tag{4.4}$$

Depending upon the parameter values, the eigenvalues are characterized in Lemma 4.1 below.

Lemma 4.1. *The eigenvalues, κ_1 and κ_2, are characterized as follows:*

- *The value of κ_1 is determined by the subjective discount factor and the parameter values related to the factor intensity from the private perspective as follows:*
  - $a_1 - a_2 < 0$ *if and only if* $\kappa_1 > 1$.
  - $0 < e\delta < a_2/(a_1-a_2)$ *if and only if* $\kappa_1 < -1$.
  - $0 < a_2/(a_1-a_2) < e\delta$ *if and only if* $-1 < \kappa_1 < 0$.
- *The value of κ_2 is determined by the parameter values related to the factor intensity from the social perspective as follows:*
  - $0 < \theta_1 < 2\theta_2$ *if and only if* $-1 < \kappa_2 < 1$.
  - $0 < 2\theta_2 < \theta_1$ *if and only if* $\kappa_2 < -1$.

Proof. See the Appendix.

We obtain various results regarding the local stability of the steady state. In particular, we focus on the case in which the steady state is totally stable and indeterminacy of equilibrium occurs so that we can consider self-fulfilling business fluctuations. It is said that equilibrium is indeterminate if there exists a continuum of competitive equilibrium. In particular, in this model, indeterminacy of equilibrium occurs if there exists a continuum of initial prices of the consumption goods for any given initial capital, z_0, such that each one of them is consistent with a competitive equilibrium.

Proposition 4.2. *It holds that the steady state, $(\bar{Y},\bar{p})$, is totally stable, and thus, equilibrium is indeterminate around the steady state if and only if* $0 < a_2/(a_1-a_2) < e\delta$ *and* $0 < \theta_1 < 2\theta_2$.

Proof. From Lemma 4.1, it follows that $-1 < \kappa_1 < 0$ and $-1 < \kappa_2 < 1$ if and only if $0 < a_2/(a_1-a_2) < e\delta$ and $0 < \theta_1 < 2\theta_2$. Therefore, the steady state, $(\bar{Y},\bar{p})$, is totally stable if and only if $0 < a_2/(a_1-a_2) < e\delta$ and $0 < \theta_1 < 2\theta_2$. Since z_0 is predetermined and p_0 can jump, if the steady state is totally stable, it follows that for any given initial capital, z_0, there exists a continuum of initial prices of the consumption goods, each one of which together with z_0 determines Y_0 by Eq. (3.8) and produces a sequence $\{Y_t, p_t\}_{t=0}^{\infty}$ in competitive equilibrium, and thus, equilibrium is indeterminate. $\square$

Proposition 4.2 implies that indeterminacy of equilibrium is more likely to occur when the factor intensity is reversed between the private and social perspectives. The intuition behind this appearance of the continuum of competitive equilibria when the factor intensity is reversed is as follows. Suppose that $\theta_2 > \theta_1$ and $a_2 < a_1$, which satisfy the condition for indeterminacy. Under this condition, the consumption sector is labor intensive from the social perspective and the intermediate sector is labor intensive from the private perspective. Consider a competitive equilibrium. To examine whether the neighborhood of this competitive equilibrium can also be another competitive equilibrium, suppose that the price of consumption goods p_τ becomes infinitesimally greater than that of the original competitive equilibrium in a certain period τ. Since the consumption sector is labor intensive from the social perspective, the greater price of consumption goods increases the wage rate and decreases the capital price (the Stolper-Samuelson property). Accordingly, as noted from Eqs. (2.11) and (2.12), the capital-labor ratios in both sectors must increase. Again, note from Eqs. (2.11) and (2.12) that since $a_2 < a_1$ and $e - a_2 = b_2 > b_1 = e - a_1$, it must hold that z_t^2/l_t^2 (the capital-labor ratio in the consumption good sector) is greater than z_t^1/l_t^1 (the capital-labor ratio in the intermediate good sector) for all $t \geq 0$. Therefore, the only way for both z_t^2/l_t^2 and z_t^1/l_t^1 to increase is that both z_t^2 and l_t^2 decrease and both z_t^1 and l_t^1 increase. Then, in period $\tau + 1$, output in the consumption good sector decreases, whereas that in the intermediate good sector increases. As a result, the price of consumption goods increases in period $\tau + 1$. Because we have assumed the infinitesimal increase in the price of consumption goods in period τ, this outcome is consistent with the dynamic equation of p_t in Eq. (4.2), and thus, the neighborhood of a competitive equilibrium can be another competitive equilibrium. In contrast, when $a_2 = \theta_2 > \theta_1 = a_1$, it must hold that z_t^2/l_t^2 is smaller than z_t^1/l_t^1 for all $t \geq 0$, and then, the only way for both z_t^2/l_t^2 and z_t^1/l_t^1 to increase is that both z_t^2 and l_t^2 increase and both z_t^1 and l_t^1 decrease. Accordingly, the output in the consumption good sector increases, whereas that in the intermediate good sector decreases in period $\tau + 1$. As a result, the price of consumption goods decreases in period $\tau + 1$. This outcome contradicts the dynamic equation of p_t in Eq. (4.2), and thus, the neighborhood of a competitive equilibrium cannot be another competitive equilibrium.

5. Self-fulfilling business fluctuations

One notes from Lemma 4.1 that the stability of the steady state is independent of the extent of financial constraints, as it is affected only by the private and social factor intensities and the subjective discount factor. Although the extent of financial constraints does not affect the stability of the steady state, it would amplify endogenous business fluctuations caused by extrinsic uncertainty. It is well known that indeterminacy equilibrium in dynamic general equilibrium models is a potential cause of sunspot fluctuations. Therefore, we focus on the case in which indeterminacy in equilibrium arises, i.e., $0 < a_2/(a_1 - a_2) < e\delta$ and $0 < \theta_1 < 2\theta_2$ as shown in Proposition 4.2.

Assumption 5.1. (i) $0 < \theta_1 < 2\theta_2$ and (ii) $0 < a_2/(a_1 - a_2) < e\delta$.

5.1. **Sunspot variable.** A sunspot variable, ϵ_t, is introduced in our model, which is assumed to follow a white noise process with mean 0 and variance σ^2 and to be independent and identically distributed across time. The support for ϵ_t is $[-\bar{\epsilon}, \bar{\epsilon}]$, where $\bar{\epsilon}$ is assumed not to be too large so that equilibrium with sunspots can exist. Each agent faces a common sunspot variable in each period, and ϵ_t is independent of $\Phi_{t'}(\omega_{t'})$ for all $t \geq 0$ and $t' \geq 0$. The history of sunspot events until period t is written as $\epsilon^t = \{\epsilon_0, \epsilon_1, ..., \epsilon_t\}$.

The sunspot variable, ϵ_t, is realized at the beginning of period t, and thus, each agent makes a decision about consumption and saving given information about the history of sunspot events, ϵ^t. Then, an agent's expected lifetime utility in period t is given by

$$E\left[\sum_{\tau=t}^{\infty} \delta^{\tau-t} c_\tau(\omega^{\tau-1}, \epsilon^t) \middle| \, \omega^{t-1}, \epsilon^t\right],$$

where $E[.| \, \omega^{t-1}, \epsilon^t]$ is the expectation operator given ω^{t-1} and ϵ^t. Although the cut-off, ϕ_t, is no longer equal to r_{t+1}/q_{t+1}, the concrete expression of ϕ_t is unnecessary. This is because from the financial market clearing condition, Proposition 3.1 still holds, and eventually, we obtain $\phi_t = \phi^*$.[8] Then, the individual return to saving, R_{t+1} in equilibrium, is given by

$$R_{t+1}(\omega_t) = \begin{cases} r_{t+1} & \text{if } \Phi_t(\omega_t) \leq \phi^* \\ \frac{q(p_{t+1})\Phi_t(\omega_t) - r_{t+1}\mu}{1-\mu} & \text{if } \Phi_t(\omega_t) > \phi^*, \end{cases} \tag{5.1}$$

The Euler equation is obtained as

$$\frac{1}{p_t} = \delta E\left[\frac{R_{t+1}(\omega_t)}{p_{t+1}} \middle| \omega^{t-1}, \epsilon^t\right]. \tag{5.2}$$

The Euler equation (5.2) can be computed as follows:

$$\frac{1}{p_t} = \frac{\delta H(\phi^*)}{1-\mu} E\left[\frac{q(p_{t+1})}{p_{t+1}} \middle| \epsilon^t\right]. \tag{5.3}$$

5.2. **Sunspot fluctuations and financial development.** From Eq. (5.3), it is assumed that the sunspot variable ϵ_{t+1} is introduced as in the following.

$$\frac{1}{p_t} = \frac{\delta H(\phi^*)}{1-\mu} \left(\frac{q(p_{t+1})}{p_{t+1}}\right) + \epsilon_{t+1}. \tag{5.4}$$

One can verify that Eq. (5.3) can be obtained by taking the expectation for both sides of Eq. (5.4) with ϵ^t given. Eq.(5.4) can be rewritten as

$$p_{t+1} = \left[\frac{\Lambda \delta H(\phi^*)}{(1-\mu)(1 - p_t \epsilon_{t+1})}\right]^{\frac{\theta_2 - \theta_1}{\theta_2}} (p_t)^{\frac{\theta_2 - \theta_1}{\theta_2}} =: s(p_t; \epsilon_{t+1}). \tag{5.5}$$

[8] For reference, we can obtain $\phi_t = r_{t+1}E[1/p_{t+1}|\epsilon^t]/E[q(p_{t+1})/p_{t+1}|\epsilon^t]$ when the agents are subject to the sunspot variable.

From (3.14) and (5.5), we obtain

$$(5.6)\quad Y_{t+1} = -\frac{H(\phi^*)q(s(p_t;\epsilon_{t+1}))}{e(1-\mu)(a_1-a_2)}(a_2Y_t - w(p_t)) + \frac{w(s(p_t;\epsilon_{t+1}))}{e} =: J(Y_t, p_t; \epsilon_{t+1}).$$

The linearization of (5.5) and (5.6) around the steady state with $\epsilon_{t+1} = 0$ leads to

$$(5.7)\quad \begin{pmatrix} Y_{t+1} - \bar{Y} \\ p_{t+1} - \bar{p} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{a_2}{\delta e(a_1-a_2)} & J_p(\bar{Y}, \bar{p}) \\ 0 & 1 - \frac{\theta_1}{\theta_2} \end{pmatrix} \begin{pmatrix} Y_t - \bar{Y} \\ p_t - \bar{p} \end{pmatrix} + \begin{pmatrix} -\Theta \left(\frac{H(\phi^*)}{1-\mu}\right)^{\frac{1+\theta_2-2\theta_1}{\theta_1}} \\ \frac{\theta_2-\theta_1}{\theta_1} \left(\frac{H(\phi^*)}{1-\mu}\right)^{\frac{\theta_2-\theta_1}{\theta_1}} \end{pmatrix} \epsilon_{t+1},$$

where

$$\Theta = \frac{\Psi(\Lambda\delta)^{\frac{1+\theta_2-2\theta_1}{\theta_1}}}{e\theta_2} \left(\frac{(a_1-a_2)(1-(1-\theta_1)e\delta) + \alpha_1}{a_2 + e\delta(a_1-a_2)}\right).$$

The sunspot variable that affects the price of consumption goods, ϵ_{t+1}, has an impact on the gross product. Formally, we have Proposition 5.2 below.

Proposition 5.2. *Given Y_t and p_t, the sunspot variable, ϵ_{t+1}, negatively affects the gross product, Y_{t+1}, in the neighborhood of the steady state, i.e., $\partial Y_{t+1}/\partial \epsilon_{t+1} < 0$.*

Proof. The claim follows from (5.7) because $\Theta > 0$. □

The sunspot variable, ϵ_{t+1}, has a negative impact on Y_{t+1} in both cases in which $\theta_2 > \theta_1$ and $\theta_2 < \theta_1$. In the case in which $\theta_2 > \theta_1$, the consumption good sector is more labor intensive than the intermediate good sector from the social perspective. In this case, if a positive extrinsic shock ($\epsilon_{t+1} > 0$) increases p_{t+1} in (5.5), the wage rate increases and the capital price decreases (the Stolper-Samuelson property). The increase in the wage rate has a positive effect on the gross product in period $t+1$ as seen in the second term of the right-hand side of Eq. (5.6), whereas the decrease in the capital price has a negative effect on the gross product, as seen in the first term of the right-hand side of Eq. (5.6) because $a_2Y_t - w(p_t)$ is negative. The latter negative effect is always stronger than the former positive effect. On the other hand, in the case in which $\theta_2 < \theta_1$, the intermediate good sector is more labor-intensive than the consumption good sector from the social perspective. Therefore, if an extrinsic shock ($\epsilon_{t+1} > 0$) increases $1/p_{t+1}$ in (5.5), which is a relative price of the intermediate goods, the wage rate increases and the capital price decreases (again, the Stolper-Samuelson property). As in the case in which $\theta_2 > \theta_1$, the increase in the wage rate has a positive effect on the gross product, whereas the decrease in the capital price has a negative effect on the gross product. Again, the latter negative effect is always stronger than the former positive effect. In either case, a positive extrinsic shock decreases the gross product.

With Y_t and p_t given, we obtain the variance of Y_{t+1} from Eq. (5.7) as follows:

$$(5.8)\quad V(Y_{t+1}) := \Theta^2 \left(\frac{H(\phi^*)}{1-\mu}\right)^{\frac{2(1+\theta_2-2\theta_1)}{\theta_1}} \sigma^2.$$

$H(\phi^*)/(1-\mu)$ is an increasing function with respect to μ as proven in the proof of Proposition 3.6. Therefore, whether financial development amplifies or contracts sunspot fluctuations depends on the factor intensity from the social perspective.

Proposition 5.3. *Suppose that Assumption 1 holds. Then, the following hold:*

- $\partial V(Y_{t+1})/\partial\mu > 0$ *if* $\theta_2 > 2\theta_1 - 1$.
- $\partial V(Y_{t+1})/\partial\mu < 0$ *if* $\theta_2 < 2\theta_1 - 1$.

Proof. $H(\phi^*)/(1-\mu)$ is an increasing function with respect to μ. Therefore, the claims follow from Eq. (5.8). □

Fig. 1 shows the regions of θ_1 and θ_2 that represent whether the effect that the sunspot variable has on the gross product is amplified or contracted. As seen in this figure, when the labor intensity in the consumption good sector from the social perspective is very large, it is more likely that the effect of the sunspot variable is magnified, whereas when the labor intensity in the intermediate good sector from the social perspective is very large, it is more likely that the effect of the sunspot variable is contracted.

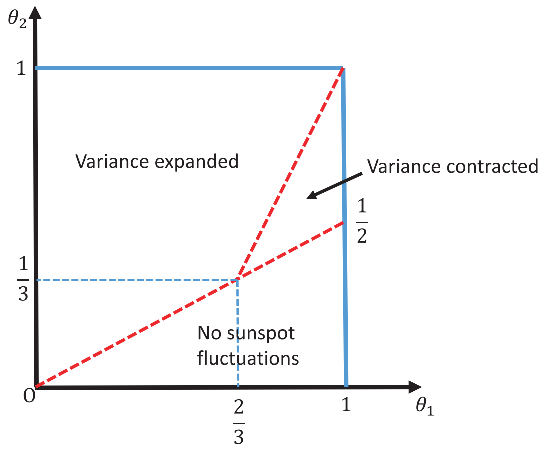

FIGURE 1. The effect of financial development on the gross product variance. Notes: In the variance-expanded region, the variance of the gross product is expanded as μ increases, whereas in the variance-contracted region, the variance of the gross product is contracted as μ increases.

The sunspot variable, ϵ_{t+1}, that appears in (5.5) perturbs both Eqs. (5.5) and (5.6), which are the dynamic equations with respect to p_t and Y_t, respectively. Our concern was whether financial development amplifies sunspot fluctuations with the perturbations being magnified. Proposition 5.3 implies that the answer to this question depends on the factor intensities from the social perspective. As far as ϵ_{t+1} is not so large, the dynamic course of (Y_t, p_t) does not diverge for any initial consumption price, p_0, and any gross product, Y_0 under Assumption 1. More precisely, (Y_t, p_t) converges to the stationary state in the neighborhood of the steady state, $(\bar{Y}, \bar{p})$, when the sunspot variable is realized in each period. Given this situation, one should check whether the dynamic behavior of (Y_t, p_t) around the steady state satisfies the transversality condition, $\lim_{\tau\to\infty} \delta^\tau E[a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})/p_{t+\tau}|\omega^{t-1}, \epsilon^t] = 0$, because the sunspot variable is extrapolated in the system. In the Appendix, we confirm that the transversality condition is actually satisfied.

6. Conclusion

Does financial development amplify sunspot fluctuations? To answer this question, we have investigated a two-sector dynamic general equilibrium model with sector-specific production externalities and financial frictions. As the financial sector is well developed, capital accumulation is promoted and the gross product in the steady state monotonically increases. On the other hand, if the consumption sector is less labor intensive from the private perspective but more labor intensive from the social perspective relative to the intermediate sector, it is more likely for indeterminacy of equilibria to occur. By introducing a sunspot variable in this situation, we demonstrate that financial development is more likely to amplify sunspot fluctuations when the labor intensity in the consumption good sector from the social perspective is very large, whereas financial development is more likely to contract sunspot fluctuations when the labor intensity in the intermediate good sector from the social perspective is very large.

One possible direction in which to extend from the current model is to investigate open economies with the basic setup remaining unchanged. Whether the results remain the same if a small open economy or a large open economy is assumed is not obvious at all because not only international trade but also financial trade across economies with financial frictions must be taken into account. This question is left for future research.

Appendix

Proof of Proposition 3.1. Inserting (2.5) into the financial market clearing condition (3.3) yields

$$\int_{\Omega^t\times\Xi_t} a_t(\omega^{t-1})dP^{t+1}(\omega^t) - \frac{\mu}{1-\mu}\int_{\Omega^t\times(\Omega\backslash\Xi_t)} a_t(\omega^{t-1})dP^{t+1}(\omega^t) = 0,$$

or equivalently,

(A.1)

$$\int_{\Xi_t}\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1})dP(\omega_t) - \frac{\mu}{1-\mu}\int_{\Omega\backslash\Xi_t}\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1})dP(\omega_t) = 0.$$

where $\Xi_t = \{\omega_t \in \Omega | \Phi_t(\omega_t) \le \phi_t\}$. The i.i.d. assumption rewrites Eq. (A.1) as

$$\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1})\int_0^{\phi_t} dG(\Phi) - \frac{\mu}{1-\mu}\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1})\int_{\phi_t}^h dG(\Phi) = 0,$$

from which we obtain

$$\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1})\left[G(\phi_t) - \frac{\mu}{1-\mu}(1-G(\phi_t))\right] = 0.$$

The last equation yields $G(\phi_t) = \mu$. $\square$

Proof of Lemma 3.2. It follows from $R_t(\omega_{t-1}) = \max\{r_t, (q(p_t)\Phi_{t-1}(\omega_{t-1}) - r_t\mu)/(1-\mu)\}$ that

$$\int_{\Omega^t} R_t(\omega_{t-1})a_{t-1}(\omega^{t-2})dP^t\omega^{t-1} = \int_{\Omega^t} \max\left\{r_t, \frac{q(p_t)\Phi_{t-1}(\omega_{t-1}) - r_t\mu}{1-\mu}\right\} \times a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}) =: I_t$$

We define $\Xi_{t-1} = \{\omega_{t-1} \in \Omega | \Phi_{t-1}(\omega_{t-1}) \le \phi_{t-1}\}$ as in the proof of Proposition 3.1. The use of $\phi_{t-1} = r_t/q(p_t)$ computes I_t as follows:

$$\begin{aligned} I_t &= \int_{\Omega^{t-1}\times\Xi_{t-1}} r_t a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}) \\ &+ \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \frac{q(p_t)\Phi_{t-1}(\omega_{t-1}) - r_t\mu}{1-\mu} a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}) \\ &= \int_{\Xi_{t-1}} \int_{\Omega^{t-1}} r_t a_{t-1}(\omega^{t-2})dP^{t-1}(\omega^{t-2})dP(\omega_{t-1}) \\ &- \int_{\Omega\backslash\Xi_{t-1}} \int_{\Omega^{t-1}} \frac{r_t\mu}{1-\mu} a_{t-1}(\omega^{t-2})dP^{t-1}(\omega^{t-2})dP(\omega_{t-1}) \\ &+ \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \frac{q(p_t)\Phi_{t-1}(\omega_{t-1})}{1-\mu} a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}). \end{aligned} \tag{B.1}$$

Because of the i.i.d. assumption, (B.1) is further computed as

$$\begin{aligned} I_t &= \int_0^{\phi_{t-1}} r_t dG(\Phi) \int_{\Omega^{t-1}} a_{t-1}(\omega^{t-2})dP^{t-1}(\omega^{t-2}) \\ &- \int_{\phi_{t-1}}^{h} \frac{r_t\mu}{1-\mu} dG(\Phi) \int_{\Omega^{t-1}} a_{t-1}(\omega^{t-2})dP^{t-1}(\omega^{t-2}) \\ &+ \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \frac{q(p_t)\Phi_{t-1}(\omega_{t-1})}{1-\mu} a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}) \\ &= \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \frac{q(p_t)\Phi_{t-1}(\omega_{t-1})}{1-\mu} a_{t-1}(\omega^{t-2})dP^t(\omega^{t-1}), \end{aligned} \tag{B.2}$$

where the second equality of (B.2) is obtained because $G(\phi_{t-1}) = \mu$. Capital producers invest $x_{t-1}(\omega^{t-1}) = a_{t-1}(\omega^{t-2})/(1-\mu)$, and then, (B.2) becomes

$$\begin{aligned} I_t &= \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \frac{q(p_t)\Phi_{t-1}(\omega_{t-1})}{1-\mu} (1-\mu)x_{t-1}(\omega^{t-1})dP^t\omega^{t-1} \\ &= q(p_t) \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \Phi_{t-1}(\omega_{t-1})x_{t-1}(\omega^{t-1})dP^t(\omega^{t-1}). \end{aligned} \tag{B.3}$$

Since $z_t = \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_{t-1})} \Phi_{t-1}(\omega_{t-1})x_{t-1}(\omega^{t-1})dP^t(\omega^{t-1})$, (B.3) becomes

$$I_t = q(p_t)z_t. \tag{6.1}$$

□

Proof of Lemma 3.3. Since $\int_{\Omega^t} \pi_t dP^t(\omega^{t-1}) = \Pi_t$, the use of Lemma 3.2 aggregates Eq. (2.6) across all agents as follows:

$$\begin{aligned}\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1}) &= \int_{\Omega^t}[R_t(\omega_{t-1})a_{t-1}(\omega^{t-2}) + w_t + \pi_t - p_t c_t(\omega^{t-1})]dP^t(\omega^{t-1}) \\ &= q(p_t)z_t + w_t + \Pi_t - p_t C_t. \qquad \text{(C.1)}\end{aligned}$$

From Eq. (2.8), we have $F^1(l_t^1, z_t^1) + p_t F^2(l_t^2, z_t^2) = q(p_t)z_t + w_t + \Pi_t$. Additionally, from the market clearing condition for consumption goods, it follows that $p_t F^2(l_t^2, z_t^2) = p_t C_t$. Then, Eq. (C.1) is transformed into

$$\int_{\Omega^t} a_t(\omega^{t-1})dP^t(\omega^{t-1}) = F^1(l_t^1, z_t^1). \qquad \text{(C.2)}$$

□

Proof of Proposition 3.4. Capital is produced by agents who draw an individual-specific productivity, $\Phi_t(\omega_t)$, greater than ϕ_t. Therefore, the i.i.d. assumption and Lemma 3.3 compute z_{t+1} as follows:

$$\begin{aligned}z_{t+1} &= \int_{\Omega^{t-1}\times(\Omega\backslash\Xi_t)} \Phi_t(\omega_t)x_t(\omega^t)dP^{t+1}(\omega^t) \\ &= \int_{\Omega\backslash\Xi_t}\int_{\Omega^{t-1}} \Phi_t(\omega_t)\frac{a_t(\omega^{t-1})}{1-\mu}dP^t(\omega^{t-1})dP(\omega_t) \\ &= \int_{\phi_t}^{h}\frac{\Phi_t(\omega_t)}{1-\mu}dG(\Phi)\int_{\Omega^{t-1}} a_t(\omega^{t-1})dP^t(\omega^{t-1}) \\ &= \frac{H(\phi^*)}{1-\mu}F^1(l_t^1, z_t^1),\end{aligned}$$

where $\Xi_t = \{\omega_t \in \Omega | \Phi_t(\omega_t) \le \phi_t\}$, and $H(\phi^*) = \int_{\phi^*}^{h} \Phi_t(\omega_t)dG(\Phi)$ because $\phi_t = \phi^*$ in equilibrium. □

Proof of Proposition 3.5. It follows from $\phi_t = r_{t+1}/q(p_{t+1})$ and $\phi_t = \phi^*$ that

$$\begin{aligned}E[R_{t+1}(\omega_t)|\omega^{t-1}] &= E\left[\max\left\{r_{t+1}, \frac{q(p_{t+1})\Phi_t(\omega_t) - r_{t+1}\mu}{1-\mu}\right\}\middle|\omega^{t-1}\right] \\ &= q(p_{t+1})E\left[\max\left\{\phi_t, \frac{\Phi_t(\omega_t) - \phi_t\mu}{1-\mu}\right\}\middle|\omega^{t-1}\right] \\ &= q(p_{t+1})\left[\int_0^{\phi^*}\phi^* dG(\Phi) + \int_{\phi^*}^{h}\frac{\Phi_t(\omega_t) - \phi_t\mu}{1-\mu}dG(\Phi)\right] \\ &= q(p_{t+1})\left[\phi^* G(\phi^*) - \frac{\phi^*\mu}{1-\mu}(1 - G(\phi^*)) + \frac{H(\phi^*)}{1-\mu}\right] \\ &= q(p_{t+1})\frac{H(\phi^*)}{1-\mu}. \qquad \text{(D.1)}\end{aligned}$$

Proposition 3.1 is used to derive the last equality. □

Proof of Proposition 3.6. From the inverse function theorem, it follows that

$$\frac{\partial}{\partial \mu}\left(\frac{H(\phi^*)}{1-\mu}\right) = \frac{-(1-\mu)\phi^* G'(\phi^*)(\partial\phi^*/\partial\mu) + H(\phi^*)}{(1-\mu)^2}$$
$$= \frac{\int_{\phi^*}^{h} \Phi_t(\omega_t) dG(\Phi) - \phi^*(1-G(\phi^*))}{(1-\mu)^2} > 0.$$

The claims follow from this equation, Eqs. (3.17), (3.18), and (3.19). □

Proof of Lemma 4.1. (i) Regarding κ_1, because $\delta \in (0,1)$ and $e \in (0,1)$, it follows that $e\delta a_1 + (1-e\delta)a_2 > 0 \Longleftrightarrow a_2 > e\delta(a_2 - a_1)$. Therefore, it holds that $a_2 - a_1 > 0 \Longleftrightarrow \kappa_1 = a_2/[e\delta(a_2-a_1)] > 1$. Obviously, it holds that $0 < e\delta < a_2/(a_1-a_2) \Longleftrightarrow \kappa_1 < -1$ and $0 < a_2/(a_1-a_2) < e\delta \Longleftrightarrow -1 < \kappa_1 < 0$. (ii) Regarding κ_2, $0 < \theta_1 < 2\theta_2 \Longleftrightarrow -1 < \kappa_2 < 1$ and $0 < 2\theta_2 < \theta_1 \Longleftrightarrow \kappa_2 < -1$. □

Confirmation of the transversality condition. The local dynamical system, (5.7), can be rewritten as

$$\begin{pmatrix} \tilde{Y}_{t+\tau} \\ \tilde{p}_{t+\tau} \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \begin{pmatrix} \tilde{Y}_{t+\tau-1} \\ \tilde{p}_{t+\tau-1} \end{pmatrix} + \begin{pmatrix} \tilde{B} \\ \tilde{D} \end{pmatrix} \epsilon_{t+1}, \tag{E.1}$$

where $\tilde{B} = -\Theta[H(\phi^*)/(1-\mu)]^{\frac{1+\theta_2-2\theta_1}{\theta_1}} + [H(\phi^*)/(1-\mu)]^{\frac{\theta_2-\theta_1}{\theta_1}}[(J_p(\bar{Y},\bar{p})(\theta_2-\theta_1)]/[\theta_1(\kappa_1-\kappa_2)]$, $\tilde{D} = [H(\phi^*)/(1-\mu)]^{\frac{\theta_2-\theta_1}{\theta_1}}[\theta_2-\theta_1]/[\theta_1(\kappa_1-\kappa_2)]$, $\tilde{Y}_t = Y_t - \bar{Y} + [J_p(\bar{Y},\bar{p})/(\kappa_1 - \kappa_2)](p_t - \bar{p})$, and $\tilde{p}_t = [1/(\kappa_1-\kappa_2)](p_t - \bar{p})$. Using (E.1), we obtain

$$\begin{aligned} Y_{t+\tau} &= \bar{Y} + \kappa_1^\tau(Y_t - \bar{Y}) + \frac{J_p(\bar{Y},\bar{p})(\kappa_1^\tau - \kappa_2^\tau)}{\kappa_1-\kappa_2}(p_t - \bar{p}) \\ &\quad - J_p(\bar{Y},\bar{p})\tilde{D}\sum_{s=1}^{\tau}\kappa_2^{\tau-s}\epsilon_{t+s} + \tilde{B}\sum_{s=1}^{\tau}\kappa_1^{\tau-s}\epsilon_{t+s} \end{aligned} \tag{E.2}$$

and

$$p_{t+\tau} = \bar{p} + \kappa_2^\tau(p_t - \bar{p}) + \frac{\theta_2-\theta_1}{\theta_1}\left(\frac{H(\phi^*)}{1-\mu}\right)^{\frac{\theta_2-\theta_1}{\theta_1}}\sum_{s=1}^{\tau}\kappa_2^{\tau-s}\epsilon_{t+s}. \tag{E.3}$$

From (2.13), (3.6), (3.8), (E.2), and (E.3), it follows that

$$\begin{aligned} &\frac{F^1(L^1_{t+\tau}, Z^1_{t+\tau})}{p_{t+\tau}} \\ &= -\frac{a_2}{a_1-a_2}\left[\frac{\bar{Y} + \kappa_1^\tau(Y_t-\bar{Y}) + \frac{\kappa_1^\tau-\kappa_2^\tau}{\kappa_1-\kappa_2}(p_t-\bar{p}) + \tilde{B}\sum_{s=1}^{\tau}\kappa_1^{\tau-s}\epsilon_{t+s}}{\bar{p} + \kappa_2^\tau(p_t-\bar{p}) + \tilde{D}(\kappa_1-\kappa_2)\sum_{s=1}^{\tau}\kappa_2^{\tau-s}\epsilon_{t+s}} - \frac{J_p(\bar{Y},\bar{p})}{\kappa_1-\kappa_2}\right] \\ &\quad + \frac{\Psi}{e}\left[\bar{p} + \kappa_2^\tau(p_t-\bar{p}) + \tilde{D}(\kappa_1-\kappa_2)\sum_{s=1}^{\tau}\kappa_2^{\tau-s}\epsilon_{t+s}\right]^{\frac{1-\theta_2}{\theta_2-\theta_1}}. \end{aligned} \tag{E.4}$$

In (E.4), note that $|\kappa_1| < 1$ and $|\kappa_2| < 1$, and the support for ϵ_{t+1} is a closed interval. Therefore, $F^1(L^1_{t+\tau}, Z^1_{t+\tau})/p_{t+\tau}$ is bounded both below and above as $\tau \to \infty$, and thus,

$$\lim_{\tau\to\infty} \delta^\tau E\left[\frac{F^1(L^1_{t+\tau}, Z^1_{t+\tau})}{p_{t+\tau}}\Big|\epsilon^t\right] = 0 \tag{E.5}$$

By applying Lemma 3.3 to (E.5), it follows that

$$\delta^\tau E\left[\frac{F^1(L^1_{t+\tau}, Z^1_{t+\tau})}{p_{t+\tau}}\Big|\epsilon^t\right] = \delta^\tau E\left[\frac{\int_{\Omega^{t+\tau}} a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})dP^{t+\tau}(\omega^{t+\tau-1})}{p_{t+\tau}}\Big|\epsilon^t\right]. \tag{E.6}$$

To compute the right-hand side of (E.6), we define $\omega^{t+\tau-1,t-1} = \{\omega_t, ..., \omega_{t+\tau-1}\} \in \Omega^{t+\tau} \setminus \Omega^t$. Then, the right-hand side of (E.6) can be computed as

$$\begin{aligned}
&\delta^\tau E\left[\frac{\int_{\Omega^{t+\tau}} a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})dP^{t+\tau}(\omega^{t+\tau-1})}{p_{t+\tau}}\Big|\epsilon^t\right]\\
&= \delta^\tau E\left[\frac{\int_{\Omega^t}\int_{\Omega^{t+\tau}\setminus\Omega^t} a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})dP^{\tau}(\omega^{t+\tau-1,t-1})dP^t(\omega^{t-1})}{p_{t+\tau}}\Big|\epsilon^t\right]\\
&= \delta^\tau \int_{\Omega^t} E\left[\frac{\int_{\Omega^{t+\tau}\setminus\Omega^t} a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})dP^{\tau}(\omega^{t+\tau-1,t-1})}{p_{t+\tau}}\Big|\epsilon^t\right] dP^t(\omega^{t-1})\\
&= \delta^\tau \int_{\Omega^t} E\left[\frac{a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})}{p_{t+\tau}}\Big|\omega^{t-1}, \epsilon^t\right] dP^t(\omega^{t-1}). \qquad \text{(E.7)}
\end{aligned}$$

From (E.5), (E.6) and (E.7), we obtain

$$\lim_{\tau\to\infty} \delta^\tau \int_{\Omega^t} E\left[\frac{a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})}{p_{t+\tau}}\Big|\omega^{t-1}, \epsilon^t\right] dP^t(\omega^{t-1}) = 0. \tag{E.8}$$

The financial constraint implies that $a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau}) = b_{t+\tau}(\omega^{t+\tau}, \epsilon^{t+\tau}) + k_{t+\tau}(\omega^{t+\tau}, \epsilon^{t+\tau}) > b_{t+\tau}(\omega^{t+\tau}, \epsilon^{t+\tau}) + \mu k_{t+\tau}(\omega^{t+\tau}, \epsilon^{t+\tau}) \geq 0$. Therefore, it follows from (E.8) that

$$\lim_{\tau\to\infty} \delta^\tau E\left[\frac{a_{t+\tau}(\omega^{t+\tau-1}, \epsilon^{t+\tau})}{p_{t+\tau}}\Big|\omega^{t-1}, \epsilon^t\right] = 0.$$

This is the transversality condition for each agent.

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References

[1] P. Aghion and A. Banerjee, *Volatility and Growth*, Oxford University Press, New York, 2005.
[2] P. Aghion, A. Banerjee and T. Piketty, *Dualism and macroeconomic volatility*, Quarterly Journal of Economics **114** (1999), 1359–1397.
[3] P. Aghion, P. Howitt and D. Mayer-Foulkes, *The effect of financial development on convergence: theory and evidence*, Quarterly Journal of Economics **120** (2005), 173–222.
[4] C. Azariadis, *Self-fulfilling prophecies*, Journal of Economic Theory **25** (1981), 380–396.

[5] J. Benhabib and R. E. Farmer, *Indeterminacy and increasing returns*, Journal of Economic Theory **63** (1994), 19–41.

[6] J. Benhabib and R. E. Farmer, *Indeterminacy and sector-specific externalities*, Journal of Monetary Economics **37** (1996), 421–443.

[7] J. Benhabib, Q. Meng and K. Nishimura, *Indeterminacy under constant returns to scale in multisector economies*, Econometrica **68** (2000), 1541–1548.

[8] J. Benhabib and K. Nishimura, *Indeterminacy and sunspots with constant returns*, Journal of Economic Theory **81** (1998), 58–96.

[9] M. Boldrin and A. Rustichini, *Growth and indeterminacy in dynamic models with externalities*, Econometrica **62** (1994), 323–342.

[10] D. Cass and K. Shell, *Do sunspots matter?*, Journal of Political Economy **91** (1983), 193–227.

[11] F. Dufourt, K. Nishimura and A. Venditti, *Indeterminacy and sunspots in two-sector RBC models with generalized no-income-effect preferences*, Journal of Economic Theory **157** (2015), 1056–1080.

[12] T. Kunieda and A. Shibata, *Asset bubbles, economic growth, and a self-fulfilling financial crisis*, Journal of Monetary Economics **82** (2016), 70–84.

[13] K. Nishimura and A. Venditti, *Indeterminacy and the role of factor substitutability*, Macroeconomic Dynamics **8** (2004), 436–465.

[14] K. Nishimura and A. Venditt, *Indeterminacy in discrete-time infinite-horizon models with non-linear utility and endogenous labor*, Journal of Mathematical Economics **43** (2007), 446–476.

[15] K. Shell, *Monnaie et allocation intertemporelle*, Mimeograph, CNRS Seminaire Roy-Malinvaud (Paris), November 21, 1977.

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