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COINCIDENCE THEOREM AND THE INNER CORE

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Abstract. We provide two coincidence theorems that are useful mathematical tools for proving the nonemptiness of the inner core. The inner core is a refinement of the core of non-transferable utility (NTU) games. Our first coincidence theorem is a synthesis of Brouwer's fixed point theorem and Debreu and Schmeidler's separation theorem for convex sets. Our second coincidence theorem is a modification of the first one in order for two correspondences to have a nonempty intersection at a strictly positive vector. Inoue's theorem on the nonemptiness of the inner core follows from our first coincidence theorem and an assumption on the efficient surface of payoff vectors for the grand coalition. Qin's theorem on the nonemptiness of the inner core is a direct consequence of our second coincidence theorem. Our coincidence theorems are suitable for proving the nonemptiness of the inner core in the sense that one of the assumptions in our first coincidence theorem is equivalence to the cardinal balancedness of an NTU game when two correspondences are defined properly.

1. Introduction

We provide mathematical tools for proving the nonemptiness of the inner core, a solution concept of a non-transferable utility (NTU) game. An NTU game consists of a finite set N of players and the set $V(S)$ of coalition S's feasible payoff vectors for every coalition $S \subseteq N$. The set $V(S)$ denotes the set of payoff vectors that can be achieved if all the members in S are in cooperation. One interpretation of $V(S)$ is that it is the set of all utility vectors when each player in S consumes his portion of the products where all the players in S worked together and produced. There exists no "money" that can transfer utilities among players, or, if money exists, players' utilities are not linear with respect to money. Thus, the summation of players' utilities does not make sense in NTU games.

A representative solution concept of an NTU game is the core. The core is the set of payoff vectors x such that x is feasible for the grand coalition N and there exists no coalition S that can achieve greater payoffs of all players in S than x. The inner core is a refinement of the core. A payoff vector x is in the inner core if it is feasible for the grand coalition and there exist strictly positive transfer rates of utilities among players such that any coalition S cannot achieve a greater weighted sum of utilities over players in S than that of x under the transfer rates. Utilities cannot be transferred in NTU games and, thus, an inner core payoff vector requires a strong stability in the sense that, even if utilities can be transferred under *fictitious*

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transfer rates, any coalition cannot achieve greater utilities of all members in the coalition than the inner core payoff vector. The fictitious transfer rates can be different among inner core payoff vectors.

The inner core is of significance in the relations to solutions of economic models with richer structures than NTU games.[1] An exchange economy or a production economy naturally generates an NTU game. The inner core of the NTU game generated from an economy is relevant to Walrasian equilibria of the original economy (see Billera [3], Qin [17], Inoue [11], and Brangewitz and Gamp [5]). Also, the inner core is relevant to the strictly inhibitive set. When a payoff vector can be improved upon by multiple coalitions, it is not clear which coalition actually improves upon the payoff vector. Myerson [15, Section 9.8] considered the situation where an improving coalition is chosen at random. The strictly inhibitive set is the set of payoff vectors that cannot be improved upon by any randomized plan. Qin [16] proved that the inner core coincides with the strictly inhibitive set in some classes of NTU games.

We provide two coincidence theorems. Our first coincidence theorem (Theorem 3.1) is a synthesis of Brouwer's fixed point theorem and Debreu and Schmeidler's separation theorem for convex sets. Our second coincidence theorem (Theorem 3.6) is a modification of the first one. If two correspondences are defined properly from a given NTU game so that those domains are the space of transfer rates of utilities, then payoff vectors in the intersection of the two correspondences are inner core payoff vectors. Hence, our coincidence theorems are useful for proving the nonemptiness of the inner core. Furthermore, our coincidence theorems are suitable in the sense that one of the assumptions in our first coincidence theorem is equivalent to the cardinal balancedness of the NTU game. The cardinal balancedness is essential for the inner core to be nonempty and it is sufficient for the core to be nonempty.

By applying our first coincidence theorem, we can prove Inoue's [12] theorem. Inoue [12] proved that, if an NTU game is cardinally balanced and if every normal vector at every individually rational and efficient payoff vector is strictly positive, then the inner core is nonempty. An NTU game generated by an exchange economy where every consumer has a continuous, concave, and strongly monotone utility function satisfies Inoue's sufficient condition for the inner core to be nonempty. In the first step of our proof of Inoue's [12] theorem, as a direct consequence of our first coincidence theorem, we can obtain a payoff vector and *nonnegative* fictitious transfer rates of utilities such that the payoff vector is stable under the transfer rates. Although transfer rates must be strictly positive at inner core payoff vectors, the transfer rates obtained in the first step need not be strictly positive. In the second step, by using the assumption of Inoue's theorem on normal vectors, we can prove that the transfer rates are actually strictly positive.[2]

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[1]Regarding economic models generating NTU games, see Inoue [12] for more details on the following literature.

[2]Inoue [12] proved the theorem by applying Qin's [18] theorem. Inoue's theorem can be proven also by the method of Aubin [2] (see Inoue [13, Appendix]).

In our second coincidence theorem, the domains of two correspondences are the set of strictly positive transfer rates of utilities. Thus, our second coincidence theorem guarantees the existence of a strictly positive vector of transfer rates where two correspondences have a nonempty intersection. Qin's [18] theorem on the nonemptiness of the inner core is a direct consequence of our second coincidence theorem.

The rest of the present paper is as follows. In Section 2, we give the precise definitions of NTU games and the inner core, and we introduce two correspondences whose intersection is a subset of the inner core. Then, we discuss the properties of these correspondences. In Section 3, we prove two our coincidence theorems. In Section 4, we first review a characterization of the efficient surface with only strictly positive normal vectors. Then, we prove that Inoue's [12] theorem follows from our first coincidence theorem and Qin's [18] theorem follows from our second coincidence theorem.

2. NTU games and the inner core

We begin with some notation. Let $N = \{1, \dots, n\}$ with $n \geq 2$ be the set of n players. Let $\mathbb{R}^N$ be the n-dimensional Euclidean space of vectors x with coordinates x_i indexed by $i \in N$. For $x, y \in \mathbb{R}^N$, let $x \cdot y = \sum_{i \in N} x_i y_i$. For $x, y \in \mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for every $i \in N$; $x \gg y$ if $x_i > y_i$ for every $i \in N$. The symbol 0 denotes the origin in $\mathbb{R}^N$ as well as the real number zero. Let $\mathbb{R}^N_{++} = \{x \in \mathbb{R}^N \mid x \gg 0\}$. For a nonempty subset S of N, let $\mathbb{R}^S = \{x \in \mathbb{R}^N \mid x_i = 0 \text{ for every } i \in N \setminus S\}$, let $\mathbb{R}^S_+ = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for every } i \in S\}$, and let $e^S \in \mathbb{R}^N$ be the characteristic vector of S, i.e., $e^S_i = 1$ if $i \in S$ and 0 otherwise. For $x \in \mathbb{R}^N$, x^S denotes the projection of x to $\mathbb{R}^S$. Let $\Delta = \{p \in \mathbb{R}^N_+ \mid \sum_{i \in N} p_i = 1\}$ and let $\Delta^\circ = \{p \in \Delta \mid p \gg 0\}$. For $A \subseteq \mathbb{R}^N$, $\mathrm{cl}(A)$ and $\mathrm{co}(A)$ denote the closure and the convex hull of set A, respectively.

Let $\mathcal{N}$ be the set of all *coalitions*, i.e., $\mathcal{N} = \{S \subseteq N \mid S \neq \emptyset\}$. A *non-transferable utility game* (NTU game, for short) with n players is a correspondence $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ such that, for every $S \in \mathcal{N}$, $V(S)$ is a nonempty subset of $\mathbb{R}^S$ with $V(S) - \mathbb{R}^S_+ = V(S)$. An NTU game is *compactly generated* if, for every $S \in \mathcal{N}$, there exists a nonempty compact subset C_S of $\mathbb{R}^S$ with $V(S) = C_S - \mathbb{R}^S_+$. In the present paper, we consider only compactly generated NTU games V with $V(N)$ convex.

The core is the set of payoff vectors which are feasible for the grand coalition N and which cannot be improved upon by any coalition. By adopting a different notion of improvement by a coalition, we can define the inner core.

Definition 2.1. (1) The *core* $C(V)$ of NTU game V is the set of payoff vectors $u \in \mathbb{R}^N$ such that $u \in V(N)$ and there exists no $S \in \mathcal{N}$ and $u' \in V(S)$ with $u'_i > u_i$ for every $i \in S$.

(2) The *inner core* $IC(V)$ of NTU game V is the set of payoff vectors $u \in \mathbb{R}^N$ such that $u \in V(N)$ and there exists $\lambda \in \mathbb{R}^N_{++}$ such that, for every $S \in \mathcal{N}$ and every $u' \in V(S)$, $\lambda^S \cdot u \geq \lambda^S \cdot u'$ holds.

By definition, $IC(V) \subseteq C(V)$ holds. The vector $\lambda \in \mathbb{R}^N_{++}$ in the definition of the inner core represents fictitious transfer rates of utilities among players. Note that we can restrict the space of fictitious transfer rates to Δ°.

Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. For every $S \in \mathcal{N}$, let C_S be a compact subset of $\mathbb{R}^S$ with $V(S) = C_S - \mathbb{R}^S_+$ and let C_N be also convex. For every $\lambda \in \Delta$ and every $S \in \mathcal{N}$, define

$$v_\lambda(S) = \max\{\lambda \cdot u \mid u \in V(S)\} = \max\{\lambda \cdot u \mid u \in C_S\}.$$

Note that, by Berge's maximum theorem, $\Delta \ni \lambda \mapsto v_\lambda(S) \in \mathbb{R}$ is continuous. For every $i \in N$, define

$$b_i = \max\{u_i \in \mathbb{R} \mid u \in V(\{i\})\},$$

the utility level that player i can achieve by himself.

Define correspondences $F : \Delta \twoheadrightarrow \mathbb{R}^N$ and $G : \Delta \twoheadrightarrow \mathbb{R}^N$ by

$$F(\lambda) = \left\{x \in \mathbb{R}^N \,\middle|\, x \geq b \text{ and } \lambda^S \cdot x \geq v_\lambda(S) \text{ for every } S \in \mathcal{N} \setminus \{N\}\right\}$$

and

$$G(\lambda) = \{y \in C_N \mid \lambda \cdot y = v_\lambda(N)\}.$$

Note that, for any $\lambda \in \Delta^\circ$, condition "$x \geq b$" in $F(\lambda)$ is redundant and condition "$y \in C_N$" in $G(\lambda)$ can be replaced by "$y \in V(N)$." These conditions are put for correspondences F and G to be uniformly bounded from below and compact-valued, respectively, on the whole domain Δ.

Note also that $IC(V) = \bigcup_{\lambda \in \Delta^\circ} (F(\lambda) \cap G(\lambda))$. Then, for the nonemptiness of the inner core, it suffices to show that two correspondences F and G have a nonempty intersection at strictly positive vector λ. Hence, coincidence theorem is a key tool.

The following properties of correspondences F and G are straightforward.

Proposition 2.2. *Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. Two correspondences $F : \Delta \twoheadrightarrow \mathbb{R}^N$ and $G : \Delta \twoheadrightarrow \mathbb{R}^N$ are defined as above. Then,*

(i) *F is nonempty-, convex-, uniformly-bounded-from-below-valued, and has a closed graph. Furthermore, for every $\lambda \in \Delta$, $F(\lambda) + \mathbb{R}^N_+ = F(\lambda)$.*

(ii) *G is nonempty-, compact-, convex-valued and upper hemi-continuous.*

Since $F(\lambda)$ is unbounded for every $\lambda \in \Delta$, the closed graph property of F does not imply the upper hemi-continuity. Actually, F does not satisfy even the upper demi-continuity. A correspondence $\varphi : \Delta \twoheadrightarrow \mathbb{R}^N$ is *upper demi-continuous* if for every $\lambda^0 \in \Delta$ and every open half-space $H = \{x \in \mathbb{R}^N \mid q \cdot x > c\}$ containing $\varphi(\lambda^0)$, there exists $U \subseteq \Delta$ such that U is open in Δ, $\lambda^0 \in U$, and $\varphi(\lambda) \subseteq H$ for every $\lambda \in U$ (see Fan [9, p.106]). Thus, $\varphi : \Delta \twoheadrightarrow \mathbb{R}^N$ is upper demi-continuous if and only if, for every $q \in \mathbb{R}^N \setminus \{0\}$ and every $c \in \mathbb{R}$, the set

$$\left\{\lambda \in \Delta \,\middle|\, \varphi(\lambda) \subseteq \{x \in \mathbb{R}^N \mid q \cdot x > c\}\right\}$$

is open in Δ. Clearly, the upper hemi-continuity implies the upper demi-continuity. In addition, if $\varphi : \Delta \twoheadrightarrow \mathbb{R}^N$ is of the form $\varphi(\lambda) = \varphi_1(\lambda) + \mathbb{R}^N_+$ for an upper demi-continuous correspondence φ_1, then φ also is upper demi-continuous.

The example below illustrates that F need not be upper demi-continuous.

Example 2.3. Let $N = \{1, 2, 3\}$. Let $C_{\{i\}} = \{(0, 0, 0)\}$ for every $i \in N$, $C_{\{1,2\}} = \{(1, 0, 0)\}$, and $C_{\{1,3\}} = C_{\{2,3\}} = \{(0, 0, 0)\}$. Since correspondence F is independent of C_N, we do not specify C_N. Then, we have

$$F(\lambda) = \left\{x \in \mathbb{R}^N \mid x \geq 0,\ \lambda_1 x_1 + \lambda_2 x_2 \geq \lambda_1\right\}$$

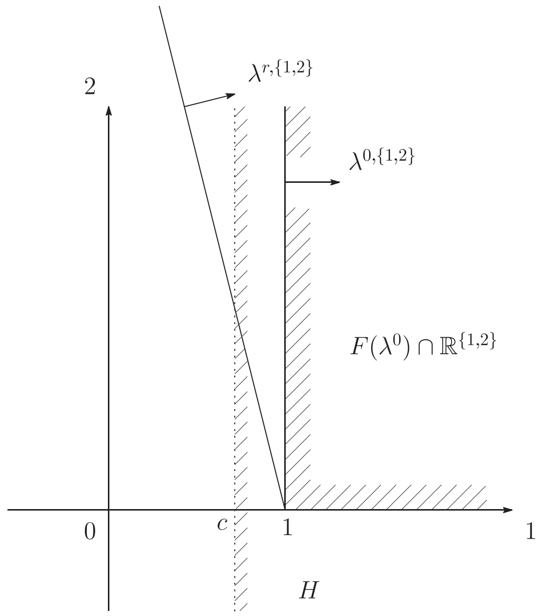

FIGURE 1. F is not upper demi-continuous

Let $\lambda^0 = (1, 0, 0)$ and $\lambda^r = (1 - 1/r, 1/r, 0)$ for every $r \in \mathbb{N}$. Let $H = \{x \in \mathbb{R}^N \mid \lambda^0 \cdot x > c\} = \{x \in \mathbb{R}^N \mid x_1 > c\}$ with $0 < c < 1$. Then, $F(\lambda^0) = \{x \in \mathbb{R}^N \mid x \geq 0,\ x_1 \geq 1\} \subseteq H$. Since $F(\lambda^r) \not\subseteq H$ for every $r \in \mathbb{N}$ and $\lambda^r \to \lambda^0$, F is not upper demi-continuous (see Figure 1).

In Example 2.3, the vector $\lambda^0 = (1, 0, 0)$ normal to the hyperplane of the open half-space H is not strictly positive. Note that, when we fix the inequality sign in the half-space $\{x \in \mathbb{R}^N \mid q{\cdot}x > c\}$, from the property that $F(\lambda)+\mathbb{R}^N_+ = F(\lambda)$, the normal vector q must be nonnegative for the set $\{\lambda \in \Delta \mid F(\lambda) \subseteq \{x \in \mathbb{R}^N \mid q \cdot x > c\}\}$ to be nonempty. The following proposition states that F has the same property as the upper demi-continuity if we restrict open half-spaces to those generated by strictly positive normal vectors.

Proposition 2.4. *Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game. Correspondence $F : \Delta \twoheadrightarrow \mathbb{R}^N$ is defined as above. Then, for every $q \in \mathbb{R}^N_{++}$ and every $c \in \mathbb{R}$, the set*

$$\left\{\lambda \in \Delta \,\middle|\, F(\lambda) \subseteq \left\{x \in \mathbb{R}^N \mid q \cdot x > c\right\}\right\}$$

is open in Δ.

Proof. Suppose, to the contrary, that there exist $q \in \mathbb{R}^N_{++}$ and $c \in \mathbb{R}$ such that the above set is not open in Δ. Then, there exist $\lambda^0 \in \Delta$ with $F(\lambda^0) \subseteq \{x \in \mathbb{R}^N \mid q{\cdot}x > c\}$ and a sequence $(\lambda^r)_r$ in Δ such that $\lambda^r \to \lambda^0$ and $F(\lambda^r) \not\subseteq \{x \in \mathbb{R}^N \mid q{\cdot}x > c\}$ for every $r \in \mathbb{N}$. Therefore, for every $r \in \mathbb{N}$, there exists $x^r \in F(\lambda^r)$ with $q{\cdot}x^r \leq c$. Since $q \in \mathbb{R}^N_{++}$ and $x^r \geq b$ for every $r \in \mathbb{N}$, the sequence $(x^r)_r$ is bounded. By passing to a subsequence if necessary, we may assume that $x^r \to x^0$. Then, $q \cdot x^0 \leq c$. Since F

has a closed graph by Proposition 2.2, we have

$$x^0 \in F(\lambda^0) \subseteq \left\{x \in \mathbb{R}^N \mid q \cdot x > c\right\},$$

a contradiction. This completes the proof. $\square$

One of the key assumptions of the theorems on the nonemptiness of the inner core is the cardinal balancedness. Let Γ be the set of *balancing vectors of weights*, i.e.,

$$\Gamma = \left\{ \gamma = (\gamma_S)_{S \in \mathcal{N}} \;\middle|\; \gamma_S \geq 0 \text{ for every } S \in \mathcal{N} \text{ and } \sum_{S \in \mathcal{N}} \gamma_S \, e^S = e^N \right\}.$$

An NTU game V is *cardinally balanced* if, for every $\gamma \in \Gamma$,

$$\sum_{S \in \mathcal{N}} \gamma_S \, V(S) \subseteq V(N).$$

This notion of balancedness is stronger than the balancedness due to Scarf [19]. Scarf's ordinal balancedness is sufficient for the nonemptiness of the core. When an NTU game is generated by an exchange economy, Scarf's ordinal balancedness holds if consumers have quasi-concave utility functions, whereas the cardinal balancedness holds if consumers have concave utility functions.

The following proposition gives conditions equivalent to the cardinal balancedness.

Proposition 2.5. *Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. Then, the following conditions are equivalent:*

(i) *V is cardinally balanced.*

(ii) *For every $\gamma \in \Gamma$ with $\gamma_N = 0$, $\sum_{S \in \mathcal{N}} \gamma_S \, V(S) \subseteq V(N)$.*

(iii) *For every $\lambda \in \Delta^\circ$ and every $\gamma \in \Gamma$ with $\gamma_N = 0$, $\sum_{S \in \mathcal{N}} \gamma_S \, v_\lambda(S) \leq v_\lambda(N)$.*

(iv) *For every $\lambda \in \Delta^\circ$,*

$$\min \left\{ \lambda \cdot x \;\middle|\; \lambda^S \cdot x \geq v_\lambda(S) \text{ for every } S \in \mathcal{N} \setminus \{N\} \right\} \leq v_\lambda(N).$$

(v) *For every $\lambda \in \Delta^\circ$, there exist $x \in F(\lambda)$ and $y \in G(\lambda)$ such that $\lambda \cdot x \leq \lambda \cdot y$.*

The equivalence (i) $\Leftrightarrow$ (ii) can be easily shown. The equivalence (ii) $\Leftrightarrow$ (iii) is due to Shapley (see Qin [18, Proposition 1]). The equivalence (iii) $\Leftrightarrow$ (iv) is essentially the same as Bondareva-Shapley theorem (Bondareva [4], Shapley [20]) and, therefore, it follows from the duality theorem of linear programming (see Qin [18, p.437]). The equivalence (iv) $\Leftrightarrow$ (v) follows from the definition of correspondences F and G.

3. Coincidence theorems

We give two coincidence theorems that are variants of Fan's coincidence theorem [9, Theorem 5]. Fan's coincidence theorem can be regarded as a synthesis of Kakutani's fixed point theorem [14] and the standard separation theorem for convex sets. Our first coincidence theorem is a synthesis of Brouwer's fixed point theorem [6] and a stronger separation theorem due to Debreu and Schmeidler [8]. Our second coincidence theorem is a mathematical theorem behind Qin's [18] proof of the nonemptiness of the inner core. Since coincidence theorems per se are of interest, we give more general forms than those needed for proving the nonemptiness of the inner core.

3.1. Coincidence theorem I.

Theorem 3.1. *Let* $\Delta = \{p \in \mathbb{R}^N_+ \mid \sum_{i \in N} p_i = 1\}$. *Let* $\varphi : \Delta \twoheadrightarrow \mathbb{R}^N$ *and* $\psi : \Delta \twoheadrightarrow \mathbb{R}^N$ *be correspondences satisfying the following conditions.*

(i) *φ is nonempty-, closed-, convex-valued, and, for every $p \in \Delta$, $\varphi(p) + \mathbb{R}^N_+ = \varphi(p)$ and $\varphi(p)$ contains no straight line;*[3]

(ii) *ψ is nonempty-, compact-, and convex-valued;*

(iii) *for every $q \in \mathbb{R}^N_{++}$ and every $c \in \mathbb{R}$, the sets*

$$\{p \in \Delta \mid \varphi(p) \subseteq \{x \in \mathbb{R}^N \mid q \cdot x > c\}\}$$

and

$$\{p \in \Delta \mid \psi(p) \subseteq \{y \in \mathbb{R}^N \mid q \cdot y < c\}\}$$

are both open in Δ; and

(iv) *for every $p \in \Delta^\circ$, there exist $x \in \varphi(p)$ and $y \in \psi(p)$ such that $p \cdot x \leq p \cdot y$.*

Then, there exists $p^* \in \Delta$ with $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$.

It is worth emphasizing that, by Proposition 2.5, condition (iv) is equivalent to the cardinal balancedness when two correspondences are F and G defined in Section 2. The proof of Theorem 3.1 will be given in the next subsection. For the purposes of comparison, we give Fan's coincidence theorem [9, Theorem 5] in the context of a Euclidean space.

Fan's coincidence theorem. *Let K be a nonempty, compact, convex subset of $\mathbb{R}^N$. Let $\varphi : K \twoheadrightarrow \mathbb{R}^N$ and $\psi : K \twoheadrightarrow \mathbb{R}^N$ be correspondences satisfying the following conditions.*

(a) *φ and ψ are nonempty-, closed-, convex-valued, and, for every $p \in K$, $\varphi(p)$ or $\psi(p)$ is compact;*

(b) *φ and ψ are upper demi-continuous; and*

(c) *for every $q \in \mathbb{R}^N$ and every $p \in K$ with $q \cdot p = \max_{p' \in K} q \cdot p'$, there exist $x \in \varphi(p)$ and $y \in \psi(p)$ such that $q \cdot x \leq q \cdot y$.*

Then, there exists $p^* \in K$ with $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$.

The domain of correspondences in our coincidence theorem is more specific than that of Fan's. In addition, we require that one correspondence ψ be compact-valued and the other correspondence be of the form $\varphi(p) + \mathbb{R}^N_+ = \varphi(p)$. Our continuity condition (iii) is weaker than Fan's continuity condition (b). Our condition (iv) is related to Fan's condition (c), because for every $p \in \Delta$ and every $p' \in \Delta$, $p \cdot (p/\|p\|) \geq p \cdot (p'/\|p'\|)$, where $\|\cdot\|$ is the Euclidean norm. Namely, every $p \in \Delta$ is normal to $\{p'/\|p'\| \mid p' \in \Delta\}$, which is homeomorphic to Δ, at $p/\|p\|$.[4]

It is known that Kakutani's fixed point theorem [14] can be proven by a simple application of Fan's coincidence theorem (see, e.g., Ichiishi [10, p.52]). By a simple

[3] A set $A \subseteq \mathbb{R}^N$ contains no straight line if, for every $z \in \mathbb{R}^N \setminus \{0\}$ and every $y \in \mathbb{R}^N$, there exists $\alpha \in \mathbb{R}$ with $y + \alpha z \notin A$.

[4] For a nonempty subset A of $\mathbb{R}^N$ and $x \in A$, $p \in \mathbb{R}^N$ is *normal* to A at x if $p \cdot x \geq p \cdot y$ for every $y \in A$.

application of our coincidence theorem, Brouwer's fixed point theorem [6] can be proven.

Proposition 3.2. *Theorem* 3.1 *implies Brouwer's fixed point theorem: every continuous function on* Δ *has a fixed point.*

Proof. Let $f : \Delta \to \Delta$ be a continuous function. Define $\varphi : \Delta \twoheadrightarrow \mathbb{R}^N$ and $\psi : \Delta \twoheadrightarrow \mathbb{R}^N$ by

$$\varphi(p) = \left\{ \frac{f(p)}{\|f(p)\|} \right\} + \mathbb{R}^N_+ \quad \text{and} \quad \psi(p) = \left\{ \frac{p}{\|p\|} \right\}.$$

It is clear that φ and ψ satisfy conditions (i)-(iii) of Theorem 3.1. Since, for every $p \in \Delta^\circ$ and every $x \in \mathbb{R}^N \setminus \{0\}$, $p \cdot (x/\|x\|) \le p \cdot (p/\|p\|)$ holds, condition (iv) is satisfied. Then, by Theorem 3.1, there exists $p^* \in \Delta$ such that $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$ and, therefore,

$$\frac{p^*}{\|p^*\|} \in \left\{ \frac{f(p^*)}{\|f(p^*)\|} \right\} + \mathbb{R}^N_+.$$

Since $f(p^*)/\|f(p^*)\|$ is a unique point in $\varphi(p^*)$ with Euclidean norm one, we have $p^*/\|p^*\| = f(p^*)/\|f(p^*)\|$. Since $p^*, f(p^*) \in \Delta$, we have

$$\frac{1}{\|p^*\|} = \sum_{i \in N} \frac{p^*_i}{\|p^*\|} = \sum_{i \in N} \frac{f_i(p^*)}{\|f(p^*)\|} = \frac{1}{\|f(p^*)\|}.$$

Thus, we have $p^* = f(p^*)$. $\square$

3.2. **Proof of Theorem 3.1.** The basic idea of the proof is the same as Fan's [9, Theorems 3 and 5]. A difference is that we rely on a stronger separation theorem for convex sets. We start with some lemmas. The first lemma is due to Debreu and Schmeidler [8, Corollary 2]. A straight line in $\mathbb{R}^N$ is a set of the form $\{y + \alpha z \mid \alpha \in \mathbb{R}\}$ for some $y \in \mathbb{R}^N$ and $z \in \mathbb{R}^N \setminus \{0\}$.

Lemma 3.3 (Debreu and Schmeidler)**.** *Let* X *be a nonempty, closed, convex subset of* $\mathbb{R}^N$ *containing no straight line. Let* $z \in \mathbb{R}^N \setminus X$. *Then, there exists a nonempty open set of normal vectors strictly separating* z *and* X, *i.e., there exists a nonempty open subset* U *of* $\mathbb{R}^N$ *such that, for every* $p \in U$,

$$p \cdot z < \inf_{x \in X} p \cdot x.$$

Lemma 3.4. *Let* X *be a nonempty, closed, convex subset of* $\mathbb{R}^N$ *containing no straight line. Let* Y *be a nonempty, bounded subset of* $\mathbb{R}^N$. *Then,* $X - Y = \{x - y \mid x \in X,\ y \in Y\}$ *contains no straight line.*

Proof. By translating X if necessary, we may assume that $0 \in X$. Since X is convex and contains no straight line, for every $z \in \mathbb{R}^N \setminus \{0\}$, there exists $\alpha_0 \in \mathbb{R}$ such that either [$\alpha \ge \alpha_0$ implies $\alpha z \notin X$] or [$\alpha \le \alpha_0$ implies $\alpha z \notin X$] holds.

Claim 1. Let $z \in \mathbb{R}^N \setminus \{0\}$. Suppose that $\alpha \ge \alpha_0$ implies $\alpha z \notin X$. Then,

$$\operatorname{dist}(\alpha z, X) = \inf\{\|\alpha z - x\| \mid x \in X\} \to \infty \quad \text{as} \quad \alpha \to \infty.$$

Proof. Let $\alpha \geq \alpha_0$. Since $\alpha z \notin X$ and X is closed convex, by the separation theorem, there exists $p \in \mathbb{R}^N \setminus \{0\}$ such that

$$p \cdot (\alpha z) < \inf_{x \in X} p \cdot x \leq 0.$$

The last inequality follows from $0 \in X$. Thus, $p \cdot (\alpha z) \to -\infty$ as $\alpha \to \infty$. Let $A = \{w \in \mathbb{R}^N \mid p \cdot w = \inf_{x \in X} p \cdot x\}$. Then,

$$\operatorname{dist}(\alpha z, X) \geq \operatorname{dist}(\alpha z, A) = \frac{|\inf_{x \in X} p \cdot x - p \cdot (\alpha z)|}{\|p\|} \to \infty \quad \text{as} \quad \alpha \to \infty.$$

□

Suppose now that $X - Y$ contains a straight line. Then, there exist $z^0 \in \mathbb{R}^N \setminus \{0\}$ and $z^1 \in \mathbb{R}^N$ such that

$$\{z^1\} + \{\alpha z^0 \mid \alpha \in \mathbb{R}\} \subseteq X - Y.$$

Then, $\{\alpha z^0 \mid \alpha \in \mathbb{R}\} \subseteq X - Y - \{z^1\}$. Since X contains no straight line, we may assume that there exists $\alpha_0 > 0$ such that for every $\alpha \geq \alpha_0$, $\alpha z^0 \notin X$ holds. (In the case where for some $\bar{\alpha} \in \mathbb{R}$, $\alpha \leq \bar{\alpha}$ implies $\alpha z^0 \notin X$, by replacing z^0 by $-z^0$, we have the above property.)

Let $(\alpha_k)_k$ be a sequence in $\mathbb{R}$ with $\alpha_k \to \infty$ as $k \to \infty$. Then, for every $k \in \mathbb{N}$, there exist $x^k \in X$ and $y^k \in Y$ such that

$$\alpha_k z^0 = x^k - y^k - z^1.$$

Thus,

$$\operatorname{dist}(\alpha_k z^0, X) \leq \|\alpha_k z^0 - x^k\| = \|y^k + z^1\| \leq \|y^k\| + \|z^1\|.$$

Since $\operatorname{dist}(\alpha_k z^0, X) \to \infty$ as $k \to \infty$ by Claim 1, we have $\|y^k\| \to \infty$ as $k \to \infty$. This contradicts that Y is bounded. Hence, $X - Y$ contains no straight line. □

Lemma 3.5. *Let X be a nonempty, closed, convex subset of $\mathbb{R}^N$ containing no straight line with the property $X + \mathbb{R}^N_+ = X$. Let Y be a nonempty, compact, convex subset of $\mathbb{R}^N$. If $X \cap Y = \emptyset$, then there exists a nonempty, open subset U of $\mathbb{R}^N_{++}$ such that, for every $p \in U$,*

$$\inf_{x \in X} p \cdot x > \sup_{y \in Y} p \cdot y.$$

Proof. Note that $X - Y$ is nonempty, closed, and convex. In addition, by Lemma 3.4, $X - Y$ contains no straight line. Since $X \cap Y = \emptyset$, we have $0 \notin X - Y$. By Lemma 3.3, there exists a nonempty, open subset U of $\mathbb{R}^N$ such that, for every $p \in U$,

$$0 < \inf_{z \in X - Y} p \cdot z.$$

Since $X + \mathbb{R}^N_+ = X$, we have $U \subseteq \mathbb{R}^N_+$. Since U is open, we have $U \subseteq \mathbb{R}^N_{++}$.

Let $p \in U$ and $c = \inf_{z \in X - Y} p \cdot z > 0$. Since, for every $x \in X$ and every $y \in Y$, $p \cdot (x - y) \geq c > 0$ holds, we have, for every $y \in Y$,

$$\inf_{x \in X} p \cdot x \geq c + p \cdot y.$$

Thus,

$$\inf_{x \in X} p \cdot x \geq c + \sup_{y \in Y} p \cdot y > \sup_{y \in Y} p \cdot y.$$

This completes the proof of Lemma 3.5. $\square$

We are now ready to prove Theorem 3.1. Suppose, to the contrary, that $\varphi(p) \cap \psi(p) = \emptyset$ for every $p \in \Delta$. Define a correspondence $\eta : \Delta \twoheadrightarrow \Delta^\circ$ by

$$\eta(p) = \left\{ q \in \Delta^\circ \;\middle|\; \inf_{x \in \varphi(p)} q \cdot x > \sup_{y \in \psi(p)} q \cdot y \right\}.$$

By Lemma 3.5, η is nonempty-valued. It is clear that η is convex-valued.

Claim 2. For every $q \in \Delta^\circ$, the set $\{p \in \Delta \,|\, q \in \eta(p)\}$ is open in Δ.

Proof. Let $q \in \Delta^\circ$ and $p^0 \in \Delta$ with $q \in \eta(p^0)$. Then, there exists $c \in \mathbb{R}$ such that

$$\inf_{x \in \varphi(p^0)} q \cdot x > c > \sup_{y \in \psi(p^0)} q \cdot y.$$

Thus,

$$\varphi(p^0) \subseteq \{x \in \mathbb{R}^N \,|\, q \cdot x > c\} \quad \text{and} \quad \psi(p^0) \subseteq \{y \in \mathbb{R}^N \,|\, q \cdot y < c\}.$$

By condition (iii) of Theorem 3.1, there exists $U \subseteq \Delta$ such that U is open in Δ, $p^0 \in U$, and, for every $p \in U$,

$$\varphi(p) \subseteq \{x \in \mathbb{R}^N \,|\, q \cdot x > c\} \quad \text{and} \quad \psi(p) \subseteq \{y \in \mathbb{R}^N \,|\, q \cdot y < c\}.$$

Thus, for every $p \in U$,

$$\inf_{x \in \varphi(p)} q \cdot x \geq c > \max_{y \in \psi(p)} q \cdot y.$$

Therefore, $p^0 \in U \subseteq \{p \in \Delta \,|\, q \in \eta(p)\}$. Hence, the set $\{p \in \Delta \,|\, q \in \eta(p)\}$ is open in Δ. $\square$

By Browder's theorem [7, Theorem 1],[5] correspondence η has a continuous selection, i.e., there exists a continuous function $h : \Delta \to \Delta^\circ$ such that $h(p) \in \eta(p)$ for every $p \in \Delta$. By Brouwer's fixed point theorem [6], there exists $\widehat{p} \in \Delta$ such that

$$\widehat{p} = h(\widehat{p}) \in \Delta^\circ.$$

Since $\widehat{p} = h(\widehat{p}) \in \eta(\widehat{p})$, we have

$$\inf_{x \in \varphi(\widehat{p})} \widehat{p} \cdot x > \sup_{y \in \psi(\widehat{p})} \widehat{p} \cdot y.$$

This contradicts condition (iv) of Theorem 3.1. Therefore, there exists $p^* \in \Delta$ with $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$. This completes the proof of Theorem 3.1.

3.3. **Coincidence theorem II.** We give another coincidence theorem behind Qin's [18, Theorem 1] proof of the nonemptiness of the inner core. In contrast to Theorem 3.1, the following theorem guarantees that two correspondences have a nonempty intersection at a *strictly positive* vector.

Theorem 3.6. *Let* $\varphi : \Delta^\circ \twoheadrightarrow \mathbb{R}^N$ *and* $\psi : \Delta^\circ \twoheadrightarrow \mathbb{R}^N$ *be correspondences satisfying conditions* (i)-(iii) *of Theorem* 3.1 *on* Δ°. *Let* $\sigma : \Delta^\circ \to \Delta^\circ$ *be a continuous function such that*

(s-1) $\operatorname{cl}(\sigma(\Delta^\circ)) \subseteq \Delta^\circ$ *and*

[5] From the proof of Theorem 1 of Browder, it follows that there exists a continuous selection.

(s-2) *for every* $p \in \Delta^\circ$, *there exist* $x \in \varphi(\sigma(p))$ *and* $y \in \psi(\sigma(p))$ *such that* $p \cdot x \leq p \cdot y$.

Then, there exists $p^* \in \Delta^\circ$ *with* $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$.

Proof. Suppose, to the contrary, that $\varphi(p) \cap \psi(p) = \emptyset$ for every $p \in \Delta^\circ$. Define a correspondence $\eta : \Delta^\circ \twoheadrightarrow \Delta^\circ$ by

$$\eta(p) = \left\{ q \in \Delta^\circ \;\middle|\; \inf_{x \in \varphi(p)} q \cdot x > \sup_{y \in \psi(p)} q \cdot y \right\}.$$

Then, by the same argument as the proof of Theorem 3.1, we can show that η satisfies all the requirements for Browder's theorem [7, Theorem 1].[6] Then, there exists a continuous function $h : \Delta^\circ \to \Delta^\circ$ such that $h(p) \in \eta(p)$ for every $p \in \Delta^\circ$.

Let K be the closed convex hull of $\sigma(\Delta^\circ)$. Then, by condition (s-1), we have $K \subseteq \Delta^\circ$. Since the composite function $\sigma \circ h : K \to K$ is continuous, by Brouwer's fixed point theorem, there exists $\widehat{p} \in K$ with $\widehat{p} = \sigma \circ h(\widehat{p})$. Since $h(\widehat{p}) \in \eta(\widehat{p})$, we have

$$\inf_{x \in \varphi(\widehat{p})} h(\widehat{p}) \cdot x > \sup_{y \in \psi(\widehat{p})} h(\widehat{p}) \cdot y.$$

Since $\widehat{p} = \sigma \circ h(\widehat{p})$, by condition (s-2), there exist $x \in \varphi(\widehat{p})$ and $y \in \psi(\widehat{p})$ such that $h(\widehat{p}) \cdot x \leq h(\widehat{p}) \cdot y$. This is a contradiction. Therefore, there exists $p^* \in \Delta^\circ$ with $\varphi(p^*) \cap \psi(p^*) \neq \emptyset$. $\square$

4. Nonemptiness of the inner core

The cardinal balancedness is not sufficient for the nonemptiness of the inner core. To guarantee the nonemptiness of the inner core, we assume that every normal vector to $V(N)$ at every individually rational and efficient payoff vector is strictly positive. In Section 4.1, we give a characterization of the efficient surface of $V(N)$ with only strictly positive normal vectors analyzed by Inoue [12]. In Section 4.2, we prove that Inoue's [12] and Qin's [18] theorems on the nonemptiness of the inner core follow from our coincidence theorems.

4.1. Characterization of efficient surface with strictly positive normal vectors.

Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game such that $V(N)$ is convex and the set $\{x \in V(N) \,|\, x \geq b\}$ is nonempty. Define

$$\begin{aligned}&\mathrm{Eff}(V(N), b) = \\ &\left\{ x \in V(N) \;\middle|\; x \geq b, \text{ there exists no } x' \in V(N) \text{ with } x' \geq x \text{ and } x' \neq x \right\}\end{aligned}$$

and

$$\mathrm{Eff_w}(V(N), b) = \left\{ x \in V(N) \;\middle|\; x \geq b, \text{ there exists no } x' \in V(N) \text{ with } x' \gg x \right\}.$$

Thus, $\mathrm{Eff}(V(N), b)$ (resp. $\mathrm{Eff_w}(V(N), b)$) is the set of individually rational efficient payoff vectors (resp. individually rational and weakly efficient payoff vectors). The set $\mathrm{Eff_w}(V(N), b)$ can be characterized by vectors normal to $V(N)$.

[6] Note that the compactness of the domain of a correspondence in Browder's theorem is dispensable, because any subset of a Euclidean space is paracompact.

Lemma 4.1.

$$\mathrm{Eff_w}(V(N), b) = \{x \in V(N) \mid x \geq b,\ \text{there exists } \lambda \in \Delta \text{ with } \lambda \cdot x = v_\lambda(N)\}.$$

For the proof of this lemma, see Inoue [12, Lemma 1].

One of the following conditions of Proposition 4.2 is assumed in Inoue's [12] theorem on the nonemptiness of the inner core.

Proposition 4.2. *The following two conditions are equivalent.*

(1) *There exists a nonempty, closed, convex subset $V'(N)$ of $\mathbb{R}^N$ such that $V(N) \subseteq V'(N)$, $V'(N)$ is generated by a compact set C'_N, $\{x \in V(N) \mid x \geq b\} = \{x \in V'(N) \mid x \geq b\}$, and*

$$\left[x \in \mathrm{Eff}(V'(N), b),\ \lambda \in \mathbb{R}^N \setminus \{0\},\ \textit{and}\ \lambda \cdot x = \max_{y \in V'(N)} \lambda \cdot y\right] \textit{ implies } \lambda \gg 0.$$

(2) *There exists a compact subset K of Δ° such that, for every $x \in \mathrm{Eff_w}(V(N), b)$, there exists $\lambda \in K$ with $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$.*

For the proof of this proposition, see Inoue [12, Proposition 3]. Clearly, condition (1) is weaker than the following condition (3).

(3) Let $x \in \mathrm{Eff}(V(N), b)$ and $\lambda \in \mathbb{R}^N \setminus \{0\}$ be such that $\lambda \cdot x = \max_{y \in V(N)} \lambda \cdot y$. Then, $\lambda \gg 0$. Namely, for every $x \in \mathrm{Eff}(V(N), b)$, the normal cone to $V(N)$ at x is a subset of $\mathbb{R}^N_{++} \cup \{0\}$.[7]

Condition (1) means that condition (3) is met for an extension $V'(N)$ of $V(N)$. Since the extension $V'(N)$ has the same set of individually rational payoff vectors as before, this extension does not make the inner core larger.

The following lemma implies that, in conditions (1) and (3), $\mathrm{Eff}(V'(N), b)$ and $\mathrm{Eff}(V(N), b)$ can be replaced by $\mathrm{Eff_w}(V'(N), b)$ and $\mathrm{Eff_w}(V(N), b)$, respectively.

Lemma 4.3. *Under condition* (3)*,* $\mathrm{Eff}(V(N), b) = \mathrm{Eff_w}(V(N), b)$ *holds.*

For the proof of this lemma, see Inoue [12, Lemma 2]. Since the set $\mathrm{Eff_w}(V(N), b)$ is always closed, condition (3) implies that the set $\mathrm{Eff}(V(N), b)$ also is closed.[8]

4.2. **Nonemptiness of the inner core.** By applying our first coincidence theorem, we provide another proof to the following theorem due to Inoue [12, Theorem 2]. Inoue [12] proved the theorem by using Qin's [18] theorem (Theorem 4.5 in the present paper).[9]

Theorem 4.4. *Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. If V is cardinally balanced and if V satisfies condition* (1) *or* (2) *of Proposition* 4.2*, then the inner core $IC(V)$ of V is nonempty.*

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[7] For a nonempty subset A of $\mathbb{R}^N$ and $x \in A$, the *normal cone* to A at x is the set of all vectors normal to A at x.

[8] In general, the set of efficient payoff vectors need not be closed. For such an example, see Arrow et al. [1].

[9] Also, Aubin [2] proved Theorem 4.4 under a slightly stronger assumption. For the proof of Aubin, see Inoue [13, Appendix].

Proof. Since conditions (1) and (2) of Proposition 4.2 are equivalent, we may assume that condition (1) holds. Let V' be an NTU game such that $V'(N)$ is the one in condition (1) and $V'(S) = V(S)$ for every $S \subsetneq N$. Then, V' is compactly generated and $V'(N)$ is convex. Since V is cardinally balanced and $V(N) \subseteq V'(N)$, V' is cardinally balanced. Thus, the new NTU game V' satisfies the same conditions as the original NTU game V. In addition, V' satisfies condition (3), a stronger condition than condition (1). Since $IC(V') \subseteq IC(V)$ holds, it suffices to prove that $IC(V') \neq \emptyset$. Therefore, we may assume that V satisfies condition (3) from the outset and it suffices to prove that $IC(V) \neq \emptyset$.

Define correspondences $F : \Delta \twoheadrightarrow \mathbb{R}^N$ and $G : \Delta \twoheadrightarrow \mathbb{R}^N$ as in Section 2, i.e.,

$$F(\lambda) = \left\{x \in \mathbb{R}^N \;\middle|\; x \geq b \text{ and } \lambda^S \cdot x \geq v_\lambda(S) \text{ for every } S \in \mathcal{N} \setminus \{N\}\right\}$$

and

$$G(\lambda) = \left\{ y \in C_N \;\middle|\; \lambda \cdot y = \max_{z \in V(N)} \lambda \cdot z \right\}.$$

Then, by Propositions 2.2, 2.4, and 2.5, F and G satisfy all the conditions of Theorem 3.1. Thus, there exists $\lambda^* \in \Delta$ with $F(\lambda^*) \cap G(\lambda^*) \neq \emptyset$.

Let $x^* \in F(\lambda^*) \cap G(\lambda^*)$. Since $\lambda^* \in \Delta$, $x^* \geq b$, $x^* \in C_N \subseteq V(N)$, and $\lambda^* \cdot x^* = \max_{z \in V(N)} \lambda^* \cdot z$, by Lemma 4.1, we have $x^* \in \mathrm{Eff}_{\mathrm{w}}(V(N), b)$. By Lemma 4.3, condition (3) implies that

$$\mathrm{Eff}_{\mathrm{w}}(V(N), b) = \mathrm{Eff}(V(N), b).$$

Therefore, $x^* \in \mathrm{Eff}(V(N), b)$. Again, by condition (3), we have $\lambda^* \gg 0$. Therefore, $x^* \in IC(V)$. □

The next theorem is due to Qin [18, Theorem 1].

Theorem 4.5 (Qin)**.** *Let $V : \mathcal{N} \twoheadrightarrow \mathbb{R}^N$ be a compactly generated NTU game with $V(N)$ convex. Correspondences F and G are defined as in Section* 2*. If there exists a continuous function $\sigma : \Delta^\circ \to \Delta^\circ$ such that*

(s-1) $\mathrm{cl}(\sigma(\Delta^\circ)) \subseteq \Delta^\circ$ *and*

(s-2) *for every $\lambda \in \Delta^\circ$, there exist $x \in F(\sigma(\lambda))$ and $y \in G(\sigma(\lambda))$ such that $\lambda \cdot x \leq \lambda \cdot y$,*

then the inner core $IC(V)$ of V is nonempty.

This can be shown by a simple application of Theorem 3.6. In the original theorem by Qin [18, Theorem 1], σ is specified as follows. For $m \in \mathbb{N}$ with $m \geq n$, let $\Delta_{1/m} = \{\lambda \in \Delta \,|\, \lambda_i \geq 1/m \text{ for every } i \in N\}$, and define continuous functions $p^m : \Delta^\circ \to [1/m, 1]^N$ and $\sigma^m : \Delta^\circ \to \Delta_{1/(m+n-1)}$ by, for every $j \in N$,

$$p_j^m(\lambda) = \begin{cases} \lambda_j & \text{if } \lambda_j \geq 1/m, \\ 1/m & \text{if } \lambda_j < 1/m, \end{cases}$$

and

$$\sigma_j^m(\lambda) = \frac{p_j^m(\lambda)}{p^m(\lambda) \cdot e^N}.$$

Then, for every $m \geq n$, function σ^m satisfies condition (s-1). Since $\sigma^m(\lambda) = \lambda$ on $\Delta_{1/m}$, condition (s-2) can be decomposed into the following (s-2.i) and (s-2.ii).

(s-2.i) for every $\lambda \in \Delta_{1/m}$, there exist $x \in F(\lambda)$ and $y \in G(\lambda)$ such that $\lambda \cdot x \leq \lambda \cdot y$.

(s-2.ii) for every $\lambda \in \Delta^\circ \setminus \Delta_{1/m}$, there exist $x \in F(\sigma^m(\lambda))$ and $y \in G(\sigma^m(\lambda))$ such that $\lambda \cdot x \leq \lambda \cdot y$.

By the duality theorem of linear programming, conditions (s-2.i) and (s-2.ii) are equivalent to conditions (i) and (ii) of Qin [18, Theorem 1], respectively.

Qin [18, Corollary 1] proved that if a compactly generated NTU game V with $V(N)$ convex is *cardinally balanced with slack*, i.e., for every $\gamma \in \Gamma$ with $\gamma_N = 0$, the set $\sum_{S\in\mathcal{N}} \gamma_S V(S)$ is in the interior of $V(N)$, then there exists $m \geq n$ such that the function σ^m above satisfies conditions (s-2.i) and (s-2.ii) and, therefore, its inner core is nonempty.

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